

BRUCKNER-GARG-TYPE RESULTS WITH RESPECT TO HAAR NULL SETS IN $C[0, 1]$

RICHÁRD BALKÁ, UDAYAN B. DARJI, AND MÁRTON ELEKES

ABSTRACT. A set $\mathcal{A} \subset C[0, 1]$ is *shy* or *Haar null* (in the sense of Christensen) if there exists a Borel set $\mathcal{B} \subset C[0, 1]$ and a Borel probability measure μ on $C[0, 1]$ such that $\mathcal{A} \subset \mathcal{B}$ and $\mu(\mathcal{B} + f) = 0$ for all $f \in C[0, 1]$. The complement of a shy set is called a *prevalent* set. We say that a set is *Haar ambivalent* if it is neither shy nor prevalent.

The main goal of the paper is to answer the following question: What can we say about the topological properties of the level sets of the prevalent/non-shy many $f \in C[0, 1]$?

The classical Bruckner-Garg Theorem characterizes the level sets of the generic (in the sense of Baire category) $f \in C[0, 1]$ from the topological point of view. We prove that the functions $f \in C[0, 1]$ for which the same characterization holds form a Haar ambivalent set.

In an earlier paper we proved that the functions $f \in C[0, 1]$ for which positively many level sets with respect to the Lebesgue measure λ are singletons form a non-shy set in $C[0, 1]$. The above result yields that this set is actually Haar ambivalent. Now we prove that the functions $f \in C[0, 1]$ for which positively many level sets with respect to the occupation measure $\lambda \circ f^{-1}$ are not perfect form a Haar ambivalent set in $C[0, 1]$.

We show that for the prevalent $f \in C[0, 1]$ for the generic $y \in f([0, 1])$ the level set $f^{-1}(y)$ is perfect.

Finally, we answer a question of Darji and White by showing that the set of functions $f \in C[0, 1]$ for which there exists a perfect $P_f \subset [0, 1]$ such that $f'(x) = \infty$ for all $x \in P_f$ is Haar ambivalent.

1. INTRODUCTION

Let G be a *Polish group*, i.e. a topological group which is separable and admits a compatible complete metric. If G is locally compact then there exists a *Haar measure* on G , that is, a left translation invariant regular Borel measure which is finite on compact sets and positive on non-empty open sets. The concept of Haar measure does not extend to groups that are not locally compact, but the notion of Haar measure zero does. The following definition is due to Christensen [4] and was rediscovered by Hunt, Sauer and York [8].

Definition 1.1. For an abelian Polish group G a set $A \subset G$ is *shy* or *Haar null* if there exists a Borel set $B \subset G$ and a Borel probability measure μ on G such that

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$A \subset B$ and $\mu(B+x) = 0$ for all $x \in G$. The complement of a shy set is called a *prevalent* set. We say that a set is *Haar ambivalent* if it is neither shy nor prevalent.

Christensen proved in [4] that shy sets form a σ -ideal and in locally compact abelian Polish groups Haar measure zero sets and shy sets coincide.

Denote by $C[0, 1]$ the Banach space of continuous functions $f: [0, 1] \rightarrow \mathbb{R}$ endowed with the supremum metric. By *Cantor set* we mean a set homeomorphic to the classical ‘middle-third’ Cantor set and *generic* is understood in the sense of Baire category. Let us recall the well-known Bruckner-Garg Theorem, see [2].

Theorem 1.2 (Bruckner-Garg). *The generic $f \in C[0, 1]$ has the property that there is a countable dense set $D_f \subset (\min f, \max f)$ such that*

- (1) $f^{-1}(y)$ is a singleton if $y \in \{\min f, \max f\}$,
- (2) $f^{-1}(y)$ is a Cantor set if $y \in (\min f, \max f) \setminus D_f$,
- (3) $f^{-1}(y)$ is the union of a Cantor set and an isolated point if $y \in D_f$.

The above theorem completely describes the level set structure of the generic $f \in C[0, 1]$ from the topological point of view. If we now replace Baire category with the measure theoretic notion of prevalence, we arrive at the the main question of the paper:

Question 1.3. *What can we say about the topological properties of the level sets of the prevalent/non-shy many $f \in C[0, 1]$?*

We say that $f \in C[0, 1]$ has the *Bruckner-Garg property* if its level sets are as described in Theorem 1.2. The following theorem shows that our situation is more complicated than the Baire category case.

Theorem 3.3. *The set*

$$\{f \in C[0, 1] : f \text{ has the Bruckner-Garg property}\}$$

is Haar ambivalent in $C[0, 1]$.

We use \exists^μ to denote positively many with respect to the measure μ . In [1] we proved the following theorem.

Theorem 1.4 (Balka-Darji-Elekes). *The set*

$$\{f \in C[0, 1] : \exists^\lambda y \in \mathbb{R} \text{ such that } f^{-1}(y) \text{ is a singleton}\}$$

is non-shy in $C[0, 1]$.

