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UNIVERSAL ALGEBRAIC BASICS FOR ALGEBRAIC LOGIC

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Hajnal Andréka, István Németi and Ildikó Sain

PRELIMINARY DRAFT VERSION

CONTENTS

1. Notation, Elementary Concepts

(Sets, Relations, Functions, First–order Logic)

1.1. Sets, classes, tuples, simple operations on sets.

Our approach to set theory is informal. We do not specify any particular set of axioms. We assume that the reader is familiar with naive set theory. If the reader would find out that s/he is unfamiliar with the material reviewed below, s/he is advised to consult e.g. Halmos [Ha60] which is an outstanding elementary introduction to set theory.

We use both *sets* and *classes*. Intuitively we think of classes as of arbitrary collections. As we will see below, sets are special classes. The objects a class is a collection of are referred to as its *elements* or *members*. Intuitive examples are: the collection of all apples (in a given moment of time) is a class, the elements of it are the apples. All the dogs living in Budapest form a class, the elements are the dogs. Classes may be elements of other classes. For example the collection of the classes in a school is a class, and the classes themselves are classes, their elements being the students. We use the notation $b \in A$ if we want to express that A is a class and b is an element of A. We write $b \notin A$ to indicate that b is not an element of A. $\in$ is called the *membership relation*.

Two classes are said to be *equal* iff they have exactly the same elements.

We assume that there is a class having no elements. Then there is exactly one such class, by our definition of equality of classes above. We call this class the *empty set*, and denote it by $\emptyset$.

Once we have some classes, we can form new classes from them. For example, if we have a class A then we can form a new class having A as its only element. A class having exactly one element is called a *singleton*. The singleton formed of a class A is denoted by $\{A\}$. For example, $\{\emptyset\}$ is the singleton having $\emptyset$ as its only element. Notice that the class A and the class $\{A\}$ are different! For example, A may have many elements while $\{A\}$ has one element only. Thus they cannot be equal according to our definition above.

Another way of forming new classes from old ones is taking *subclasses* of a class. If A and B are classes then A is a subclass of B iff every element of A is an element of B as well. $A \subseteq B$ is our notation for A is a subclass of B. For example, the class of red apples is a subclass of the class of all apples; the class of puppies living in Budapest is a subclass of the dogs living in Budapest. Notice that every class is a subclass of itself; further, the empty set $\emptyset$ is a subclass of every class. $\subseteq$ is called the *inclusion* relation. We denote *proper inclusion* by $\subsetneqq$, that is, $A \subsetneqq B$ iff $A \subseteq B$ but $A \neq B$.

A third example for forming new classes is taking the *union* of two classes, e.g., we can think of a class containing all the red apples and the dogs living in Budapest and nothing else. The union of classes A and B is denoted by $A \cup B$.

Another example for forming new classes from old ones is the following. If both A and B are classes then $\{A, B\}$ is a class again, where the elements of $\{A, B\}$ are A and B (and nothing else).

A very natural questions is: *Are all collections classes?* The answer is *no*. A classical example for a collection which is not a class is the *collection of all classes*. From the assumption that the collection of all classes is a class, one can derive a logical contradiction. (This is called the Russel paradox, see more on this in Exercise 1.1 below.) There are many examples of this nature. The common thing in them is that we want to collect "too much" in one single step.

Another example for a collection which is not a class is the *collection of all subclasses* of a class. This collection is not a class in general. However, there are special classes called *sets* for which the collection of their subclasses does form a class (in particular, a set).

Sets are more "coherent" than other kinds of classes. By "coherent" we mean that when we are forming new collections from "coherent" ones, we always get classes again. For example, the collection of all functions going from a "coherent" class (set) into another "coherent" class (set) is again a class. (This is not true for classes in general.)

Remember that every set is a class, while some classes are not sets. Those classes which are not sets are referred to as *proper classes*. Typical examples for proper classes in mathematics are: the class of all Boolean algebras, the class of all semigroups.

There are many well–known examples for sets in mathematics. For example, we have a fairly intuitive knowledge of the set ω of all natural numbers (defined later below, though), the set $\mathbb{Z}$ of all integers, the set $\mathbb{Q}$ of all rational numbers and the set $\mathbb{R}$ of all real numbers.

Every class with finitely many elements is a set (proper classes are "very large"). Sometimes we denote classes by enumerating their elements within braces, like $\{a, b, c, d\}$. We use a similar notation when, though we cannot enumerate the elements of a class, but we can clearly indicate the elements of the class. E.g., if n is an arbitrary but fixed natural number ($n \in \omega$) then $\{a_1, \ldots, a_n\}$ denotes the set consisting of the elements a_1, a_2, etc. until a_n; and for every natural number n, A_n is an element of the class $\{A_1, A_2, A_3, \ldots\}$, and it has no other elements.

We note that any subclass of a set is a set (this follows from the usual axioms of set theory).

Next we are looking at one of the most basic "class forming" constructions. Let us look at the following "all encompassing" way of defining a collection. Let $\varphi(x)$ be a statement about x such that $\varphi(x)$ expresses some property of x (whichever object x may be). Consider the collection of all x's which have property $\varphi(x)$. Let $\{x : \varphi(x)\}$ denote this collection. It is known from set theory that the collection of only those x's with property $\varphi(x)$ which are elements of *some* sets, forms a class. That is,

$$\{x : \varphi(x) \text{ and } x \text{ is an element of some set}\} \tag{1.1}$$

is a class. If we omit the extra requirement that x must be an element of some set, then we cannot be sure that what we get is a class. Therefore we require this whenever we use the "$\{x : \varphi(x)\}$" construct. Therefore, instead of writing the longer expression (1.1), we will often simply write $\{x : \varphi(x)\}$, but we always mean (1.1). But sometimes we write $\{x \in A : \varphi(x)\}$ (when such a set A is known).

A basic difference between sets and classes is that classes can be defined in such an "all encompassing" way as described above, while sets cannot. For example, it is not true in general that $\{x : \varphi(x)$ and x is an element of some set$\}$ would be a set. However, if we know that A is a set then, since every subclass of a set is a set, we also know that the collection $\{x \in A : \varphi(x)\}$ is also a set.

There are some more ways of forming new classes from old ones. We do not go into giving more of these.

Besides sets and classes, we will speak about (ordered) *pairs* and (ordered) *n–tuples* (n a natural number). A pair is an object consisting of a *first member* and a *second member*. We denote a pair with first member a and second member b as $\langle a, b\rangle$. Two pairs are equal iff their first members as well as their second members coincide. That is, $\langle a, b\rangle = \langle c, d\rangle$ iff $a = c$ and $b = d$. The concept of an *n–tuple* is a natural generalization of that of a pair. An n–tuple ($n \in \omega$) has a first member, a second member, ..., n–th member. $\langle a_1, \dots, a_n\rangle$ denotes the n–tuple the first member of which is a_1, ..., the n–th member of which is a_n. $\langle a_1, \dots, a_n\rangle = \langle b_1, \dots, b_n\rangle$ iff $a_1 = b_1$, ..., $a_n = b_n$. If $n = 3$ or 4 then we call an n–tuple, respectively, a *triple* or a *quadruple*. In set theory it is shown that pairs (as well as n–tuples) can be coded by sets. Here we do not need (and do not care) how to do this coding.

The (set theoretic) *difference* of two sets A and B is defined as

$$A \smallsetminus B \stackrel{\text{def}}{=} \{a \in A : a \notin B\}\,.$$

If A and B are sets then their *union* (already mentioned above) and *intersection* are, respectively,

$$\begin{aligned} A \cup B &\stackrel{\text{def}}{=} \{a : a \in A \text{ or } a \in B\} \quad \text{and} \\ A \cap B &\stackrel{\text{def}}{=} \{a : a \in A \text{ and } a \in B\}\,. \end{aligned}$$

If $A \cap B = \emptyset$ then we say that A and B are *disjoint*. The symbol $\dot\cup$ stands for *disjoint union*, that is, $H = A \,\dot\cup\, B$ iff $H = A \cup B$ and $A \cap B = \emptyset$, for any set H.

For any set A,

$$\bigcup A \stackrel{\text{def}}{=} \{a : a \in B \text{ for some } B \in A\}\,,$$

and

$$\bigcap A \stackrel{\text{def}}{=} \{a : a \in B \text{ for every } B \in A\}\,.$$

See Figure 1.1 below.

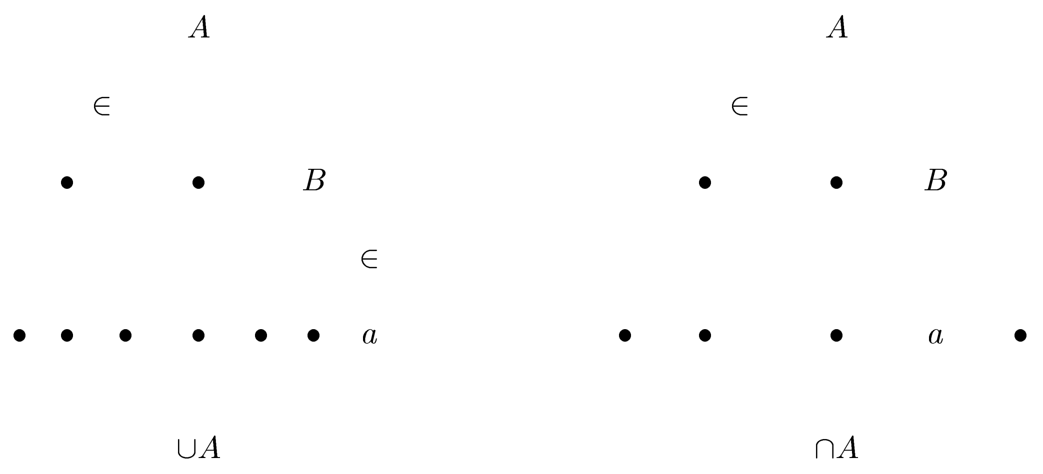


Figure 1.1

Exercise 1.1 (Russel's paradox for sets): Prove that if $H = \{x : x \text{ is a set}\}$ then H is *not* a set.

Hint: Assume H is a set, and consider the set $H' \stackrel{\text{def}}{=} \{x \in H : x \notin x\}$. Prove that both assumptions $H' \in H'$ and $H' \notin H'$ lead to contradictions. ◀

1.2. Binary relations, equivalence relation, functions.

For any two sets A and B, $A \times B$ denotes the *Cartesian product* (or *direct product*) of A and B, and it is defined as follows:

$$A \times B \stackrel{\text{def}}{=} \{\langle a, b\rangle : a \in A \text{ and } b \in B\}.$$

If $R \subseteq A \times B$ for some sets A and B then R is called a *binary relation.* If R is a binary relation then $Dom(R)$ and $Rng(R)$ denote its *domain* and *range* respectively, that is,

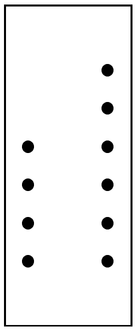
$$Dom(R) = \{a : \langle a, b\rangle \in R \text{ for some } b\} \quad \text{and} \quad Rng(R) = \{b : \langle a, b\rangle \in R \text{ for some } a\}.$$

On Figure 1.2 we show two ways of "drawing" (or illustrating) binary relations. R on Figure 1.2 was drawn in "coordinate system style": a horizontal line and a vertical line (two "coordinate axes") represent, respectively, (sets containing) $Dom(R)$ and $Rng(R)$. A point of the plain determined by the two lines represents an ordered pair the first member of which comes from $Dom(R)$, the second one from $Rng(R)$ (see, e.g., the point $\langle a, b\rangle$ on Figure 1.2). Thus R itself is represented by a collection of points of the plain (see shaded area).

The second way we draw binary relations differs from the first one in that we represent ordered pairs differently: an ordered pair is represented by two points connected by a line segment; we assume that the left–hand–side point is the first member of the pair. Then the relation is represented by a collection of pairs drawn as line segments connecting points (see the relation S on Figure 1.2).

	−−	−|−	−|−	−−	−−	−−	−|−
		|	|			R	|
$Rng(R)$	b	−|−	−|−	−−	$\langle a,b\rangle$		|
		|	|		|		|
	−−	−|−	−|−	−−	−|−	−−	−|−
		−|−	−|−	−−	−|−	−−	−|−
			|		a		|

$Dom(R)$



S

Figure 1.2

To illustrate composition, converse and identity, let $A = 4 = \{0,1,2,3\}$, and let $R, S \subseteq A \times A$ as given below.

3	3	3	3
2	2	2	2
1	1	1	1
0	0	0	0

$R = \{\langle 0,1\rangle, \langle 2,1\rangle, \langle 3,2\rangle\}$ $S = \{\langle 0,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,2\rangle\}$

Now:

3	3	3	3	3	3
2	2	2	2	2	2
1	1	1	1	1	1
0	0	0	0	0	0

$R \circ S =$
$\{\langle 0,2\rangle, \langle 0,3\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ $R^{\smile} = \{\langle 1,0\rangle, \langle 1,2\rangle, \langle 2,3\rangle\}$ $\mathsf{Id}_4 = \{\langle n,n\rangle : n < 4\}$

Figure 1.3

If R is a binary relation then instead of writing $\langle a, b\rangle \in R$, we will often write $R(a, b)$ or aRb (the latter form is called *infix notation*). $a \not R\, b$ stands for $\langle a, b\rangle \notin R$.

