HAUSDORFF AND PACKING DIMENSION OF FIBERS AND GRAPHS OF PREVALENT CONTINUOUS MAPS

RICHÁRD BALKA, UDAYAN B. DARJI, AND MÁRTON ELEKES

Abstract. The notions of shyness and prevalence generalize the property of being zero and full Haar measure to arbitrary (not necessarily locally compact) Polish groups. The main goal of the paper is to answer the following question: What can we say about the Hausdorff and packing dimension of the fibers of prevalent continuous maps?

Let K be an uncountable compact metric space. We prove that the prevalent $f \in C(K, \mathbb{R}^d)$ has many fibers with almost maximal Hausdorff dimension. This generalizes a theorem of Dougherty and yields that the prevalent $f \in C(K, \mathbb{R}^d)$ has graph of maximal Hausdorff dimension, generalizing a result of Bayart and Heurteaux. We obtain similar results for the packing dimension.

We show that for the prevalent $f \in C([0,1]^m, \mathbb{R}^d)$ the set of $y \in f([0,1]^m)$ for which $\dim_H f^{-1}(y) = m$ contains a dense open set having full measure with respect to the occupation measure $\lambda^m \circ f^{-1}$, where $\dim_H$ and λ^m denote the Hausdorff dimension and the m-dimensional Lebesgue measure, respectively. We also prove an analogous result when $[0,1]^m$ is replaced by any self-similar set satisfying the open set condition.

We cannot replace the occupation measure with Lebesgue measure in the above statement: We show that the functions $f \in C[0,1]$ for which positively many level sets are singletons form a non-shy set in $C[0,1]$. In order to do so, we generalize a theorem of Antunović, Burdzy, Peres and Ruscher.

We also prove sharper results in which large Hausdorff dimension is replaced by positive measure with respect to generalized Hausdorff measures, which answers a problem of Fraser and Hyde.

Contents

2010 *Mathematics Subject Classification.* Primary: 28A78, 28C10, 46E15, 60B05; Secondary: 54E52.

Key words and phrases. Haar null, shy, prevalent, Hausdorff dimension, packing dimension, level set, fiber, graph, continuous map, Hölder map, Lipschitz map, ultrametric space, Baire category, generic, occupation measure, Brownian motion.

The first author was supported by the Hungarian Scientific Research Fund grants no. 72655 and 104178. The third author was supported by the Hungarian Scientific Research Fund grants no. 72655, 83726 and 104178.

1. Introduction

Let G be a *Polish group*, that is, a separable topological group which is endowed with a compatible complete metric. If G is locally compact then there exists a *Haar measure* on G, that is, a left translation invariant regular Borel measure which is finite on compact sets and positive on non-empty open sets. The concept of Haar measure does not extend to groups that are not locally compact, but the idea of Haar measure zero sets does. The following definition is due to Christensen [8] and was rediscovered by Hunt, Sauer and York [20].

Definition 1.1. For an abelian Polish group G a set $A \subset G$ is *shy* or *Haar null* if there exists a Borel set $B \subset G$ and a Borel probability measure μ on G such that $A \subset B$ and $\mu(B + x) = 0$ for all $x \in G$. The complement of a shy set is called a *prevalent* set.

Christensen proved in [8] that shy sets form a σ-ideal and in locally compact abelian Polish groups Haar measure zero sets and shy sets coincide. Later Topsøe and Hoffmann-Jørgensen [39] and Mycielski [31] extended the definition to all Polish groups, but here we consider only the abelian case.

Notation 1.2. The Hausdorff and packing dimension of a metric space X is denoted by $\dim_H X$ and $\dim_P X$. We use the convention $\dim_H \emptyset = \dim_P \emptyset = -1$. For a compact metric space K let us denote by $C(K, \mathbb{R}^d)$ the set of continuous functions from K to $\mathbb{R}^d$ endowed with the supremum metric. Then $C(K, \mathbb{R}^d)$ is an abelian Polish group. We simply write $C[0,1] = C([0,1], \mathbb{R})$.

Over the last 25 years there has been a large interest in studying dimensions of various sets related to 'typical' continuous functions. If typical means generic in the sense of Baire category, then the following theorem about level sets is folklore.

Theorem 1.3. *For the generic $f \in C[0,1]$ for all $y \in f([0,1])$*

$$\dim_H f^{-1}(y) = 0.$$

Mauldin and Williams [27] proved the next theorem.

Theorem 1.4 (Mauldin-Williams)**.** *For the generic $f \in C[0,1]$*

$$\dim_H \operatorname{graph}(f) = 1.$$

As for the higher dimensional analogues, the next result was obtained by Kirchheim [25].

Theorem 1.5 (Kirchheim)**.** *If $m, d \in \mathbb{N}^+$ and $m \geq d$ then for the generic $f \in C([0,1]^m, \mathbb{R}^d)$ for all $y \in \operatorname{int} f([0,1]^m)$*

$$\dim_H f^{-1}(y) = m - d.$$

Now let K be an arbitrary compact metric space. In order to determine the Hausdorff dimension of the level sets of the generic $f \in C(K, \mathbb{R})$, we need a new notion of dimension, the *topological Hausdorff dimension*, see [3] and [4]. More generally, the right concept to describe the Hausdorff dimension of the fibers of the generic $f \in C(K, \mathbb{R}^d)$ is the so-called d^{th} *inductive topological Hausdorff dimension*, see [2].

The case of graphs is much simpler, the strategy of Mauldin and Williams actually easily yields the following general result.

Theorem 1.6. *Let K be an uncountable compact metric space and $d \in \mathbb{N}^+$. Then for the generic $f \in C(K, \mathbb{R}^d)$*

$$\dim_H \operatorname{graph}(f) = \dim_H K.$$

These theorems indicate that the generic $f \in C[0,1]$ behaves quite regularly in a sense, e.g. its level sets and graph have minimal Hausdorff dimension, similarly to the case of smooth functions. It is quite natural to expect more chaotic behavior from typical continuous functions, which is already a reason to replace genericity with another notion. Moreover, since these problems are measure theoretic in nature, it is natural to replace Baire category by the more measure theoretic concept of prevalence.

In contrast to Theorem 1.3, we show that the prevalent $f \in C[0,1]$ has fibers of maximal Hausdorff dimension. Let us denote by λ the one-dimensional Lebesgue measure. (Note that $\dim_H X \le \dim_P X$ for every metric space X, so the packing dimension analogue of the following statement would be weaker.)

Corollary 4.6. *For the prevalent $f \in C[0,1]$ there is an open set $U_f \subset \mathbb{R}$ such that $\lambda(f^{-1}(U_f)) = 1$ (hence U_f is dense in $f([0,1])$) and for all $y \in U_f$*

$$\dim_H f^{-1}(y) = 1.$$

In general, prevalent continuous maps have many fibers of cardinality continuum, for the following theorem see [9, Thm. 11] and the remark following its proof.

Theorem 1.7 (Dougherty)**.** *Let K be an uncountable compact metric space*[1] *and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\operatorname{int} f(K) \neq \emptyset.$$

Moreover, there is a non-empty open set $U_f \subset \mathbb{R}^d$ such that for all $y \in U_f$

$$\# f^{-1}(y) = 2^{\aleph_0}.$$

The next theorem widely generalizes Corollary 4.6 and Theorem 1.7 in Euclidean spaces. We can find many fibers not only of cardinality continuum, but also with almost maximal Hausdorff and packing dimension. (For the definition of dimensions of measures and their properties see the Preliminaries section.)

Theorem 4.13 (Main Theorem, simplified version)**.** *Let $m, d \in \mathbb{N}^+$. Let $K \subset \mathbb{R}^m$ be compact and let μ be a continuous, finite Borel measure on K. Then for the*

[1]Dougherty proved this result only if K is the triadic Cantor set. As every uncountable compact metric space contains a subset homeomorphic to the triadic Cantor set by [23, Cor. 6.5], Corollary 3.11 yields this more general result.

prevalent $f \in C(K, \mathbb{R}^d)$ there is an open set $U_f \subset \mathbb{R}^d$ such that $\mu(f^{-1}(U_f)) = \mu(K)$ and for all $y \in U_f$

$$\dim_H f^{-1}(y) \geq \dim_H \mu \quad \text{and} \quad \dim_P f^{-1}(y) \geq \dim_P \mu.$$

After some technical lemmas in Section 3, we prove the above theorem for $K \subset \mathbb{R}$ and $\mu = \lambda$ in Subsection 4.1, which is the most subtle proof of the paper. In Subsection 4.2 we prove this result for ultrametric spaces using ideas from [24]. In Subsection 4.3 we finish the proof of the Main Theorem, we trace back the case of general compact spaces to ultrametric ones by using a theorem of Zindulka [41]. Let us denote by λ^m the m-dimensional Lebesgue measure.

Corollary 4.14. *Let $m, d \in \mathbb{N}^+$. Then for the prevalent $f \in C([0,1]^m, \mathbb{R}^d)$ there is an open set $U_f \subset \mathbb{R}^d$ such that $\lambda^m(f^{-1}(U_f)) = 1$ (hence U_f is dense in $f([0,1]^m)$) and for all $y \in U_f$*

$$\dim_H f^{-1}(y) = m.$$

Corollary 4.15 (simplified version)**.** *Let $m, d \in \mathbb{N}^+$ and let $K \subset \mathbb{R}^m$ be an uncountable compact set. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ for all $s < \dim_P K$ there is a non-empty open set $U_{f,s} \subset \mathbb{R}^d$ such that for all $y \in U_{f,s}$*

$$\dim_P f^{-1}(y) \geq s.$$

Corollary 4.16 (simplified version)**.** *Let $m, d \in \mathbb{N}^+$ and let $K \subset \mathbb{R}^m$ be compact. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\sup\{\dim_P f^{-1}(y) : y \in \mathbb{R}^d\} = \dim_P K.$$

In the case of Hausdorff dimension we prove more general versions of the above two corollaries based on a deep theorem of Mendel and Naor [29].

Theorem 4.18. *Let K be an uncountable compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ for all $s < \dim_H K$ there is a non-empty open set $U_{f,s} \subset \mathbb{R}^d$ such that for all $y \in U_{f,s}$*

$$\dim_H f^{-1}(y) \geq s.$$

Corollary 4.19. *Let K be a compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}^d\} = \dim_H K.$$

The supremum is not necessarily attained in Corollaries 4.16 and 4.19.

Theorem 4.22. *There is a compact set $K \subset \mathbb{R}$ such that $\dim_H K = \dim_P K = 1$ and*

$$\{f \in C(K, \mathbb{R}) : \dim_H f^{-1}(y) \leq \dim_P f^{-1}(y) < 1 \text{ for all } y \in \mathbb{R}\}$$

is non-shy in $C(K, \mathbb{R})$.

If K is 'large in its dimension' then we can say more. The Main Theorem implies the next corollary, which we now formalize only for the Hausdorff dimension.

Corollary 4.23 (simplified version)**.** *Let $m, d \in \mathbb{R}^d$ and let $K \subset \mathbb{R}^m$ be compact. Let μ be a continuous, finite Borel measure on K such that $\dim_H \mu = \dim_H K$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists an open set $U_f \subset \mathbb{R}^d$ such that $\mu(f^{-1}(U_f)) = \mu(K)$ and for all $y \in U_f$*

$$\dim_H f^{-1}(y) = \dim_H K.$$

For sufficiently homogeneous spaces we can generalize the Main Theorem. Let us denote by $\mathcal{H}^s$ and $\mathcal{P}^s$ the s-dimensional Hausdorff and packing measure, respectively. For the definitions of packing measure, self-similar set, and open set condition see [13].

Corollary 4.27. *Let $m, d \in \mathbb{N}^+$ and let $K \subset \mathbb{R}^m$ be a self-similar set satisfying the open set condition. It is well-known that $\dim_H K = \dim_P K = s$ and $0 < \mathcal{H}^s(K), \mathcal{P}^s(K) < \infty$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists an open set $U_f \subset \mathbb{R}^d$ such that $\mathcal{H}^s(f^{-1}(U_f)) = \mathcal{H}^s(K)$ (hence U_f is dense in $f(K)$) and*

$$\dim_H f^{-1}(y) = s \text{ for all } y \in U_f.$$

Similarly, for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists an open set $V_f \subset \mathbb{R}^d$ such that $\mathcal{P}^s(f^{-1}(V_f)) = \mathcal{P}^s(K)$ (hence V_f is dense in $f(K)$) and

$$\dim_P f^{-1}(y) = s \text{ for all } y \in V_f.$$

For other results in sufficiently homogeneous spaces see Subsection 4.5, where we describe the compact metric spaces K for which $\dim_H f^{-1}(y) = \dim_H K$ for the prevalent $f \in C(K, \mathbb{R}^d)$ and the generic $y \in f(K)$. The characterization is independent of d.

Corollary 4.6 yields for the prevalent $f \in C[0,1]$ that $\{y : \dim_H f^{-1}(y) = 1\}$ is co-meager in $f([0,1])$ with full $\lambda \circ f^{-1}$ measure. As the main result of Section 5, we show that this does not remain true if we replace the occupation measure $\lambda \circ f^{-1}$ by the Lebesgue measure λ on $f([0,1])$. Let $\exists^\lambda$ denote positively many with respect to λ.

Theorem 5.5. *The set*

$$\{f \in C[0,1] : \exists^\lambda y \in \mathbb{R} \text{ such that } f^{-1}(y) \text{ is a singleton}\}$$

is non-shy in $C[0,1]$.

Let $\mathcal{Z}(f) = \{x \in [0,1] : f(x) = 0\}$, the next theorem is [1, Prop. 3.3].

Theorem 1.8 (Antunović-Burdzy-Peres-Ruscher)**.** *Let μ be the Wiener measure on $C[0,1]$. Then there exists a function $g \in C[0,1]$ such that*

$$\mu(\{f \in C[0,1] : \mathcal{Z}(f-g) \setminus \{0\} \text{ is a singleton}\}) > 0.^2$$

The next theorem generalizes Theorem 1.8 and easily implies Theorem 5.5.

Theorem 5.2. *Let μ be a Borel probability measure on $C[0,1]$. Then there exists a function $g \in C[0,1]$ such that*

$$\mu(\{f \in C[0,1] : \mathcal{Z}(f-g) \text{ is a singleton}\}) > 0.$$

Theorem 5.2 also yields the following corollary.

Corollary 5.3. *The set $\{f \in C[0,1] : \mathcal{Z}(f)$ is a singleton$\}$ is non-shy.*

As a complement to Theorem 5.5, we prove that all non-extremal level sets can be large.

Theorem 5.7. *The set*

$$\{f \in C[0,1] : \dim_H f^{-1}(y) = 1 \text{ for all } y \in (\min f, \max f)\}$$

is non-shy in $C[0,1]$.

[2]It is easy to see that if $g(x) = x^{1/3}$ then $\mathcal{Z}(f-g) = \{0\}$ for positively many f with respect to the Wiener measure. In order to avoid this degenerate case, we remove the origin.

Recently, describing the various fractal dimensions of graphs of prevalent continuous functions has attracted notable attention, this is the topic of Section 6.

First McClure [28] proved that the packing dimension, and thus the upper box dimension of the graph of the prevalent $f \in C[0,1]$ is 2. The analogous result for the lower box dimension was proved in [14], and [37], independently.

Fraser and Hyde [15] generalized the above results by showing that the prevalent $f \in C[0,1]$ has graph of Hausdorff dimension 2. In contrast to Theorem 1.4 this means that the prevalent value of $\dim_H \operatorname{graph}(f)$ is as large as possible.

Theorem 1.9 (Fraser-Hyde)**.** *For the prevalent $f \in C[0,1]$*

$$\dim_H \operatorname{graph}(f) = 2.$$

The next result was proved by Bayart and Heurteaux, see [6, Thm. 3].

Theorem 1.10 (Bayart-Heurteaux)**.** *If $K \subset \mathbb{R}^m$ is compact with $\dim_H K > 0$ then for the prevalent $f \in C(K, \mathbb{R})$*

$$\dim_H \operatorname{graph}(f) = \dim_H K + 1.$$

The proof of Theorem 1.10 is based on potential theoretic methods, they give a lower estimate for the Hausdorff dimension of $\operatorname{graph}(X + f)$, where $X \colon K \to \mathbb{R}$ is a so-called additive fractional Brownian motion and $f \in C(K, \mathbb{R})$ is a continuous drift. Note that if $X \colon K \to \mathbb{R}^d$ is a fractional Brownian motion restricted to some $K \subset [0,1]$ and $f \in C(K, \mathbb{R}^d)$ then Peres and Sousi [34] determined the almost sure Hausdorff dimension of $\operatorname{graph}(X + f)$ in terms of f and the Hurst index of X. It is not difficult to extend the proof of [6, Thm. 3] to vector valued functions, and Theorem 1.7 handles the case $\dim_H K = 0$. These yield the following theorem.

Theorem 1.11. *Let $m, d \in \mathbb{N}^+$ and let $K \subset \mathbb{R}^m$ be an uncountable compact set. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\dim_H \operatorname{graph}(f) = \dim_H K + d.$$

We will show that Theorem 4.18 also easily implies the above theorem. Moreover, the condition $K \subset \mathbb{R}^m$ is superfluous.

Theorem 6.5. *Let K be an uncountable compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\dim_H \operatorname{graph}(f) = \dim_H K + d.$$

Much less was known about the prevalent value of the packing dimension of the graphs. Corollary 4.15 implies the packing dimension analogue of Theorem 1.11.

Theorem 6.6 (simplified version)**.** *Let $m, d \in \mathbb{N}^+$ and let $K \subset \mathbb{R}^m$ be an uncountable compact set. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\dim_P \operatorname{graph}(f) = \dim_P K + d.$$

In Section 7 we indicate how to obtain stronger forms of the main results by replacing large dimension by positive measure with respect to generalized Hausdorff measures. Finally, in Section 8 we pose some open problems.

2. Preliminaries

Let (X,d) be a metric space. For $A,B\subseteq X$ let us define $\operatorname{dist}(A,B)=\inf\{d(x,y): x\in A,\ y\in B\}$. Let $B(x,r)$ and $U(x,r)$ be the closed and open ball of radius r centered at x, respectively. Set $B(A,r)=\{x\in X:\operatorname{dist}(\{x\},A)\le r\}$. We denote by $\operatorname{cl}A$, $\operatorname{int}A$ and ∂A the closure, interior and boundary of A, respectively. The diameter of a set A is denoted by $\operatorname{diam}A$. We use the conventions $\operatorname{diam}\emptyset=0$ and $\inf\emptyset=\infty$. For two metric spaces (X,d_X) and (Y,d_Y) a map $f\colon X\to Y$ is *s-Hölder* for an $s>0$ if there is a constant $c\in\mathbb{R}$ such that $d_Y(f(x_1),f(x_2))\le c(d_X(x_1,x_2))^s$ for all $x_1,x_2\in X$. A map $f\colon X\to Y$ is *Lipschitz* if it is 1-Hölder, and the smallest c in the definition is called the Lipschitz constant of f and is denoted by $\operatorname{Lip}(f)$. We say that f is *bi-Lipschitz* if it is one-to-one and both f and f^{-1} are Lipschitz.

Let $s\ge 0$. The *s-dimensional Hausdorff measure* of a metric space X is

$$\mathcal{H}^s(X)=\lim_{\delta\to 0+}\mathcal{H}^s_\delta(X),\text{ where}$$
$$\mathcal{H}^s_\delta(X)=\inf\left\{\sum_{i=1}^\infty(\operatorname{diam}X_i)^s : X\subset\bigcup_{i=1}^\infty X_i,\ \forall i\operatorname{diam}X_i\le\delta\right\}.$$

Let $\dim_H\emptyset=-1$. The *Hausdorff dimension* of a non-empty X is defined as

$$\dim_H X=\inf\{s\ge 0:\mathcal{H}^s(X)=0\},$$

for more information on these concepts see [11] or [26]. Now we define the packing dimension. If X is non-empty and totally bounded then for all $\delta>0$ let $N_\delta(X)$ be the smallest number of closed balls of radius δ whose union cover X. Then the *upper box dimension* of X is defined as

$$\overline{\dim}_B(X)=\limsup_{\delta\to 0+}\frac{\log N_\delta(X)}{\log(1/\delta)}.$$

Let $\overline{\dim}_B\emptyset=-1$ and let $\overline{\dim}_B(X)=\infty$ if X is not totally bounded. The *packing dimension* of X is defined as

$$\dim_P X=\inf\left\{\sup_i\overline{\dim}_B X_i : X\subset\bigcup_{i=1}^\infty X_i\right\}.$$

Then clearly $\dim_P\emptyset=-1$. Since we do not need the packing measure, it was more convenient for us to define the packing dimension as the modified upper box dimension. Note that it is the same as the dimension defined using the so-called radius-based packing measure, see [30, Section 10.2] for the definition and the equivalence. The following fact is an easy consequence of the definitions.