In fact, Theorem 3.3 yields that the term non-shy can be replaced by Haar ambivalent in the above theorem. For $f \in C[0, 1]$ let $\lambda_f = \lambda \circ f^{-1}$ be the *occupation measure corresponding to f* . In Theorem 1.4 one cannot replace Lebesgue measure with occupation measure, because we showed in [1] that for the prevalent $f \in C[0, 1]$ for λ_f almost every $y \in \mathbb{R}$ the level set $f^{-1}(y)$ has Hausdorff dimension 1. The following theorem yields that some of these level sets are not perfect.

Theorem 4.3. *The set*

$$\{f \in C[0, 1] : \exists^{\lambda_f} y \in \mathbb{R} \text{ such that } f^{-1}(y) \text{ is not perfect}\}$$

is Haar ambivalent in $C[0, 1]$.

Next we consider maps from an uncountable compact metric spaces K to \mathbb{R}^d . Let us denote by $C(K, \mathbb{R}^d)$ the set of continuous functions from K to \mathbb{R}^d endowed with the supremum metric. Prevalent continuous maps have many fibers of cardinality continuum, the following theorem is essentially [6, Thm. 11] and the remark following its proof.

Theorem 1.5 (Dougherty). *Let K be the middle-third Cantor set and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists a non-empty open set $U_f \subset \mathbb{R}^d$ such that for all $y \in U_f$*

$$\#f^{-1}(y) = 2^{\aleph_0}.$$

Applying the above theorem we will show that generic fibers are perfect.

Theorem 5.1. *Let K be a compact metric space without isolated points and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ for the generic $y \in f(K)$*

$$f^{-1}(y) \text{ is perfect.}$$

Corollary 5.2. *For the prevalent $f \in C[0, 1]$ for the generic $y \in f([0, 1])$*

$$f^{-1}(y) \text{ is perfect.}$$

In [5] the following theorem was proved.

Theorem 1.6 (Darji-White). *Let $P \subseteq [0, 1]$ be perfect. Then the set*

$$\mathcal{D}_P = \{f \in C[0, 1] : \forall x \in P, f'(x) = \infty \text{ for all } x \in P\}$$

is shy.

Darji and White asked in [5] whether the above theorem holds if P is allowed to vary with f . We answer their question in the negative.

Theorem 6.1. *The set*

$$\mathcal{D} = \{f \in C[0, 1] : \exists \text{ perfect set } P_f \text{ such that } f'(x) = \infty \text{ for all } x \in P_f\}$$

is Haar ambivalent.

2. PRELIMINARIES

Let (X, d) be a metric space. For $x \in X$ and $r > 0$ let $B(x, r)$ and $U(x, r)$ be the closed and open balls of radius r centered at x , respectively. For $A \subset X$ we denote by $\text{cl } A$, $\text{int } A$ and ∂A the closure, interior and boundary of A , respectively. The diameter of a A is denoted by $\text{diam } A$. For $A, B \subseteq X$ let us define $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Let X be a complete metric space. A set is *somewhere dense* if it is dense in a non-empty open set, and otherwise it is called *nowhere dense*. We say that $M \subset X$ is *meager* if it is a countable union of nowhere dense sets, and a set is called *co-meager* if its complement is meager. We say that the *generic* element $x \in X$ has property \mathcal{P} if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. A metric space X is *Polish* if it is complete and separable. See e.g. [10] for more on these concepts. A set is *perfect* if it is closed and has no isolated points.

For a measure μ we use \exists^μ to denote positively many with respect to μ . Let λ be the one-dimensional Lebesgue measure and for all $f \in C[0, 1]$ let $\lambda_f = \lambda \circ f^{-1}$ be the *occupation measure corresponding to f* .

For the following lemma see [6, Prop. 8].

Lemma 2.1. *Let G, H be abelian Polish groups and let $\Phi: G \rightarrow H$ be a continuous epimorphism. If $S \subset H$ is prevalent then so is $\Phi^{-1}(S) \subset G$.*

The next corollary follows from Lemma 2.1 and the fact that the Tietze Extension Theorem holds in \mathbb{R}^d .

Corollary 2.2. *Let $K_1 \subset K_2$ be compact metric spaces, let $d \in \mathbb{N}^+$ and define*

$$R: C(K_2, \mathbb{R}^d) \rightarrow C(K_1, \mathbb{R}^d), \quad R(f) = f|_{K_1}.$$

If $\mathcal{A} \subset C(K_1, \mathbb{R}^d)$ is prevalent then so is $R^{-1}(\mathcal{A}) \subset C(K_2, \mathbb{R}^d)$.

The next theorem follows from Theorem 1.5, Corollary 2.2 and the fact that every uncountable Polish space contains a compact set homeomorphic to the middle-third Cantor set, see [10, Cor. 6.5].