For two binary relations R and S, their *composition* $R \circ S$ is defined by

$$R \circ S \stackrel{\text{def}}{=} \{\langle a, b\rangle : \langle a, c\rangle \in R \text{ and } \langle c, b\rangle \in S \text{ for some set } c\}\,.$$

See the illustration of $R \circ S$ on Figure 1.3 above. The *converse* (or *inverse*) $R^{\smile}$ of R is:

$$R^{\smile} \stackrel{\text{def}}{=} \{\langle a, b\rangle : \langle b, a\rangle \in R\}\,.$$

If A is a set then the *identity relation* Id_A on A is defined as follows:

$$\mathsf{Id}_A \stackrel{\text{def}}{=} \{\langle a, a\rangle : a \in A\}\,.$$

Let $R \subseteq U \times U$ for some set U. R is called an *equivalence relation* iff (i)–(iii) below hold.

(i) R is *reflexive*, that is, $\mathsf{Id}_{Dom(R)\cup Rng(R)} \subseteq R$;
(ii) R is *symmetric*, that is, $R^{\smile} \subseteq R$;
(iii) R is *transitive*, that is, $R \circ R \subseteq R$.

R is called an equivalence relation on U iff $U = Dom(R) \cup Rng(R)$.

For every $u \in U$ we let

$$u/R \stackrel{\text{def}}{=} \{v \in U \;:\; \langle u, v\rangle \in R\}\,.$$

The set u/R is called the R*–equivalence class* of u. Any element of an equivalence class is called a *representative* (or representing element) of that class.

Exercise 1.2 (partition): Prove that if R is an equivalence relation on U then R determines a *partition* of U, that is, there are sets U_i ($i \in I$ for some set I) satisfying (i)–(iii) below.

(i) $\bigcup\{U_i : i \in I\} = U$
(ii) $(\forall i, j \in I)(i \neq j \Longrightarrow U_i \cap U_j = \emptyset)$
(iii) $(\forall u_1, u_2 \in U)\big[\langle u_1, u_2\rangle \in R \Longleftrightarrow (\exists i \in I)(u_1 \in U_i \;\;\&\;\; u_2 \in U_i)\big]$.

Hint: Use the equivalence classes defined above. ◀

Let R be an equivalence relation on U. Then the *partition* U/R *determined by* R is defined as

$$U/R \stackrel{\text{def}}{=} \{u/R \;:\; u \in U\}\,.$$

A binary relation f is called a *function* iff $(\langle a,b\rangle, \langle a,c\rangle \in f \implies b = c)$. E.g., Id_A defined above is a function for any set A; on Figure 1.3, R and Id_4 are functions, but S, $R \circ S$, $R^{\smile}$ are not functions. For any $x \in Dom(f)$ of a function f, $f(x)$ denotes the unique element y for which $\langle x, y\rangle \in f$. Instead of $f(x)$, sometimes we write fx or f_x. When defining a function, say f, we often use the following notation.

$$f = \langle f(x) : x \in Dom(f)\rangle = \langle f(x)\rangle_{x\in Dom(f)}$$

and some slight variations of these. An example for using this notation: Consider the traditional definition "f is a function, $Dom(f) = \omega$, and $(\forall n \in \omega) f(n) \overset{\text{def}}{=} n + 3$". Instead of this formulation we will write concisely "$f = \langle n + 3 : n \in \omega\rangle$". The similar notation $g \overset{\text{def}}{=} \langle n + 3 : n \in \omega \text{ and } n \text{ is even}\rangle$ says that g is the restriction of the function $\langle n + 3 : n \in \omega\rangle$ to the even numbers. That is,

$$\langle n + 3 : n \in \omega, n \text{ is even}\rangle = \{\langle 2k, 2k + 3\rangle : k \in \omega\}\,.$$

For a function f and sets A, B, "$f : A \longrightarrow B$" means that $Dom(f) = A$ and $Rng(f) \subseteq B$. Sometimes we write $A \xrightarrow{f} B$ instead of $f : A \longrightarrow B$. If $f : A \longrightarrow B$ and $C \subseteq A$ then $f \restriction C$ denotes the *restriction* of f to C, that is,

$$f \restriction C \overset{\text{def}}{=} \langle f(x) : x \in C\rangle = \{\langle x, y\rangle \in f : x \in C\}\,.$$

A function $f : A \longrightarrow B$ is called *surjective* (or *onto*) iff $Rng(f) = B$; *injective* (or *one–one*) iff $(\forall a, b \in A)[f(a) = f(b) \implies a = b]$; *bijective* iff it is both surjective and injective. We introduce the following notation for surjective, injective and bijective functions. Each of

$$f : A \twoheadrightarrow B \qquad \text{and} \qquad A \overset{f}{\twoheadrightarrow} B$$

denotes that f is a surjective function (*surjection*) from A onto B; each of

$$f : A \rightarrowtail B \qquad \text{and} \qquad A \overset{f}{\rightarrowtail} B$$

denotes that f is an injective function (*injection*) from A into B; finally, each of

$$f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B \qquad \text{and} \qquad A \overset{f}{\rightarrowtail\!\!\!\!\twoheadrightarrow} B$$

denotes that f is a bijective function (*bijection*) from A otno B.

Let A, B be sets. Then

$${}^{A}B \overset{\text{def}}{=} \{f : f \text{ is a function with } Dom(f) = A \text{ and } Rng(f) \subseteq B\}\,.$$

Thus ${}^{\emptyset}B = \{\emptyset\} = 1$ and ${}^{A}\emptyset = \emptyset$ if $A \neq \emptyset$.

If $f \in {}^A B$ and $g \in {}^B C$ for some sets A, B, C then, since functions are binary relations, it is meaningful to compose them using relation composition as defined below Figure 1.3. It is easy to see that $(f \circ g)(x) = g(f(x))$ for every $x \in Dom(f)$, and $f \circ g : A \longrightarrow C$. As opposed to *relation* composition $f \circ g$, we will speak about their *function–composition* as well. The function–composition fg of f and g is defined as

$$fg \stackrel{\text{def}}{=} g \circ f\,.$$

Hence $(g \circ f)(x) = fg(x) = f(g(x))$.

Exercises 1.3: Prove that if both f and g are injective then so is $f \circ g$. Assume $f \circ g$ is injective. Does this imply that so is f? Does this imply that so is g? ◀

Let f be a function and $X \subseteq Dom(f)$. Then the *f–image* $f[X]$ *of* X taken pointwise is defined as

$$f[X] \stackrel{\text{def}}{=} \{f(y) : y \in X\}\,.$$

Remark 1.4: We use a notation for $f[X]$ different from $f(X)$ because it may happen that both $X \subseteq Dom(f)$ and $X \in Dom(f)$, but $f[X] \neq f(X)$. ◀

Exercises 1.5:

(1) Define a function f and a set X such that both $f[X]$ and $f(X)$ exist and they differ.

Hint: If $Dom(f) = \omega$ then $3 \in \omega$ and $3 = \{0, 1, 2\} \subseteq \omega$ is a good start.

(2) Prove that $f[X] = Rng(f \restriction X)$. ◀

Synonyms for "function" are *operation, mapping, map, system* and *family*. The word "system" and "family" are used when we, intuitively, think of the values of the function as collections of elements. E.g., we speak sometimes about a system or family $\langle A_i : i \in I\rangle$ of sets with index set I — this is just a function with domain I.

Sometimes we also call functions *sequences*, we will return to this in subsection 1.4.

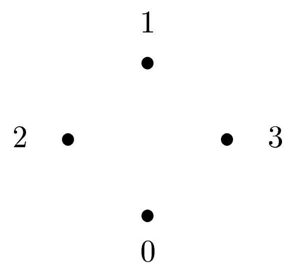
1.3. Orderings, ordinals, cardinals.

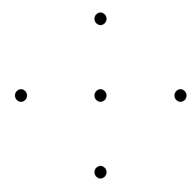
Let $R \subseteq A \times A$ for some set A. This binary relation R is called a *partial ordering* of A iff R is transitive, reflexive and anti–symmetric (i.e., $R \cap R^{\smile} = \emptyset$). E.g., every set of sets is partially ordered by the inclusion relation $\subseteq$. Partial orderings are often denoted by $\leqslant$. Instead of $a \leqslant b$, we sometimes write $b \geqslant a$. Both $a < b$ and $b > a$ abbreviate ($a \leqslant b$ and $a \neq b$). If $\leqslant$ is a partial ordering of some nonempty set A then the pair $\langle A, \leqslant\rangle$ is called a *partially ordered set* (*poset* for short).

Let $\langle A, \leqslant\rangle$ be a poset, and let $a, b \in A$. We say that b covers a, in symbols $a \prec b$, iff $a < b$ and $\{c \in A : a < c < b\} \neq \emptyset$. By drawing the covering relation $\prec$ in the style the relation S was drawn on Figure 1.2 or all relations on Figure 1.3, it is possible to draw diagrams of finite posets and certain infinite posets (this technique has been developed and successfully used in lattice theory). In more detail, a diagram of a poset $\langle A, \leqslant\rangle$ is obtained by arranging the elements of A as points on the plane in such a way that if $a < b$ then the point representing b is above the point representing a. Then a line segment is drawn between any two points a and b whenever $a \prec b$ or $b \prec a$. On Figure 1.4 we give examples for poset diagrams.

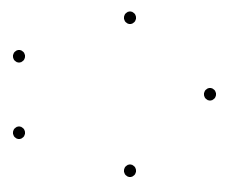
$\mathfrak{A} = \langle\{0,1\}, \mathsf{Id}_A \cup \{\langle 0,1\rangle\}\rangle$

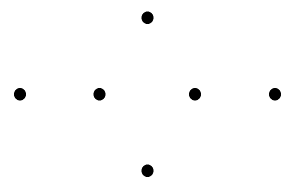
$\mathfrak{B} = \langle\{0,1,2\}, \mathsf{Id}_B \cup \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}\rangle$

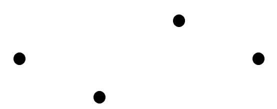




$\mathfrak{C} = \langle\{0,1,2,3\}, \mathsf{Id}_C \cup \{\langle 0,a\rangle : a \in C\} \cup \{\langle 2,1\rangle, \langle 3,1\rangle\}\rangle$







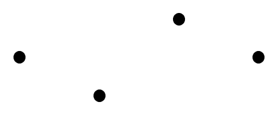


Figure 1.4

By a *linear* (or *simple* or *total*) *ordering* of a set A we mean a partial ordering R of A which is connected (i.e., $A \times A \subseteq R \cup R^{\smile}$). If $\leqslant$ is a linear ordering of A then $\langle A, \leqslant\rangle$ is called a *linearly ordered set*. By a *well–ordered set* we mean a linearly ordered set $\langle A, \leqslant\rangle$ such that every nonempty subset $B \subseteq A$ has a least element ($\ell \in B$ is a least element of B iff $\ell \leqslant x$ for all $x \in B$). A subset $C \subseteq A$ of a poset $\langle A, \leqslant\rangle$ is called a *chain* in the poset iff $\langle C, \leqslant \cap (C \times C)\rangle$ is a linearly ordered set. By an *upper bound* of a chain C we mean an element $u \in A$ for which $c \leqslant u$ for every $c \in C$.

Zorn's Lemma is the statement that if $\langle A, \leqslant\rangle$ is a poset in which every chain has an upper bound, then $\langle A, \leqslant\rangle$ has a maximal element (m is a maximal element of $\langle A, \leqslant\rangle$ iff ($m \in A$ and $m \leqslant x \in A$) implies $m = x$).

We take this statement as an axiom.

By the *successor* of a set x we mean $S(x) = x \cup \{x\}$. *Ordinals* are generated from the empty set $\emptyset$ using the operations of successor S and union $\bigcup$ (the union of any *set* of ordinals is an ordinal). The *finite ordinals*, also called *natural numbers*, are: $0 = \emptyset$, $1 = S(0)$, $2 = S(1)$, $\ldots$. Every set of ordinals is well–ordered by the following ordering $\leqslant$. For any ordinals α and β we set $\alpha \leqslant \beta$ iff ($\alpha = \beta$ or $\alpha \in \beta$). Then $0 \leqslant 1 \leqslant 2 \ldots$, and we have $n = \{0, 1, \ldots, n-1\}$ for each finite ordinal $n \geqslant 1$. The least *infinite ordinal* is

$$\omega = \bigcup\{\alpha : \alpha \text{ is a finite ordinal}\} = \{0, 1, 2, \ldots\}\,.$$

Now, clearly, $n \in \omega$ iff $n \leqslant \omega$ for any set n.