Fact 2.1. *If X,Y are non-empty metric spaces and $f\colon X\to Y$ is s-Hölder then*

$$\dim_H f(X)\le\frac{\dim_H X}{s}\quad\text{and}\quad\dim_P f(X)\le\frac{\dim_P X}{s}.$$

Let K be a compact metric space an let μ be a finite Borel measure on K, where we always assume that $\mu(K)>0$. We define

$$\dim_H\mu=\inf\{\dim_H B : B\subset K\text{ is Borel and }\mu(B)>0\},$$
$$\dim_P\mu=\inf\{\dim_P B : B\subset K\text{ is Borel and }\mu(B)>0\}.$$

For the following theorem see [13, Prop. 10.2] when K is a subset of a Euclidean space. In fact, the proof of [13, Prop. 2.2] with the covering theorem [26, Thm. 2.1] works in an arbitrary compact metric space.

Theorem 2.2. *If μ is a finite Borel measure on a compact metric space K then*

$$\dim_H \mu = \sup\left\{ s \geq 0 : \limsup_{r\to 0+} \frac{\mu(B(x,r))}{r^s} < \infty \text{ for } \mu\text{-a.e. } x \in K \right\},$$

$$\dim_P \mu = \sup\left\{ s \geq 0 : \liminf_{r\to 0+} \frac{\mu(B(x,r))}{r^s} < \infty \text{ for } \mu\text{-a.e. } x \in K \right\}.$$

The next theorem states that we can approximate the dimension of a compact metric space K by the dimension of measures supported within it. For the proof see the theorem above with Frostman's lemma [26, Thm. 8.17] or [21] in the case of the Hausdorff or the packing dimension, respectively. Moreover, we may assume that the measures are Hausdorff or packing measures restricted to a compact subset of K, see [19] or [22]. See also [13, Prop. 10.1] for the Euclidean case.

Theorem 2.3. *If K is a compact metric space then*

$$\dim_H K = \sup\{\dim_H \mu : \mu \text{ is a finite Borel measure on } K\},$$

$$\dim_P K = \sup\{\dim_P \mu : \mu \text{ is a finite Borel measure on } K\}.$$

If K is uncountable then we may assume that the above measures μ are continuous.

The metric space (X,d) is called *ultrametric* if the triangle inequality is replaced with the stronger inequality $d(x,y) \leq \max\{d(x,z),d(y,z)\}$ for all $x,y,z \in X$.

Fact 2.4. *If X is ultrametric then for all $x,y \in X$ and $r > 0$ either $B(x,r) \cap B(y,r) = \emptyset$ or $B(x,r) = B(y,r)$.*

Let X be a *complete* metric space. A set is *somewhere dense* if it is dense in a non-empty open set, and otherwise it is called *nowhere dense*. We say that $A \subset X$ is *meager* if it is a countable union of nowhere dense sets, and a set is called *co-meager* if its complement is meager. By Baire's category theorem a set is co-meager iff it contains a dense G_δ set. We say that the *generic* element $x \in X$ has property $\mathcal{P}$ if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. Our main example will be $X = C(K,\mathbb{R}^d)$. See e.g. [23] for more on these concepts.

A metric space X is a *Polish space* if it is complete and separable. We say that $A \subset X$ *analytic* if it is a continuous image of a Polish space, and *co-analytic* if its complement is analytic. A Borel subset of a Polish space is analytic [23, 13.7]. Continuous images, countable unions and countable intersections of analytic sets are analytic [23, 14.4]. For more on these concepts see [23].

Let μ be a σ-finite Borel measure on a Polish space X. Then μ can be extended to the σ-algebra of the μ-measurable sets as a complete measure, see [16, 113C]. Analytic and co-analytic sets are μ-measurable [16, 434D (c)]. We denote by $\operatorname{supp}\mu$ the *support of* μ, the minimal closed subset F of X so that $\mu(X \setminus F) = 0$. The measure μ is called *continuous* is $\mu(\{x\}) = 0$ for all $x \in X$. For the following classical theorems see [16, 433C] and [18, Thm. A, p. 54.], respectively.

Theorem 2.5. *If X is a Polish space and μ is a σ-finite Borel measure on X then there is a compact set $K \subset X$ with $\mu(K) > 0$.*

Theorem 2.6 (Carathéodory's extension theorem)**.** *Any σ-finite measure defined on an algebra $\mathcal{A}$ can be* uniquely *extended to the σ-algebra generated by $\mathcal{A}$.*

Let G be an abelian Polish group and let μ, ν be σ-finite Borel measures on G. For a Borel set $A \subset G$ let us define

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(x,y) \in G \times G : x + y \in A\}),$$

where $\mu \times \nu$ is the product measure on $G \times G$. Then $\mu * \nu$ is a σ-finite Borel measure on G called the *convolution of μ and ν*.

For all $f \in C[0,1]$ let $\mathcal{Z}(f) = \{x \in [0,1] : f(x) = 0\}$. If $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ then the *maximum norm* of x is defined as $||x|| = \max_{1 \le i \le d} |x_i|$. Let χ_A be the characteristic function of the set A. If $A \subset \mathbb{R}$ then let $\operatorname{conv}(A)$ be the convex hull of A. We denote by Pr, $\mathbb{E}$ and Var the probability, expected value and variance, respectively.

3. Technical lemmas

Our definition of prevalence follows Hunt, Sauer and York [20] and differs from Christensen [8] in which the definition is given for so-called universally measurable sets (without the Borel hulls). These definitions are equivalent for Borel sets, but they differ in general, see [10]. The following theorem states that the definitions are also equivalent for co-analytic sets, see [38, Prop. (i)] for the proof.

Theorem 3.1 (Solecki)**.** *Let G be an abelian Polish group and let $A \subset G$ be a co-analytic set. If there exists a Borel probability measure μ on G such that $\mu(A+g) = 1$ for all $g \in G$ then A is prevalent.*

The following lemma is basically [4, Lemma 2.11]. It is only stated there in the special case $d = 1$, but the proof works verbatim for all $d \in \mathbb{N}^+$.

Lemma 3.2. *Let K be a compact metric space, let $d \in \mathbb{N}^+$ and $c \in \mathbb{R}$. Then*

$$\Delta = \left\{(f, y) \in C(K, \mathbb{R}^d) \times \mathbb{R}^d : \dim_H f^{-1}(y) < c\right\}$$

is a Borel set in $C(K, \mathbb{R}^d) \times \mathbb{R}^d$.

Lemma 3.3. *Let $K \subset \mathbb{R}$ be compact, let $d \in \mathbb{N}^+$ and $c \in \mathbb{R}$. Then*

$$\begin{aligned}\mathcal{A} = \{f \in C(K, \mathbb{R}^d) : \exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that} \\ \lambda\left(f^{-1}(U_f)\right) = \lambda(K) \text{ and } \dim_H f^{-1}(y) \ge c \text{ for all } y \in U_f\}\end{aligned}$$

is co-analytic in $C(K, \mathbb{R}^d)$.

Proof. Let $\mathcal{V}$ be a countable basis of $\mathbb{R}^d$ and let $\mathcal{U}$ be the family of finite unions of elements of $\mathcal{V}$. Clearly $\mathcal{U}$ is countable and $\mathcal{A} = \bigcup_{U \in \mathcal{U}} \bigcap_{n=1}^{\infty} \mathcal{A}_{n,U}$, where

$$\begin{aligned}\mathcal{A}_{n,U} = \{f \in C(K, \mathbb{R}^d) : \lambda(f^{-1}(U)) > \lambda(K) - 1/n \\ \text{and } \dim_H f^{-1}(y) \ge c \text{ for all } y \in U\}.\end{aligned}$$

As co-analytic sets are closed under countable union and countable intersection, it is enough to prove that the $\mathcal{A}_{n,U}$ are co-analytic. Fix $n \in \mathbb{N}^+$ and $U \in \mathcal{U}$ and let

$$\begin{aligned}\mathcal{B} &= \{f \in C(K, \mathbb{R}^d) : \dim_H f^{-1}(y) \ge c \text{ for all } y \in U\}, \\ \mathcal{C} &= \{f \in C(K, \mathbb{R}^d) : \lambda\left(f^{-1}(U)\right) > \lambda(K) - 1/n\}.\end{aligned}$$

Since $\mathcal{A}_{n,U} = \mathcal{B} \cap \mathcal{C}$, it is enough to prove that $\mathcal{B}$ and $\mathcal{C}$ are co-analytic.

First we show that $\mathcal{B}$ is co-analytic. By Lemma 3.2 the set

$$\Delta = \left\{(f, y) \in C(K, \mathbb{R}^d) \times \mathbb{R}^d : \dim_H f^{-1}(y) < c\right\}$$

is Borel. Define $\operatorname{pr}\colon C(K, \mathbb{R}^d) \times \mathbb{R}^d \to C(K, \mathbb{R}^d)$ as $\operatorname{pr}(f, y) = f$. Then

$$\mathcal{B} = \left(\operatorname{pr}\left(\Delta \cap (C(K, \mathbb{R}^d) \times U)\right)\right)^c$$

is the complement of the projection of a Borel set. Hence $\mathcal{B}$ is co-analytic.

Finally, we prove that $\mathcal{C}$ is Borel. For all $r \in \mathbb{R}$ let

$$\mathcal{C}(r) = \{f \in C(K, \mathbb{R}^d) : \lambda\left(f^{-1}(U)\right) > r\}.$$

It is enough to prove that the $\mathcal{C}(r)$ are open. Fix $r \in \mathbb{R}$ and assume that $f \in \mathcal{C}(r)$, that is, $\lambda(f^{-1}(U)) > r$. We need to find an $\varepsilon > 0$ such that $U(f, \varepsilon) \subset \mathcal{C}(r)$. The regularity of the Lebesgue measure implies that there is a compact set $C \subset f^{-1}(U)$ with $\lambda(C) > r$. As $f(C) \subset U$ is compact, we can define $\varepsilon = \operatorname{dist}(f(C), \mathbb{R}^d \setminus U) > 0$. Clearly $g(C) \subset U$ for every $g \in U(f, \varepsilon)$, thus $\lambda(g^{-1}(U)) \geq \lambda(C) > r$. Hence $U(f, \varepsilon) \subset \mathcal{C}(r)$, and the proof is complete. □

Definition 3.4. Let $\{a_n\}_{n\in\mathbb{N}^+}$ be a sequence of positive integers. A compact set $K \subset \mathbb{R}$ is an (a_n)-*type fat Cantor set* if $\lambda(K) > 0$ and it is of the form

$$K = \bigcap_{n=1}^{\infty} \left(\bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} K_{i_1 \ldots i_n} \right), \tag{3.1}$$

where $K_{i_1 \ldots i_n} \subset K$ are compact sets such that for every $n \in \mathbb{N}^+$ and for each distinct $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \prod_{k=1}^{n} \{1, \ldots, a_k\}$ we have

(i) $\operatorname{conv}(K_{i_1 \ldots i_n}) \cap \operatorname{conv}(K_{j_1 \ldots j_n}) = \emptyset$,
(ii) $K_{i_1 \ldots i_{n+1}} \subset K_{i_1 \ldots i_n}$,
(iii) $\lambda(K_{i_1 \ldots i_n}) = \frac{\lambda(K)}{a_1 \cdots a_n}$.

We say that the $K_{i_1 \ldots i_n}$ are the *elementary pieces of* K.

Definition 3.5. Let $\{a_n\}_{n\in\mathbb{N}^+}$, $\{b_n\}_{n\in\mathbb{N}^+}$ be sequences of positive integers such that $a_n \geq b_n$ for all $n \in \mathbb{N}^+$. A compact set $C \subset \mathbb{R}$ is an (a_n, b_n)-*type Cantor set* if it is of the form

$$C = \bigcap_{n=1}^{\infty} \left(\bigcup_{i_1=1}^{b_1} \cdots \bigcup_{i_n=1}^{b_n} C_{i_1 \ldots i_n} \right),$$

where $C_{i_1 \ldots i_n} \subset \mathbb{R}$ are compact sets and there is an (a_n)-type fat Cantor set $K \subset \mathbb{R}$ of the form (3.1) such that for all $n \in \mathbb{N}^+$ and $(i_1, \ldots, i_n) \in \prod_{k=1}^{n} \{1, \ldots, b_k\}$

$$\emptyset \neq C_{i_1 \ldots i_{n+1}} \subset C_{i_1 \ldots i_n} \subset K_{i_1 \ldots i_n}. \tag{3.2}$$

The compact set $C \subset \mathbb{R}$ is an (a_n, b_n)-*type compact set* if it satisfies the above definition after replacing (i) by the weaker property

(1) $\operatorname{conv}(K_{i_1 \ldots i_n}) \cap \operatorname{conv}(K_{j_1 \ldots j_n})$ is either empty or a singleton.

For the following well-known lemma see e.g. [30, Thm. 4.19].

Lemma 3.6 (Mass distribution principle)**.** *Let μ be a Borel probability measure on a metric space X. Assume that there are $c, s, \delta \in \mathbb{R}^+$ such that $\mu(B) \leq c(\operatorname{diam} B)^s$ for every Borel set $B \subset X$ with $\operatorname{diam} B \leq \delta$. Then $\dim_H X \geq s$.*

Lemma 3.7. *Let $C \subset \mathbb{R}$ be an (a_n, b_n)-type compact set such that for all $n \in \mathbb{N}^+$*

$$a_n \geq \left(\frac{a_1 \cdots a_{n+1}}{b_1 \cdots b_{n+1}} \right)^{n+1}. \tag{3.3}$$

Then $\dim_H C = 1$.

Proof. Let $C_{i_1\dots i_n}$ be the compact sets corresponding to Definition 3.5. Let $K \subset \mathbb{R}$ be a compact set with elementary pieces $K_{i_1\dots i_n}$ associated to C. By considering similar copies of C and K we may assume that $\lambda(K) = 1$.

For all $n \in \mathbb{N}^+$ let $\mathcal{I}_n = \prod_{k=1}^{n}\{1,\dots,b_k\}$. We may suppose that $C_{i_1\dots i_n} \subset C$ for all $n \in \mathbb{N}^+$ and $(i_1,\dots,i_n) \in \mathcal{I}_n$, otherwise we intersect them with C. Now we construct a Borel probability measure μ supported on C such that for all $n \in \mathbb{N}^+$ and $(i_1,\dots,i_n) \in \mathcal{I}_n$ we have

$$\mu(C_{i_1\dots i_n}) = \frac{1}{b_1\cdots b_n}. \tag{3.4}$$

Choose $x_{i_1\dots i_n} \in C_{i_1\dots i_n}$ for all $n \in \mathbb{N}^+$ and $(i_1,\dots,i_n) \in \mathcal{I}_n$. Define the probability measures

$$\mu_n = \sum_{(i_1,\dots,i_n)\in\mathcal{I}_n} (b_1\dots b_n)^{-1}\delta_{x_{i_1\dots i_n}},$$

where δ_x denotes the Dirac measure concentrated on $\{x\}$. Let F_n be the distribution function of μ_n. The definitions of C and μ_n easily yield that F_n converges (uniformly) to a continuous distribution function F. Let μ be the Borel probability measure associated with F. Then μ_n converges weakly to μ by [30, Thm. 12.7], so [30, Thm. 12.6] yields that for all $n \in \mathbb{N}^+$ and $(i_1,\dots,i_n) \in \mathcal{I}_n$ we have

$$\mu(C_{i_1\dots i_n}) \geq \limsup_{k\to\infty} \mu_k(C_{i_1\dots i_n}) = \frac{1}{b_1\cdots b_n}.$$

As μ is continuous, we have $\sum_{(i_1,\dots,i_n)\in\mathcal{I}_n} \mu(C_{i_1\dots i_n}) \leq 1$. These imply that (3.4) holds and μ is supported on C.

Fix an arbitrary $k \in \mathbb{N}^+$ and a Borel set $B \subset C$ with $\operatorname{diam} B \leq (a_1\cdots a_k)^{-1}$. We can choose $n > k$ and $t \in \{1,\dots,a_n - 1\}$ such that

$$\frac{t}{a_1\cdots a_n} \leq \operatorname{diam} B \leq \frac{t+1}{a_1\cdots a_n}. \tag{3.5}$$

Property (iii) yields that for all $n \in \mathbb{N}^+$ and $(i_1,\dots,i_n) \in \prod_{k=1}^{n}\{1,\dots,a_k\}$

$$\operatorname{diam}(\operatorname{conv}(K_{i_1\dots i_n})) \geq \lambda(K_{i_1\dots i_n}) = \frac{1}{a_1\cdots a_n},$$

thus property (1) and (3.5) yield that B can intersect at most $t+3$ sets of the form $K_{i_1\dots i_n}$. Since $C_{i_1\dots i_n} \subset K_{i_1\dots i_n}$ for all $n \in \mathbb{N}^+$ and $(i_1,\dots,i_n) \in \mathcal{I}_n$, we obtain that B can intersect at most $t+3$ sets of the form $C_{i_1\dots i_n}$. Therefore

$$\mu(B) \leq \frac{t+3}{b_1\dots b_n}. \tag{3.6}$$

Inequalities (3.3) and $t+1 \leq a_n$ yield

$$\left(\frac{a_1\cdots a_n}{b_1\cdots b_n}\right)^n \leq a_{n-1} \leq \frac{a_1\cdots a_n}{t+1}. \tag{3.7}$$

Inequalities (3.6), (3.7), (3.5) and $n > k$ with $\operatorname{diam} B \leq 1$ imply

$$\begin{aligned}\mu(B) &\leq \frac{t+3}{b_1 \dots b_n} \\ &\leq \frac{4t}{a_1 \cdots a_n} \cdot \frac{a_1 \cdots a_n}{b_1 \cdots b_n} \\ &\leq 4(\operatorname{diam} B) \left(\frac{a_1 \cdots a_n}{t+1} \right)^{1/n} \\ &\leq 4(\operatorname{diam} B)(\operatorname{diam} B)^{-1/n} \\ &\leq 4(\operatorname{diam} B)^{1-1/k}.\end{aligned}$$

Thus Lemma 3.6 yields that $\dim_H K \geq 1 - 1/k$. As $k \in \mathbb{N}^+$ was arbitrary, we obtain that $\dim_H K = 1$. The proof is complete. □

Lemma 3.8. *Let $C \subset \mathbb{R}$ be compact with $\lambda(C) > 0$ and let $\{a_n\}_{n \in \mathbb{N}^+}$ be an arbitrary sequence of positive integers. Then there is an (a_n)-type fat Cantor set $K \subset C$.*

Proof. It is straightforward to construct an (a_n)-type fat Cantor set $D \subset [0,1]$, let $D_{i_1 \dots i_n} \subset D$ be its elementary pieces. By considering a similar copy of C we may assume that $\lambda(C) = 1$. Let $\phi \colon C \to [0,1]$ be the onto map defined as

$$\phi(x) = \lambda((-\infty, x] \cap C).$$

For every Borel set $B \subset [0,1]$ we have

$$\lambda(\phi^{-1}(B)) = \lambda(B), \tag{3.8}$$

since (3.8) holds for intervals in $[0,1]$ by the definition of ϕ, thus Carathéodory's extension theorem yields that the Borel measures $\lambda \circ \phi^{-1}$ and $\lambda|_{[0,1]}$ coincide.

Let us define $K = \phi^{-1}(D) \subset C$ and $K_{i_1 \dots i_n} = \phi^{-1}(D_{i_1 \dots i_n})$ for all $n \in \mathbb{N}^+$ and $(i_1, \dots, i_n) \in \prod_{k=1}^{n} \{1, \dots, a_k\}$. Applying that ϕ preserves the order $\leq$ and (3.8) yields that K is an (a_n)-type fat Cantor set with elementary pieces $K_{i_1 \dots i_n}$. □

Corollary 3.9. *Let $K \subset \mathbb{R}$ be a compact set with $\lambda(K) > 0$ and let $\{a_n\}_{n \in \mathbb{N}^+}$ be an arbitrary sequence of positive integers. Then there exist (a_n)-type fat Cantor sets $K_i \subset K$ such that $\lambda(\bigcup_{i=1}^{\infty} K_i) = \lambda(K)$.*

Lemma 3.10. *Let G, H be abelian Polish groups and let $\Phi \colon G \to H$ be a continuous onto homomorphism. If $S \subset H$ is prevalent then so is $\Phi^{-1}(S) \subset G$.*

For the proof of the above lemma see [9, Prop. 8.]. The following corollary follows from Lemma 3.10 and the fact that Tietze's extension theorem holds in $\mathbb{R}^d$.

Corollary 3.11. *Let $K_1 \subset K_2$ be compact metric spaces and let $d \in \mathbb{N}^+$. Define*

$$R \colon C(K_2, \mathbb{R}^d) \to C(K_1, \mathbb{R}^d), \quad R(f) = f|_{K_1}.$$

If $\mathcal{A} \subset C(K_1, \mathbb{R}^d)$ is prevalent then so is $R^{-1}(\mathcal{A}) \subset C(K_2, \mathbb{R}^d)$.