Theorem 2.3. *Let K be an uncountable compact metric space and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ there exists a non-empty open set $U_f \subset \mathbb{R}^d$ such that for all $y \in U_f$*

$$\#f^{-1}(y) = 2^{\aleph_0}.$$

3. BRUCKNER-GARG-TYPE THEOREM FOR PREVALENT CONTINUOUS FUNCTIONS

The goal of this section is to prove our main theorem, Theorem 3.3.

Definition 3.1. We say that $f \in C[0, 1]$ has the *Bruckner-Garg property* if its level sets are as described in Theorem 1.2.

Definition 3.2. We say that $f \in C[0, 1]$ is *non-decreasing at a point* $x \in [0, 1]$ if there exists $\varepsilon > 0$ such that $\frac{f(z)-f(x)}{z-x} \geq 0$ for all $z \in [0, 1]$ with $0 < |z - x| < \varepsilon$. A function $f \in C[0, 1]$ is *non-increasing at x* if $-f$ is non-decreasing at x , and f is *monotone at x* if f is either non-decreasing or non-increasing at x .

Theorem 3.3. *The set*

$$\mathcal{A} = \{f \in C[0, 1] : f \text{ has the Bruckner-Garg property}\}$$

is Haar ambivalent in $C[0, 1]$.

Before proving Theorem 3.3 we need three lemmas. For the following one see the proof of [2, Thm. 3.3].

Lemma 3.4. *If $f \in C[0, 1]$ is not monotone at any point and one-to-one on its local extremum points then f has the Bruckner-Garg property.*

Applying [5, Thm. 1.1] for $A = B = \emptyset$ yields the following lemma.

Lemma 3.5. *The functions $f \in C[0, 1]$ that are not monotone at any point form a non-shy set in $C[0, 1]$.*

Lemma 3.6. *The prevalent $f \in C[0, 1]$ is one-to-one on its local extremum points.*

Proof. Let

$$\mathcal{A} = \{f \in C[0, 1] : f \text{ is not one-to-one on its local extremum points}\}.$$

Let \mathcal{I} be the family of closed rational subintervals of $[0, 1]$. For all disjoint $I, J \in \mathcal{I}$ let $\mathcal{A}_{I,J}$ be the set of functions $f \in C[0, 1]$ for which the maximum or the minimum of f on I is equal to either the maximum or the minimum of f on J . Clearly

$$\mathcal{A} = \bigcup_{I, J \in \mathcal{I}, I \cap J = \emptyset} \mathcal{A}_{I, J}.$$

Since the countable union of shy sets is shy, it is enough to prove that the sets $\mathcal{A}_{I,J}$ are all shy.

Let us fix disjoint $I, J \in \mathcal{I}$, and we need to prove that $\mathcal{A}_{I,J}$ is shy. As $\mathcal{A}_{I,J}$ is clearly closed, it is Borel. We may assume that $I = [a, b]$ and $J = [c, d]$ such that $b < c$. For all $u \in [0, 1]$ let $g_u: [0, 1] \rightarrow \mathbb{R}$ be defined as

$$g_u(x) = \begin{cases} 0 & \text{if } x \in [0, b], \\ u & \text{if } x \in [c, 1], \\ \text{affine} & \text{if } x \in [b, c]. \end{cases}$$

Let us define the continuous map

$$\phi: [0, 1] \rightarrow C[0, 1], \quad \phi(u) = g_u,$$

and consider the Borel probability measure $\mu = \lambda \circ \phi^{-1}$ on $C[0, 1]$. Note that the support of our measure satisfies $\text{supp}(\mu) = \{g_u : u \in [0, 1]\}$.

Now it is sufficient to show that $\mu(\mathcal{A}_{I,J} + f) = 0$ for every $f \in C[0, 1]$. But for a fixed $f \in C[0, 1]$ it is easy to see that $(\mathcal{A}_{I,J} + f) \cap \text{supp}(\mu)$ is actually finite, since there are at most four u such that $g_u - f \in \mathcal{A}_{I,J}$. As finite sets are μ -null, this completes the proof. \square

Proof of Theorem 3.3. Theorem 1.4 implies that the functions $f \in C[0, 1]$ having the Bruckner-Garg property does not form a prevalent set in $C[0, 1]$.

Now we prove that they form a non-shy set. Let

$$\begin{aligned} \mathcal{A} &= \{f \in C[0, 1] : f \text{ is not monotone at any point}\}, \\ \mathcal{B} &= \{f \in C[0, 1] : f \text{ is one-to-one on its local extremum points}\}. \end{aligned}$$

Lemma 3.4 implies that it is enough to prove that $\mathcal{A} \cap \mathcal{B}$ is non-shy. The set \mathcal{A} is non-shy by Lemma 3.5 and \mathcal{B} is prevalent by Lemma 3.6. Therefore $\mathcal{A} \cap \mathcal{B}$ is non-shy, and the proof is complete. \square

4. LEVEL SETS WITH RESPECT TO THE OCCUPATION MEASURE

The main goal of this section is to prove Theorem 4.3. First we need some preparation.

Lemma 4.1. *The set*

$$\mathcal{A} = \{f \in C[0, 1] : \lambda_f \text{ is absolutely continuous with respect to } \lambda\}$$

is Borel.