Two sets A and B are said to have the *same cardinality* iff there is a bijection from A onto B. The *cardinals* are those ordinals κ for which we have that no ordinal $\beta < \kappa$ has the same cardinality as κ. The *finite cardinals* are just the finite ordinals, and ω is the smallest *infinite cardinal*.

The **Well Ordering Theorem** is the statement that every set has the same cardinality as some ordinal.

We take it as an axiom.

The *cardinality* of a set A is the (unique) cardinal κ such that A and κ have the same cardinality. We denote the cardinality of A by $|A|$. We say that A is *finite* iff $|A|$ is a finite cardinal, that is, iff $|A| \in \omega$ that is, iff $|A| < \omega$.

The *power set* $\mathcal{P}(A)$ of a set A is defined as

$$\mathcal{P}(A) \stackrel{\text{def}}{=} \{B : B \subseteq A\}\,.$$

$\mathcal{P}(A)$ has the same cardinality as ${}^{A}2$ (hence the term "power set").

We do not recall the operations of multiplication, addition and exponentiation of cardinals. Instead, we recall some simple facts concerning these, as follows.

• Addition and multiplication of cardinals are rather trivial when infinite cardinals are involved. For example, if κ, λ are cardinals, $0 < \kappa \leqslant \lambda$ and $\omega \leqslant \lambda$ then $\kappa + \lambda = \kappa \cdot \lambda = \lambda$.

• For any sets A and B, $|A| \cdot |B| = |A \times B|$, $|A| + |B| = |A \cup B|$ if A and B are disjoint, and $|A|^{|B|} = |{}^{B}A|$.

Exercise 1.6: Let $R \subseteq A \times A$ for some set A. Prove equivalences (1)–(5) below.

(1) R is reflexitve iff xRx for every $x \in A$.
(2) R is symmetric iff xRy implies yRx for every $x, y \in A$.
(3) R is transitive iff xRy and yRz imply xRz for every $x, y, z \in A$.
(4) R is connected iff for all $x, y \in A$, xRy or yRx.
(5) R is anti–symmetric iff xRy implies $y \not R x$ for all $x, y \in A$. ◀

1.4. Sequences.

As we already mentioned, functions are sometimes called *sequences*. In more detail, if $I = Dom(f)$ then we sometimes call f an I*–sequence*. In particular, if $Dom(f) = \omega$ then we call f an ω*–sequence*; and if $Dom(f) = n$ for some $n \in \omega$ then we call f an n*–sequence* or a *sequence of length* n. A function f is called a *finite sequence* iff it is an n–sequence for some $n \in \omega$. (The latter two examples, ω–sequences and finite sequences, motivate the use of the word "sequence" — an ω–sequence or n–sequence can really be pictured as a concrete sequence or series $\langle f_0, f_1, f_2, \ldots \rangle$ resp. $\langle f_0, f_1, f_2, \ldots, f_{n-1} \rangle$ of values.)

For any set X, X^* denotes the set of all *finite sequences over* X, defined as follows:

$$X^* \stackrel{\text{def}}{=} \{f : Dom(f) \in \omega \text{ and } Rng(f) \subseteq X\} = \bigcup_{n \in \omega} ({}^{n}X) \, .$$

Throughout this paper, we identify finite sequences of length n with n–tuples discussed in subsection 1.1. This convention will make our text simpler, and will cause no confusion. Accordingly, we will use the same notation for sequences of length n and n–tuples, e.g. $\langle a_0, \ldots, a_{n-1} \rangle$ may denote an n–tuple as well as an n–sequence. Sometimes we will use the "vector notation" $\bar{a}$ for $\langle a_0, \ldots, a_{n-1} \rangle$.

1.5. Direct product of families of sets.

In subsection 1.2 we introduced the direct product of two sets. Generalizing this concept, we define the *Cartesian or direct product* $\mathbf{P}A$ *of a system of sets* $A = \langle A_i : i \in I \rangle$ as follows.

A function f with $Dom(f) = I$ and $(\forall i \in I) f(i) \in A_i$ is called a *choice function* for A. Now

$$\mathbf{P}A = \mathbf{P}\langle A_i : i \in I \rangle = \mathbf{P}_{i \in I} A_i \stackrel{\text{def}}{=} \{f : f \text{ is a choice function for } A\} \, .$$

See Figure 1.5.

Notice that if $(\exists i \in I) A_i = \emptyset$ then $\mathbf{P}A = \emptyset$. Also, if $I = \emptyset$ then $|\mathbf{P}A| = 1$ (the only element of $\mathbf{P}A$ being the empty function $\emptyset$).

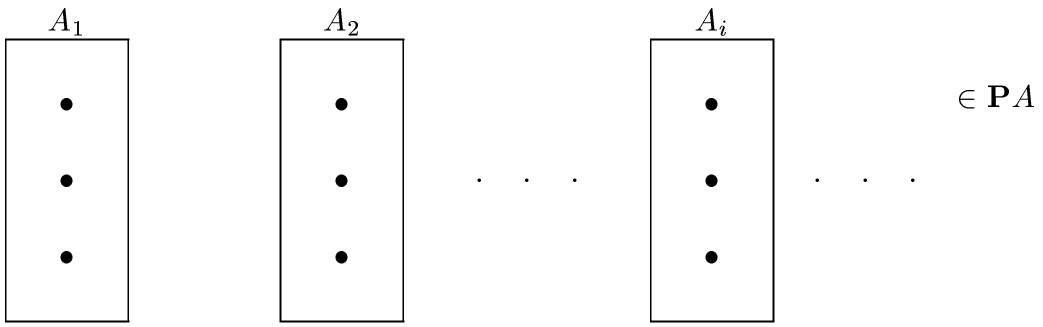


Figure 1.5

A variant of Figure 1.5 is Figure 1.6 below: In the latter, a set is represented by a (vertical) line segment instead of a box (as in Figure 1.5). On Figure 1.6, the index set I is represented by a horizontal line.

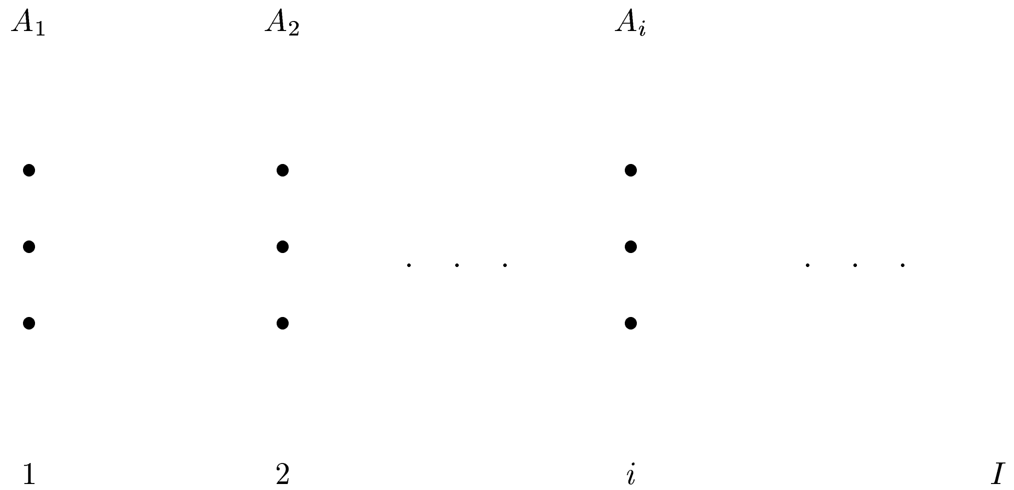


Figure 1.6

Exercise 1.7: Let $\langle A_i : i \in I \rangle$ be a system of sets. Assume that $(\forall i \in I) A_i = B$ for some set B. Prove that $\mathbf{P}\langle A_i : i \in I \rangle = {}^I B$. ◀

In this context, we call ${}^I B$ the (Cartesian or) *direct power* of the set B. In particular, if $n \in \omega$ and U is any set then $\mathbf{P}\langle U : i < n \rangle = {}^n U$. For $\mathbf{P}\langle U : i < n \rangle$ we sometimes write $\underbrace{U \times \cdots \times U}_{n \text{ times}}$.

Let $A = \langle A_i : i \in I \rangle$ be a system of sets, and let $j \in I$. Then the j–th *projection* function $p_j : \mathbf{P}A \longrightarrow A_j$ is defined as follows:

$$(\forall f \in \mathbf{P}A)\, p_j(f) \stackrel{\text{def}}{=} f(j)\,.$$

The **Axiom of Choice** (AC for short) is the statement that if $\langle A_i : i \in I\rangle$ is any system of sets with $A_i \neq \emptyset$ for all $i \in I$ then $\mathbf{P}_{i\in I} A_i \neq \emptyset$.

We take this as an axiom. We note that AC, Zorn's Lemma and the Well Ordering Theorem are (mutually) equivalent.

1.6. Relations of higher ranks.

In this subsection we generalize the concept of a binary relation introduced in subsection 1.2.

If $n \in \omega \smallsetminus \{0\}$ and R is a set of n–sequences then we say that R is an *n–ary relation*. We refer to this n as the *rank* or *arity* of R. Clearly, an n–ary relation R is a subset of the Cartesian product $U_0 \times U_1 \times \cdots \times U_{n-1}$ of the following sets U_i $(i < n)$:

$$\begin{aligned} U_0 &= \{u_0 : \langle u_0, u_1, \dots, u_{n-1}\rangle \in R \text{ for some } u_1, \dots, u_{n-1}\}\,, \\ U_1 &= \{u_1 : \langle u_0, u_1, \dots, u_{n-1}\rangle \in R \text{ for some } u_1, \dots, u_{n-1}\}\,, \\ &\ \ \vdots \\ U_{n-1} &= \{u_{n-1} : \langle u_0, u_1, \dots, u_{n-1}\rangle \in R \text{ for some } u_1, \dots, u_{n-1}\}\,. \end{aligned}$$

Also, $R \subseteq {}^nU$ for $U = \bigcup\{U_i : i < n\}$. It is clear that a binary relation as defined previously is a 2–ary relation in the present sense.

If $n \in \omega$ and U is a set then a function f with $Dom(f) = {}^nU$ and $Rng(f) \subseteq U$ is called an *n–ary function* (or *operation*) on U, and we say that the *rank* (or *arity*) *of the function* f is n. We often identify n–ary functions with $n{+}1$–ary relations, associating to an n–ary function f the $n+1$–ary relation $\{\langle u_0, u_1, \dots, u_{n-1}, u_n\rangle : \langle\langle u_0, u_1, \dots, u_{n-1}\rangle u_n\rangle \in f\}$.

A synonym for "1–ary function (relation)" is "unary function (relation)". Similarly, for "2–ary" and "3–ary" we say "binary" and "ternary", respectively.

If R is a unary relation then we identify R with the set $\{u : \langle u\rangle \in R\}$. Similarly, if f is a 0–ary function on some nonempty set U then $f : {}^0U \longrightarrow U$ (where ${}^0U = 1 = \{\emptyset\}$), and we identify f with $f(\emptyset) \in U$. 0–ary functions are called *constants* (or constant functions).

For being able to handle relations of different ranks in a unified framework, we will need so called *ω–ary relations*, too. R is called an ω–ary relation iff $R \subseteq {}^\omega U$ for some set U. Examples for ω–ary relations are:

1) The set of all convergent ω–sequences over $\mathbb{R}$ (in the usual sense of calculus) is clearly an ω–ary relation.
2) The universe of an ω–dimensional vector space is an ω–ary relation. So are its subspaces.

3) If U is a set then ${}^{\omega}U$ is called the *Cartesian space* with base U and dimension ω. If $p \in {}^{\omega}U$, we set

$$ {}^{\omega}U^{(p)} \stackrel{\text{def}}{=} \{q \in {}^{\omega}U : \{i \in \omega : q_i \neq p_i\} \text{ is finite}\} \ , $$

and we call ${}^{\omega}U^{(p)}$ the *weak Cartesian space* with base U and dimension ω determined by p. Clearly, ${}^{\omega}U$ and ${}^{\omega}U^{(p)}$, for any U and $p \in {}^{\omega}U$, are ω–ary relations.

After the first pages of subsection 1.1, we always spoke about sets and did not mention classes. However, many of the above constructs generalize from sets to proper classes, and we will need some of these in this paper. For example, it is meaningful to speak about functions and relations defined on proper classes instead of on sets. Also, sometimes we want to consider ordered tuples $\langle A, B, C, \ldots\rangle$, where some of $A, B, C, \ldots$ may be proper classes. We will use such general constructs the way it is customary in the literature of algebra and model theory. In this paper we do not want to go into explaining how to make sure that we cause no set theoretic problems when using such general constructs. (The interested reader can find a compact but thorough exposition to the basics of set theoretic foundations in Chang–Keisler [ChK90, Appendix A].)