Lemma 3.12. *Let K be a compact metric space and let μ be a finite Borel measure on K. Let $K_n \subset K$ be compact sets with $\mu(K) = \mu(\bigcup_{n=1}^{\infty} K_n)$. If Δ is an upward closed family of subsets of K and for all $n \in \mathbb{N}^+$ the*

$$\begin{aligned}\mathcal{A}_n = \{f \in C(K_n, \mathbb{R}^d) : &\exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that} \\ &\mu(f^{-1}(U_f)) = \mu(K_n) \text{ and } f^{-1}(y) \in \Delta \text{ for all } y \in U_f\}\end{aligned}$$

are prevalent then so is

$$\mathcal{A} = \{f \in C(K, \mathbb{R}^d) : \exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that } \mu(f^{-1}(U_f)) = \mu(K) \text{ and } f^{-1}(y) \in \Delta \text{ for all } y \in U_f\}.$$

Proof. For all $n \in \mathbb{N}^+$ let

$$R_n \colon C(K, \mathbb{R}^d) \to C(K_n, \mathbb{R}^d), \quad R_n(f) = f|_{K_n}.$$

Corollary 3.11 implies that the $R_n^{-1}(\mathcal{A}_n)$ are prevalent in $C(K, \mathbb{R}^d)$. As a countable intersection of prevalent sets, $\bigcap_{n=1}^{\infty} R_n^{-1}(\mathcal{A}_n)$ is also prevalent in $C(K, \mathbb{R}^d)$. Thus it is enough to prove that $\bigcap_{n=1}^{\infty} R_n^{-1}(\mathcal{A}_n) \subset \mathcal{A}$. For all $f \in \bigcap_{n=1}^{\infty} R_n^{-1}(\mathcal{A}_n)$ let $U_f = \bigcup_{n=1}^{\infty} U_{f|_{K_n}}$. Then $U_f \subset \mathbb{R}^d$ is open and for all $y \in U_f$ there is an $n \in \mathbb{N}^+$ such that $y \in U_{f|_{K_n}}$. Then $f^{-1}(y) \supset (f|_{K_n})^{-1}(y) \in \Delta$ implies that $f^{-1}(y) \in \Delta$.

Finally, we need to show that $\mu(f^{-1}(U_f)) = \mu(K)$. By $\mu(K) = \mu(\bigcup_{n=1}^{\infty} K_n)$ it is enough to prove that $\mu(f^{-1}(U_f) \cap K_n) = \mu(K_n)$ for an arbitrary fixed $n \in \mathbb{N}^+$. The definitions of U_f and $\mathcal{A}_n$ yield that

$$\mu\left(f^{-1}(U_f) \cap K_n\right) \geq \mu\left((f|_{K_n})^{-1}(U_{f|_{K_n}})\right) = \mu(K_n),$$

and the proof is complete. □

We will apply the above lemma for families of the form $\Delta = \{A : \dim_H A \geq c\}$ and $\Delta = \{A : \dim_H A \geq c_1 \text{ and } \dim_P A \geq c_2\}$.

Lemma 3.13. *Let $u, v \in \mathbb{N}^+$ and $0 < p \leq 1/v$. Assume that $\xi_1, \ldots, \xi_u$ are independent random variables such that $\Pr(\xi_i = j) = p$ for all $i \in \{1, \ldots, u\}$ and $j \in \{1, \ldots, v\}$. Then*

$$\Pr\left(\#\{i : \xi_i = j\} < up/2 \text{ for some } j \in \{1, \ldots, v\}\right) \leq \frac{4v}{up}.$$

Proof. Let us fix $j \in \{1, \ldots, v\}$ arbitrarily and for all $i \in \{1, \ldots, u\}$ let $X_i = 1$ if $\xi_i = j$, and let $X_i = 0$ otherwise. Set $X = \sum_{i=1}^{u} X_i$. Then $\mathbb{E}(X_i) = p$ and $\mathrm{Var}(X_i) = p - p^2 < p$ for all $i \in \{1, \ldots, u\}$, thus $\mathbb{E}(X) = up$ and the independence of X_i yields $\mathrm{Var}(X) = \sum_{i=1}^{u} \mathrm{Var}(X_i) < up$. Then Chebyshev's inequality [7, (5.32)] implies

$$\begin{aligned}\Pr(\#\{i : \xi_i = j\} < up/2) &= \Pr(X < up/2) \\ &\leq \Pr\left(|X - \mathbb{E}(X)| > \mathbb{E}(X)/2\right) \\ &\leq \mathrm{Var}(X)/(\mathbb{E}(X)/2)^2 \leq \frac{4}{up}.\end{aligned}$$

Hence

$$\Pr\left(\#\{i : \xi_i = j\} < up/2 \text{ for some } j \in \{1, \ldots, v\}\right) \leq \frac{4v}{up},$$

and this concludes the proof. □

Lemma 3.14. *If X, Y are independent $\mathbb{R}^d$-valued random variables and $r > 0$ then*

$$\Pr(|X - Y| \leq r) \leq \sup_{y \in \mathbb{R}^d} \Pr(|X - y| \leq r).$$

Proof. Let μ_X, μ_Y and $\mu_{X,Y}$ be the distribution measure of X, Y and (X,Y), respectively. The independence of X and Y yields $\mu_{X,Y} = \mu_X \times \mu_Y$, thus

$$\begin{aligned}\Pr(|X-Y| \le r) &= \iint_{\mathbb{R}^{2d}} \chi_{\{(x,y):\,|x-y|\le r\}}\,\mathrm{d}\mu_{X,Y}(x,y)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_{\{(x,y):\,|x-y|\le r\}}\,\mathrm{d}\mu_X(x)\,\mathrm{d}\mu_Y(y)\\ &= \int_{\mathbb{R}^d} \Pr(|X-y|\le r)\,\mathrm{d}\mu_Y(y)\\ &\le \sup_{y\in\mathbb{R}^d} \Pr(|X-y|\le r).\end{aligned}$$

The proof is complete. □

4. Dimensions of fibers of prevalent continuous maps

4.1. **The real case.** First we prove the Main Theorem for $K \subset \mathbb{R}$ and $\mu = \lambda$.

Theorem 4.1. *Let $K \subset \mathbb{R}$ be a compact set with $\lambda(K) > 0$ and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K,\mathbb{R}^d)$ there exists a non-empty open set $U_f \subset \mathbb{R}^d$ such that $\lambda(f^{-1}(U_f)) = \lambda(K)$ and for all $y \in U_f$*

$$\dim_H f^{-1}(y) = 1.$$

Proof. Consider

$$\begin{aligned}\mathcal{A} = \{f \in C(K,\mathbb{R}^d) : \exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that}\\ \lambda(f^{-1}(U_f)) = \lambda(K) \text{ and } \dim_H f^{-1}(y) = 1 \text{ for all } y \in U_f\}.\end{aligned}$$

Lemma 3.3 with $c = 1$ yields that $\mathcal{A}$ is co-analytic. By Theorem 3.1 it is enough to show that there exists a Borel probability measure μ on $C(K,\mathbb{R}^d)$ such that $\mu(\mathcal{A} - g) = 1$ for all $g \in C(K,\mathbb{R}^d)$.

Now we construct the measure μ. Let us endow $\mathbb{R}^d$ with the maximum norm, which we simply denote by $|\cdot|$. Let $s = 2^d$ and let $S_n = \{-2^{-n}, 2^{-n}\}^d$ for all $n \in \mathbb{N}^+$, then $\#S_n = s$. Clearly for all $z \in \mathbb{R}^d$ and $n \in \mathbb{N}$

$$B(z, 2^{-n}) = \bigcup_{y\in S_{n+1}} B(z+y, 2^{-(n+1)}). \tag{4.1}$$

For all $n \in \mathbb{N}^+$ let us define the positive integers a_n and b_n by

$$a_n = (2s)^{4^n} \quad \text{and} \quad b_n = (2s)^{-(n+3)} a_n,$$

easy calculations show that there is an $n_0 \in \mathbb{N}^+$ such that for all $n \ge n_0$ we have

$$a_n \ge \max\left\{(2s)^{8n}(a_1\cdots a_{n-1}), \left(\frac{a_1\cdots a_{n+1}}{b_1\cdots b_{n+1}}\right)^{n+1}\right\}. \tag{4.2}$$

For all $n \in \mathbb{N}^+$ let

$$\mathcal{I}_n = \prod_{i=1}^{n} \{1,\dots,a_i\}.$$

Let us recall Definition 3.4. Corollary 3.9 implies that there exist (a_n)-type fat Cantor sets $K_i \subset K$ such that $\lambda(\bigcup_{i=1}^{\infty} K_i) = \lambda(K)$. Therefore, by Lemma 3.12 we may assume that K is an (a_n)-type fat Cantor set with elementary pieces $K_{i_1\dots i_n}$.

By considering a similar copy of K we may suppose that $\lambda(K) = 1$. Then for all $n \in \mathbb{N}^+$ and $(i_1, \dots, i_n) \in \mathcal{I}_n$ we have

$$\lambda(K_{i_1 \dots i_n}) = \frac{1}{a_1 \cdots a_n}.$$

For all Borel sets $A, B \subset K$ with $\lambda(B) > 0$ let us use the notation

$$\lambda(A \,|\, B) = \frac{\lambda(A \cap B)}{\lambda(B)}.$$

For all $n \in \mathbb{N}^+$ and $(i_1, \dots, i_n) \in \mathcal{I}_n$ let us define countable many independent random variables $X_{i_1 \dots i_n}$ and $Y_{i_1 \dots i_n}$ such that for all $y \in S_n$

$$\Pr(X_{i_1 \dots i_n} = y) = \Pr(Y_{i_1 \dots i_n} = y) = 1/s. \tag{4.3}$$

For every $n \in \mathbb{N}^+$ and $x \in K$ there exists a unique $(i_1, \dots, i_n) \in \mathcal{I}_n$ for which $x \in K_{i_1 \dots i_n}$. Then let us define the random function $f_n \in C(K, \mathbb{R}^d)$ as

$$f_n(x) = X_{i_1 \dots i_n} - Y_{i_1 \dots i_n}.$$

Note that the dependence of the right hand side on x is simply that the indices depend on x. Let $\mathbb{P}_n$ be the probability measure on $C(K, \mathbb{R}^d)$ corresponding to this method of randomly choosing f_n, and let $\mathcal{S}_n \subset C(K, \mathbb{R}^d)$ be its finite support. Clearly for all $f_n \in \mathcal{S}_n$ and $x \in K$

$$|f_n(x)| \le 2^{1-n}. \tag{4.4}$$

Thus $\sum_{n=1}^{\infty} f_n$ always converges uniformly. Let $\mathbb{P} = \prod_{n=1}^{\infty} \mathbb{P}_n$ be a probability measure on the Borel subsets of $\mathcal{S} = \prod_{n=1}^{\infty} \mathcal{S}_n$ and let

$$\pi \colon \mathcal{S} \to C(K, \mathbb{R}^d), \quad \pi((f_n)) = \sum_{n=1}^{\infty} f_n.$$

Let us define

$$\mu = \mathbb{P} \circ \pi^{-1}.$$

Now we prove that $\mu(\mathcal{A} - g) = 1$ for all $g \in C(K, \mathbb{R}^d)$. Let $g \in C(K, \mathbb{R}^d)$ and $\varepsilon > 0$ be arbitrarily fixed, it is enough to show that $\mu(\mathcal{A} - g) \ge 1 - \varepsilon$. As $g(K)$ is compact, we can fix an integer $m > n_0$ such that $2^m > 1/\varepsilon$ and $g(K)$ can be covered by 2^m closed balls of radius 1, it is sufficient to prove that

$$\mu(\mathcal{A} - g) = \mathbb{P}(\pi^{-1}(\mathcal{A} - g)) \ge 1 - 2^{-m}. \tag{4.5}$$

For all $n \in \mathbb{N}$ consider

$$h_n = g + \sum_{i=1}^{m+n} f_i,$$
$$r_n = (2s)^{-(m+n+2)},$$
$$p_n = \frac{(2s)^{m+n+5}}{a_{m+n+1}}.$$

Statement 4.2. *Let $n \in \mathbb{N}$ and assume that $z \in \mathbb{R}^d$ and for all $i \in \{1, \dots, m+n\}$ the functions $f_i \in \mathcal{S}_i$ and $\sigma \in \mathcal{I}_{m+n}$ are fixed. Let $A \subset h_n^{-1}(B(z, 2^{-(m+n)}))$ with*

$$\lambda(A \,|\, K_\sigma) \ge r_n.$$

For all $y \in S_{m+n+1}$ let us define the random set $I(y) \subset \{1, \dots, a_{m+n+1}\}$ as

$$I(y) = \left\{ i : \lambda\left(A \cap h_{n+1}^{-1}\left(B\left(z + y, 2^{-(m+n+1)} \right) \right) \mid K_{\sigma i} \right) \geq r_{n+1} \right\},$$

where σi is the concatenation of σ and i. Then

$$\mathbb{P}_{m+n+1}\left(\#I(y) < b_{m+n+1} \text{ for some } y \in S_{m+n+1} \right) \leq p_n.$$

Proof of Statement 4.2. Let us define $I \subset \{1, \dots, a_{m+n+1}\}$ as

$$I = \{ i : \lambda(A \mid K_{\sigma i}) \geq r_n/2 \}.$$

First we prove that

$$\#I \geq \frac{r_n a_{m+n+1}}{2}. \tag{4.6}$$

Our assumption and the definition of I imply that

$$\begin{aligned} r_n \lambda(K_\sigma) \leq \lambda(A \cap K_\sigma) &= \sum_{i=1}^{a_{m+n+1}} \lambda(A \cap K_{\sigma i}) \\ &\leq \sum_{i=1}^{a_{m+n+1}} \frac{r_n}{2} \lambda(K_{\sigma i}) + \sum_{i \in I} \lambda(K_{\sigma i}) \\ &= \frac{r_n}{2} \lambda(K_\sigma) + (\#I) \frac{\lambda(K_\sigma)}{a_{m+n+1}}, \end{aligned}$$

which easily yields (4.6). Then $A \subset h_n^{-1}(B(z, 2^{-(m+n)}))$ and (4.1) imply that

$$A \subset \bigcup_{y \in S_{m+n+1}} h_n^{-1}\left(B\left(z + y, 2^{-(m+n+1)} \right) \right).$$

Thus the definition of I and $r_{n+1} = r_n/(2s)$ yield that for all $i \in I$ there exists $y(i) \in S_{m+n+1}$ such that

$$\lambda\left(A \cap h_n^{-1}\left(B\left(z + y(i), 2^{-(m+n+1)} \right) \right) \mid K_{\sigma i} \right) \geq r_{n+1}. \tag{4.7}$$

Let $S_{m+n+1} = \{ y_j : 1 \leq j \leq s \}$. Define for all $i \in I$ independent random variables

$$\xi_i = \begin{cases} j \text{ if } X_{\sigma i} = y_j \text{ and } Y_{\sigma i} = y(i), \\ 0 \text{ otherwise.} \end{cases} \tag{4.8}$$

For all $j \in \{1, \dots, s\}$ let us define the random set

$$I_j = \{ i \in I : \xi_i = j \}.$$

Now we show that for all $j \in \{1, \dots, s\}$ we have

$$I_j \subset I(y_j). \tag{4.9}$$

Assume that $i \in I_j$ and $x \in K_{\sigma i}$, then

$$h_{n+1}(x) = h_n(x) + f_{n+1}(x) = h_n(x) + X_{\sigma i} - Y_{\sigma i} = h_n(x) + y_j - y(i),$$

therefore

$$h_n^{-1}\left(B\left(z + y(i), 2^{-(m+n+1)} \right) \right) \cap K_{\sigma i} \subset h_{n+1}^{-1}\left(B\left(z + y_j, 2^{-(m+n+1)} \right) \right). \tag{4.10}$$

Formulae (4.10) and (4.7) imply

$$\lambda\left(A \cap h_{n+1}^{-1}\left(B\left(z + y_j, 2^{-(m+n+1)} \right) \right) \mid K_{\sigma i} \right) \geq r_{n+1}, \tag{4.11}$$

thus $i \in I(y_j)$, so (4.9) holds. The definitions yield that

$$\frac{r_n a_{m+n+1}}{4s^2} = b_{m+n+1} \quad \text{and} \quad \frac{8s^3}{r_n a_{m+n+1}} = p_n. \tag{4.12}$$

Clearly, we have $\Pr(\xi_i = j) = 1/s^2$ for all $i \in I$ and $j \in \{1, \dots, s\}$. We apply Lemma 3.13 for ξ_i with $u = \#I$, $v = s$ and $p = 1/s^2$. Then (4.6) and the first part of (4.12) yield that $b_{m+n+1} \le up/2$. Therefore (4.9), Lemma 3.13, (4.6) and the second part of (4.12) imply that

$$\begin{aligned}
&\mathbb{P}_{m+n+1}\left(\#I(y) < b_{m+n+1} \text{ for some } y \in S_{m+n+1}\right)\\
&\qquad \le \Pr\left(\#I_j < b_{m+n+1} \text{ for some } j \in \{1, \dots, s\}\right)\\
&\qquad \le \Pr\left(\#\{i \in I : \xi_i = j\} < up/2 \text{ for some } j \in \{1, \dots, v\}\right)\\
&\qquad \le \frac{4v}{up} \le \frac{8s^3}{r_n a_{m+n+1}} = p_n.
\end{aligned}$$

The proof of the statement is complete. □

Now we return to the proof of Theorem 4.1. For all $k \in \mathbb{N}$ let $\mathcal{Y}_{k,k} = \mathcal{J}_{k,k} = \{\emptyset\}$. For all natural numbers $k < n$ let

$$\mathcal{Y}_{k,n} = \prod_{i=k+1}^{n} S_{m+i} \quad \text{and} \quad \mathcal{J}_{k,n} = \prod_{i=k+1}^{n} \{1, \dots, b_{m+i}\}.$$

Let $(f_1, \dots, f_m) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_m$ be arbitrary. By the definition of m we can cover $B(g(K), 2)$ by 2^m closed balls of radius 3, and so by $2^m s^{m+2} < (2s)^{m+2}$ closed balls of radius 2^{-m}. Let $T_m \subset \mathbb{R}^d$ be the set of their centers, that is, $\#T_m \le (2s)^{m+2}$ and $B(g(K), 2) \subset \bigcup_{y \in T_m} B(y, 2^{-m})$. Let

$$\mathcal{T}_0 = \{y_0 \in T_m : \lambda(h_0^{-1}(B(y_0, 2^{-m}))) \ge r_0\}.$$

As (4.4) yields $h_0 = g + \sum_{i=1}^{m} f_i \in B(g, 2)$, we have $h_0(K) \subset \bigcup_{y \in T_m} B(y, 2^{-m})$. Therefore $r_0 = (2s)^{-(m+2)}$ and $\#T_m \le (2s)^{m+2}$ imply that $\mathcal{T}_0 \ne \emptyset$. Let us fix $(f_1, \dots, f_m)$ and for all $y_0 \in \mathcal{T}_0$ fix $\sigma_{y_0} \in \mathcal{I}_m$ such that

$$\lambda(h_0^{-1}(B(y_0, 2^{-m})) \,|\, K_{\sigma_{y_0}}) \ge r_0.$$

Let $n \in \mathbb{N}$ and suppose that $f_i \in \mathcal{S}_i$ are fixed for all $i \in \{1, \dots, m+n\}$. Assume by induction that for all $k \in \{0, \dots, n\}$ the sets $\mathcal{T}_k \subset T_m \times \prod_{i=1}^{k} S_{m+i}$ and for all $(y_0, \dots, y_k) \in \mathcal{T}_k$, $(y_{k+1}, \dots, y_n) \in \mathcal{Y}_{k,n}$ and $(j_{k+1}, \dots, j_n) \in \mathcal{J}_{k,n}$ the index sets

$$\sigma = \sigma_{y_0 \dots y_k}(y_{k+1}, \dots, y_n; j_{k+1}, \dots, j_n) \in \mathcal{I}_{m+n} \tag{4.13}$$

are already defined (we use the convention that $\sigma_{y_0 \dots y_n}(\emptyset; \emptyset) = \sigma_{y_0 \dots y_n}$) such that

$$\begin{aligned}
&\lambda(A_k^n(y_0, \dots, y_n) \,|\, K_\sigma) \ge r_n, \text{ where}\\
&A_k^n(y_0, \dots, y_n) = \bigcap_{i=k}^{n} h_i^{-1}\left(B\left((y_0 + \cdots + y_i), 2^{-(m+i)}\right)\right).
\end{aligned} \tag{4.14}$$