Proof. Let \mathcal{S} be the family of all finite collections of pairwise disjoint open rational intervals of \mathbb{R} . Then \mathcal{S} is countable. For $n \in \mathbb{N}^+$ and $S \in \mathcal{S}$ let

$$\begin{aligned} \mathcal{S}_n &= \{S \in \mathcal{S} : \lambda(\cup S) < 1/n\}, \\ \mathcal{A}_{n,S} &= \{f \in C[0, 1] : \lambda_f(\cup S) < 1/n\}. \end{aligned}$$

As λ_f is absolutely continuous with respect to λ iff the function $x \mapsto \lambda_f((-\infty, x))$ is absolutely continuous, we obtain that

$$\mathcal{A} = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{S \in \mathcal{S}_k} \mathcal{A}_{n,S}.$$

Thus it is enough to prove that $\mathcal{A}_{n,S}$ is Borel for an arbitrarily fixed $n \in \mathbb{N}^+$ and $S = \{I_1, \dots, I_m\} \in \mathcal{S}$. For each open set $U \subset \mathbb{R}$ consider

$$\Phi_U: C[0, 1] \rightarrow [0, 1], \quad \Phi_U(f) = \lambda(f^{-1}(U)).$$

It is easy to see that

$$\mathcal{A}_{n,S} = \left\{ f \in C[0, 1] : \sum_{i=1}^m \Phi_{I_i}(f) < \frac{1}{n} \right\},$$

therefore it is enough to prove that $\sum_{i=1}^m \Phi_{I_i}$ is Borel measurable. Thus it suffices to show that Φ_U is Borel measurable for every open set $U \subset \mathbb{R}$. Fix an arbitrary open set $U \subset \mathbb{R}$ and $r > 0$, we will check that $\Phi_U^{-1}((r, \infty))$ is open. Pick $f \in \Phi_U^{-1}((r, \infty))$, we need to find $\varepsilon > 0$ such that $U(f, \varepsilon) \subset \Phi_U^{-1}((r, \infty))$. Since $\Phi_U(f) = \lambda(f^{-1}(U)) > r$, the regularity of Lebesgue measure implies that there is a compact set $K \subset f^{-1}(U)$ such that $\lambda(K) > r$. As $f(K) \subset U$ is compact, we can define $\varepsilon = \text{dist}(f(K), \mathbb{R} \setminus U) > 0$. Clearly $g(K) \subset U$ for all $g \in U(f, \varepsilon)$, thus $\lambda(g^{-1}(U)) \geq \lambda(K) > r$. Hence $U(f, \varepsilon) \subset \Phi_U^{-1}((r, \infty))$, and the proof is complete. \square

The following theorem is essentially known, however, for the sake of completeness we point out how standard arguments concerning the Brownian motion yield this result.

Theorem 4.2. *For the prevalent $f \in C[0, 1]$ the occupation measure λ_f is absolutely continuous with respect to the Lebesgue measure λ .*

Proof. Consider

$$\mathcal{A} = \{f \in C[0, 1] : \lambda_f \text{ is absolutely continuous with respect to } \lambda\},$$

then \mathcal{A} is Borel by Lemma 4.1. Let μ be the Wiener measure on $C[0, 1]$ and let $\{B(s) : s \in [0, 1]\}$ be the standard one-dimensional Brownian motion. It is enough to prove that $\mu(\mathcal{A} - f) = 1$ for all $f \in C[0, 1]$, that is, λ_{B+f} is almost surely absolutely continuous with respect to λ . Now one can repeat the proof of [13, Thm. 3.26] with $t = 1$, using also that for all $s_1, s_2, r \in [0, 1]$

$$(4.1) \quad \Pr(|(B+f)(s_1) - (B+f)(s_2)| \leq r) \leq \Pr(|B(s_1) - B(s_2)| \leq r),$$

which we verify next. Let X be the standard normal random variable. We may assume that $s_1 \neq s_2$ and let us consider $a = \sqrt{|s_1 - s_2|}$, $b = \frac{r}{a}$ and $c = \frac{f(s_1) - f(s_2)}{a}$. As $B(s_1) - B(s_2)$ has the same distribution as that of aX and the density function of X is even and monotone decreasing on $[0, \infty)$, we obtain

$$\begin{aligned} \Pr(|(B+f)(s_1) - (B+f)(s_2)| \leq r) &= \Pr(X \in [-b - c, b - c]) \\ &\leq \Pr(X \in [-b, b]) \\ &= \Pr(|B(s_1) - B(s_2)| \leq r), \end{aligned}$$

thus (4.1) holds. \square

Theorem 4.3. *The set*

$$\mathcal{A} = \{f \in C[0, 1] : \exists \lambda^r y \in \mathbb{R} \text{ such that } f^{-1}(y) \text{ is not perfect}\}$$

is Haar ambivalent in $C[0, 1]$.