1.7. First–order logic.

In this subsection we recall the definition of *first–order logic*. We assume that the reader is familiar with this concept, we recall it here only for fixing our notation.

Definition 1.8 (first–order logic):

(1) We call a function t a *similarity type* (or *signature* or *ranked alphabet*) iff (i)–(iii) below hold.

(i) $Rng(t) \subseteq \omega$.
(ii) $t = Fns_t \,\dot{\cup}\, Rls_t$ for some sets Fns_t and Rls_t.
(iii) If $r \in Rls_t$ then $t(r) \in \omega \smallsetminus \{0\}$.

The sets Fns_t and Rls_t are called, respectively, the set of *function symbols* or *operation symbols* and the set of *relation symbols* of t. For any $s \in Dom(t)$, $t(s)$ is called the *rank* or *arity* of s; if $t(s) = 0$ then we call s a *constant symbol*; if $s \in Fns_t$ then we call s a *unary* (*binary*) function symbol iff $t(s) = 1$ ($t(s) = 2$).

From now on until the end of this subsection, t stands for an arbitrary but fixed similarity type.

(2) Let V be an arbitrary set but such that $V \cap Dom(t) = \emptyset$. We define the set $Trm_t(V)$ of *terms* of similarity type t with *variables* from V and the set $Fml_t(V)$ of *formulas* of similarity type t with variables from V by recursion, as follows. $Trm_t(V)$

is defined to be the smallest set satisfying (i) and (ii) below, and $Fml_t(V)$ is defined to be the smallest set satisfying (iii) and (iv) below.

(i) $V \cup \{c \in Fns_t : t(c) = 0\} \subseteq Trm_t(V)$.

(ii) $\{f(\tau_1, \dots, \tau_n) : f \in Fns_t, t(f) = n \neq 0 \text{ and } \tau_1, \dots, \tau_n \in Trm_t(V)\} \subseteq Trm_t(V)$.

(iii) $\{r(\tau_1, \dots, \tau_n) : r \in Rls_t, t(r) = n \text{ and } \tau_1, \dots, \tau_n \in Trm_t(V)\} \cup \{\tau = \sigma : \tau, \sigma \in Trm_t(V)\} \subseteq Fml_t(V)$. The formulas belonging to the left–hand–side set are called *atomic formulas*.

(iv) $\{\neg\varphi : \varphi \in Fml_t(V)\} \cup \{(\varphi \wedge \psi) : \varphi, \psi \in Fml_t(V)\} \cup \{\exists v\varphi : v \in V \text{ and } \varphi \in Fml_t(V)\} \subseteq Fml_t(V)$.

The logical connectives $\neg, \wedge, \exists v$ are called, respectively, *negation, disjunction* and *existential quantifier*, and are read as *not, and, there exists v such that*, respectively.

From $Trm_t(V)$ and $Fml_t(V)$ we will omit t or V and write simply, e.g., Trm_t, $Fml(V)$ or even Trm and Fml when there is no danger of confusion.

(3) By a *model* of similarity type t we mean a pair $\langle M, m\rangle$ satisfying (i)–(ii) below.

(i) M is a non–empty set (called the *universe* of $\langle M, m\rangle$).

(ii) m is a function with $Dom(m) = Dom(t)$ such that
- if $c \in Fns_t$ and $t(c) = 0$ (that is, if c is a constant symbol) then $m(c) \in M$ (that is, $m(c)$ is a constant on M);
- if $f \in Fns_t$ and $t(f) = n \neq 0$ then $m(f) : {}^nM \longrightarrow M$;
- if $r \in Rls_t$ and $t(r) = n$ then $m(r) \subseteq {}^nM$.

The functions $m(c), m(f)$ and the relations $m(r)$ are called the *interpretations* of the symbols c, f and r. They are also referred to as *fundamental operations* or *fundamental relations*, respectively.

Concerning models, we use the following notation. The meta–variables we use for models are the German capitals $\mathfrak{M}, \mathfrak{N}, \mathfrak{W}$, and these with indices like e.g. $\mathfrak{M}_3$. The universe of a model is denoted by the corresponding Roman capital, that is, the universes of $\mathfrak{M}, \mathfrak{N}, \mathfrak{W}, \mathfrak{M}_3$ are respectively M, N, W, M_3. If $\mathfrak{M} = \langle M, m\rangle$, $f \in Fns_t$, $r \in Rls_t$ then $f^{\mathfrak{M}}$ and $r^{\mathfrak{M}}$ stand, respectively, for $m(f)$ and $m(r)$. We may denote $\mathfrak{M}$ as $\langle M, f^{\mathfrak{M}}, r^{\mathfrak{M}}\rangle_{f \in Fns_t, r \in Rls_t}$ as well. If t is finite, for example t consists of a function symbol $+$ and a relation symbol $\leqslant$, then we sometimes denote $\mathfrak{M}$ as $\langle M, +^{\mathfrak{M}}, \leqslant^{\mathfrak{M}}\rangle$. The class of all models of similarity type t is denoted by $\mathbf{Mod}_t$.

(4) Let $\mathfrak{M} \in \mathbf{Mod}_t$ and let V be an arbitrary set of variables (e.g., $V = \{v_i : i \in \omega\}$). A function $k : V \longrightarrow M$ is called a *valuation* of the variables from V in $\mathfrak{M}$.

Let k be an arbitrary but fixed valuation of the variables in $\mathfrak{M}$. We define when a formula $\varphi \in Fml_t(V)$ is *true in* $\mathfrak{M}$ *at valuation k of the variables*, in symbols $\mathfrak{M} \models \varphi[k]$, by recursion, as follows. First we define the value $\tau^{\mathfrak{M}}[k]$ of any term $\tau \in Trm_t(V)$ at k in $\mathfrak{M}$ as:

- $v^{\mathfrak{M}}[k] \overset{\text{def}}{=} k(v)$ if $v \in V$,
- $c^{\mathfrak{M}}[k] \overset{\text{def}}{=} c^{\mathfrak{M}}$ if $t(c) = 0$,
- $(f(\tau_1, \dots, \tau_n))^{\mathfrak{M}} \overset{\text{def}}{=} f^{\mathfrak{M}}\left(\tau_1^{\mathfrak{M}}[k], \dots, \tau_n^{\mathfrak{M}}[k]\right)$ if $f \in Fns_t$, $t(f) = n$, $\tau_1, \dots, \tau_n \in Trm_t$.

Now

- for atomic formulas $r(\tau_1, \ldots, \tau_n)$,

$$\mathfrak{M} \models r(\tau_1, \ldots, \tau_n)[k] \overset{\text{def}}{\iff} \langle \tau_1^{\mathfrak{M}}[k], \ldots, \tau_n^{\mathfrak{M}}[k] \rangle \in r^{\mathfrak{M}},$$

  for atomic formulas $\tau = \sigma$,

$$\mathfrak{M} \models \tau = \sigma[k] \overset{\text{def}}{\iff} \tau^{\mathfrak{M}}[k] = \sigma^{\mathfrak{M}}[k],$$

- for negated formulas $\neg\varphi$,

$$\mathfrak{M} \models \neg\varphi[k] \overset{\text{def}}{\iff} \text{it is not the case that } \mathfrak{M} \models \varphi[k],$$

- for conjunctions $(\varphi \wedge \psi)$,

$$\mathfrak{M} \models (\varphi \wedge \psi)[k] \overset{\text{def}}{\iff} \mathfrak{M} \models \varphi[k] \text{ and } \mathfrak{M} \models \psi[k],$$

- for quantified formulas $\exists v\varphi$,

$$\mathfrak{M} \models \exists v\varphi[k] \overset{\text{def}}{\iff} \mathfrak{M} \models \varphi[k'] \text{ for some valuation } k' \text{ such that } k \restriction (V \smallsetminus \{v\}) = k' \restriction (V \smallsetminus \{v\}).$$

By these, $\mathfrak{M} \models \varphi[k]$ has been defined for any $\varphi \in \mathit{Fml}_t$.

We say that φ is *valid* in $\mathfrak{M}$ or that $\mathfrak{M}$ is a *model of* φ, in symbols $\mathfrak{M} \models \varphi$, iff $\mathfrak{M} \models \varphi[k]$ for every valuation k.

Notation: If $\mathfrak{M} \in \mathbf{Mod}_t$, $k : V \longrightarrow M$, $\varphi \in \mathit{Fml}_t(V)$, and $v_{i_1}, \ldots, v_{i_n}$ are all the variables occurring freely in φ then, instead of $\mathfrak{M} \models \varphi[k]$ we sometimes write $\mathfrak{M} \models \varphi[k(v_{i_1}), \ldots, k(v_{i_n})]$.

(5) Let $\mathsf{K} \subseteq \mathbf{Mod}_t$ and $\Sigma \subseteq \mathit{Fml}_t$.
If $\varphi \in \mathit{Fml}_t$ and $\mathfrak{M} \in \mathbf{Mod}_t$ then

$$\begin{aligned} \mathsf{K} \models \varphi &\quad \text{abbreviates that } \mathfrak{M} \models \varphi \text{ for every } \mathfrak{M} \in \mathsf{K}, \text{ and} \\ \mathfrak{M} \models \Sigma &\quad \text{abbreviates that } \mathfrak{M} \models \varphi \text{ for every } \varphi \in \Sigma. \end{aligned}$$

$\mathsf{K} \models \Sigma$ means that $\mathfrak{M} \models \varphi$ for every $\mathfrak{M} \in \mathsf{K}$ and $\varphi \in \Sigma$.

The *first–order theory of* K is defined as:

$$\mathit{Th}(\mathsf{K}) \overset{\text{def}}{=} \{\varphi \in \mathit{Fml}_t : \mathsf{K} \models \varphi\}.$$

The class of *models of* Σ is defined as:

$$\mathbf{Mod}(\Sigma) \overset{\text{def}}{=} \{\mathfrak{M} \in \mathbf{Mod}_t : \mathfrak{M} \models \Sigma\}.$$

(6) If $\Sigma \subseteq Fml_t$ and $\varphi \in Fml_t$ then we say that φ is a *semantical consequence of* Σ, in symbols $\Sigma \models \varphi$, iff

$$\text{for every } \mathfrak{M} \in \mathbf{Mod}_t, \qquad \mathfrak{M} \models \Sigma \Longrightarrow \mathfrak{M} \models \varphi\,.$$

If Σ is a singleton, that is if $\Sigma = \{\psi\}$ for some $\psi \in Fml_t$ then we simply write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$.

(7) We will use the following abbreviations:

$$\begin{array}{rll} (\varphi \vee \psi) & \text{stands for} & \neg(\neg\varphi \wedge \neg\psi), \\ (\varphi \to \psi) & \text{stands for} & \neg\varphi \vee \psi, \\ (\varphi \leftrightarrow \psi) & \text{stands for} & (\varphi \to \psi) \wedge (\psi \to \varphi), \\ \forall v\varphi & \text{stands for} & \neg\exists v(\neg\varphi). \end{array}$$

The derived logical connective $\vee$ is called *conjunction* and is read as *or*; $\to$ is called *implication* and is read as *implies*; $\leftrightarrow$ is called *equivalence* and is read as *if and only if* or *iff* for short; $\forall v$ is called *universal quantifier* and is read as *for all v such that*.

(8) We define some distinguished subclasses of Fml_t. We assume that the reader is familiar with the concepts of "free occurrence of a variable in a formula", and "a variable occurs under the scope of a quantifier".

- $\varphi \in Fml_t$ is called a *sentence* iff every variable v occurring in it occurs under the scope of a quantifier.
- $\varphi \in Fml_t$ is called a *quantifier–free formula* iff no quantifiers occur in φ. φ is called a *universal formula* iff it is of form $Q_0 \ldots Q_n\psi$ where ψ is quantifier–free, $n \in \omega$, $Q_i (i \leqslant n)$ is a universal quantifier ($\forall v$ for some $v \in V$).
- $\varphi \in Fml_t$ is called an *equation* iff it is of the form $\sigma = \tau$, where $\sigma, \tau \in Trm_t$.
- $\varphi \in Fml_t$ is called a *quasi–equation* iff it is of the form

  $$(e_1 \wedge \cdots \wedge e_n) \to e\,,$$

  where $e, e_1, \ldots, e_n$ $(n \in \omega)$ are equations. ◀

If $\mathfrak{M}$ is a model, then sometimes we say that "φ is a formula in the language of $\mathfrak{M}$". By this we mean that φ is a first–order formula of similarity type of that of $\mathfrak{M}$.

Exercise 1.9: Prove that $K \subseteq \mathbf{Mod}(Th(K))$ for any class K of similar models. ◀

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2. Basics from Universal Algebra

2.1. Examples for algebras.

Informally speaking, by an algebra we mean a model in which there are no relations. In more detail, recall from Definition 1.8 (1) (in subsection 1) that,in general, the domain of a similarity type consists of both function symbols and relation symbols. By an *algebraic similarity type* we mean a similarity type the domain of which does not contain any relation symbols (i.e., the set of relation symbols is empty), but is may contain function symbols.