Let us consider the functions $f_{m+n+1} \in \mathcal{S}_{m+n+1}$ for which for every σ from (4.13) and for every $y_{n+1} \in S_{m+n+1}$ and $j_{n+1} \in \{1, \dots, b_{m+n+1}\}$ we can define i as the j_{n+1}st smallest element of $\{1, \dots, a_{m+n+1}\}$ that satisfies

$$\lambda\left(A_k^n(y_0, \dots, y_n) \cap h_{n+1}^{-1}\left(B\left((y_0 + \cdots + y_{n+1}), 2^{-(m+n+1)}\right)\right) \,|\, K_{\sigma i}\right) \ge r_{n+1},$$

and let us define

$$\sigma_{y_0\dots y_k}(y_{k+1},\dots,y_{n+1};j_{k+1},\dots,j_{n+1})=\sigma i\in\mathcal{I}_{m+n+1}.$$

Statement 4.2 implies that the $\mathbb{P}_{m+n+1}$-probability that $f_{m+n+1}\in\mathcal{S}_{m+n+1}$ does not have this property is at most

$$\begin{aligned}\sum_{k=0}^{n}(\#\mathcal{T}_k)(\#\mathcal{Y}_{k,n})(\#\mathcal{J}_{k,n})p_n &\le \sum_{k=0}^{n}(2s)^{m+2}s^n(b_{m+k+1}\cdots b_{m+n})p_n\\ &\le (2s)^{m+n+2}(b_1\cdots b_{m+n})p_n\\ &\le (2s)^{7(m+n+1)}(a_1\cdots a_{m+n})(a_{m+n+1})^{-1}\\ &\le 2^{-(m+n+1)},\end{aligned}\tag{4.15}$$

where we used $n+1\le 2^n$, $b_n\le a_n$ and (4.2). Let us fix f_{m+n+1} with the above property and define $\mathcal{T}_{n+1}\subset T_m\times\prod_{i=1}^{n+1}S_{m+i}$ as

$$\mathcal{T}_{n+1}=\left\{(y_0,\dots,y_{n+1}):\lambda\left(h_{n+1}^{-1}\left(B\left((y_0+\dots+y_{n+1}),2^{-(m+n+1)}\right)\right)\right)\ge r_{n+1}\right\},$$

and for all $(y_0,\dots,y_{n+1})\in\mathcal{T}_{n+1}$ let us fix $\sigma_{y_0\dots y_{n+1}}\in\mathcal{I}_{m+n+1}$ such that

$$\lambda\left(h_{n+1}^{-1}\left(B\left((y_0+\dots+y_{n+1}),2^{-(m+n+1)}\right)\right)\mid K_{\sigma_{y_0\dots y_{n+1}}}\right)\ge r_{n+1}.$$

Let $\mathcal{F}$ be the set of sequences $(f_i)\in\mathcal{S}$ which can be defined by this process. Then (4.15) yields

$$\mathbb{P}(\mathcal{F})\ge 1-\sum_{n=0}^{\infty}2^{-(m+n+1)}=1-2^{-m}.\tag{4.16}$$

For all $(f_i)\in\mathcal{S}$ let $h=g+\sum_{i=1}^{\infty}f_i$ and for all $(f_i)\in\mathcal{F}$ let

$$U_h=\bigcup_{k=0}^{\infty}\bigcup_{(y_0,\dots,y_k)\in\mathcal{T}_k}U\big((y_0+\dots+y_k),2^{-(m+k)}\big).$$

Now we are ready to prove (4.5). By (4.16) it is enough to show that for $\mathbb{P}$-almost every $(f_i)\in\mathcal{F}$ we have $(f_i)\in\pi^{-1}(\mathcal{A}-g)$. Therefore it is sufficient to prove that $\dim_H h^{-1}(y)=1$ for every $(f_i)\in\mathcal{F}$ and $y\in U_h$, and $\lambda(h^{-1}(U_h))=\lambda(K)$ for $\mathbb{P}$-almost every $(f_i)\in\mathcal{F}$. Therefore the following three lemmas will complete the proof of Theorem 4.1.

Lemma 4.3. *For all $(f_i)\in\mathcal{F}$ and $y\in U_h$ we have $\dim_H h^{-1}(y)=1$.*

Proof. Fix $k\in\mathbb{N}$ and $(y_0,\dots,y_k)\in\mathcal{T}_k$ such that $y\in U((y_0+\dots+y_k),2^{-(m+k)})$. Then (4.1) yields that for all $n\in\mathbb{N}^+$ we can fix $y_{k+n}\in S_{m+k+n}$ such that

$$y\in B\big((y_0+\dots+y_{k+n}),2^{-(m+k+n)}\big).\tag{4.17}$$

For all $n\in\mathbb{N}^+$ and $(j_1,\dots,j_n)\in\mathcal{J}_{k,k+n}=\prod_{i=1}^{n}\{1,\dots,b_{m+k+i}\}$ let

$$\sigma(j_1,\dots,j_n)=\sigma_{y_0\dots y_k}(y_{k+1},\dots,y_{k+n};j_1,\dots,j_n)\in\mathcal{I}_{m+k+n}$$

and

$$C_{j_1\dots j_n}=K_{\sigma(j_1,\dots,j_n)}\cap A_k^{k+n}(y_0,\dots,y_{k+n}),$$

where we recall (4.13) and (4.14). Let us define

$$C = \bigcap_{n=1}^{\infty} \left(\bigcup_{j_1=1}^{b_{m+k+1}} \cdots \bigcup_{j_n=1}^{b_{m+k+n}} C_{j_1 \dots j_n} \right).$$

Then (4.14) yields that $h_{k+n}(C_{j_1 \dots j_n}) \subset B((y_0 + \cdots + y_{k+n}), 2^{-(m+k+n)})$ for all $n \in \mathbb{N}^+$ and $(j_1, \dots, j_n) \in \mathcal{J}_{k,k+n}$. Thus (4.17) and uniform convergence imply $h(x) = y$ for all $x \in C$, so

$$C \subset h^{-1}(y). \tag{4.18}$$

Let us recall Definitions 3.4 and 3.5. Let $c_n = a_{m+k+n}$ and $d_n = b_{m+k+n}$ for all $n \in \mathbb{N}^+$. As K is an (a_n)-type fat Cantor set, the $(m+k)$th level elementary pieces of K are (c_n)-type fat Cantor sets. Inequality (4.14) yields that $C_{j_1 \dots j_n} \neq \emptyset$ for all $n \in \mathbb{N}^+$ and $(j_1, \dots, j_n) \in \prod_{i=1}^{n} \{1, \dots, d_i\}$, thus K witnesses that C is a (c_n, d_n)-type Cantor set. Then (4.2) easily implies that

$$c_n \geq \left(\frac{c_1 \cdots c_{n+1}}{d_1 \cdots d_{n+1}} \right)^{n+1},$$

therefore Lemma 3.7 yields that $\dim_H C = 1$. Hence (4.18) implies that

$$\dim_H h^{-1}(y) \geq \dim_H C = 1, \tag{4.19}$$

which completes the proof. □

Lemma 4.4. *Let $(f_i) \in \mathcal{F}$ be fixed such that $\lambda \circ h^{-1}$ is absolutely continuous with respect to λ^d. Then $\lambda(h^{-1}(U_h)) = \lambda(K)$.*

Proof. There exists a measurable function $\varphi \colon \mathbb{R}^d \to [0, \infty)$ such that for every Borel set $B \subset \mathbb{R}^d$

$$\lambda(h^{-1}(B)) = \int_B \varphi(y) \, \mathrm{d}\lambda^d(y). \tag{4.20}$$

Assume to the contrary that $\lambda(h^{-1}(h(K) \setminus U_h)) > 0$. Then there exist a measurable set $E \subset h(K) \setminus U_h$ and $c > 0$ such that $\lambda^d(E) > 0$ and $\varphi|_E \geq c$. By Lebesgue's density theorem [16, 261D] we can fix a density point $z \in E$. As z is a density point of E, there is an $n \in \mathbb{N}^+$ such that $r = 2^{-(m+n)}$ satisfies $r \leq c$ and if $B \subset B(z, 2r)$ is a ball with radius at least $r/4$ then

$$\lambda^d(B \cap E) \geq \lambda^d(B)/2. \tag{4.21}$$

As $z \in E \subset h(K) \subset B(g(K), 2)$, there exists $(y_0, \dots, y_n) \in T_m \times S_{m+1} \times \cdots \times S_{m+n}$ with $z \in B((y_0 + \cdots + y_n), r)$. Let $y = y_0 + \cdots + y_n$ and $D = B(y, r/2)$, then we have $B(D, r/2) = B(y, r) \subset B(z, 2r)$. Then (4.20), the definition of c and (4.21) yield

$$\lambda(h^{-1}(D)) = \int_D \varphi(y) \, \mathrm{d}\lambda^d(y) \geq c\lambda^d(D \cap E) \geq (c/2)\lambda^d(D). \tag{4.22}$$

Let $\mathcal{B} = \{B(y + y_{n+1} + y_{n+2}, r/4) : y_{n+i} \in S_{m+n+i}\}$, clearly $\#\mathcal{B} = s^2$. If $x \in K$ and $h(x) \in D$ then (4.4) and the definition of D imply

$$h_{n+2}(x) \in B(h(x), 2^{-(m+n+1)}) \subset B(D, r/2) = B(y, r) = \bigcup \mathcal{B}.$$

Hence there exists a $B \in \mathcal{B}$ such that $\lambda(h_{n+2}^{-1}(B)) \geq s^{-2}\lambda(h^{-1}(D))$. Then (4.22), $\lambda^d(D) = s^{-(m+n)}$ and $c \geq 2^{-(m+n)}$ yield

$$\lambda(h_{n+2}^{-1}(B)) \geq \frac{\lambda(h^{-1}(D))}{s^2} \geq \frac{c\lambda^d(D)}{2s^2} > (2s)^{-(m+n+4)} = r_{n+2}.$$

Let $y_0+\cdots+y_{n+2}$ be the center of B. The definition of $\mathcal{T}_{n+2}$ and $\lambda(h_{n+2}^{-1}(B)) > r_{n+2}$ imply that $(y_0, \dots, y_{n+2}) \in \mathcal{T}_{n+2}$. Hence the definition of U_h yields that $B \subset U_h$. Then (4.21) implies that $\lambda^d(B \cap E) \geq \lambda^d(B)/2 > 0$, thus $E \cap B \neq \emptyset$, so $E \cap U_h \neq \emptyset$. This contradicts the definition of E, thus $\lambda(h^{-1}(U_h)) = \lambda(K)$. □

Lemma 4.5. *For $\mathbb{P}$-almost every $(f_i) \in \mathcal{S}$ the measure $\lambda \circ h^{-1}$ is absolutely continuous with respect to λ^d.*

Proof. For all $(f_i) \in \mathcal{S}$ and $n \in \mathbb{N}^+$ let $F_n = \sum_{i=n}^{\infty} f_i$. For all distinct $x, z \in K$ let $i(x,z)$ be the minimal number i such that x and z are in different ith level elementary pieces of K. Fix $n \in \mathbb{N}^+$ and $x, z \in K$ with $i(x,z) = n$ and also fix $r > 0$. Pick indexes i_k such that $x \in K_{i_1 \dots i_k}$ for all $k \in \mathbb{N}^+$ and define

$$X_n(x) = \sum_{k=n}^{\infty} X_{i_1 \dots i_k} \quad \text{and} \quad Y_n(x) = \sum_{k=n}^{\infty} Y_{i_1 \dots i_k},$$

where we recall (4.3). Then clearly $X_n(x)$ is uniformly distributed in $B(\mathbf{0}, 2^{1-n})$, therefore for all $y \in \mathbb{R}^d$ we have

$$\mathbb{P}(|X_n(x) - y| \leq r) \leq (r2^{n-1})^d \leq (r2^n)^d. \tag{4.23}$$

Then $i(x,z) = n$ implies that $X_n(x)$ and $Y_n(x) - g(x) + F_n(z) + g(z)$ are independent. Since $f_i(x) = f_i(z)$ for all $i < n$, the difference of the above two variables equals $h(x) - h(z)$. Thus Lemma 3.14 and (4.23) imply

$$\begin{aligned} \mathbb{P}(|h(x) - h(z)| \leq r) &= \mathbb{P}(|X_n(x) - (Y_n(x) - g(x) + F_n(z) + g(z))| \leq r) \\ &\leq \sup_{y \in \mathbb{R}^d} \mathbb{P}(|X_n(x) - y| \leq r) \leq (r2^n)^d. \end{aligned} \tag{4.24}$$

Let $A_n = \{(x,z) \in K \times K : i(x,z) = n\}$ for all $n \in \mathbb{N}^+$, then

$$(\lambda \times \lambda)(A_n) = \frac{a_1 \cdots a_n(a_n - 1)}{(a_1 \cdots a_n)^2} \leq \frac{1}{a_1 \cdots a_{n-1}}, \tag{4.25}$$

where $a_1 \cdots a_0 = 1$ by convention. Let us use the notation $\lambda_h = \lambda \circ h^{-1}$ and define the random function $q \colon \mathbb{R}^d \to [0, \infty]$ as

$$q(y) = \liminf_{r \to 0+} \frac{\lambda_h(B(y,r))}{\lambda^d(B(y,r))}.$$

By [26, Thm. 2.12] it is enough to show that $q(y) < \infty$ for λ_h almost every $y \in \mathbb{R}^d$. Therefore it is enough to prove that the following expected value is finite. Applying Fatou's lemma, the substitution formula $\int_{\mathbb{R}^d} \psi \, \mathrm{d}\lambda_h = \int_K \psi \circ h \, \mathrm{d}\lambda$, Fubini's theorem,

(4.24), (4.25) and $a_n \geq s^{2n} = 2^{2dn}$ yield

$$\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^d} q(y)\,\mathrm{d}\lambda_h(y) &\leq \liminf_{r\to 0+}\frac{1}{(2r)^d}\mathbb{E}\int_{\mathbb{R}^d}\lambda_h(B(y,r))\,\mathrm{d}\lambda_h(y)\\
&= \liminf_{r\to 0+}\frac{1}{(2r)^d}\int_K\int_K \mathbb{P}(|h(x)-h(z)|\leq r)\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(z)\\
&= \liminf_{r\to 0+}\frac{1}{(2r)^d}\sum_{n=1}^{\infty}\iint_{A_n}\mathbb{P}(|h(x)-h(z)|\leq r)\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(z)\\
&\leq \liminf_{r\to 0+}\frac{1}{(2r)^d}\sum_{n=1}^{\infty}\frac{(r2^n)^d}{a_1\cdots a_{n-1}}\\
&= \sum_{n=0}^{\infty}\frac{2^{dn}}{a_1\cdots a_n}\leq\sum_{n=0}^{\infty}2^{-dn}<\infty,
\end{aligned}$$

and the proof is complete. □

Therefore the proof of Theorem 4.1 is also complete. □

Corollary 4.6. *For the prevalent $f\in C[0,1]$ there is an open set $U_f\subset\mathbb{R}$ such that $\lambda(f^{-1}(U_f))=1$ (hence U_f is dense in $f([0,1])$) and for all $y\in U_f$*

$$\dim_H f^{-1}(y)=1.$$

4.2. **The ultrametric case.** Before turning to the general case, we prove the Main Theorem for ultrametric spaces. The key step is the following lemma, where the construction of the map h is based on the proof of [24, Thm. 2.1].

Lemma 4.7. *Let (K,d) be a compact ultrametric space and let μ be a continuous, finite Borel measure on K. Then there is a map $h\colon K\to[0,1]$ and there are compact sets $K_n\subset K$ such that if $h_n=h|_{K_n}$ and $D_n=h(K_n)$ then for all $n\in\mathbb{N}^+$*

(i) $\mu(\bigcup_{n=1}^{\infty}K_n)=\mu(K)$ and $\lambda(D_n)>0$;
(ii) the maps $h_n\colon K_n\to D_n$ are homeomorphisms;
(iii) for every Borel set $B\subset D_n$ we have $\mu(h_n^{-1}(B))=\lambda(B)$;
(iv) for every non-empty $B\subset D_n$ we have $\dim_H h_n^{-1}(B)\geq\dim_H\mu\cdot\dim_H B$;
(v) for every non-empty $B\subset D_n$ we have $\dim_P h_n^{-1}(B)\geq\dim_P\mu\cdot\dim_H B$.

Proof. We may assume that $\mu(K)=1$. By [24, Lemma 2.3] we obtain that K is a 1-monotone metric space, that is, there exists a linear order $\prec$ on K such that $\operatorname{diam}[a,b]=d(a,b)$ for all $a,b\in K$, where $[a,b]$ denotes the closed interval $\{x\in K: a\preceq x\preceq b\}$. It is easy to show (see also in [32]) that any open interval $(a,b)=\{x\in K: a\prec x\prec b\}$ and every open half-line $(-\infty,x)=\{z\in K: z\prec x\}$, $(x,\infty)=\{z\in K: x\prec z\}$ is open in K, so every interval is Borel. Let us define

$$h\colon K\to[0,1],\quad h(x)=\mu((-\infty,x)).$$

Then for every Borel set $B\subset[0,1]$

$$\mu(h^{-1}(B))=\lambda(B), \tag{4.26}$$

since it holds for intervals in $[0,1]$ by the definition of h, so Carathéodory's extension theorem yields that the Borel measures $\mu\circ h^{-1}$ and $\lambda|_{[0,1]}$ coincide.

Now we show that $h(K)=[0,1]$. The continuity of μ implies that h is continuous, so $h(K)$ is compact. Since K is compact and the intervals of the form $(-\infty,b)$ and (a,∞) are open, there exists a minimal element x_- and a maximal element

x_+ in $(K, \prec)$. Then $h(x_-) = \mu(\emptyset) = 0$, and the continuity of μ yields $h(x_+) = \mu((-\infty, x_+)) = \mu(K \setminus \{x_+\}) = \mu(K) = 1$. Hence $\{0,1\} \subset h(K)$ and $h(K)$ is a subset of $[h(x_-), h(x_+)] = [0,1]$. Thus it is enough to prove that there are no $u, v \in h(K)$ with $u < v$ and $(u,v) \cap h(K) = \emptyset$. Assume to the contrary that there exist such u and v. Let S be a countable dense subset of K and let $S_1 = \{s \in S : h(s) \leq u\}$ and $S_2 = \{t \in S : h(t) \geq v\}$. Then $(u,v) \cap h(K) = \emptyset$ implies that $S = S_1 \cup S_2$. Since S_1 and S_2 are countable and $h(x) = \mu((-\infty, x))$, we obtain $\mu(\bigcup_{s \in S_1}(-\infty, s)) \leq u$ and $\mu(\bigcap_{t \in S_2}(-\infty, t)) \geq v$. Let $E = \bigcap_{t \in S_2}(-\infty, t) \setminus \bigcup_{s \in S_1}(-\infty, s)$, then we have $\mu(E) \geq v - u > 0$. But E can contain at most two points: If $a, b, c \in E$ and $a \prec b \prec c$, then (a, c) would be a non-empty open set not containing any point of the dense set $S = S_1 \cup S_2$. Then the continuity of μ implies that $\mu(E) = 0$, which is a contradiction. Thus $h(K) = [0,1]$.

We prove that $Y = \{y \in [0,1] : \#h^{-1}(y) \geq 2\}$ is countable. For all $n \in \mathbb{N}^+$ let

$$Y_n = \{y \in [0,1] : \operatorname{diam} h^{-1}(y) \geq 2/n\}.$$

As $Y = \bigcup_{n=1}^{\infty} Y_n$, it is enough to show that Y_n is finite for all $n \in \mathbb{N}^+$. Let us fix n and for all $y \in Y_n$ pick $a_y, b_y \in h^{-1}(y)$ such that $a_y \prec b_y$ and $d(a_y, b_y) > 1/n$. Let us say that a set is $1/n$-separated if the distance between every pair of its points is at least $1/n$. Since a compact metric space does not have an infinite $1/n$-separated subspace, it is enough to prove that $A_n = \{a_y : y \in Y_n\}$ is $1/n$-separated. Let $y, w \in Y_n$ such that $y < w$. Then the definition of h yields $a_y \prec b_y \preceq a_w$. Thus the definition of the order $\prec$ implies

$$1/n < d(a_y, b_y) = \operatorname{diam}[a_y, b_y] \leq \operatorname{diam}[a_y, a_w] = d(a_y, a_w),$$

so A_n is $1/n$-separated.

Let us define

$$X = \left\{ x \in K : \limsup_{r \to 0+} \frac{\mu(B(x,r))}{r^s} < \infty \text{ for all } s < \dim_H \mu \right\},$$
$$Z = \left\{ x \in K : \liminf_{r \to 0+} \frac{\mu(B(x,r))}{r^s} < \infty \text{ for all } s < \dim_P \mu \right\}.$$

Theorem 2.2 yields that $\mu(X \cap Z) = 1$. Since Y is countable and μ is continuous, (4.26) implies that $\mu(h^{-1}(Y)) = 0$. Therefore we can choose compact sets $K_n \subset (X \cap Z) \setminus h^{-1}(Y)$ such that $\mu(K_n) > 0$ for all $n \in \mathbb{N}^+$ and $\mu(\bigcup_{n=1}^{\infty} K_n) = 1$. Clearly, the h_n are one-to-one, so (ii) holds. Then (4.26) yields (iii) and $\lambda(D_n) > 0$, thus (i) is satisfied.