Proof. Theorem 3.3 easily yields that \mathcal{A} is not prevalent, so it is enough to prove that \mathcal{A} is non-shy. For all $f \in C[0, 1]$ consider

$$S_f = \{y \in \mathbb{R} : f^{-1}(y) \cap [0, 1/2] \text{ is a singleton}\}.$$

Then S_f is Borel, because it is easy to see that $\{y \in \mathbb{R} : \#(f^{-1}(y) \cap [0, 1/2]) \geq 1\}$ and $\{y \in \mathbb{R} : \#(f^{-1}(y) \cap [0, 1/2]) \geq 2\}$ are F_σ sets. Theorem 1.4 and symmetry imply that

$$\mathcal{B} = \{f \in C[0, 1] : \lambda(S_f) > 0\}$$

is non-shy. Theorem 4.2 yields that

$$\mathcal{C} = \{f \in C[0, 1] : \lambda_f \text{ is absolutely continuous with respect to } \lambda\}$$

is prevalent, so $\mathcal{B} \cap \mathcal{C}$ is non-shy. Assume that $f \in \mathcal{B} \cap \mathcal{C}$. Then $\lambda(S_f) > 0$, so the Lebesgue Density Theorem [7, 223B] implies that $S_f + \mathbb{Q}$ has full Lebesgue measure. Therefore the absolute continuity of λ_f with respect to λ yields that $\lambda_f(S_f + \mathbb{Q}) = 1$. Thus there exists a $q(f) \in \mathbb{Q}$ such that

$$(4.2) \quad \lambda(f^{-1}(S_f + q(f)) \cap [2/3, 1]) > 0.$$

As shy sets form a σ -ideal, there is a $q \in \mathbb{Q}$ such that

$$\mathcal{D} = \{f \in \mathcal{B} \cap \mathcal{C} : q(f) = q\}$$

is non-shy. Set $I = [0, 1/2]$ and $J = [2/3, 1]$ and define $g \in C[0, 1]$ as

$$g(x) = \begin{cases} 0 & \text{if } x \in I, \\ q & \text{if } x \in J, \\ \text{affine} & \text{otherwise.} \end{cases}$$

Since \mathcal{D} is non-shy and shy sets are invariant under translations, $\mathcal{D} - g$ is also non-shy. Thus it is enough to prove that $\mathcal{D} - g \subset \mathcal{A}$. Let us fix $f \in \mathcal{D}$, we prove that $f - g \in \mathcal{A}$. It is sufficient to show that $\lambda_{f-g}(S_{f-g}) > 0$. Then $g|_I \equiv 0$, $g|_J = q$ and $q(f) = q$ imply that $S_{f-g} = S_f$ and

$$\begin{aligned} (f - g)^{-1}(S_{f-g}) \cap J &= (f - g)^{-1}(S_f) \cap J \\ &= f^{-1}(S_f + q(f)) \cap J. \end{aligned}$$

The above equation and (4.2) yield

$$\begin{aligned} \lambda_{f-g}(S_{f-g}) &\geq \lambda((f - g)^{-1}(S_{f-g}) \cap J) \\ &= \lambda(f^{-1}(S_f + q(f)) \cap J) > 0. \end{aligned}$$

This concludes the proof. \square

5. GENERIC LEVEL SETS ARE PERFECT

The aim of this section is to prove the following theorem.

Theorem 5.1. *Let K be a compact metric space without isolated points and let $d \in \mathbb{N}^+$. Then for the prevalent $f \in C(K, \mathbb{R}^d)$ for the generic $y \in f(K)$*

$$f^{-1}(y) \text{ is perfect.}$$

Proof. First we prove that

$$\mathcal{A} = \{f \in C(K, \mathbb{R}^d) : \exists \text{ an open set } U_f \subset \mathbb{R}^d \text{ such that} \\ U_f \text{ is a dense subset of } f(K) \text{ and } \#f^{-1}(y) = 2^{\aleph_0} \text{ for all } y \in U_f\}$$

is prevalent in $C(K, \mathbb{R}^d)$. Let $\mathcal{V} = \{V_n : n \in \mathbb{N}^+\}$ be a countable basis of K consisting of non-empty open sets. For all $n \in \mathbb{N}^+$ let $K_n = \text{cl } V_n$ and consider

$$\mathcal{A}_n = \{f \in C(K_n, \mathbb{R}^d) : \exists \text{ a non-empty open set } U_f \subset \mathbb{R}^d \\ \text{such that } \#f^{-1}(y) = 2^{\aleph_0} \text{ for all } y \in U_f\}.$$

Since K has no isolated points, the same holds for all K_n , hence they are uncountable by [10, Cor. 6.3]. Thus Theorem 2.3 implies that the \mathcal{A}_n are prevalent. For all $n \in \mathbb{N}^+$ let us define

$$R_n : C(K, \mathbb{R}^d) \rightarrow C(K_n, \mathbb{R}^d), \quad R(f) = f|_{K_n}.$$