To distinguish between algebras and arbitrary models in our notation, for denoting algebras, we use German capitals from the beginning of the alphabet, e.g. $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$; while $\mathfrak{M}, \mathfrak{N}, \mathfrak{W}$ stand for models of a possibly non–algebraic similarity type (cf. our notational conventions in Definition 1.8 (3)). When we deal with algebras, we prefer using $X = \{x_i : i \in \omega\}$ for our set of variables of the first–order language of algebras, while in case of general models, we prefer using the set of variables $V = \{v_i : i \in \omega\}$. Otherwise we use the notation introduced for models in case of algebras as well. E.g., if $\mathfrak{A}$ is an algebra of similarity type t then $\mathfrak{A} = \langle A, f^{\mathfrak{A}}\rangle_{f\in Dom(t)}$, further we call A the universe of $\mathfrak{A}$, and $f^{\mathfrak{A}}$ a fundamental operation of $\mathfrak{A}$ or the interpretation of f in $\mathfrak{A}$.

Convention 2.1.1: Throughout the rest of this work, unless otherwise specified, by a similarity type we always mean an algebraic one, that is, one the set of relation symbols of which is empty. ◀

Definition 2.1.2 (similarity class etc.):

(i) Let t be an arbitrary (algebraic) similarity type. Then the *similarity class* Alg_t associated to t is defined as follows.

$$\mathsf{Alg}_t \stackrel{\text{def}}{=} \{\mathfrak{A} : \mathfrak{A} \text{ is an algebra of similarity type } t\}\,.$$

(ii) We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *similar* iff both $\mathfrak{A}$ and $\mathfrak{B}$ belong to the same similarity class (Alg_t for some t).

(iii) Let $\mathfrak{A}$ be an algebra, with universe A. We say that $\mathfrak{A}$ is *finite* (*infinite*) iff $|A| < \omega$ ($|A| \geqslant \omega$). We say that $\mathfrak{A}$ is *trivial* iff $|A| = 1$. ◀

Examples 2.1.3:

1. Let $t = \{\langle s, 1\rangle\}$, that is, t consists of one unary function symbol. The algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ illustrated on Figure 2.1.1 belong to Alg_t.

The definition of $\mathfrak{A} = \langle A, s^{\mathfrak{A}} \rangle$ is this:

$$A \stackrel{\text{def}}{=} \mathbb{Z} \text{ (the set of all integers), and } (\forall z \in \mathbb{Z}) s^{\mathfrak{A}}(z) \stackrel{\text{def}}{=} z + 1\,.$$

$\mathfrak{B} \stackrel{\text{def}}{=} \langle \omega, s^{\mathfrak{B}} \rangle$, where $s^{\mathfrak{B}}(n) = s^{\mathfrak{A}}(n)$ for every $n \in \omega$. $\mathfrak{C} \stackrel{\text{def}}{=} \langle 6, s^{\mathfrak{C}} \rangle$, where $s^{\mathfrak{C}}(n) = n + 1 \pmod 6$. $\mathfrak{D}$ is a trivial algebra of similarity type t.

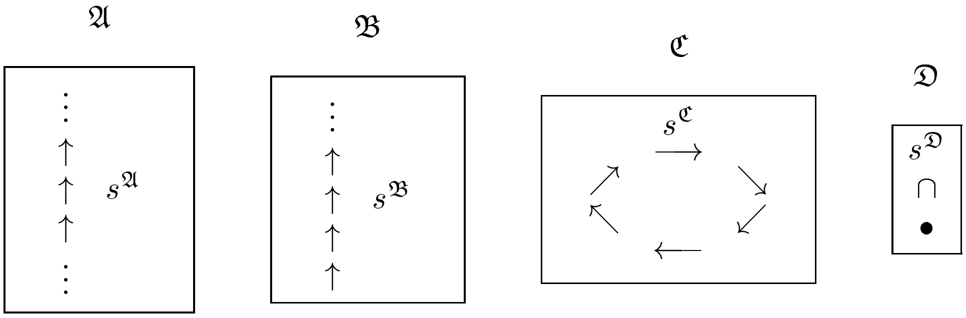
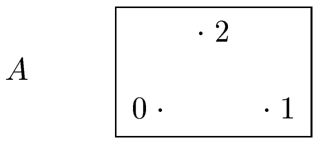


Figure 2.1.1

2. Let $p = \{\langle +, 2 \rangle\}$. Then $\langle \mathbb{Z}, +^{\mathbb{Z}} \rangle$ with $+^{\mathbb{Z}}$ the usual addition on the integers, and $\mathfrak{A}$ given on Figure 2.1.2 clearly belong to Alg_p.



$\mathfrak{A} = \langle 3, +(\text{mod } 3) \rangle$

$+^{\mathfrak{A}}$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

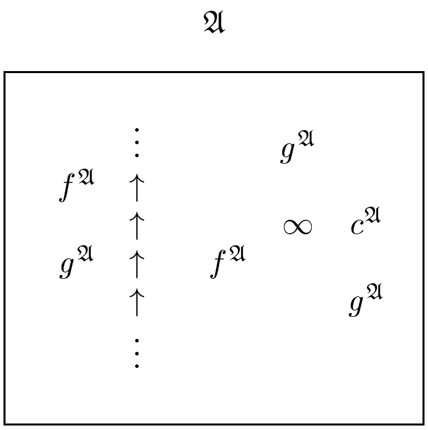
addition table for $\mathfrak{A}$

Figure 2.1.2

3. Let $t = \{\langle g, 2 \rangle, \langle f, 1 \rangle, \langle c, 0 \rangle\}$. Then

$$\mathfrak{A} = \langle \mathbb{Z} \cup \{\infty\}, g^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}} \rangle$$

given and illustrated on Figure 2.1.3 belongs to Alg_t.



$$c^{\mathfrak{A}} = \infty\,, \qquad f^{\mathfrak{A}}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{Z} \\ x & \text{if } x = \infty\,, \end{cases}$$

$$g^{\mathfrak{A}}(x,y) = g^{\mathfrak{A}}(y,x) = \begin{cases} \infty & \text{if } \infty \in \{x,y\} \\ x & \text{if } x \text{ can be reached in finitely many } f^{\mathfrak{A}}\text{–steps from } y\,. \end{cases}$$

Figure 2.1.3

4. We define $ba \stackrel{\text{def}}{=} \{\langle\vee,2\rangle,\langle -,1\rangle\}$. We call ba the *similarity type of Boolean algebras*, we call the function symbols $\vee$ and $-$ *join* and *minus*, respectively. Next we define a distinguished subclass of Alg_{ba} called the class of *powerset Boolean algebras*. Recall from subsection 1.3 that for a set U, $\mathcal{P}(U)$ denotes the powerset of U. For an arbitrary set U, we define the operation *complementation* $_U-$ *relative to* U as follows. For any set X, $_U-(X) \stackrel{\text{def}}{=} U \smallsetminus X$. Often we omit the subscript $_U$ from $_U-$, and thus, ambiguously, we use the same (meta–)symbol $-$ for referring to both the function symbol minus (from ba) and the set theoretic operation complementation. Context will help avoiding confusion.

Now, given a nonempty set U, the powerset Boolean algebra $\mathfrak{P}(U)$ over U is defined to be the algebra with universe $\mathcal{P}(U)$ and fundamental operations $\cup$ and $-$ (interpreting $\vee$ and $-$, respectively). Formally:

$$\mathfrak{P}(U) \stackrel{\text{def}}{=} \langle \mathcal{P}(U), \cup, - \rangle\,,$$

$\mathfrak{P}(U)$ is really an algebra, because $\mathcal{P}(U)$ is closed under both $\cup$ and $-$ (i.e., for any $X, Y \in \mathcal{P}(U)$, $X \cup Y, -X \in \mathcal{P}(U)$).

When working in the similarity class Alg_{ba}, we use the following derived operations:

$$\begin{aligned} x \wedge y &\stackrel{\text{def}}{=} -(-x \vee -y) && (\wedge \text{ is called } \textit{meet}) \\ 0 &\stackrel{\text{def}}{=} x \wedge -x && (\textit{zero} \text{ constant}) \\ 1 &\stackrel{\text{def}}{=} x \vee -x && (\textit{one} \text{ constant}) \\ x - y &\stackrel{\text{def}}{=} x \wedge -y && (\textit{binary minus} \text{ operation}). \end{aligned}$$

It is easy to check that the corresponding "set–operations" in the powerset Boolean algebra over some set U as follows:

$$\begin{array}{ll@{\qquad\qquad}ll} \vee & \cup & 0 & \emptyset \\ - & _U- & 1 & U \\ \wedge & \cap & - & \smallsetminus\,. \end{array}$$

This checking is left to the reader.

We will often use one more derived operation, called *symmetric difference* and denoted by $\otimes$. In this case, similarly to the case of minus and complementation, we use the same symbol, moreover, the same name for these (corresponding) operations both in the "abstract case" and in the "concrete case" (i.e., powerset Boolean algebra case). This will not cause confusion, because context (and a very strong so called representation theorem to be formulated in section 3) will always help. The definition of symmetric difference is:

$$x \otimes y \stackrel{\text{def}}{=} (x - y) \vee (y - x) \qquad \text{in the "abstract case"},$$
$$x \otimes y \stackrel{\text{def}}{=} (x \smallsetminus y) \cup (y \smallsetminus x) \qquad \text{in powerset Boolean algebras.}$$

We refer to the operations $\vee, -, \wedge, 0, 1, \otimes$ (and also to others derivable from these) as *Boolean operations*.

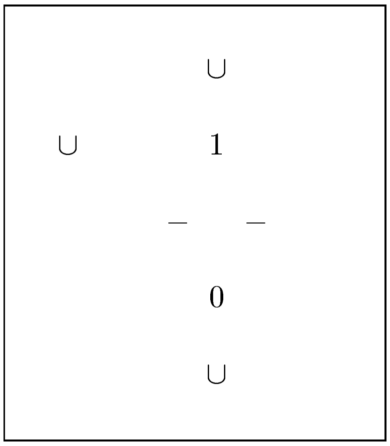
One more kind of examples for algebras are the $\cup$*–reducts* $\langle \mathcal{P}(U), \cup \rangle$ *of powerset Boolean algebras* $\mathfrak{P}(U)$. For any nonempty set U, the algebra $\langle \mathcal{P}(U), \cup \rangle$ and the poset $\langle \mathcal{P}(U), \subseteq \rangle$ are inter–definable in the following sense. In $\langle \mathcal{P}(U), \cup \rangle$, $\subseteq$ can be defined using the fundamental operation $\cup$ as $X \subseteq Y \iff X \cup Y = Y$ for every $X, Y \in \mathcal{P}(U)$. On the other hand, $\cup$ can be captured in $\langle \mathcal{P}(U), \subseteq \rangle$ as $X \cup Y = \sup^{\subseteq}(X, Y)$, where

$$\begin{aligned} \sup{}^{\subseteq}(X,Y) = Z \ \stackrel{\text{def}}{\iff}\ & [Z \in \mathcal{P}(U), X \subseteq Z, Y \subseteq Z, \text{ and} \\ & (\forall W \in \mathcal{P}(U))([X \subseteq W \,\&\, Y \subseteq W] \Rightarrow Z \subseteq W)]\,. \end{aligned}$$

We showed that the partial order $\langle \mathcal{P}(U), \subseteq \rangle$ is determined by the algebra $\langle \mathcal{P}(U), \cup \rangle$ and vice versa. This is why we use two ways of drawing powerset Boolean algebras: one indicating $\cup$, the other indicating $\subseteq$ (the latter are poset diagrams augmented with the function $-$, cf. the beginning of subsection 1.3), see Figures 2.1.4, …, 2.1.7.

Examples for powerset Boolean algebras:

4.1. $\underset{\sim}{2} \stackrel{\text{def}}{=} \mathfrak{P}(1)$.



or

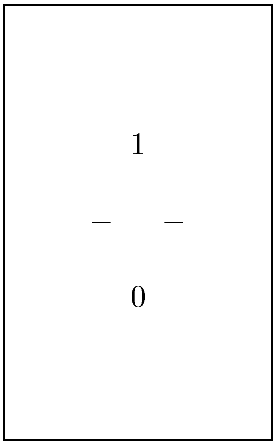


Figure 2.1.4

4.2. $\mathfrak{P}(2)$.

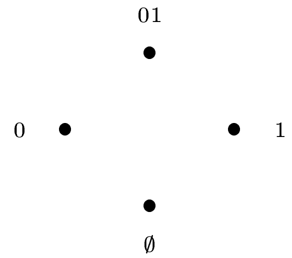


Figure 2.1.5

4.3. $\mathfrak{P}(3)$.