Now we prove (iv). We may assume that $\dim_H \mu > 0$, otherwise we are done. Let us fix $0 < s < \dim_H \mu$. As $K_n \subset X$ for all $n \in \mathbb{N}^+$, it is enough to show that for each non-empty $A \subset X$ we have

$$\dim_H h(A) \leq \frac{\dim_H A}{s}, \tag{4.27}$$

then letting $s \nearrow \dim_H \mu$ yields (iv). Fix a non-empty $A \subset X$ and for all $i \in \mathbb{N}^+$ let

$$X_i = \{x \in X : \mu(B(x,r)) \leq i r^s \text{ for all } r \geq 0\}.$$

The definition of X clearly implies that $X = \bigcup_{i=1}^{\infty} X_i$. The definitions of h and the linear order $\prec$ yield that for all $x, z \in X_i$ with $x \preceq z$ we have

$$|h(x) - h(z)| = \mu([x,z)) \leq \mu(B(x, d(x,z))) \leq i(d(x,z))^s,$$

so the $h|_{X_i}$ are s-Hölder. By Fact 2.1 we have $\dim_H h(A\cap X_i) \le (\dim_H(A\cap X_i))/s$ for all $i \in \mathbb{N}^+$. Therefore $A \subset X = \bigcup_{i=1}^{\infty} X_i$ and the countable stability of the Hausdorff dimension imply that

$$\dim_H h(A) = \sup_{i\in\mathbb{N}^+} \dim_H h(A\cap X_i) \le \sup_{i\in\mathbb{N}^+} \frac{\dim_H(A\cap X_i)}{s} = \frac{\dim_H A}{s},$$

thus (4.27) holds. Hence (iv) is satisfied.

Finally, we show (v). We may assume that $\dim_P \mu > 0$, otherwise we are done. Let us fix an arbitrary $0 < s < \dim_P \mu$. As $K_n \subset Z$ for all $n \in \mathbb{N}^+$, it is enough to show that for each fixed non-empty $A \subset Z$ we have $\dim_H h(A) \le (\dim_P A)/s$, then letting $s \nearrow \dim_P \mu$ yields (v). We may suppose that $\dim_P A < \infty$, otherwise we are done. It is enough to show that for each fixed $t > \dim_P A$ we have

$$\dim_H h(A) \le t/s, \tag{4.28}$$

then letting $t \searrow \dim_P A$ finishes the proof. The definition of the packing dimension implies that there are sets A_i such that $A = \bigcup_{i=1}^{\infty} A_i$ and $\overline{\dim}_B A_i < t$ for all $i \in \mathbb{N}^+$. For all $j \in \mathbb{N}^+$ let

$$Z_j = \{x \in Z : \mu(B(x, 2^{-n})) \le j2^{-ns} \text{ for infinitely many } n \in \mathbb{N}^+\}.$$

The definition of Z implies that $Z = \bigcup_{j=1}^{\infty} Z_j$. Let us fix $i, j \in \mathbb{N}^+$ and let $D = A_i \cap Z_j$. Now we show that

$$\dim_H h(D) \le t/s. \tag{4.29}$$

Since $\overline{\dim}_B D < t$, we can fix u such that $\overline{\dim}_B D < u < t$. For all $n \in \mathbb{N}^+$ let

$$\mathcal{N}_n = \{B(x, 2^{-n}) : x \in D\},$$
$$\mathcal{S}_n = \{B(x, 2^{-n}) : x \in D \text{ and } \mu(B(x, 2^{-n})) \le j2^{-ns}\}.$$

Then clearly $\mathcal{S}_n \subset \mathcal{N}_n$. Fact 2.4 and the compactness of K yield that $\mathcal{N}_n$ consists of finitely many pairwise disjoint balls, so $\#\mathcal{N}_n = N_{2^{-n}}(D)$ for all $n \in \mathbb{N}^+$, where we recall that $N_{2^{-n}}(D)$ is the smallest number of closed balls with radius 2^{-n} whose union cover D. Thus $\overline{\dim}_B D < u$ and the definition of the upper box dimension yields that for all $n \in \mathbb{N}^+$

$$\#\mathcal{S}_n \le N_{2^{-n}}(D) \le c_1 2^{nu}, \tag{4.30}$$

where $c_1 \in \mathbb{R}^+$. Let $S \in \mathcal{S}_n$ for some $n \in \mathbb{N}^+$. For all $x, z \in S$ with $x \preceq z$ the definition of $\prec$ implies that $[x, z) \subset B(x, d(x, z)) \subset B(x, 2^{-n})$. Thus the definition of h and $x \in Z_j$ yield

$$|h(x) - h(z)| = \mu([x, z)) \le \mu(B(x, 2^{-n})) \le c_2 2^{-ns},$$

where $c_2 = j$. Therefore for all $n \in \mathbb{N}^+$ and $S \in \mathcal{S}_n$

$$\operatorname{diam} h(S) \le c_2 2^{-ns}. \tag{4.31}$$

Since $D \subset Z_j$, we have $D \subset \bigcup_{n=N}^{\infty} \bigcup_{S\in\mathcal{S}_n} S$ for all $N \in \mathbb{N}^+$. Thus for all $N \in \mathbb{N}^+$

$$h(D) \subset \bigcup_{n=N}^{\infty} \bigcup_{S\in\mathcal{S}_n} h(S). \tag{4.32}$$

For all $N \in \mathbb{N}^+$ let $\delta_N = j2^{-Ns}$. Then (4.32), (4.30), (4.31) and $u < t$ imply

$$\begin{aligned}\mathcal{H}^{t/s}_{\delta_N}(h(D)) &\le \sum_{n=N}^{\infty} \sum_{S\in\mathcal{S}_n} (\operatorname{diam} h(S))^{t/s} \\ &\le \sum_{n=N}^{\infty} c_1 2^{nu}(c_2 2^{-ns})^{t/s} = c_3 2^{N(u-t)},\end{aligned}$$

where $c_3 = c_1 {c_2}^{t/s}(1-2^{u-t})^{-1}$. Then $u < t$ yields $\lim_{N\to\infty} \mathcal{H}^{t/s}_{\delta_N}(h(D)) = 0$, so $\mathcal{H}^{t/s}(h(D)) = 0$. Therefore $\dim_H h(D) \le t/s$, thus (4.29) holds.

As $A = \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}(A_i \cap Z_j)$, the countable stability of the Hausdorff dimension and (4.29) imply

$$\dim_H h(A) = \sup_{i,j\in\mathbb{N}^+} \dim_H h(A_i \cap Z_j) \le t/s,$$

thus (4.28) holds. This implies (v), and the proof is complete. □

Theorem 4.8. *Let K be a compact ultrametric space and let μ be a continuous, finite Borel measure on K. If $d \in \mathbb{N}^+$ then for the prevalent $f \in C(K,\mathbb{R}^d)$ there is an open set $U_f \subset \mathbb{R}^d$ such that $\mu(f^{-1}(U_f)) = \mu(K)$ and for all $y \in U_f$*

$$\dim_H f^{-1}(y) \ge \dim_H \mu \quad \textit{and} \quad \dim_P f^{-1}(y) \ge \dim_P \mu.$$

Proof. For all $n \in \mathbb{N}^+$ let us choose compact sets $K_n \subset K$ and $D_n \subset \mathbb{R}$ and homeomorphisms $h_n\colon K_n \to D_n$ according to Lemma 4.7. As $\mu(\bigcup_{n=1}^{\infty} K_n) = \mu(K)$, Lemma 3.12 yields that it is enough to prove that the sets

$$\begin{aligned}\mathcal{A}_n = \{f \in C(K_n,\mathbb{R}^d) : \exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that } \mu(f^{-1}(U_f)) = \mu(K_n) \\ \text{and } \dim_H f^{-1}(y) \ge \dim_H \mu \text{ and } \dim_P f^{-1}(y) \ge \dim_P \mu \text{ for all } y \in U_f\}\end{aligned}$$

are prevalent in $C(K_n,\mathbb{R}^d)$. As $\lambda(D_n) > 0$ by (i), Theorem 4.1 implies that

$$\begin{aligned}\mathcal{B}_n = \{f \in C(D_n,\mathbb{R}^d) : \exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that} \\ \lambda(f^{-1}(U_f)) = \lambda(D_n) \text{ and } \dim_H f^{-1}(y) = 1 \text{ for all } y \in U_f\}\end{aligned}$$

are prevalent in $C(D_n,\mathbb{R}^d)$. Fix $n \in \mathbb{N}^+$ and define

$$H_n\colon C(K_n,\mathbb{R}^d) \to C(D_n,\mathbb{R}^d), \quad H_n(f) = f \circ h_n^{-1}.$$

Then H_n is a continuous group isomorphism, so Lemma 3.10 yields that $H_n^{-1}(\mathcal{B}_n)$ is prevalent in $C(K_n,\mathbb{R}^d)$. Therefore it is enough to prove that $H_n^{-1}(\mathcal{B}_n) \subset \mathcal{A}_n$. Let us fix $f \in H_n^{-1}(\mathcal{B}_n)$, we need to prove that $f \in \mathcal{A}_n$. Let $g = H_n(f) = f \circ h_n^{-1} \in \mathcal{B}_n$, then there exists a non-empty open set $U_g \subset \mathbb{R}^d$ such that $\lambda(g^{-1}(U_g)) = \lambda(D_n)$ and $\dim_H g^{-1}(y) = 1$ for all $y \in U_g$. Let $U_f = U_g$. Applying (iii) twice with $K_n = h_n^{-1}(D_n)$ implies that

$$\mu(f^{-1}(U_f)) = \mu(h_n^{-1}(g^{-1}(U_g))) = \lambda(g^{-1}(U_g)) = \lambda(D_n) = \mu(K_n).$$

Let us fix $y \in U_f$. Then (iv) and (v) yield that

$$\begin{aligned}\dim_H f^{-1}(y) = \dim_H h_n^{-1}(g^{-1}(y)) \ge \dim_H \mu \cdot \dim_H g^{-1}(y) = \dim_H \mu, \\ \dim_P f^{-1}(y) = \dim_P h_n^{-1}(g^{-1}(y)) \ge \dim_P \mu \cdot \dim_H g^{-1}(y) = \dim_P \mu.\end{aligned}$$

These imply that $f \in \mathcal{A}_n$, and the proof is complete. □

4.3. **The Main Theorem.** We prove the Main Theorem after some preparation.

Definition 4.9. Let X be a metric space. For all $r > 0$ let $N(r)$ be the minimal number such that every closed ball $B(x, r)$ can be covered by $N(r)$ closed balls of radius $r/2$. Then X is called *doubling* if $\sup\{N(r) : r > 0\} < \infty$. We say that X is *non-exploding* if

$$\lim_{r\to 0+} \frac{\log N(r)}{\log r} = 0.$$

Every subspace of $\mathbb{R}^m$ is doubling, and every doubling space is non-exploding.

Definition 4.10. Let X, Y be metric spaces. A map $f\colon X \to Y$ is called *nearly Lipschitz* if f is s-Hölder for all $s < 1$. We say that f is *nearly bi-Lipschitz* if it is one-to-one and both f and f^{-1} are nearly Lipschitz. The spaces X and Y are *nearly Lipschitz equivalent* if there exists a nearly bi-Lipschitz onto map $f\colon X \to Y$.

Fact 4.11. *If X, Y are metric spaces and $f\colon X \to Y$ is a nearly bi-Lipschitz map then $\dim_H f(A) = \dim_H A$ and $\dim_P f(A) = \dim_P A$ for all $A \subset X$.*

The above fact follows from Fact 2.1. For the following theorem see [41].

Theorem 4.12 (Zindulka)**.** *Let K be a non-exploding compact metric space and let μ be a finite Borel measure on K. Then there exists a compact set $C \subset K$ such that $\mu(C) > 0$ and C is nearly bi-Lipschitz equivalent to an ultrametric space.*

Theorem 4.13 (Main Theorem)**.** *Let K be a non-exploding compact metric space, let μ be a continuous, finite Borel measure on K and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there is an open set $U_f \subset \mathbb{R}^d$ such that $\mu(f^{-1}(U_f)) = \mu(K)$ and for all $y \in U_f$*

$$\dim_H f^{-1}(y) \geq \dim_H \mu \quad \text{and} \quad \dim_P f^{-1}(y) \geq \dim_P \mu.$$

Proof. By Theorem 4.12 there exist compact sets $K_n \subset K$ such that $\mu(K_n) > 0$ for all $n \in \mathbb{N}^+$ and $\mu(\bigcup_{n=1}^{\infty} K_n) = \mu(K)$, and there are compact ultrametric spaces C_n and nearly bi-Lipschitz onto maps $h_n\colon K_n \to C_n$. Define the finite Borel measures $\mu_n = \mu|_{K_n}$ on K_n and $\nu_n = \mu \circ h_n^{-1}$ on C_n for all $n \in \mathbb{N}^+$. Then $\dim_H \mu_n \geq \dim_H \mu$ and $\dim_P \mu_n \geq \dim_P \mu$ by the definition, thus Lemma 3.12 yields that it is enough to prove that the sets

$$\begin{aligned}\mathcal{A}_n = \{f \in C(K_n, \mathbb{R}^d) : \exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that } \mu_n(f^{-1}(U_f)) = \mu_n(K_n)\\ \text{and } \dim_H f^{-1}(y) \geq \dim_H \mu_n \text{ and } \dim_P f^{-1}(y) \geq \dim_P \mu_n \text{ for all } y \in U_f\}\end{aligned}$$

are prevalent in $C(K_n, \mathbb{R}^d)$. Clearly, the ν_n are continuous Borel measures such that $\nu_n(C_n) = \mu(K_n) > 0$, thus Theorem 4.8 implies that the sets

$$\begin{aligned}\mathcal{B}_n = \{f \in C(C_n, \mathbb{R}^d) : \exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that } \nu_n(f^{-1}(U_f)) = \nu_n(C_n)\\ \text{and } \dim_H f^{-1}(y) \geq \dim_H \nu_n \text{ and } \dim_P f^{-1}(y) \geq \dim_P \nu_n \text{ for all } y \in U_f\}\end{aligned}$$

are prevalent in $C(C_n, \mathbb{R}^d)$. Fix $n \in \mathbb{N}^+$ and define $H_n\colon C(K_n, \mathbb{R}^d) \to C(C_n, \mathbb{R}^d)$ as $H_n(f) = f \circ h_n^{-1}$. Then H_n is a continuous isomorphism, so Lemma 3.10 yields that $H_n^{-1}(\mathcal{B}_n)$ is prevalent in $C(K_n, \mathbb{R}^d)$. Since h_n is nearly bi-Lipschitz, by Fact 4.11 we have $\dim_H h_n^{-1}(B) = \dim_H B$ and $\dim_P h_n^{-1}(B) = \dim_P B$ for all $B \subset C_n$. Hence $\mu \circ h_n^{-1} = \nu_n$ yields that $\dim_H \mu_n = \dim_H \nu_n$ and $\dim_P \mu_n = \dim_P \nu_n$. These easily imply that $H_n^{-1}(\mathcal{B}_n) = \mathcal{A}_n$, so $\mathcal{A}_n$ is prevalent. The proof is complete. □

Corollary 4.14. *Let $m, d \in \mathbb{N}^+$. Then for the prevalent $f \in C([0,1]^m, \mathbb{R}^d)$ there is an open set $U_f \subset \mathbb{R}^d$ such that $\lambda^m(f^{-1}(U_f)) = 1$ (hence U_f is dense in $f([0,1]^m)$) and for all $y \in U_f$*

$$\dim_H f^{-1}(y) = m.$$

Corollary 4.15. *Let K be an uncountable, non-exploding compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ for all $s < \dim_P K$ there is a non-empty open set $U_{f,s} \subset \mathbb{R}^d$ such that for all $y \in U_{f,s}$*

$$\dim_P f^{-1}(y) \geq s.$$

Proof. Fix $s = 0$ if $\dim_P K = 0$ and $s \in (0, \dim_P K)$ if $\dim_P K > 0$. Theorem 2.3 implies that there is a continuous, finite Borel measure μ on K with $\dim_P \mu \geq s$. Applying the Main Theorem for μ yields that

$$\mathcal{A}(s) = \{f \in C(K, \mathbb{R}^d) : \exists \text{ a non-empty open set } U_f \subset \mathbb{R}^d \\ \text{such that } \dim_P f^{-1}(y) \geq s \text{ for all } y \in U_f\}$$

is prevalent in $C(K, \mathbb{R}^d)$. If $\dim_P K = 0$ then we are done, otherwise choose a sequence $s_n \nearrow \dim_P K$, then $\bigcap_{n=1}^{\infty} \mathcal{A}(s_n)$ is the desired prevalent set. □

Corollary 4.16. *Let K be a non-exploding compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\sup\{\dim_P f^{-1}(y) : y \in \mathbb{R}^d\} = \dim_P K.$$

In the case of Hausdorff dimension we generalize the above two corollaries. For the following theorem see [29, Thm. 1.4].

Theorem 4.17 (Mendel-Naor)**.** *If K is a compact metric space and $s < \dim_H K$ then there exists a compact set $C \subset K$ such that $\dim_H C > s$ and C is bi-Lipschitz equivalent to an ultrametric space.*

Theorem 4.18. *Let K be an uncountable compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ for all $s < \dim_H K$ there is a non-empty open set $U_{f,s} \subset \mathbb{R}^d$ such that for all $y \in U_{f,s}$*

$$\dim_H f^{-1}(y) \geq s.$$

Proof. By Theorem 1.7 we may assume $\dim_H K > 0$. Fix $s \in (0, \dim_H K)$ and let

$$\mathcal{A} = \{f \in C(K, \mathbb{R}^d) : \exists \text{ a non-empty open set } U_f \subset \mathbb{R}^d \\ \text{such that } \dim_H f^{-1}(y) \geq s \text{ for all } y \in U_f\}.$$

It is enough to prove that $\mathcal{A} = \mathcal{A}(s)$ is prevalent, since we can choose a sequence $s_n \nearrow \dim_H K$ and $\bigcap_{n=1}^{\infty} \mathcal{A}(s_n)$ will be a prevalent set in $C(K, \mathbb{R}^d)$ satisfying the theorem. By Theorem 4.17 there is a compact set $C \subset K$ such that $\dim_H C > s$ and there exist a compact ultrametric space D and a bi-Lipschitz onto map $h \colon C \to D$. By Fact 4.11 we have $\dim_H D > s$, so Theorem 2.3 yields that there exists a continuous, finite measure μ on D such that $\dim_H \mu \geq s$. Therefore Theorem 4.8 implies that

$$\mathcal{B} = \{g \in C(D, \mathbb{R}^d) : \exists \text{ a non-empty open set } U_g \subset \mathbb{R}^d \\ \text{such that } \dim_H g^{-1}(y) \geq s \text{ for all } y \in U_g\}$$

is prevalent in $C(D,\mathbb{R}^d)$. Consider

$$R\colon C(K,\mathbb{R}^d)\to C(D,\mathbb{R}^d),\quad R(f)=f|_C\circ h^{-1}.$$

As h is a homeomorphism, R is a composition of two continuous onto homomorphisms, so R is a continuous onto homomorphism. Thus Lemma 3.10 implies that $R^{-1}(\mathcal{B})\subset C(K,\mathbb{R}^d)$ is prevalent, so it is enough to prove that $R^{-1}(\mathcal{B})\subset\mathcal{A}$. Let us fix $f\in R^{-1}(\mathcal{B})$, we need to prove that $f\in\mathcal{A}$. Let $g=R(f)=f|_C\circ h^{-1}\in\mathcal{B}$, then there exists a non-empty open set $U_g\subset\mathbb{R}^d$ such that $\dim_H g^{-1}(y)\geq s$ for all $y\in U_g$. Let $U_f=U_g$ and fix $y\in U_f$. Then $h^{-1}(g^{-1}(y))=(f|_C)^{-1}(y)\subset f^{-1}(y)$, so applying Fact 4.11 for the bi-Lipschitz map h yields that

$$\dim_H f^{-1}(y)\geq\dim_H h^{-1}(g^{-1}(y))=\dim_H g^{-1}(y)\geq s.$$

Hence $f\in\mathcal{A}$, and the proof is complete. □

Corollary 4.19. *Let K be a compact metric space and let $d\in\mathbb{N}^+$. Then for the prevalent $f\in C(K,\mathbb{R}^d)$*

$$\sup\{\dim_H f^{-1}(y)\colon y\in\mathbb{R}^d\}=\dim_H K.$$

4.4. **Fibers of maximal dimension.** First we prove that one cannot replace supremum with maximum in Corollaries 4.16 and 4.19. For the following well-known lemmas see e.g. [41, Lemma 4] and [5], respectively. Note that Lemma 4.21 is stated in [5] only in the case $K=[0,1]$, but the proof works verbatim for all K.