Corollary 2.2 implies that $R_n^{-1}(\mathcal{A}_n)$ are prevalent in $C(K, \mathbb{R}^d)$. As a countable intersection of prevalent sets, $\bigcap_{n=1}^{\infty} R_n^{-1}(\mathcal{A}_n)$ is also prevalent in $C(K, \mathbb{R}^d)$. Thus it is enough to prove that $\bigcap_{n=1}^{\infty} R_n^{-1}(\mathcal{A}_n) \subset \mathcal{A}$. Fix $f \in \bigcap_{n=1}^{\infty} R_n^{-1}(\mathcal{A}_n)$ and define

$$U_f = \bigcup_{n=1}^{\infty} U_{f|_{K_n}}.$$

Then $U_f \subset \mathbb{R}^d$ is open and is a dense subset of $f(K)$. Clearly for all $y \in U_f$ there is an $n \in \mathbb{N}^+$ such that $y \in U_{f|_{K_n}}$, so

$$\#f^{-1}(y) \geq \#(f|_{K_n})^{-1}(y) = 2^{\aleph_0}.$$

Hence \mathcal{A} is prevalent in $C(K, \mathbb{R}^d)$.

Now for all $n \in \mathbb{N}^+$ let

$$\mathcal{B}_n = \{f \in C(K_n, \mathbb{R}^d) : \exists \text{ an open set } W_f \subset \mathbb{R}^d \text{ such that} \\ W_f \text{ is a dense subset of } f(K_n) \text{ and } \#f^{-1}(y) = 2^{\aleph_0} \text{ for all } y \in W_f\}.$$

Since the K_n have no isolated points, the \mathcal{B}_n are prevalent as above. Corollary 2.2 implies that the $R_n^{-1}(\mathcal{B}_n)$ are prevalent in $C(K, \mathbb{R}^d)$. As a countable intersection of prevalent sets, $\mathcal{B} = \bigcap_{n=1}^{\infty} R_n^{-1}(\mathcal{B}_n)$ is also prevalent in $C(K, \mathbb{R}^d)$. For all $f \in \mathcal{B}$ and $n \in \mathbb{N}^+$ let

$$W_{f,n} = W_{f|_{K_n}} \cup (f(K) \setminus f(K_n))$$

and

$$W_f = \bigcap_{n=1}^{\infty} W_{f,n}.$$

As a countable intersection of dense relatively open sets, W_f is co-meager in $f(K)$. Let us fix $f \in \mathcal{B}$ and $y \in W_f$, it is enough to prove that $f^{-1}(y)$ is perfect. By definition, $y \in W_{f,n}$ for all $n \in \mathbb{N}^+$. If $y \in f(K) \setminus f(K_n)$ then $f^{-1}(y) \cap K_n = \emptyset$, if $y \in W_{f|_{K_n}}$ then $\#(f^{-1}(y) \cap K_n) = 2^{\aleph_0}$. Thus $\#(f^{-1}(y) \cap K_n) \neq 1$ for all $n \in \mathbb{N}^+$, hence $f^{-1}(y)$ has no isolated point. Therefore $f^{-1}(y)$ is perfect and the proof is complete. \square

Corollary 5.2. *For the prevalent $f \in C[0, 1]$ for the generic $y \in f([0, 1])$*

$$f^{-1}(y) \text{ is perfect.}$$

6. INFINITE DERIVATIVE ON PERFECT SETS

The main goal of this section is to prove the following theorem.

Theorem 6.1. *The set*

$$\mathcal{D} = \{f \in C[0, 1] : \exists \text{ perfect set } P_f \text{ such that } f'(x) = \infty \text{ for all } x \in P_f\}$$

is Haar ambivalent.

We need some preparation before we prove the theorem. The next lemma is well-known, see e.g. [14, Lemma 4] for the proof.

Lemma 6.2. *Let G be an abelian Polish group and let $A \subset G$. If for all compact set $K \subset G$ there exists a $g \in G$ such that $K + g \subset A$ then A is non-shy.*

The following lemma is probably known, but we could not find a reference, so we outline its short proof.

Lemma 6.3. *Let $\mathcal{K} \subset C[0, 1]$ be a compact set. Then there is a strictly increasing subadditive function $h \in C[0, 1]$ such that $h(0) = 0$ and for all $f \in \mathcal{K}$ and $x, z \in [0, 1]$, $x \neq z$*

$$|f(x) - f(z)| < h(|x - z|).$$

Proof. Let us define $g \in C[0, 1]$ as

$$g(t) = \sup_{f \in \mathcal{K}} M(f, t), \text{ where}$$

$$M(f, t) = \sup\{|f(x) - f(z)| : x, z \in [0, 1], |x - z| \leq t\}.$$

By the Arzelá-Ascoli Theorem \mathcal{K} is bounded and equicontinuous. Boundedness implies that $g(t) < \infty$ for all $t \in [0, 1]$. Clearly, g is non-decreasing and for all $f \in C[0, 1]$ and $s, t \in [0, 1]$ we have $M(f, t + s) \leq M(f, t) + M(f, s)$, so $g(t + s) \leq g(t) + g(s)$. Hence g is subadditive. By equicontinuity we obtain $\lim_{t \rightarrow 0^+} g(t) = 0$, so the subadditivity of g yields

$$\lim_{s \rightarrow t} |g(t) - g(s)| \leq \lim_{s \rightarrow t} g(|t - s|) = 0,$$

thus g is continuous. Finally, let us define $h \in C[0, 1]$ as

$$h(t) = g(t) + t.$$

The definition and properties of g imply that h satisfies the required conditions. \square