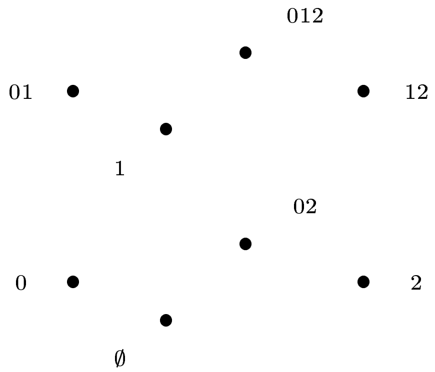


Figure 2.1.6

4.4. $\mathfrak{P}(4)$.

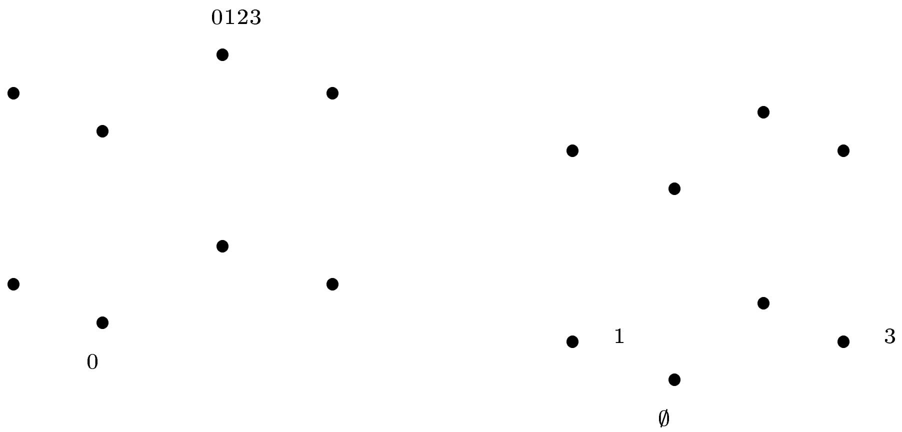


Figure 2.1.7

◀

Remark 2.1.4: Recall that we introduced the logical connectives conjunction and disjunction in Definition 1.8 (2) and (7) in subsection 1.7. Notice that we used the same symbol $\vee$ for denoting both join and conjunction, and we used $\wedge$ for denoting both meet and disjunction. This is not just a coincidence. To the Boolean operations one can naturally associate logical connectives. ◀

æ

2.2. Building new algebras from old ones (operations on algebras).

We want to learn new things from algebras via the old (Greek) method of *analysis and synthesis*. When using this method, one investigates something in such a way that first one takes the thing to (relatively) simple pieces; then one investigates the simple pieces instead of the original thing; finally, one puts the new information obtained about the simple pieces together, gaining this way new knowledge about the whole thing itself. This method, when applied to algebras, assumes the existence of certain ways of obtaining new algebras from already existing ones, namely, two kinds of such ways: ways for gaining simpler algebras from existing ones, and ways of building new, bigger, more complex algebras from existing ones. These ways are performed by introducing *operations on algebras*. Next we will introduce such operations. The ones we will call taking *subalgebras*, *homomorphic images*, forming *direct decompositions* and *subdirect decompositions* give us algebras smaller, simpler than their arguments, while *direct products*, *subdirect products*, *reduced products*, *ultraproducts* produce bigger, more complex, more complicated algebras from a number of simpler, smaller ones.

When decomposing or reducing algebras, it is very natural to ask the question: Can we decompose an algebra to non–decomposable, irreducible, "atomic" fragments? To answer this question, we will introduce *minimal* algebras, *simple* algebras, *directly indecomposable* algebras and *subdirectly irreducible* algebras.

SUBALGEBRA

Let f be an operation of rank n on the nonempty set A, that is, $f : {}^nA \longrightarrow A$, and let $X \subseteq A$. We say that X is *closed w.r.t.* f iff

$$f(a_1, \ldots, a_n) \in X \quad \text{for every } a_1, \ldots, a_n \in X\,.$$

According to this, if f is a constant then X is closed w.r.t. f iff $f \in X$. Thus the empty set $\emptyset$ is closed w.r.t. every operation on A of positive rank, but it is not closed w.r.t. any operation of rank 0.

Example: Taking A to be $\mathbb{Z}$, we see that the set of odd integers is closed w.r.t. multiplication but not w.r.t addition.

Definition 2.2.1 (subalgebra, subalgebra generated by a set etc.):
Let $\mathfrak{A}$ be an algebra.

(i) Let $X \subseteq A$ be a nonempty set. X is called a *subuniverse* of $\mathfrak{A}$ iff X is closed w.r.t. each fundamental operation of $\mathfrak{A}$.

(ii) An algebra $\mathfrak{B}$ is called a *subalgebra* of $\mathfrak{A}$, in symbols $\mathfrak{B} \subseteq \mathfrak{A}$, iff $\mathfrak{A}$ and $\mathfrak{B}$ are similar, B is a subuniverse of $\mathfrak{A}$, and for each function symbol f of $\mathfrak{A}$, $f^{\mathfrak{B}} = f^{\mathfrak{A}} \restriction$

B. We say that $\mathfrak{B}$ is a *proper subalgebra* of $\mathfrak{A}$ iff it is a subalgebra of $\mathfrak{A}$ different from $\mathfrak{A}$, that is, iff $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B} \neq \mathfrak{A}$.

(iii) Let $X \subseteq A$. The *subuniverse of* $\mathfrak{A}$ *generated by* X is

$$\mathrm{Sg}^{\mathfrak{A}}(X) \stackrel{\text{def}}{=} \bigcap\{B : X \subseteq B \text{ and } B \text{ is a subuniverse of } \mathfrak{A}\}.$$

The *subalgebra of* $\mathfrak{A}$ *generated by* X is

$$\mathfrak{Sg}^{\mathfrak{A}}(X) \stackrel{\text{def}}{=} \left\langle \mathrm{Sg}^{\mathfrak{A}}(X), f^{\mathfrak{A}} \restriction \mathrm{Sg}^{\mathfrak{A}}(X) \right\rangle_{f \in Dom(t)}.$$

(iv) If K is a class of similar algebras then

$$\mathsf{SK} \stackrel{\text{def}}{=} \{\mathfrak{B} : \mathfrak{B} \text{ is a subalgebra of some } \mathfrak{A} \in \mathsf{K}\}.$$

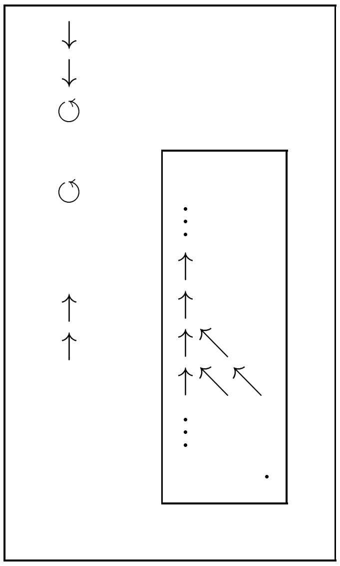
Clearly $\mathsf{K} \subseteq \mathsf{SK}$, because $\mathfrak{A}$ is a subalgebra of itself, for any algebra $\mathfrak{A}$. If $\mathsf{K} = \{\mathfrak{A}\}$ for some algebra $\mathfrak{A}$ then $\mathbf{S}\mathfrak{A} \stackrel{\text{def}}{=} \mathbf{S}\{\mathfrak{A}\}$, that is, $\mathbf{S}\mathfrak{A}$ denotes the set of all subalgebras of $\mathfrak{A}$.

(v) We say that $\mathfrak{A}$ is a *minimal algebra* iff $\mathfrak{A}$ has no proper subalgebras (that is, iff $\mathfrak{B} \subseteq \mathfrak{A} \Longrightarrow \mathfrak{B} = \mathfrak{A}$ for every algebra $\mathfrak{B}$). ◀

Notice that, on Figure 2.1.1, $\mathfrak{C}$ and $\mathfrak{D}$ are minimal but $\mathfrak{A}$ and $\mathfrak{B}$ are not. We note that $\mathfrak{A}$ is minimal iff $\mathfrak{A}$ is generated by any of its subsets.

Remark 2.2.2: The "act" of taking a subalgebra of an algebra $\mathfrak{A}$ can be visualized as "cutting out" a *coherent* part of A (not just any part!). If we imagine that the operations of $\mathfrak{A}$ are drawn as arrows (like on Figure 2.2.1 below), then we can say that a part of A is coherent if no arrows "go out of" it. ◀

$\mathfrak{A}$



Cutting out one part without cutting "outgoing" arrows

Figure 2.2.1

Definition 2.2.3 (Boolean set algebra, SetBA): By a *Boolean set algebra* we mean an element of the class

$$\mathsf{SetBA} \stackrel{\text{def}}{=} \mathbf{S}\{\mathfrak{B} : \mathfrak{B} \text{ is a powerset Boolean algebra}\} = \mathbf{S}\{\mathfrak{P}(U) : U \text{ is a set}\}. \quad \blacktriangleleft$$

Exercise 2.2.4: How many subalgebras does $\mathfrak{P}(3)$ have? ($\mathfrak{P}(U)$ was defined in Exercise 2.1.3 (4).) Answer the same question for $\mathfrak{P}(1)$, $\mathfrak{P}(2)$, $\mathfrak{P}(4)$ instead of $\mathfrak{P}(3)$ as well. ◀

Exercise 2.2.5: For any algebra $\mathfrak{A} \in \mathsf{Alg}_t$ and set $X \subseteq A$, we let

$$\begin{aligned} E^{\mathfrak{A}}(X) &\stackrel{\text{def}}{=} X \cup \{f^{\mathfrak{A}}(a_0, \dots, a_{t(f)-1}) : f \in Dom(t),\, a_0, \dots, a_{t(f)-1} \in X\} \\ E_0^{\mathfrak{A}}(X) &\stackrel{\text{def}}{=} X \\ E_{n+1}^{\mathfrak{A}}(X) &\stackrel{\text{def}}{=} E^{\mathfrak{A}}\big(E_n^{\mathfrak{A}}(X)\big) \quad \text{for every } n \in \omega. \end{aligned}$$

Prove that

$$\mathrm{Sg}^{\mathfrak{A}}(X) = \bigcup \big\{E_n^{\mathfrak{A}}(X) : n \in \omega\big\}. \quad \blacktriangleleft$$

Earlier we said that one possible step in the analytic investigation of algebras is taking subalgebras. It is natural to ask how refined this analysis can be. The following question is a special — and natural — one in this line. Is it true that any algebra $\mathfrak{A}$ has subalgebras of arbitrary sizes (smaller than the size of $\mathfrak{A}$)? The following proposition gives an answer to this question for the case of infinite algebras with finite similarity type.

PROPOSITION 2.2.6. *Let $\mathfrak{A} \in \mathsf{Alg}_t$, $|A| = \kappa \geqslant \omega$, $|Dom(t)| < \omega$. Then*

$$\forall\beta[\omega \leqslant \beta < \kappa \Longrightarrow (\exists \mathfrak{B} \in \mathbf{S}\mathfrak{A})|B| = \beta].$$

Sketch of proof: Let $\mathfrak{A}, t, \kappa$ be as in the formulation of the proposition. Let β be such that $\omega \leqslant \beta < \kappa$, and let $G \subseteq A$, $|G| = \beta$. We let $\mathfrak{B} \stackrel{\text{def}}{=} \mathrm{Sg}^{\mathfrak{A}}(G)$. Then $|B| = \beta$ can be seen as follows.

First one shows that for any set X, $|X| = \beta \Rightarrow |E^{\mathfrak{A}}(X)| = \beta$ ($E^{\mathfrak{A}}(X)$ was defined in Exercise 2.2.5). Here we use the condition $|Dom(t)| < \omega$. From this $(\forall n \in \omega)|E_n^{\mathfrak{A}}(G)| = \beta$ follows. Now, using Exercise 2.2.5,

$$|\mathrm{Sg}^{\mathfrak{A}}(G)| = |\cup \big\{E_n^{\mathfrak{A}}(G) : n \in \omega\big\}| = \cup\{\beta : n \in \omega\} = \beta. \quad \blacksquare$$

HOMOMORPHIC IMAGE

Definition 2.2.7 (homomorphism): Let t be an arbitrary similarity type, let $\mathfrak{A}, \mathfrak{B} \in \mathsf{Alg}_t$, and let $h : A \longrightarrow B$ be an arbitrary function.

(i) Let $f \in Dom(t)$ (that is, f is a function symbol of t). h is said to *respect the interpretation of f* iff

$$h\left(f^{\mathfrak{A}}(a_1, \ldots, a_{t(f)})\right) = f^{\mathfrak{B}}\left(h(a_1), \ldots, h(a_{t(f)})\right)$$

for every $a_1, \ldots, a_{t(f)} \in A$.

(ii) The function h is called a *homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$ iff h respects the interpretation of every function symbol $f \in Dom(t)$.