Lemma 4.20. *Let G be an abelian Polish group and let $A\subset G$. If for all compact set $K\subset G$ there exists a $g\in G$ such that $K+g\subset A$ then A is non-shy.*

Lemma 4.21. *Let $K\subset[0,1]$ and $\mathcal{K}\subset C(K,\mathbb{R})$ be compact sets. Then there is a strictly increasing function $h\in C[0,1]$ such that $h(0)=0$ and for all $f\in\mathcal{K}$ and $x,z\in K$, $x\neq z$ we have*

$$|f(x)-f(z)|<h(|x-z|).$$

Theorem 4.22. *There is a compact set $K\subset\mathbb{R}$ such that $\dim_H K=\dim_P K=1$ and*

$$\mathcal{A}=\{f\in C(K,\mathbb{R})\colon\dim_H f^{-1}(y)\leq\dim_P f^{-1}(y)<1\text{ for all }y\in\mathbb{R}\}$$

is non-shy in $C(K,\mathbb{R})$.

Proof. For all $n\in\mathbb{N}^+$ let $K_n\subset[0,1/n]$ be compact sets such that $\dim_H K_n=\dim_P K_n=1-1/n$ and let $K=\bigcup_{n=1}^{\infty}K_n\cup\{0\}$. Then $K\subset[0,1]$ is compact and $\dim_H K=\dim_P K=1$. Define

$$\mathcal{B}=\{f\in C(K,\mathbb{R})\colon f^{-1}(f(0))=\{0\}\}.$$

Now we show that $\mathcal{B}\subset\mathcal{A}$, that is, $\dim_H f^{-1}(y)\leq\dim_P f^{-1}(y)<1$ for every $f\in\mathcal{B}$ and $y\in\mathbb{R}$. The first inequality clearly holds, so it is enough to prove that $\dim_P f^{-1}(y)<1$. Let us fix $f\in\mathcal{B}$, by definition $f^{-1}(f(0))=\{0\}$. For all $y\in\mathbb{R}\setminus\{f(0)\}$ the level set $f^{-1}(y)\subset K\setminus\{0\}$ is compact, thus it can be covered by finitely many sets K_n. Therefore the countable stability of packing dimension yields that $\dim_P f^{-1}(y)<1$.

Finally, it is enough to show that $\mathcal{B}$ is non-shy in $C(K,\mathbb{R})$. Let $\mathcal{K}\subset C(K,\mathbb{R})$ be an arbitrary compact set, by Lemma 4.20 it is enough to prove that there exists a $g\in C(K,\mathbb{R})$ with $\mathcal{K}+g\subset\mathcal{B}$. By Lemma 4.21 there is a strictly increasing

function $h \in C[0,1]$ such that $h(0) = 0$ and $|f(x) - f(0)| < h(x)$ for all $f \in \mathcal{K}$ and $x \in K \setminus \{0\}$. Let $g = h|_K$, then for all $f \in \mathcal{K}$ and $x \in K \setminus \{0\}$ we have

$$f(x) + g(x) = f(x) + h(x) > f(0) = f(0) + h(0) = f(0) + g(0),$$

thus $f + g \in \mathcal{B}$. This completes the proof. □

If K is 'large in its dimension' then the maximum is attained in Corollaries 4.16 and 4.19. The Main Theorem easily implies the next corollaries.

Corollary 4.23. *Let K be a non-exploding compact metric space and let $d \in \mathbb{N}^+$. Let μ be a continuous, finite Borel measure on K such that $\dim_H \mu = \dim_H K$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists an open set $U_f \subset \mathbb{R}^d$ such that $\mu(f^{-1}(U_f)) = \mu(K)$ and for all $y \in U_f$*

$$\dim_H f^{-1}(y) = \dim_H K.$$

Corollary 4.24. *Let K be a non-exploding compact metric space and let $d \in \mathbb{N}^+$. Let μ be a continuous, finite Borel measure on K such that $\dim_P \mu = \dim_P K$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists an open set $U_f \subset \mathbb{R}^d$ such that $\mu(f^{-1}(U_f)) = \mu(K)$ and for all $y \in U_f$*

$$\dim_P f^{-1}(y) = \dim_P K.$$

Remark 4.25. Note that the compact set K in Theorem 4.22 can be decomposed as $K = \bigcup_{n=1}^{\infty} A_n$ such that $\dim_H A_n < \dim_H K$. Assume that K is a non-exploding compact metric space with $\dim_H K < \infty$ and such a decomposition does not exist, then we sketch that K satisfies the assumption of Corollary 4.23. By [36, Thm. 2] there is a gauge function (see Section 7 for the definition) $\varphi\colon [0,\infty) \to [0,\infty)$ such that

$$\liminf_{r \to 0+} \frac{\log \varphi(r)}{\log r} = \dim_H K \quad \text{and} \quad \mathcal{H}^{\varphi}(K) > 0,$$

where $\mathcal{H}^{\varphi}$ denotes the φ-Hausdorff measure (see Section 7 for the definition). As $\dim_H K < \infty$, we may assume that φ is of finite order, that is, $\varphi(2r) \leq c\varphi(r)$ for some $c \in \mathbb{R}^+$ and for all $r \in [0,\infty)$. Thus [19] yields that there is a compact set $C \subset K$ such that $0 < \mathcal{H}^{\varphi}(C) < \infty$. Then $\mu = \mathcal{H}^{\varphi}|_C$ is a finite Borel measure on K with $\dim_H \mu = \dim_H K$. Therefore Corollary 4.23 holds for K.

4.5. **The homogeneous case.** Let us now consider continuous, finite Borel measures μ on K such that $\operatorname{supp} \mu = K$. Then the larger $\dim_H \mu$ or $\dim_P \mu$ can be, the more homogeneous K is. The Main Theorem yields the following corollary.

Corollary 4.26. *Let K be a non-exploding compact metric space and let $d \in \mathbb{N}^+$. Let μ be a continuous, finite Borel measure on K such that $\operatorname{supp} \mu = K$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists an open set $U_f \subset \mathbb{R}^d$ such that $\mu(f^{-1}(U_f)) = \mu(K)$ (hence U_f is dense in $f(K)$) and for all $y \in U_f$*

$$\dim_H f^{-1}(y) \geq \dim_H \mu \quad \text{and} \quad \dim_P f^{-1}(y) \geq \dim_P \mu.$$

If $K \subset \mathbb{R}^m$ is a self-similar set satisfying the open set condition, we can say more.

Corollary 4.27. *Let $m, d \in \mathbb{N}^+$ and let $K \subset \mathbb{R}^m$ be a self-similar set satisfying the open set condition. It is well-known that $\dim_H K = \dim_P K = s$ and $0 <$*

$\mathcal{H}^s(K), \mathcal{P}^s(K) < \infty$. *Then for the prevalent* $f \in C(K, \mathbb{R}^d)$ *there exists an open set* $U_f \subset \mathbb{R}^d$ *such that* $\mathcal{H}^s(f^{-1}(U_f)) = \mathcal{H}^s(K)$ *(hence* U_f *is dense in* $f(K)$*) and*

$$\dim_H f^{-1}(y) = s \text{ for all } y \in U_f.$$

Similarly, for the prevalent $f \in C(K, \mathbb{R}^d)$ *there exists an open set* $V_f \subset \mathbb{R}^d$ *such that* $\mathcal{P}^s(f^{-1}(V_f)) = \mathcal{P}^s(K)$ *(hence* V_f *is dense in* $f(K)$*) and*

$$\dim_P f^{-1}(y) = s \text{ for all } y \in V_f.$$

Proof. For the fact that $\dim_H K = \dim_P K = s$ and $0 < \mathcal{H}^s(K), \mathcal{P}^s(K) < \infty$ see [13, Thm. 2.7]. Applying the Main Theorem for $\mu = \mathcal{H}^s$ and $\mu = \mathcal{P}^s$ concludes the proof, we need to show only that U_f and V_f are dense in $f(K)$. Let U be a non-empty open set in K, then it is enough to prove that $\mathcal{H}^s(U) > 0$ and $\mathcal{P}^s(U) > 0$. As K is self-similar, U contains a similar copy of K, and the above statement follows. □

Finally, we prove two characterization theorems.

Theorem 4.28. *If* K *is a compact metric space and* $d \in \mathbb{N}^+$ *then the following statements are equivalent:*

(i) $\dim_H f^{-1}(y) = \dim_H K$ *for the prevalent* $f \in C(K, \mathbb{R}^d)$ *and for the generic* $y \in f(K)$*;*
(ii) $\dim_H U = \dim_H K$ *for every non-empty open set* $U \subset K$*.*

Proof. $(ii) \Rightarrow (i)$: We may assume that $\dim_H K > 0$, otherwise the statement is trivial. Choose a positive sequence $s_i \nearrow \dim_H K$ and let $\mathcal{V} = \{V_n : n \in \mathbb{N}^+\}$ be a countable basis of K consisting of non-empty open sets. For all $n \in \mathbb{N}^+$ let $K_n = \operatorname{cl} V_n$, then $\dim_H K_n = \dim_H K$. For all $i, n \in \mathbb{N}^+$ consider

$$\mathcal{A}_{i,n} = \{f \in C(K_n, \mathbb{R}^d) : \exists \text{ a non-empty open set } U_{f,s_i} \subset \mathbb{R}^d \text{ such that } \dim_H f^{-1}(y) \geq s_i \text{ for all } y \in U_{f,s_i}\}.$$

Theorem 4.18 implies that the $\mathcal{A}_{i,n}$ are prevalent. For all $n \in \mathbb{N}^+$ let us define

$$R_n \colon C(K, \mathbb{R}^d) \to C(K_n, \mathbb{R}^d), \quad R_n(f) = f|_{K_n}.$$

Corollary 3.11 yields that the $R_n^{-1}(\mathcal{A}_{i,n})$ are prevalent in $C(K, \mathbb{R}^d)$ for all $i, n \in \mathbb{N}^+$. As a countable intersection of prevalent sets, $\mathcal{A} = \bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} R_n^{-1}(\mathcal{A}_{i,n})$ is also prevalent in $C(K, \mathbb{R}^d)$. For all $f \in \mathcal{A}$ let

$$U_f = \bigcap_{i=1}^{\infty} \left(\bigcup_{n=1}^{\infty} U_{f|_{K_n}, s_i} \right).$$

As a countable intersection of dense open sets, U_f is co-meager in $f(K)$. Let us fix $f \in \mathcal{A}$ and $y \in U_f$, it is enough to prove that $\dim_H f^{-1}(y) = \dim_H K$. For all $i \in \mathbb{N}^+$ there is an $n \in \mathbb{N}^+$ such that $y \in U_{f|_{K_n}, s_i}$, therefore

$$\dim_H f^{-1}(y) \geq \dim_H (f|_{K_n})^{-1}(y) \geq s_i.$$

As $s_i \nearrow \dim_H K$, we obtain that $\dim_H f^{-1}(y) = \dim_H K$. Hence (i) holds.

$(i) \Rightarrow (ii)$: Assume to the contrary that there exist $x \in K$ and $r > 0$ such that $\dim_H U(x, r) < \dim_H K$. Tietze's extension theorem implies that there is a $g \in C(K, \mathbb{R}^d)$ such that $g(K \setminus U(x, r))$ and $g(B(x, r/2))$ are distinct points in $\mathbb{R}^d$. Then there exist an $\varepsilon > 0$ and an open set $U \subset \mathbb{R}^d$ such that $f(B(x, r/2)) \subset U$ and $f(K \setminus U(x, r)) \subset \mathbb{R}^d \setminus U$ for all $f \in B(g, \varepsilon)$. Clearly, $B(g, \varepsilon)$ is non-shy in $C(K, \mathbb{R}^d)$.

If $f \in B(g,\varepsilon)$ then $U \cap f(K)$ is a non-empty open set in $f(K)$ such that for all $y \in U$ we have $\dim_H f^{-1}(y) \leq \dim_H U(x,r) < \dim_H K$, which contradicts (i). □

Theorem 4.29. *If K is a non-exploding compact metric space and $d \in \mathbb{N}^+$ then the following statements are equivalent:*

(i) $\dim_P f^{-1}(y) = \dim_P K$ *for the prevalent $f \in C(K,\mathbb{R}^d)$ and for the generic $y \in f(K)$;*

(ii) $\dim_P U = \dim_P K$ *for every non-empty open set $U \subset K$.*

Proof. Repeat the proof of Theorem 4.28, only replace Hausdorff dimension with packing dimension, and apply Corollary 4.15 instead of Theorem 4.18. □

5. Positively many level sets can be singletons

The main goal of this section is to prove Theorem 5.5. The heart of the proof is the following theorem, which generalizes a result of Antunović, Burdzy, Peres and Ruscher [1, Prop. 3.3], see also Theorem 1.8. First we need an easy lemma. Recall that $\mathcal{Z}(f) = \{x \in [0,1] : f(x) = 0\}$.

Lemma 5.1. *The set $\mathcal{A} = \{f \in C[0,1] : \mathcal{Z}(f)$ is a singleton$\}$ is Borel.*

Proof. Clearly $\mathcal{A} = \mathcal{B}_1 \setminus \mathcal{B}_2$, where $\mathcal{B}_i = \{f \in C[0,1] : \#\mathcal{Z}(f) \geq i\}$. Since $\mathcal{B}_1$ is closed, it is enough to prove that $\mathcal{B}_2$ is Borel. If $\mathcal{I}$ denotes the pairs of disjoint closed rational subintervals of $[0,1]$, then $\mathcal{B}_2 = \bigcup_{(I_1,I_2)\in\mathcal{I}} \mathcal{B}_{I_1,I_2}$, where

$$\mathcal{B}_{I_1,I_2} = \{f \in C[0,1] : 0 \in f(I_1) \cap f(I_2)\}.$$

Clearly $\mathcal{B}_{I_1,I_2}$ are closed, therefore $\mathcal{B}_2$ is F_σ, thus Borel. □

Theorem 5.2. *Let μ be a Borel probability measure on $C[0,1]$. Then there exists a function $g \in C[0,1]$ such that*

$$\mu(\{f \in C[0,1] : \mathcal{Z}(f-g) \text{ is a singleton}\}) > 0.$$

Proof. Lemma 5.1 implies that $\mathcal{A} = \{f \in C[0,1] : \mathcal{Z}(f)$ is a singleton$\}$ is Borel, so $\{f \in C[0,1] : \mathcal{Z}(f-g)$ is a singleton$\} = \mathcal{A} + g$ is Borel for all $g \in C[0,1]$.

By Theorem 2.5 we may assume by shifting, restricting and normalizing μ that there is a compact set $\mathcal{K} \subset C[0,1]$ such that $\mu(\mathcal{K}) = 1$ and $f(x) \in [0,1]$ for all $f \in \mathcal{K}$ and $x \in [0,1]$. For each compact set $\Gamma \subset [0,1]^2$ let us define the compact set

$$\pi(\Gamma) = \{f \in \mathcal{K} : \exists (x,y) \in \Gamma \text{ such that } f(x) = y\}.$$

First assume that there exists a point $(x_0,y_0) \in [0,1]^2$ with $\mu(\pi(\{(x_0,y_0)\})) > 0$. By Lemma 4.21 there is a strictly increasing function $h \in C[0,1]$ such that $h(0) = 0$ and for all $f \in \mathcal{K}$ and $x,z \in [0,1]$, $x \neq z$

$$|f(x) - f(z)| < h(|x-z|). \tag{5.1}$$

Let us define $g \in C[0,1]$ as $g(x) = y_0 + h(|x-x_0|)$. Then clearly $\mathcal{Z}(f-g) = \{x_0\}$ for all $f \in \pi(\{(x_0,y_0)\})$, thus

$$\mu(\{f \in \mathcal{K} : \mathcal{Z}(f-g) \text{ is a singleton}\}) \geq \mu(\pi(\{(x_0,y_0)\})) > 0,$$

which concludes the proof.

Therefore we may assume that $\mu(\pi(\Delta)) = 0$ for every finite set $\Delta \subset [0,1]^2$. We prove that it is enough to find a function $g \in C[0,1]$ such that

$$\mu(\{f \in \mathcal{K} : \mathcal{Z}(f-g) \text{ has an isolated point}\}) > 0. \tag{5.2}$$

Indeed, by (5.2) we may assume that there are rationals $0 \le a < b \le 1$ with

$$\mu(\{f \in \mathcal{K} : \mathcal{Z}(f-g) \cap [a,b] \text{ is a singleton, } g(a) \le f(a),\ f(b) \le g(b)\}) > 0, \tag{5.3}$$

since the other three cases with reversed inequalities are similar. Define $\widehat{g} \in C[0,1]$ as

$$\widehat{g}(x) = \begin{cases} g(a) - h(a-x) & \text{if } x \in [0,a), \\ g(x) & \text{if } x \in [a,b], \\ g(b) + h(x-b) & \text{if } x \in (b,1]. \end{cases}$$

Then for all $f \in \mathcal{K}$ we have $\widehat{g}(x) < f(x)$ if $x \in [0,a)$ and $f(x) < \widehat{g}(x)$ if $x \in (b,1]$. Thus $\mathcal{Z}(f-\widehat{g}) = \mathcal{Z}(f-g) \cap [a,b]$ for all $f \in \mathcal{K}$, so (5.3) implies that

$$\mu(\{f \in C[0,1] : \mathcal{Z}(f-\widehat{g}) \text{ is a singleton}\}) > 0,$$

and we are done.

Now we show (5.2). First we define the function $g \in C[0,1]$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of positive reals such that

$$\alpha_0 = 1/2 \text{ and } \alpha_{n+1} \le \alpha_n/2 \text{ for all } n \in \mathbb{N}, \tag{5.4}$$

the exact values will be given later. For all $n \in \mathbb{N}^+$ and $(k_1,\dots,k_n) \in \{-1,1\}^n$ let

$$z_{k_1\dots k_n} = \frac{1}{2} + \sum_{i=1}^{n} k_i \alpha_i.$$

Consider

$$Z = \{0\} \cup \left\{ z_{k_1\dots k_n} : (k_1,\dots,k_n) \in \{-1,1\}^n,\ n \in \mathbb{N}^+ \right\}.$$

Let $g(0) = 0$, and for all $n \in \mathbb{N}^+$ and $(k_1,\dots,k_n) \in \{-1,1\}^n$ let

$$g(z_{k_1\dots k_n}) = \frac{1}{2} + \sum_{i=1}^{n} \frac{k_i}{2^{i+1}}. \tag{5.5}$$

For all $x \in [0,1]$ let

$$g(x) = \sup\{g(z) : z \in Z,\ z \le x\}. \tag{5.6}$$

Then (5.4) and (5.5) easily imply that $g|_Z$ is well-defined and non-decreasing, so g is also well-defined and non-decreasing. Therefore the definitions yield $g([0,1]) \subset [g(0), g(1)] = [0,1]$. As a monotone function can have only jump discontinuities and $g(Z)$ is dense in $[0,1]$ by (5.5), we obtain that $g\colon [0,1] \to [0,1]$ is continuous.