Lemma 6.4. *Let $h \in C[0, 1]$ be a strictly increasing subadditive function with $h(0) = 0$ and extend h to $[-1, 1]$ by $h(-x) = -h(x)$ for all $x \in [0, 1]$. Then there is a non-empty perfect $P \subset [0, 1]$ and a non-decreasing $g \in C[0, 1]$ such that for all $p \in P$*

$$\lim_{x \rightarrow p} \frac{g(x) - g(p)}{h(x - p)} = \infty.$$

Proof. We may assume that $h(x) \geq x$ for all $x \in [0, 1]$, otherwise we may add the identity function to it. For all $n \in \mathbb{N}^+$ let

$$(6.1) \quad l_n = h^{-1}(5^{-n}),$$

then $l_1 \leq 1/5$. Since h is subadditive on $[0, 1]$, we obtain that $l_{n+1} \leq l_n/5$ for all $n \in \mathbb{N}^+$. Therefore we can define for all $n \in \mathbb{N}^+$ and $\sigma \in \{0, 1, 2, 3\}^n$ closed intervals $I_\sigma \subset [0, 1]$ such that

- (1) $\lambda(I_\sigma) = l_n$,
- (2) $I_{\sigma i} \subset I_\sigma$ for all $i \in \{0, 1, 2, 3\}$,
- (3) The intervals $\{I_\sigma\}_{\sigma \in \{0,1,2,3\}^n}$ are pairwise disjoint and they are placed according to the lexicographical ordering of the indexes σ .

Let us define $g \in C[0, 1]$ as

$$g(x) = \int_{[0,x]} f, \text{ where}$$

$$f = \sum_{n=1}^{\infty} \sum_{\sigma \in \{0,1,2,3\}^n} n \cdot \frac{h(l_n)}{l_n} \cdot \chi_{I_\sigma}.$$

Note that f is integrable so g is well-defined, because (6.1) and (1) yield the estimate

$$\int_{[0,1]} f = \sum_{n=1}^{\infty} \sum_{\sigma \in \{0,1,2,3\}^n} n \cdot \frac{h(l_n)}{l_n} \lambda(I_\sigma) = \sum_{n=1}^{\infty} 4^n n 5^{-n} < \infty.$$

Let

$$P = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in \{1,2\}^n} I_\sigma.$$

It is clear that P is a non-empty perfect set. Now we show that P satisfies the desired property. Fix $p \in P$, $x \in [0, 1]$ and $n \in \mathbb{N}^+$ such that $l_{n+1} < |x - p| \leq l_n$, it is enough to prove that

$$(6.2) \quad \frac{g(x) - g(p)}{h(x - p)} \geq \frac{n}{25}.$$

Then $|x - p| > l_{n+1}$ and (3) imply that there is a $\sigma \in \{0, 1, 2, 3\}^{n+2}$ such that I_σ is between x and p . The definition of g , (1), (6.1) and the monotonicity of h yield

$$|g(x) - g(p)| \geq \int_{I_\sigma} f \geq (n+2)h(l_{n+2}) = \frac{n+2}{25}h(l_n) \geq \frac{n+2}{25}h(|x - p|).$$

Clearly $g(x) - g(p)$ and $h(x - p)$ have the same sign, thus the above inequality implies (6.2), which concludes the proof. \square

Proof of Theorem 6.1. Applying [5, Thm. 1.1] for $A = B = \emptyset$ yields that \mathcal{D} is not prevalent. We next show that \mathcal{D} is not shy. Let $\mathcal{K} \subset C[0, 1]$ be a given compact set, by Lemma 6.2 it is enough to prove that $\mathcal{K} + g \subset \mathcal{D}$ for some $g \in C[0, 1]$. Let $h \in C[0, 1]$ be a strictly increasing function guaranteed by Lemma 6.3. We may assume that $h(x) \geq x$ for all $x \in [0, 1]$. Extend h to $[-1, 1]$ so that $h(-x) = -h(x)$ for all $x \in [0, 1]$. Choose a non-empty perfect set $P \subset [0, 1]$ and a non-decreasing function $g \in C[0, 1]$ according to Lemma 6.4. The definitions of h and g imply that for all $f \in \mathcal{K}$ and $p \in P$

$$\begin{aligned} \liminf_{x \rightarrow p} \frac{(f + g)(x) - (f + g)(p)}{x - p} &= \liminf_{x \rightarrow p} \left(\frac{f(x) - f(p)}{h(x - p)} + \frac{g(x) - g(p)}{h(x - p)} \right) \frac{h(x - p)}{x - p} \\ &\geq \liminf_{x \rightarrow p} \left(-1 + \frac{g(x) - g(p)}{h(x - p)} \right) \frac{h(x - p)}{x - p} \\ &\geq (-1 + \infty) \cdot 1 = \infty. \end{aligned}$$