(iii) $Hom(\mathfrak{A}, \mathfrak{B})$ denotes the set of all homomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$. ◀

Exercise 2.2.8: Prove that for any set X, if $\mathfrak{A} = \mathfrak{Sg}^{\mathfrak{A}}(X)$, $h, k \in Hom(\mathfrak{A}, \mathfrak{B})$ for some algebra $\mathfrak{B}$, and h, k are such that $h \restriction X = k \restriction X$, then $h = k$. ◀

We distinguish several kinds of homomorphisms and use the following notation for them. Each of

$$h : \mathfrak{A} \longrightarrow \mathfrak{B}, \qquad \mathfrak{A} \xrightarrow{h} \mathfrak{B}, \qquad h \in Hom(\mathfrak{A}, \mathfrak{B})$$

denotes that h is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. Both

$$h : \mathfrak{A} \rightarrowtail \mathfrak{B} \qquad \text{and} \qquad \mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{B}$$

denote that h is a one–one homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. We call such homomorphisms *embeddings*. If $\mathfrak{A}$ and $\mathfrak{B}$ are such that $\mathfrak{A} \overset{h}{\rightarrowtail} \mathfrak{B}$ for some embedding h then we say that $\mathfrak{A}$ is *embeddable* into $\mathfrak{B}$.

Similarly, both

$$h : \mathfrak{A} \twoheadrightarrow \mathfrak{B} \qquad \text{and} \qquad \mathfrak{A} \overset{h}{\twoheadrightarrow} \mathfrak{B}$$

denote that h is a homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, and in this case we say that $\mathfrak{B}$ is the *homomorphic image* of $\mathfrak{A}$ under h. Further, each of

$$h : \mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{B}, \qquad \mathfrak{A} \overset{h}{\rightarrowtail\!\!\!\!\twoheadrightarrow} \mathfrak{B}, \qquad \mathfrak{A} \overset{h}{\cong} \mathfrak{B}$$

denotes that h is a one-one homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. We call such a homomorphism an *isomorphism*. $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic*, which we denote by $\mathfrak{A} \cong \mathfrak{B}$, iff there is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. A homomorphism from $\mathfrak{A}$ into $\mathfrak{A}$ is called an *endomorphism* of $\mathfrak{A}$, and an isomorphism from $\mathfrak{A}$ onto $\mathfrak{A}$ is called an *automorphism* of $\mathfrak{A}$.

Remark 2.2.9: Taking a homomorphic image of an algebra is a new example for an operation on algebras. This operation produces a smaller, simpler algebra from its argument (cf. the introductory paragraphs of the present section 2.2).

Taking a homomorphic image of an algebra $\mathfrak{A}$ involves *identifying* certain elements of $\mathfrak{A}$. In the homomorphic image, we cannot "see" certain details which were still clearly "visible" in $\mathfrak{A}$. See Figure 2.2.2.

If h is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, then a simplified image of $\mathfrak{A}$ appears in $\mathfrak{B}$. Putting this another way, magnified versions of certain parts of $\mathfrak{B}$ can be found in $\mathfrak{A}$ (moving backwards via h). See Figure 2.2.3. ◀

Definition 2.2.10 ($\mathbf{I}, \mathbf{H}$): For a class K of similar algebras,

$$\mathbf{I}\mathsf{K} \stackrel{\text{def}}{=} \{\mathfrak{A} : \mathfrak{A} \text{ is isomorphic to some } \mathfrak{B} \in \mathsf{K}\}, \quad \text{and}$$

$$\mathbf{H}\mathsf{K} \stackrel{\text{def}}{=} \{\mathfrak{A} : \mathfrak{A} \text{ is a homomorphic image of some } \mathfrak{B} \in \mathsf{K}\}.$$

Similarly to our convention in case of $\mathbf{S}$, we use the notation

$$\mathbf{H}\mathfrak{A} \stackrel{\text{def}}{=} \mathbf{H}\{\mathfrak{A}\}, \qquad \mathbf{I}\mathfrak{A} \stackrel{\text{def}}{=} \mathbf{I}\{\mathfrak{A}\}.$$ ◀

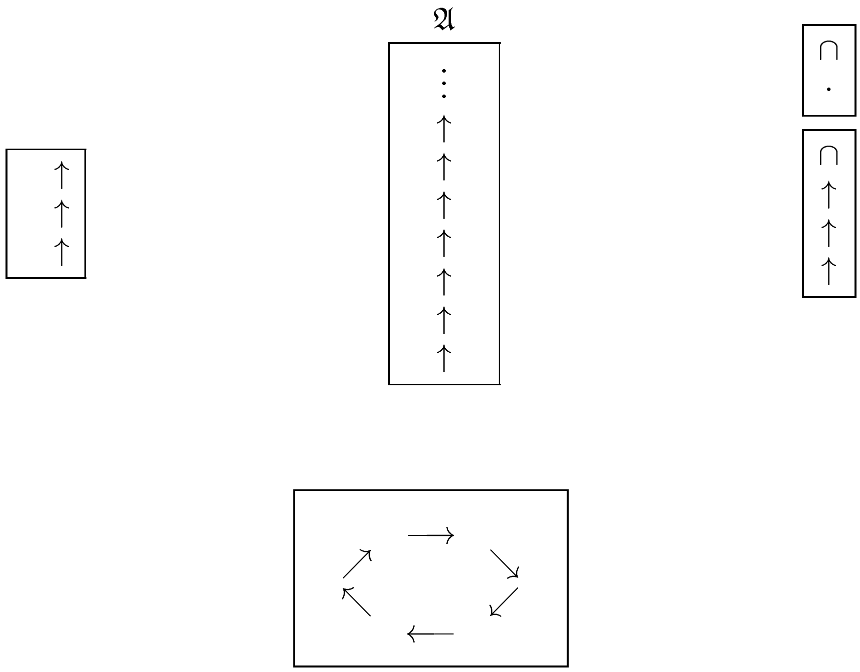


Figure 2.2.2

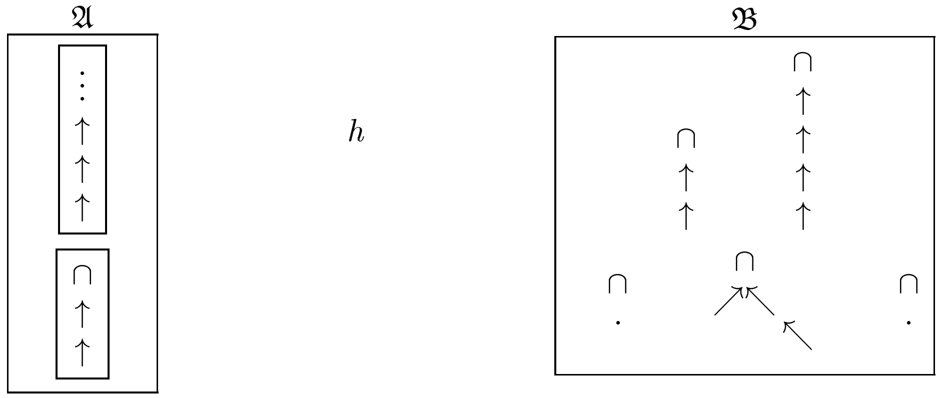


Figure 2.2.3

Principle of identifying isomorphic objects: According to our definition above, an isomorphism is a one–one correspondence between the elements of two algebras that respects the interpretation of each function symbol. Therefore, with regard to a host of properties, isomorphic algebras are indistinguishable from each other. This applies to most of the properties with which we shall deal; if they are true in a given algebra, then they are true for all isomorphic images or isomorphic "copies" of that algebra as well. Such properties are called *algebraic* or *abstract* properties.

If φ is a property of algebras and we say that "property φ holds for algebra $\mathfrak{A}$ *up to isomorphism*" then we mean that φ holds for any isomorphic image of $\mathfrak{A}$. For example, $\mathfrak{P}(3)$ has 5 subalgebras but it has only 3 subalgebras up to isomorphism (because some subalgebras of $\mathfrak{P}(3)$ are isomorphic). ◀

Definition 2.2.11 (Boolean algebra, BA): Recall the definition of Boolean set algebras from Definition 2.2.3. The class BA of all *Boolean algebras* is defined as:

$$\mathsf{BA} \stackrel{\text{def}}{=} \mathbf{I}\mathsf{SetBA}\,. \qquad ◀$$

As we have seen, unlike the formation of subalgebras, the formation of homomorphic images involved external considerations. Despite of this, we will show that all the homomorphic images of an algebra can still be captured completely internally, too. To do this, we introduce the concepts of a *congruence relation* and that of a *quotient algebra*.

Definition 2.2.12 (congruence relation, quotient algebra):

(i) Let A be a set and f an n–ary operation on A, that is, $f : {}^nA \longrightarrow A$. Let $\theta \subseteq A \times A$ be an equivalence relation. We say that θ has the *substitution property* w.r.t. f iff for every $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, the following implication holds.

$$[\forall i(0 < i \leqslant n \Rightarrow a_i\ \theta\ b_i)] \Longrightarrow f(a_1, \ldots, a_n)\ \theta\ f(b_1, \ldots, b_n)\,.$$

(ii) Let $\mathfrak{A}$ be an algebra. A binary relation $\theta \subseteq A \times A$ is called a *congruence relation* (or briefly, *congruence*) *on* $\mathfrak{A}$, iff it is an equivalence relation on A and θ has the substitution property w.r.t. every fundamental operation of $\mathfrak{A}$. $Con(\mathfrak{A})$ denotes the set of all congruence relations on $\mathfrak{A}$.

(iii) If $\mathfrak{A}$ is an algebra then to every $\theta \in Con(\mathfrak{A})$ we associate a new algebra $\mathfrak{A}/\theta$ similar to $\mathfrak{A}$, as follows. The universe of $\mathfrak{A}/\theta$ is the partition A/θ determined by θ (see the definition of a partition after Exercise 1.2 in subsection 1.2). (Thus an element of the universe of $\mathfrak{A}/\theta$ is an equivalence class a/θ for some $a \in A$.) For any function symbol f of $\mathfrak{A}$ of rank n, the fundamental operation $f^{\mathfrak{A}/\theta}$ of $\mathfrak{A}/\theta$ is defined as follows. For any $a_1, \ldots, a_n \in A$,

$$f^{\mathfrak{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = f^{\mathfrak{A}}(a_1, \ldots, a_n)/\theta\,.$$

The algebra $\mathfrak{A}/\theta$ is called the *quotient algebra of* $\mathfrak{A}$ *w.r.t.* θ. ◀

Exercises 2.2.13: Prove that the above definition of a quotient algebra is sound, that is, the definition of its functions does not depend on the choice of the representing elements of the equivalence classes. ◀

Notice that to every algebra $\mathfrak{A}$, $Con(\mathfrak{A})$ has a minimal element and a maximal element w.r.t. inclusion $\subseteq$ as ordering; these are Id_A and $A \times A$.

Definition 2.2.14 (kernel): Let $h \in Hom(\mathfrak{A}, \mathfrak{B})$. Then

$$\ker(h) \stackrel{\text{def}}{=} \{\langle a, b\rangle \in {}^2A : h(a) = h(b)\}$$

is called the *kernel* of h. ◀

Exercises 2.2.15 (kernels and homomorphisms): Let $\mathfrak{A}$ be an arbitrary algebra.

(1) Prove that for any homomorphism h of $\mathfrak{A}$, $\ker(h) \in Con(\mathfrak{A})$.

(2) Let θ be an equivalence relation on A. We define the function $q : A \longrightarrow A/\theta$ as follows:

$$(\forall a \in A) q(a) \stackrel{\text{def}}{=} a/\theta \quad \text{and} \quad \ker(q) = \theta\,.$$

Prove that if $\theta \in Con(\mathfrak{A})$ then $q \in Hom(\mathfrak{A}, \mathfrak{A}/\theta)$.

(3) Let θ be an equivalence relation on A. Prove that $\theta \in Con(\mathfrak{A})$ iff for each function symbol f of $\mathfrak{A}$ of rank n, $\{\langle a_1/\theta, \ldots, a_n/\theta, f^{\mathfrak{A}}(a_1, \ldots, a_n)/\theta\rangle : a_1, \ldots, a_n \in A\}$ is an operation (an n–ary function) on A/θ. ◀

The homomorphism q defined in Exercise 2.2.15 (2) is called the *quotient map* from $\mathfrak{A}$ onto $\mathfrak{A}/\theta$.

COROLLARY 2.2.16.
$Con(\mathfrak{A}) = \{\ker(h) : h$ is a homomorphism with $Dom(h) = A\}$. ∎

Exercise 2.2.17:
Prove that if $\mathfrak{A}$ is an algebra and $\emptyset \neq C \subseteq Con(\mathfrak{A})$ then $\bigcap C \in Con(\mathfrak{A})$. ◀

THEOREM 2.2.18 (Homomorphism Theorem). *Let $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, $\theta = \ker(h)$, and let $q : \mathfrak{A} \longrightarrow \mathfrak{A}/\theta$ be the quotient map. Then the unique function $i : A/\theta \twoheadrightarrow B$ satisfying $q \circ i = h$ is an isomorphism from $\mathfrak{A}/\theta$ onto $\mathfrak{B}$.*

Proof: The proof is illustrated on Figure 2.2.4.