We prove that if the numbers α_n are small enough then g satisfies (5.2). For all $n \in \mathbb{N}^+$ and $i \in \{1,\dots,2^n\}$ let $\phi(i,n)$ be the ith element of $\{-1,1\}^n$ with respect to the lexicographical ordering. Note that $\phi(i,n)$ precedes $\phi(j,n)$ with respect to this ordering iff $z_{\phi(i,n)} < z_{\phi(j,n)}$ with respect to the usual ordering of the real numbers. Let $C_0 = [0,1]$ and $\Gamma_0 = \{1/2\} \times [0,1]$. For the definition of h recall (5.1). Assume by induction that α_n, C_n and Γ_n are already defined for some $n \in \mathbb{N}$ and let

$$C_{n+1} = C_n \setminus \bigcup_{i=0}^{2^{n+1}} U\left(\frac{i}{2^{n+1}}, h(4\alpha_{n+1})\right), \tag{5.7}$$

$$\Gamma_{n+1} = \bigcup_{i=1}^{2^{n+1}} \left(\left\{ z_{\phi(i,n+1)} \right\} \times \left(\left[\frac{i-1}{2^{n+1}}, \frac{i}{2^{n+1}} \right] \cap C_{n+1} \right) \right). \tag{5.8}$$

We show that if $\alpha_{n+1} \in (0, \alpha_n/2]$ is small enough then

$$\mu(\pi(\Gamma_{n+1}) \cap \pi(\Gamma_n)) \geq \mu(\pi(\Gamma_n)) - \frac{1}{2^{n+2}}. \tag{5.9}$$

For all $m \in \mathbb{N}^+$ let C_{n+1}^m and Γ_{n+1}^m be the sets C_{n+1} and Γ_{n+1} defined by the value $\alpha_{n+1} = 1/m$. For (5.9) it is enough to prove that

$$\lim_{m\to\infty} \mu(\pi(\Gamma_{n+1}^m) \cap \pi(\Gamma_n)) = \mu(\pi(\Gamma_n)). \tag{5.10}$$

Observe that Γ_n consists of finitely many vertical line segments and let

$$\Delta_n = \{(x, y) \in \Gamma_n : 2^{n+1}y \in \mathbb{N} \text{ or } (x, y) \text{ is an endpoint of a segment of } \Gamma_n\}.$$

As Δ_n is finite, $\mu(\pi(\Delta_n)) = 0$ by our assumption. Since $h(4/m) \to 0$ as $m \to \infty$, it is easy to see that for every given $(x, y) \in \Gamma_n \setminus \Delta_n$ there is a constant $c = c(x, y) > 0$ such that if m is large enough then Γ_{n+1}^m contains either the vertical line segment $\{x-1/m\}\times[y-c, y+c]$ or $\{x+1/m\}\times[y-c, y+c]$. Thus for every $f \in \pi(\Gamma_n)\setminus\pi(\Delta_n)$ there exists an $M(f) \in \mathbb{N}^+$ such that for all $m \geq M(f)$ we have $f \in \pi(\Gamma_{n+1}^m)$. Let us define for all $m \in \mathbb{N}^+$

$$\mathcal{A}_m = \{f \in (\pi(\Gamma_{n+1}^m) \cap \pi(\Gamma_n)) \setminus \pi(\Delta_n) : M(f) \leq m\},$$

then clearly for all $m \in \mathbb{N}^+$ we have $\mathcal{A}_m \subset \pi(\Gamma_{n+1}^m) \cap \pi(\Gamma_n)$. The definition of $M(f)$ implies that $\mathcal{A}_m \subset \mathcal{A}_{m+1}$ and the existence of $M(f)$ yields that $\bigcup_{m=1}^\infty \mathcal{A}_m = \pi(\Gamma_n) \setminus \pi(\Delta_n)$. Therefore $\mu(\pi(\Delta_n)) = 0$ implies that

$$\liminf_{m\to\infty} \mu(\pi(\Gamma_{n+1}^m) \cap \pi(\Gamma_n)) \geq \lim_{m\to\infty} \mu(\mathcal{A}_m) = \mu(\pi(\Gamma_n) \setminus \pi(\Delta_n)) = \mu(\pi(\Gamma_n)),$$

so (5.10) is satisfied. Thus α_n, C_n and Γ_n can be defined for all $n \in \mathbb{N}$ such that (5.7), (5.8) and (5.9) hold. Then (5.9) and $\mu(\pi(\Gamma_0)) = 1$ imply that

$$\mu\left(\bigcap_{n=1}^\infty \pi(\Gamma_n)\right) \geq 1 - \sum_{n=1}^\infty \frac{1}{2^{n+1}} = \frac{1}{2} > 0. \tag{5.11}$$

Let us consider

$$C = \bigcap_{n=1}^\infty C_n \quad \text{and} \quad K = g^{-1}(C).$$

We show that for all $f \in \mathcal{K}$

$$\mathcal{Z}(f - g) \cap K \subset \{\text{isolated points of } \mathcal{Z}(f - g)\}. \tag{5.12}$$

First we check that if for all $n \in \mathbb{N}^+$ and $i \in \{1, \dots, 2^n\}$

$$I_{i,n} = \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right),$$

then

$$\operatorname{diam} g^{-1}(I_{i,n}) = 2\sum_{k=n+1}^\infty \alpha_k \leq 4\alpha_{n+1}. \tag{5.13}$$

The inequality is clearly implied by (5.4), so we only need to check the equality. Note that it is important that the $I_{i,n}$ are open, otherwise the constant pieces of the graph would make the pre-image much bigger. If $y = \frac{2i-1}{2^{n+1}}$ is the midpoint of $I_{i,n}$ then it is easy to see that $g^{-1}(y) = \{z_{k_1\dots k_n}\}$ for some $(k_1, \dots, k_n) \in \{-1, 1\}^n$. Then $I_{i,n} = U(y, \sum_{k=n+1}^\infty \frac{1}{2^{k+1}})$, and one can check using the definition of g that this corresponds to $g^{-1}(I_{i,n}) = U(z_{k_1\dots k_n}, \sum_{k=n+1}^\infty \alpha_k)$, which proves (5.13).

Let $x \in \mathcal{Z}(f-g) \cap K$, it is enough to show that if $z \in [0,1]$ with $0 < |x-z| \le 4\alpha_1$ then $f(z) \neq g(z)$. As $\alpha_n \searrow 0$, there exists a unique number $n \in \mathbb{N}^+$ such that

$$4\alpha_{n+1} < |x-z| \le 4\alpha_n. \tag{5.14}$$

Since $x \in K$ implies that $g(x) \in C_n$, there exists an $i \in \{1, \dots, 2^n\}$ such that

$$g(x) \in I_{i,n} \text{ and } \operatorname{dist}(\{g(x)\}, \partial I_{i,n}) \ge h(4\alpha_n). \tag{5.15}$$

Then (5.13) and (5.14) imply that $|x-z| > \operatorname{diam} g^{-1}(I_{i,n})$, so $x \in g^{-1}(I_{i,n})$ yields that $z \notin g^{-1}(I_{i,n})$. Therefore $g(z) \notin I_{i,n}$ and (5.15) imply that

$$|g(x) - g(z)| \ge \operatorname{dist}(\{g(x)\}, \partial I_{i,n}) \ge h(4\alpha_n). \tag{5.16}$$

The monotonicity of h and (5.14) yield that

$$h(|x-z|) \le h(4\alpha_n). \tag{5.17}$$

Therefore $f(x) = g(x)$, the triangle inequality, (5.16), (5.1) and (5.17) imply that

$$\begin{aligned} |f(z) - g(z)| &= |(g(x) - g(z)) - (f(x) - f(z))| \\ &\ge |g(x) - g(z)| - |f(x) - f(z)| \\ &> h(4\alpha_n) - h(|x-z|) \ge 0, \end{aligned}$$

thus $f(z) \neq g(z)$. Hence (5.12) holds.

By (5.11) and (5.12) it is enough to show that

$$\bigcap_{n=1}^{\infty} \pi(\Gamma_n) \subset \{f \in \mathcal{K} : \mathcal{Z}(f-g) \cap K \neq \emptyset\}. \tag{5.18}$$

Let us fix $f \in \bigcap_{n=1}^{\infty} \pi(\Gamma_n)$, we need to find an $x \in K$ such that $f(x) = g(x)$. For every $n \in \mathbb{N}^+$ we can select points $(x_n, f(x_n)) \in \Gamma_n$. We can choose a convergent subsequence $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in [0,1]$. Then (5.8) yields that for every $k \in \mathbb{N}^+$ we have $x_{n_k} = z_{\phi(i_k,n_k)}$ and $f(x_{n_k}) \in I_{i_k,n_k}$ for some $i_k \in \{1, \dots, 2^{n_k}\}$. The definition of g implies that $g(x_{n_k})$ is the midpoint of I_{i_k,n_k}, so $|f(x_{n_k}) - g(x_{n_k})| \le 2^{-n_k} \le 2^{-k}$ for all $k \in \mathbb{N}^+$. Therefore

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} g(x_{n_k}) = g(x).$$

Then (5.8) yields $f(x_{n_k}) \in C_{n_k}$ for all $k \in \mathbb{N}^+$. By (5.7) we have $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}^+$, so $f(x) \in \bigcap_{k=1}^{\infty} C_{n_k} = \bigcap_{n=1}^{\infty} C_n = C$. Thus $x = g^{-1}(f(x)) \in g^{-1}(C) = K$, hence (5.18) holds. The proof is complete. □

Hence the definition of shyness readily implies the following.

Corollary 5.3. *The set $\{f \in C[0,1] : \mathcal{Z}(f)$ is a singleton$\}$ is non-shy and Borel.*

Now we turn to the question concerning positively many level sets.

Lemma 5.4. *Let Δ be a family of subsets of $[0,1]$ and consider*

$$\mathcal{A} = \{f \in C[0,1] : f^{-1}(0) \in \Delta\},$$
$$\mathcal{B} = \{f \in C[0,1] : \exists^{\lambda} y \in \mathbb{R} \text{ such that } f^{-1}(y) \in \Delta\}.$$

If $\mathcal{A}$ is a non-shy Borel set then $\mathcal{B}$ is a non-shy Borel set, too.

Proof. Let $\mathcal{L}$ be the one-dimensional Lebesgue measure defined on the constant functions of $C[0,1]$, then clearly

$$\mathcal{B} = \{f \in C[0,1] : \mathcal{L}(\mathcal{A} - f) > 0\}. \tag{5.19}$$

As $\mathcal{A}$ is a Borel set, the function $f \mapsto \mathcal{L}(\mathcal{A}-f)$ is Borel measurable by [16, 417A], so $\mathcal{B}$ is Borel. Assume to the contrary that $\mathcal{B}$ is shy. Then there is a Borel probability measure μ on $C[0,1]$ such that $\mu(\mathcal{B} - g) = 0$ for all $g \in C[0,1]$, so (5.19) yields

$$\mu(\{f \in C[0,1] : \mathcal{L}(\mathcal{A} + g - f) > 0\}) = \mu(\mathcal{B} - g) = 0.$$

Therefore we obtain by Fubini's theorem that for all $g \in C[0,1]$

$$(\mu * \mathcal{L})(\mathcal{A} + g) = \int_{C[0,1]} \mathcal{L}(\mathcal{A} + g - f)\, \mathrm{d}\mu(f) = 0.$$

Although $\mu * \mathcal{L}$ is only σ-finite, by restricting and normalizing it we obtain a probability measure witnessing the shyness of $\mathcal{A}$. This is a contradiction, thus the proof is complete. □

Theorem 5.5. *The set*

$$\mathcal{B} = \{f \in C[0,1] : \exists^{\lambda} y \in \mathbb{R} \text{ such that } f^{-1}(y) \text{ is a singleton}\}$$

is non-shy in $C[0,1]$.

Proof. Corollary 5.3 yields that $\mathcal{A} = \{f \in C[0,1] : \#f^{-1}(0) = 1\}$ is a non-shy Borel set. Lemma 5.4 with $\Delta = \{\{x\} : x \in [0,1]\}$ implies that $\mathcal{B}$ is also a non-shy Borel set. □

Remark 5.6. Similar arguments yield that for all $n \in \mathbb{N}^+$ the sets

$$\mathcal{A}_n = \{f \in C[0,1] : \#f^{-1}(0) = n\},$$
$$\mathcal{B}_n = \{f \in C[0,1] : \exists^{\lambda} y \in \mathbb{R} \text{ with } \#f^{-1}(y) = n\}$$

are non-shy, the details are left to the reader. The sets $\mathcal{A}_n$ are pairwise disjoint.

For the prevalent $f \in C[0,1]$ the sets $f^{-1}(\min f)$ and $f^{-1}(\max f)$ are singletons, see e.g. [5]. The next theorem states that all other non-empty level sets can be large, which can be considered as a complementary result to Theorem 5.5.

Theorem 5.7. *The set*

$$\mathcal{C} = \{f \in C[0,1] : \dim_H f^{-1}(y) = 1 \text{ for all } y \in (\min f, \max f)\}$$

is non-shy in $C[0,1]$.

Proof. Let $\mathcal{K} \subset C[0,1]$ be an arbitrarily fixed compact set. By Lemma 4.20 it is enough to construct a $g \in C[0,1]$ such that $\mathcal{K} + g \subset \mathcal{C}$. First we construct g. By Lemma 4.21 there is a strictly increasing function $h \in C[0,1]$ such that $h(0) = 0$ and for all $f \in \mathcal{K}$ and $x, z \in [0,1]$ we have

$$|f(x) - f(z)| \leq h(|x - z|).$$

For all $n \in \mathbb{N}^+$ fix positive integers $a_n > b_n$ such that

$$a_n \geq \max\left\{\frac{1}{h^{-1}(2^{-(n+2)})}, 2^{5n^2}\right\} \quad \text{and} \quad b_n = \left\lceil \frac{a_n}{32} \right\rceil, \tag{5.20}$$

where $\lceil \cdot \rceil$ denotes the upper integer part. For all $n \in \mathbb{N}^+$ let

$$p_n = \frac{1}{a_1 \cdots a_n}.$$

Define $g_n \in C[0,1]$ for all $n \in \mathbb{N}^+$ as

$$g_n(x) = \begin{cases} (-1)^i 2^{-n} & \text{if } x = ip_n \text{ and } 0 \le i \le a_1 \cdots a_n, \\ \text{affine} & \text{on } [(i-1)p_n, ip_n] \text{ for all } 1 \le i \le a_1 \cdots a_n. \end{cases}$$

Let us define $g \in C[0,1]$ and $G_n \in C[0,1]$ for all $n \in \mathbb{N}^+$ as

$$g = \sum_{i=1}^{\infty} g_i \quad \text{and} \quad G_n = \sum_{i=1}^{n} g_i.$$

Now we prove that $\mathcal{K}+g \subset \mathcal{C}$. Let us fix $f \in \mathcal{K}$ and $y \in (\min(f+g), \max(f+g))$, we need to prove that $\dim_H (f+g)^{-1}(y) = 1$. As $f + G_n$ uniformly converges to $f+g$, the intermediate value theorem implies that there is an $m \in \mathbb{N}^+$ and $x_\emptyset \in [0,1]$ such that $(f+G_m)(x_\emptyset) = y$. For all $n \in \mathbb{N}$ let

$$\mathcal{I}_n = \prod_{i=1}^{n} \{1, \dots, b_{m+i}\} \quad \text{and}$$
$$\mathcal{J}_n = \{[(i-1)p_{m+n}, ip_{m+n}] : 1 \le i \le a_1 \cdots a_{m+n}\},$$

where we use the convention $\mathcal{I}_0 = \{\emptyset\}$ and $(i_1, \dots, i_0) = \emptyset$. Let $I_\emptyset \in \mathcal{J}_0$ such that $x_\emptyset \in I_\emptyset$. We construct for each $n \in \mathbb{N}$ and $(i_1, \dots, i_n) \in \mathcal{I}_n$ an interval $I_{i_1 \dots i_n} \in \mathcal{J}_n$ and a point $x_{i_1 \dots i_n} \in I_{i_1 \dots i_n}$ such that for all $(i_1, \dots, i_{n+1}) \in \mathcal{I}_{n+1}$

(1) $I_{i_1 \dots i_{n+1}} \subset I_{i_1 \dots i_n}$,
(2) $(f + G_{m+n})(x_{i_1 \dots i_n}) = y$.

Then $x_\emptyset$ and $I_\emptyset$ are already constructed such that (2) holds. Assume by induction that for some fixed $n \in \mathbb{N}$ for each $(i_1, \dots, i_n) \in \mathcal{I}_n$ the interval $I_{i_1 \dots i_n}$ and the point $x_{i_1 \dots i_n} \in I_{i_1 \dots i_n}$ are defined. Let us fix $(i_1, \dots, i_n) \in \mathcal{I}_n$. We can choose b_{m+n+1} distinct elements of $\mathcal{J}_{n+1}$ which are subsets of $I_{i_1 \dots i_n} \cap U(x_{i_1 \dots i_n}, p_{m+n}/16)$, let us enumerate them as $I_{i_1 \dots i_{n+1}}$ $(1 \le i_{n+1} \le b_{m+n+1})$, then (1) holds. Fix $i_{n+1} \in \{1, \dots, b_{m+n+1}\}$ and define $u_{i_1 \dots i_{n+1}}, v_{i_1 \dots i_{n+1}} \in I_{i_1 \dots, i_{n+1}}$ such that

$$g_{m+n+1}(u_{i_1 \dots i_{n+1}}) = -2^{-(m+n+1)} \quad \text{and} \quad g_{m+n+1}(v_{i_1 \dots i_{n+1}}) = 2^{-(m+n+1)}.$$

It is enough to prove that

$$(f + G_{m+n+1})(u_{i_1 \dots i_{n+1}}) \le y \le (f + G_{m+n+1})(v_{i_1 \dots i_{n+1}}), \tag{5.21}$$

then by the intermediate value theorem we can choose an $x_{i_1 \dots i_{n+1}} \in I_{i_1 \dots i_{n+1}}$ satisfying (2). We prove the second inequality of (5.21) only, the proof of the first one is analogous. As $(f + G_{m+n})(x_{i_1 \dots i_n}) = y$ and $g_{m+n+1}(v_{i_1 \dots i_{n+1}}) = 2^{-(m+n+1)}$, it is enough to prove that

$$|(f + G_{m+n})(x_{i_1 \dots i_n}) - (f + G_{m+n})(v_{i_1 \dots i_{n+1}})| \le 2^{-(m+n+1)}. \tag{5.22}$$

Since $x_{i_1 \dots i_n}, v_{i_1 \dots i_{n+1}} \in I_{i_1 \dots i_n}$, the definition of h, p_{m+n} and a_{m+n} imply that

$$|f(x_{i_1 \dots i_n}) - f(v_{i_1 \dots i_{n+1}})| \le h(p_{m+n}) \le h(1/a_{m+n}) \le 2^{-(m+n+2)}. \tag{5.23}$$

It is easy to show that for all $i \in \mathbb{N}^+$ the function g_i is Lipschitz and $\mathrm{Lip}(g_i) = 2^{1-i}p_i^{-1} = 2^{1-i}a_1 \cdots a_i$. Thus G_{m+n} is Lipschitz and $a_{m+n} \geq 2^{m+n}$ implies that

$$\begin{aligned}\mathrm{Lip}(G_{m+n}) &\leq \sum_{i=1}^{m+n} \mathrm{Lip}(g_i) = \sum_{i=1}^{m+n} 2^{1-i}a_1 \cdots a_i \\ &\leq 2a_1 \cdots a_{m+n-1} + 2^{1-(m+n)}a_1 \cdots a_{m+n} \\ &\leq 2^{2-(m+n)}a_1 \cdots a_{m+n} = \frac{2^{2-(m+n)}}{p_{m+n}}.\end{aligned}$$

Therefore $I_{i_1 \ldots i_n} \subset U(x_{i_1 \ldots i_n}, p_{m+n}/16)$ yields that

$$\begin{aligned}|G_{m+n}(x_{i_1 \ldots i_n}) - G_{m+n}(v_{i_1 \ldots i_{n+1}})| &\leq \mathrm{Lip}(G_{m+n})|x_{i_1 \ldots i_n} - v_{i_1 \ldots i_{n+1}}| \\ &\leq \frac{2^{2-(m+n)}}{p_{m+n}} \cdot \frac{p_{m+n}}{16} = 2^{-(m+n+2)}.\end{aligned} \tag{5.24}$$

Equations (5.23) and (5.24) imply (5.22), and the induction is complete. For all $n \in \mathbb{N}^+$ let $c_n = a_{m+n}$ and $d_n = b_{m+n}$. Set

$$C = \bigcap_{n=1}^{\infty} \left(\bigcup_{i_1=1}^{d_1} \cdots \bigcup_{i_n=1}^{d_n} I_{i_1 \cdots i_n} \right).$$

Then C is a (c_n, d_n)-type compact set, see Definition 3.5. Then (5.20) implies that $a_i \geq 2^{5i^2}$ and $32 \geq a_i/b_i$, so for all $n \in \mathbb{N}^+$ we have

$$c_n \geq 2^{5(m+n)^2} \geq 2^{5(n+1)^2} \geq \left(\frac{c_1 \cdots c_{n+1}}{d_1 \cdots d_{n+1}} \right)^{n+1}.$$

Therefore Lemma 3.7 implies that $\dim_H C = 1$.

Finally, in order to prove $\dim_H (f+g)^{-1}(y) = 1$, it is enough to show that $C \subset (f+g)^{-1}(y)$. Let us fix $x \in C$, we prove that $f(x) = y$. For all $n \in \mathbb{N}^+$ pick indices $i_n \in \{1, \ldots, d_n\}$ such that $x \in I_{i_1 \ldots i_n}$, then clearly $\lim_{n\to\infty} x_{i_1 \ldots i_n} = x$. As $f + G_{m+n}$ converges uniformly to $f+g$, property (2) implies that

$$(f+g)(x) = \lim_{n\to\infty} (f + G_{m+n})(x_{i_1 \ldots i_n}) = y,$$

and the proof is complete. □

Corollary 5.8. *The sets*

$$\mathcal{B} = \{f \in C[0,1] : \exists^{\lambda} y \in \mathbb{R} \text{ such that } f^{-1}(y) \text{ is a singleton}\},$$
$$\mathcal{C} = \{f \in C[0,1] : \dim_H f^{-1}(y) = 1 \text{ for all } y \in (\min f, \max f)\}$$

are disjoint non-shy sets in $C[0,1]$, so they are neither shy nor prevalent.