Thus $(f + g)'(p) = \infty$ for all $f \in \mathcal{K}$ and $p \in P$, therefore $\mathcal{K} + g \subset \mathcal{D}$. \square

7. OPEN PROBLEMS

Problem 7.1. *What can we say about the topological properties of the level sets of the prevalent/non-shy many $f \in C([0, 1]^d)$ for $d \geq 2$?*

Problem 7.2. *Krasinkiewicz [11] and Levin [12] independently showed that if K is a compact metric space then the generic $f \in C(K)$ has the property that each component of each of its fibers is hereditarily indecomposable. What can we say from the point of view of prevalence?*

Problem 7.3. *Buczolich and Darji [3] showed that if K is a non-degenerate continuum then the generic $f \in C(K)$ has the property that the Bruckner-Garg Theorem holds when $f^{-1}(y)$ is replaced by $\text{Comp}(f^{-1}(y))$, the space whose elements are the components of $f^{-1}(y)$ and the topology is the so called upper semicontinuous topology (that is, we consider the factor topology on $f^{-1}(y)$ where the equivalence classes are the components). What can we say from the point of view of prevalence?*

Problem 7.4. *Buczolich and Darji [3] examined the fiber structure of the generic map $f \in C(\mathbb{S}^2)$, where \mathbb{S}^2 is the two-dimensional sphere. What can we say from the point of view of prevalence?*

Problem 7.5. *What can we say if we replace $C(K)$ by $C(K, \mathbb{R}^d) = \{f: K \rightarrow \mathbb{R}^d : f \text{ is continuous}\}$?*

For the generic version of this last questions see e.g. [9].

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REFERENCES

- [1] R. Balka, U. B. Darji, M. Elekes, The Hausdorff dimension of fibers of prevalent continuous maps, in preparation.
- [2] A. M. Bruckner, K. M. Garg, The level set structure of a residual set of continuous functions, *Trans. Amer. Math. Soc.* **232** (1977), 307–321.
- [3] Z. Buczolich, U. Darji, Pseudoarcs, pseudocircles, Lakes of Wada and generic maps of S^2 , *Topology Appl.* **150** (2005), 223–254.
- [4] J. P. R. Christensen, On sets of Haar measure zero in abelian Polish groups, *Israel J. Math.* **13** (1972), 255–260.
- [5] U. B. Darji, S. C. White, Haar ambivalent sets in the space of continuous functions, *Acta Math. Hungar.* **126** (2010), no. 3, 230–240.
- [6] R. Dougherty, Examples of non-shy sets, *Fund. Math.* **144** (1994), 73–88.
- [7] D. H. Fremlin, *Measure Theory*, Vol. 4, *Topological measure spaces*, Torres Fremlin, 2003.
- [8] B. Hunt, T. Sauer, J. Yorke, Prevalence: a translation-invariant “almost every” on infinite-dimensional spaces, *Bull. Amer. Math. Soc.* **27** (1992), 217–238.
- [9] H. Kato, Higher-dimensional Bruckner-Garg type theorem, *Topology Appl.* **154** (2007), no. 8, 1690–1702.
- [10] A. S. Kechris, *Classical descriptive set theory*, Springer-Verlag, 1995.
- [11] J. Krasinkiewicz, On mappings with hereditarily indecomposable fibers, *Bull. Polish Acad. Sci. Math.* **44** (1996), no. 2, 147–156.
- [12] M. Levin, Bing maps and finite dimensional maps, *Fund. Math.* **151** (1996), no. 1, 47–52.
- [13] P. Mörters, Y. Peres, *Brownian motion*, Cambridge University Press, 2010.
- [14] L. Zajíček, On differentiability properties of typical continuous functions and Haar null sets, *Proc. Amer. Math. Soc.* **134** (2005), no. 4, 1143–1151.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, PO BOX 127,
1364 BUDAPEST, HUNGARY AND UNIVERSITY OF WASHINGTON, DEPARTMENT OF MATHEMATICS,
BOX 354350 SEATTLE, WA 98195-4350, USA

E-mail address: `balka.richard@renyi.mta.hu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISVILLE, LOUISVILLE, KY 40292, USA

E-mail address: `ubdarj01@louisville.edu`

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, PO BOX
127, 1364 BUDAPEST, HUNGARY AND EÖTVÖS LORÁND UNIVERSITY, INSTITUTE OF MATHEMATICS,
PÁZMÁNY PÉTER S. 1/C, 1117 BUDAPEST, HUNGARY

E-mail address: `elekes.marton@renyi.mta.hu`