6. Dimensions of graphs of prevalent continuous maps

By product of two metric spaces (X, d_X) and (Y, d_Y) we will always mean the l^2-product, that is,

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)}.$$

For $E \subset X \times Y$ and $y \in Y$ let $E^y = \{x \in X : (x, y) \in E\}$.

The following lemma is basically [26, Thm. 7.7]. It is only stated there in the special case $X = A \subseteq \mathbb{R}^m$, but the proof works verbatim for all metric spaces X.

Lemma 6.1. *Let X be a metric space and let $d \in \mathbb{N}^+$. If $f\colon X \to \mathbb{R}^d$ is Lipschitz and $t \geq d$ then*

$$\int_{\mathbb{R}^d}^{\star} \mathcal{H}^{t-d}(f^{-1}(y))\,\mathrm{d}\lambda^d(y) \leq c(d)\operatorname{Lip}(f)^d\mathcal{H}^t(X),$$

where $\int^\star$ denotes the upper integral and $c(d)$ is a finite constant depending on d only.

Let $E \subset X \times \mathbb{R}^d$ and define $f\colon E \to \mathbb{R}^d$ as $f(x,y) = y$. Applying Lemma 6.1 for f yields the following lemma.

Lemma 6.2. *Let X be a metric space and let $d \in \mathbb{N}^+$. If $E \subset X \times \mathbb{R}^d$ then for λ^d-almost every $y \in \mathbb{R}^d$*

$$\dim_H(E^y) \leq \max\{0, \dim_H E - d\}.$$

Recently Orponen [33, Cor. 1.2] has shown that Lemma 6.1 does not remain true if we replace Hausdorff measures by packing measures. The analogous version of Lemma 6.2 holds, see the proof of [12, Lemma 5] with the natural modifications.

Lemma 6.3. *Let X be a metric space and let $d \in \mathbb{N}^+$. If $E \subset X \times \mathbb{R}^d$ then for λ^d-almost every $y \in \mathbb{R}^d$*

$$\dim_P(E^y) \leq \max\{0, \dim_P E - d\}.$$

For the following lemma see [26, Thm. 8.10]. It is only stated there for subsets of Euclidean spaces, but the same proof works here as well.

Lemma 6.4. *If X, Y are non-empty metric spaces then*

$$\dim_H(X \times Y) \leq \dim_H X + \dim_P Y,$$
$$\dim_P(X \times Y) \leq \dim_P X + \dim_P Y.$$

Theorem 6.5. *Let K be an uncountable compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\dim_H \operatorname{graph}(f) = \dim_H K + d.$$

Proof. Lemma 6.4 and $\dim_P \mathbb{R}^d = d$ yield that for all $f \in C(K, \mathbb{R}^d)$

$$\dim_H \operatorname{graph}(f) \leq \dim_H(K \times \mathbb{R}^d) \leq \dim_H K + d,$$

so it is enough to prove the opposite inequality for the prevalent f.

If $\dim_H K = 0$ then Theorem 1.7 yields that for the prevalent $f \in C(K, \mathbb{R}^d)$ we have $\operatorname{int} f(K) \neq \emptyset$, so $\dim_H f(K) = d$. As $f(K)$ is a Lipschitz image of $\operatorname{graph}(f)$ and Hausdorff dimension cannot increase under a Lipschitz map, we obtain

$$\dim_H \operatorname{graph}(f) \geq \dim_H f(K) = d = \dim_H K + d,$$

and we are done. Hence we may assume that $\dim_H K > 0$. Consider

$$\mathcal{A} = \{f \in C(K, \mathbb{R}^d) : \text{for all } s < \dim_H K \text{ there exists a non-empty}$$
$$\text{open set } U_{f,s} \subset \mathbb{R}^d \text{ such that } \dim_H f^{-1}(y) > s \text{ for all } y \in U_{f,s}\}.$$

Theorem 4.18 yields that $\mathcal{A}$ is prevalent in $C(K, \mathbb{R}^d)$, so it is enough to show that $\dim_H \operatorname{graph}(f) \geq s + d$ for an arbitrary given $f \in \mathcal{A}$ and $s \in (0, \dim_H K)$. Let $E = \operatorname{graph}(f) \subset K \times \mathbb{R}^d$, then for all $y \in U_{f,s}$ we have $\dim_H E^y = \dim_H f^{-1}(y) \geq s$. As $\lambda^d(U_{f,s}) > 0$ and $s > 0$, Lemma 6.2 implies that $\dim_H \operatorname{graph}(f) = \dim_H E \geq s + d$. The proof is complete. □

Theorem 6.6. *Let K be an uncountable, non-exploding compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$*

$$\dim_P \operatorname{graph}(f) = \dim_P K + d.$$

Proof. We can repeat the proof of Theorem 6.5, only replace Hausdorff dimension with packing dimension, and apply Corollary 4.15 and Lemma 6.3 instead of Theorem 4.18 and Lemma 6.2, respectively. □

7. Finer results with generalized Hausdorff measures

In this section we indicate how to obtain sharper versions of the main results. Since the proofs were quite technical already, we decided not to include these stronger forms in the main body of the paper, only give a brief sketch in this separate section.

A function $\varphi\colon [0,\infty) \to [0,\infty)$ is defined to be a *gauge function* if it is non-decreasing and $\varphi(0) = 0$. For a metric space X let

$$\mathcal{H}^\varphi(X) = \lim_{\delta \to 0+} \mathcal{H}^\varphi_\delta(X), \text{ where}$$

$$\mathcal{H}^\varphi_\delta(X) = \inf\left\{ \sum_{i=1}^\infty \varphi(\operatorname{diam} A_i) : X \subseteq \bigcup_{i=1}^\infty A_i,\ \forall i \operatorname{diam} A_i \leq \delta \right\}.$$

We call $\mathcal{H}^\varphi$ the *φ-Hausdorff measure*, which extends the concept of classical Hausdorff measure. There are examples when this finer notion of measure is needed, this is the case when we want to measure the level sets of the linear Brownian motion or the range of a d-dimensional Brownian motion. For more information see [35].

Let $\mathcal{G}$ be the set of gauge functions and for all $s > 0$ let

$$\mathcal{G}(s) = \left\{ \varphi \in \mathcal{G} : \lim_{r \to 0+} \frac{\varphi(r)}{r^s} = \infty \right\}.$$

Now we show how to generalize Theorem 4.1, Theorem 5.7 and Theorem 1.9. First we need to extend Lemma 3.7.

Lemma 7.1. *Let $\varphi \in \mathcal{G}(1)$ be a gauge function. Let us define the non-decreasing function $\Phi\colon [1,\infty) \to [1,\infty)$ as*

$$\Phi(x) = \sup\{r \in \mathbb{R}^+ : r\varphi(1/r) \leq x\} + 1, \tag{7.1}$$

where $\sup \emptyset = 0$ by convention. Let $C \subset \mathbb{R}$ be an (a_n, b_n)-type compact set such that for all $n \in \mathbb{N}^+$

$$a_n \geq \Phi\left(\frac{a_1 \cdots a_{n+1}}{b_1 \cdots b_{n+1}} \right).$$

Then $\mathcal{H}^\varphi(C) > 0$.

Proof. Let μ be the same measure as in the proof of Lemma 3.7, then similar arguments yield that all Borel sets $B \subset C$ with $\operatorname{diam} B \leq 1$ satisfy

$$\mu(B) \leq 4\varphi(\operatorname{diam} B).$$

Therefore the mass distribution principle for generalized Hausdorff measures implies that $\mathcal{H}^\varphi(C) > 0$, see also [30, Prop. 6.44 (i)]. □

Instead of Theorem 4.1 we can prove the following stronger form.

Theorem 7.2. *Let $K \subset \mathbb{R}$ be a compact set with $\lambda(K) > 0$ and let $d \in \mathbb{N}^+$. Let $\varphi \in \mathcal{G}(1)$ be a gauge function. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists a non-empty open set $U_f \subset \mathbb{R}^d$ such that $\lambda(f^{-1}(U_f)) = \lambda(K)$ and for all $y \in U_f$*

$$\mathcal{H}^{\varphi}(f^{-1}(y)) > 0.$$

Proof. Let $\Phi\colon [1,\infty) \to [1,\infty)$ be the function defined in (7.1). Let us follow the proof of Theorem 4.1, the only difference is that we define the numbers a_n by induction such that $b_n = (2s)^{-(n+3)}a_n$ are integers and for all $n \in \mathbb{N}^+$ we have

$$a_n \geq \max\left\{(2s)^{8n}(a_1 \cdots a_{n-1}), \Phi\left(\frac{a_1 \cdots a_{n+1}}{b_1 \cdots b_{n+1}}\right)\right\}.$$

Then applying Lemma 7.1 instead of Lemma 3.7 concludes the proof. □

Instead of Theorem 5.7 we can prove the following stronger form.

Theorem 7.3. *Let $\varphi \in \mathcal{G}(1)$ be a gauge function. The set*

$$\mathcal{C} = \{f \in C[0,1] : \mathcal{H}^{\varphi}(f^{-1}(y)) > 0 \text{ for all } y \in (\min f, \max f)\}$$

is non-shy in $C[0,1]$.

Proof. Let $\Phi\colon [1,\infty) \to [1,\infty)$ be the function defined in (7.1). Let us follow the proof of Theorem 5.7, the only difference is that in (5.20) we replace 2^{5n^2} by $\Phi(2^{5n})$ and we apply Lemma 7.1 instead of Lemma 3.7. □

Fraser and Hyde proved in [15] that the prevalent $C[0,1]$ has graph of Hausdorff dimension 2. They observed that $\mathcal{H}^2(\mathrm{graph}(f)) = 0$ for all $f \in C[0,1]$ by Fubini's theorem, and raised the problem what we can say using different gauge functions. The following theorem solves this problem by stating that the graph of the prevalent $f \in C[0,1]$ is as large as possible according to this finer scale, too.

Theorem 7.4. *Let $d \in \mathbb{N}^+$ and let $\psi \in \mathcal{G}(d+1)$ be a gauge function. Then for the prevalent $f \in C([0,1], \mathbb{R}^d)$ we have*

$$\mathcal{H}^{\psi}(\mathrm{graph}(f)) > 0.$$

Before sketching the proof of Theorem 7.4 we need two lemmas.

Lemma 7.5. *Let $d \in \mathbb{N}^+$ and let $\psi \in \mathcal{G}(d+1)$ be a gauge function. Then there is a gauge function $\varphi \in \mathcal{G}(1)$ such that for all $r \in [0,1]$ we have*

$$\varphi(r)r^d \leq \psi(r).$$

Proof. Let $\varphi(0) = 0$ and $\varphi(r) = \psi(1)$ for all $r > 1$. Define $\varphi(r) = \inf_{s\in[r,1]} \psi(s)s^{-d}$ if $0 < r \leq 1$. Then clearly $\varphi(r)r^d \leq \psi(r)$ for all $r \in [0,1]$, and it is easy to check that φ is a gauge function with $\varphi \in \mathcal{G}(1)$. □

For the following lemma see the proof of [26, Thm. 7.7] with the natural modifications.

Lemma 7.6. *Let X be a metric space and let $d \in \mathbb{N}^+$. Let φ, σ be gauge functions such that $\sigma(r) = \varphi(r)r^d$ for all $r \geq 0$. If $g\colon X \to \mathbb{R}^d$ is Lipschitz then*

$$\int_{\mathbb{R}^d}^{\star} \mathcal{H}^{\varphi}(g^{-1}(y))\,\mathrm{d}\lambda^d(y) \leq c(d)\,\mathrm{Lip}(g)^d\mathcal{H}^{\sigma}(X),$$

where $\int^{\star}$ denotes the upper integral and $c(d)$ is a finite constant depending on d only.

Proof of Theorem 7.4. By Lemma 7.5 there is a gauge function $\varphi \in \mathcal{G}(1)$ such that $\varphi(r)r^d \leq \psi(r)$ for all $r \in [0,1]$. Let us define $\sigma(r) = \varphi(r)r^d$ for all $r \geq 0$. Consider

$$\mathcal{A} = \{f \in C([0,1], \mathbb{R}^d) : \text{ there exists a non-empty open set } U_f \subset \mathbb{R}^d \text{ such that } \mathcal{H}^{\varphi}(f^{-1}(y)) > 0 \text{ for all } y \in U_f\}.$$

Theorem 7.2 yields that $\mathcal{A}$ is prevalent in $C(K, \mathbb{R}^d)$. Let us fix $f \in \mathcal{A}$, it is enough to prove that $\mathcal{H}^{\psi}(\text{graph}(f)) > 0$. Let $g\colon [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, $g(x,y) = y$ be the natural projection onto $\mathbb{R}^d$ and let $X = \text{graph}(f)$. Applying Lemma 7.6 for X and $g|_X$ implies that $\mathcal{H}^{\sigma}(\text{graph}(f)) > 0$. Since $\sigma(r) \leq \psi(r)$ for all $r \in [0,1]$, we obtain that $\mathcal{H}^{\psi}(\text{graph}(f)) > 0$. The proof is complete. □

8. Open problems

The following problem asks whether the Wiener measure witnesses the prevalence of a somewhat weaker statement than Corollary 4.6. The motivation comes from [1], where the zero set of Brownian motion with variable drift is investigated.

Problem 8.1. *Let $\{B(t) : t \in [0,1]\}$ be the standard one-dimensional Brownian motion and let $f \in C[0,1]$. Does there exist almost surely an open set $U_{B+f} \subset \mathbb{R}$ such that $\lambda((B+f)^{-1}(U_{B+f})) = 1$ and for all $y \in U_{B+f}$*

$$\dim_H (B+f)^{-1}(y) \geq 1/2?$$

Problem 8.2. *Can we omit the condition that K is non-exploding from the Main Theorem, or more generally, from Theorem 4.12?*

We would like to describe the compact metric spaces K for which Corollary 4.19 can be strengthened. Here we consider only the one-dimensional case.

Problem 8.3. *Let us characterize the compact sets $K \subset \mathbb{R}$ such that for the prevalent $f \in C(K, \mathbb{R})$ there exists a non-empty open set $U_f \subset \mathbb{R}$ so that $\dim_H f^{-1}(y) = \dim_H K$ for all $y \in U_f$.*

Problem 8.4. *Let us characterize the compact sets $K \subset \mathbb{R}$ such that for the prevalent $f \in C(K, \mathbb{R})$ there exists a $y_f \in \mathbb{R}$ such that $\dim_H f^{-1}(y_f) = \dim_H K$.*

Acknowledgments. We are indebted to Y. Peres, M. Vizer and O. Zindulka for numerous illuminating conversations.

References

[1] T. Antunović, K. Burdzy, Y. Peres, J. Ruscher, Isolated zeros for Brownian motion with variable drift, *Electron. J. Probab.* **16** (2011), no. 65, 1793–1814.
[2] R. Balka, Inductive topological Hausdorff dimensions and fibers of generic continuous functions, *Monatsh. Math.* 174 (2014), no. 1, 1–28.
[3] R. Balka, Z. Buczolich, M. Elekes, A new fractal dimension: The topological Hausdorff dimension, submitted, arXiv:1108.4292.
[4] R. Balka, Z. Buczolich, M. Elekes, Topological Hausdorff dimension and level sets of generic continuous functions on fractals, *Chaos Solitons Fractals* **45** (2012), no. 12, 1579–1589.
[5] R. Balka, U. B. Darji, M. Elekes, Bruckner-Garg-type results with respect to Haar null sets in $C[0,1]$, submitted, arXiv:1311.5293.
[6] F. Bayart, Y. Heurteaux, On the Hausdorff dimension of graphs of prevalent continuous functions on compact sets, In: *Further Developments in Fractals and Related Fields*, edited by Julien Barral and Stéphane Seuret, Springer, 2013, 25–34.
[7] P. Billingsley, *Probability and measure*, Third edition, John Wiley & Sons, 1995.

[8] J. P. R. Christensen, On sets of Haar measure zero in abelian Polish groups, *Israel J. Math.* **13** (1972), 255–260.
[9] R. Dougherty, Examples of non-shy sets, *Fund. Math.* **144** (1994), 73–88.
[10] M. Elekes, Z. Vidnyánszky, Haar null sets without G_δ hulls, to appear in *Israel J. Math.*, arXiv:1312.7667.
[11] K. Falconer, *Fractal geometry: Mathematical foundations and applications*, Second Edition, John Wiley & Sons, 2003.
[12] K. Falconer, Sets with large intersection properties, *J. London Math. Soc.* **49** (1994), no. 2, 267–280.
[13] K. Falconer, *Techniques in fractal geometry*, John Wiley & Sons, 1997.
[14] K. J. Falconer, J. M. Fraser, The horizon problem for prevalent surfaces, *Math. Proc. Cambridge Philos. Soc.* **151** (2011), 355–372.
[15] J. M. Fraser, J. T. Hyde, The Hausdorff dimension of graphs of prevalent continuous functions, *Real Anal. Exchange* **37** (2011), no. 2, 333–352.
[16] D. H. Fremlin, *Measure Theory, vol. 4, Topological Measure Spaces*, Torres Fremlin, 2003.
[17] V. Gruslys, J. Jonušas, V. Mijovič, O. Ng, L. Olsen, I. Petrykiewicz, Dimensions of prevalent continuous functions, *Monatsh. Math.* **166** (2012), 153–180.
[18] P. R. Halmos, *Measure theory*, Springer-Verlag, 1974.
[19] J. D. Howroyd, On dimension and on the existence of sets of finite positive Hausdorff measure, *Proc. London Math. Soc.* **70** (1995), no. 3, 581–604.
[20] B. Hunt, T. Sauer, J. Yorke, Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces, *Bull. Amer. Math. Soc.* **27** (1992), 217–238.
[21] S. Ikeda, M. Tamashiro, Frostman theorem for the packing measure, *C. R. Math. Acad. Sci. Paris* **320** (1995), no. 1, 1445–1448.
[22] H. Joyce, D. Preiss, On the existence of subsets of finite positive packing measure, *Mathematika* **42** (1995), 15–24.
[23] A. S. Kechris, *Classical descriptive set theory*, Springer-Verlag, 1995.
[24] T. Keleti, A. Máthé, O. Zindulka, Hausdorff dimension of metric spaces and Lipschitz maps onto cubes, *Int. Math. Res. Not. IMRN*, **2014** (2014), no. 2, 289–302.
[25] B. Kirchheim, Hausdorff measure and level sets of typical continuous mappings in Euclidean spaces, *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1763–1777.
[26] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics No. 44, Cambridge University Press, 1995.
[27] R. D. Mauldin, S. C. Williams, On the Hausdorff dimension of some graphs, *Trans. Amer. Math. Soc.* **298** (1986), 793–803.
[28] M. McClure, The prevalent dimension of graphs, *Real Anal. Exchange* **23** (1997), 241–246.
[29] M. Mendel, A. Naor, Ultrametric subsets with large Hausdorff dimension, *Invent. Math.* **192** (2013), no. 1, 1–54.
[30] P. Mörters, Y. Peres, *Brownian Motion*, with an appendix by Oded Schramm and Wendelin Werner, Cambridge University Press, 2010.
[31] J. Mycielski, Some unsolved problems on the prevalence of ergodicity, instability and algebraic independence, *Ulam Quart.* **1** (1992), no. 3, 30-37.
[32] A. Nekvinda, O. Zindulka, Monotone metric spaces, *Order* **29** (2012), no. 3, 545–558.
[33] T. Orponen, On the packing measure of slices of self-similar sets, preprint, arXiv:1309.3896.
[34] Y. Peres, P. Sousi, Dimension of fractional Brownian motion with variable drift, preprint, arXiv:1310.7002.
[35] C. A. Rogers, *Hausdorff measures*, Cambridge University Press, 1970.
[36] C. A. Rogers, Sets non-σ-finite for Hausdorff measures, *Mathematika* **9** (1962), no. 2, 95–103.
[37] A. Shaw, Prevalence, *M.Math Dissertation*, University of St. Andrews, 2010.
[38] S. Solecki, On Haar null sets, *Fund. Math.* **149** (1996), 205–210.
[39] F. Topsøe, J. Hoffmann-Jørgensen, Analytic spaces and their application, In: *Analytic Sets*, edited by C. A. Rogers et al., Academic Press, London, 1980, 317–401.
[40] L. Zajíček, On differentiability properties of typical continuous functions and Haar null sets, *Proc. Amer. Math. Soc.* **134** (2005), no. 4, 1143–1151.
[41] O. Zindulka, Mapping Borel sets onto balls and self-similar sets by Lipschitz and nearly Lipschitz maps, in preparation.

Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195-4350, USA and Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, PO Box 127, 1364 Budapest, Hungary
E-mail address: balka@math.washington.edu

Department of Mathematics, University of Louisville, Louisville, KY 40292, USA
E-mail address: ubdarj01@louisville.edu

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, PO Box 127, 1364 Budapest, Hungary and Eötvös Loránd University, Institute of Mathematics, Pázmány Péter s. 1/c, 1117 Budapest, Hungary
E-mail address: elekes.marton@renyi.mta.hu