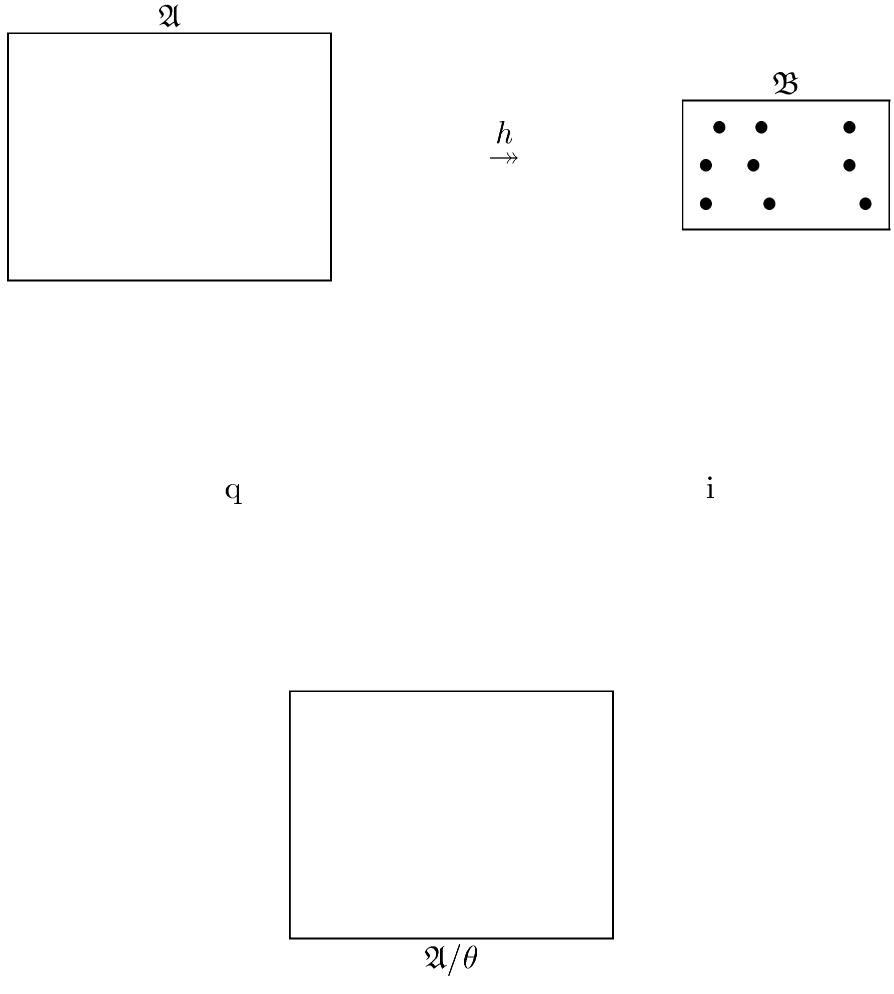


Figure 2.2.4

We define:

$$i(a/\theta) \stackrel{\text{def}}{=} h(a) \text{ for every } a \in A.$$

- This definition is sound: Suppose $a\theta b$. Then $h(a) = h(b)$ since $\theta = \ker(h)$. Thus $i(a/\theta) = h(a) = h(b) = i(b/\theta)$.

- i is one–one:

$$\begin{aligned} i(a/\theta) = i(b/\theta) &\Longrightarrow h(a) = h(b) \\ &\Longrightarrow a/\theta = b/\theta \qquad \text{by } \ker(h) = \theta. \end{aligned}$$

- i is onto because h is onto, and $q \circ i = h$.
- i is a homomorphism:
  If f is an n–ary function symbol and $a_1, \ldots, a_n \in A$ then

$$\begin{aligned} i\left(f^{\mathfrak{A}/\theta}(a_1/\theta, \ldots, a_n/\theta)\right) &= i\left(f^{\mathfrak{A}/\theta}(q(a_1), \ldots, q(a_n))\right) && \text{by def. of } q \\ &= i\left(q\left(f^{\mathfrak{A}}(a_1), \ldots, f^{\mathfrak{A}}(a_n)\right)\right) && q \text{ is a homom.} \\ &= h\left(f^{\mathfrak{A}}(a_1), \ldots, f^{\mathfrak{A}}(a_n)\right) && \text{by } q \circ i = h \\ &= f^{\mathfrak{B}}\left(h(a_1), \ldots, h(a_n)\right) && h \text{ is a homom.} \\ &= f^{\mathfrak{B}}\left(i(q(a_1)), \ldots, i(q(a_n))\right) && \text{by } q \circ i = h \\ &= f^{\mathfrak{B}}\left(i(a_1/\theta), \ldots, i(a_n/\theta)\right) && \text{by def. of } q\,. \end{aligned}$$

∎

Definition 2.2.19: An algebra $\mathfrak{A}$ is called *simple* iff $\mathfrak{A}$ is non–trivial, and

$$\forall \mathfrak{B} \left(\forall (\mathfrak{A} \xrightarrow{h} \mathfrak{B})\right) (h \text{ is one–one} \quad \text{or} \quad |Rng(h)| = 1)\,.$$

Smp denotes the class of all simple algebras. If K is a class of algebras then

$$\mathsf{SmpK} = \mathsf{Smp}(\mathsf{K}) \stackrel{\text{def}}{=} \mathsf{Smp} \cap \mathsf{K}\,.$$

That is, $\mathsf{Smp}(\mathsf{K})$ is the class of all simple algebras of K. ◀

Notice that $\mathfrak{A} \in \mathsf{Smp} \Longrightarrow |Con(\mathfrak{A})| = 2$.

Examples 2.2.20 (concerning simple algebras):
1. Every two–element algebra is simple. Thus so is $\underset{\sim}{2} (= \mathfrak{P}(1))$, see on Figure 2.1.4.
2. $\mathfrak{A}$ on Figure 2.1.2 is simple.
3. Let $\mathfrak{A} = \langle \omega, \mathsf{pred} \rangle$, where $(\forall n > 0)\mathsf{pred}(n) = n - 1$ and $\mathsf{pred}(0) = 0$, see Figure 2.2.5.

$\mathfrak{A}$

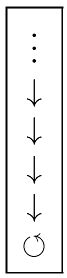


Figure 2.2.5

Then we have that

$$\forall \mathfrak{B} \left(\forall (\mathfrak{A} \overset{h}{\twoheadrightarrow} \mathfrak{B}) \right) (\mathfrak{A} \cong \mathfrak{B} \quad \text{or} \quad |Rng(h)| = 1) . \tag{2.2.1}$$

But $\mathfrak{A}$ is not simple. Algebras satisfying condition (2.2.1) are called *pseudo–simple* algebras. ◀

DIRECT (CARTESIAN) PRODUCT, DIRECT DECOMPOSITION

Recall the concept of a Cartesian (or direct) product of a system of sets from subsection 1.5. Now we introduce the direct product of a system of algebras. Before giving the general definition, we describe some special cases.

Let $\mathfrak{A} = \langle A, f^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle B, f^{\mathfrak{B}} \rangle$ be two algebras with f unary (i.e., $f^{\mathfrak{A}} : A \to A$). Their *Cartesian* (or *direct*) *product* $\mathfrak{A} \times \mathfrak{B}$ has universe $A \times B$ and is defined as follows. $\mathfrak{A} \times \mathfrak{B} = \langle A \times B, f \rangle$, where $f(\langle a, b \rangle) = \langle f^{\mathfrak{A}}(a), f^{\mathfrak{B}}(b) \rangle$ for all $\langle a, b \rangle \in A \times B$. See Figure 2.2.6.

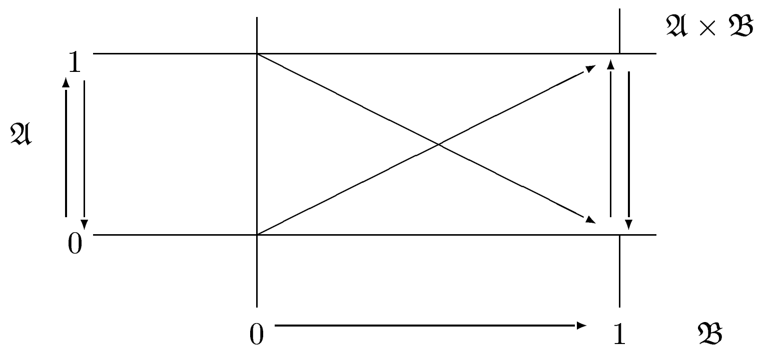


Figure 2.2.6

A straightforward generalization of a product of two algebras is a product of an arbitrary sequence (or family) $\langle \mathfrak{A}_i : i \in I \rangle$ of algebras, as follows. Let $\mathfrak{A}_i = \langle A_i, f^i \rangle$ for $i \in I$. Then $\mathbf{P}_{i \in I} \mathfrak{A}_i = \langle \mathbf{P}_{i \in I} A_i, f \rangle$, where $f(\langle a_i : i \in I \rangle) = \langle f^i(a_i) : i \in I \rangle$.

The definition for n-ary operations $f : {}^n A \to A$ is completely analogous. So is the definition to arbitrary algebras, as follows.

Definition 2.2.21 (direct [Cartesian] product): Let I be a set, t an arbitrary similarity type and $\{\mathfrak{A}_i : i \in I\} \subseteq \mathsf{Alg}_t$. The *direct* (or *Cartesian*) *product* of the system $\langle \mathfrak{A}_i : i \in I \rangle$ of similar algebras, denoted by $\mathbf{P}\langle \mathfrak{A}_i : i \in I \rangle$, is defined to be the algebra of type t with universe $\mathbf{P}_{i\in I}A_i$ and fundamental operations defined as follows. If $t(f) = n$ and $a_1, \ldots, a_n \in \mathbf{P}_{i\in I}A_i$ then

$$\left(f^{\mathbf{P}\mathfrak{A}}(a_1, \ldots, a_n)\right)_i = f^{\mathfrak{A}_i}(a_1(i), \ldots, a_n(i))$$

for all $i \in I$. ◀

$\mathbf{P}\langle\mathfrak{A}_i : i \in I\rangle$ is also denoted as $\mathbf{P}_{i\in I}\mathfrak{A}_i$ or $\underset{I}{\mathbf{P}}\mathfrak{A}$. If, for some algebra $\mathfrak{B}$, $\mathfrak{A}_i = \mathfrak{B}$ for all $i \in I$ then we write ${}^I\mathfrak{B}$ for the direct product, and call it a *direct power* of $\mathfrak{B}$.

Exercises 2.2.22:

(1) Show that $\mathbf{P}_{i\in I}\mathfrak{A}_i$ is indeed a generalization of $\mathfrak{A} \times \mathfrak{B}$.
Hint: Choose first $I = 2$ ($= \{0, 1\}$). Then choose $I = 3$. Then go on. Make drawings like Figure 2.2.5.

(2) Prove that if $I = \emptyset$ then $\mathbf{P}_{i\in I}\mathfrak{A}_i$ is a trivial algebra. ◀

LEMMA 2.2.23 (on projections). *We use the notation introduced in Definition 2.2.21 above and the notion of a projection function p_i introduced in subsection 1.5.*

(i) $(\forall i \in I)p_i$ *is a homomorphism.*

(ii) $(\forall i \in I)p_i$ *is onto* A_i.

(iii) $(\forall a, b \in A, a \neq b)(\exists i \in I)p_i(a) \neq p_i(b)$.

The **proof** is left to the reader. ∎

If I is a set, $\mathfrak{A}_i$ $(i \in I)$ and $\mathfrak{B}$ are similar algebras and $h_i : \mathfrak{B} \longrightarrow \mathfrak{A}_i$ are homomorphisms for every $i \in I$, then we call the system $\langle h_i : i \in I\rangle$ of homomorphisms a *cone*. We also use the notation $\langle h_i\rangle_{i\in I}$ as well as $\langle\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i\rangle_{i\in I}$ for denoting this cone. The cone $\langle\mathbf{P}_{i\in I}\mathfrak{A}_i \xrightarrow{p_i} \mathfrak{A}_i\rangle_{i\in I}$ of projection homomorphisms is called the *projection cone of the product* $\mathbf{P}_{i\in I}\mathfrak{A}_i$.

LEMMA 2.2.24 ("universal property" of direct product). *Let $\langle\mathfrak{A}_i : i \in I\rangle$ be an indexed system of similar algebras, and let $\langle\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i\rangle_{i\in I}$ be a cone, for some algebras $\mathfrak{B}$ and homomorphisms $\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i$ $(i \in I)$. Then there exists a unique homomorphism $k : \mathfrak{B} \longrightarrow \mathbf{P}_{i\in I}\mathfrak{A}_i$ such that for every $i \in I$, $k \circ p_i = h_i$, where p_i denotes the i–th projection function. Concisely, our lemma states:*

$$\left(\forall\langle\mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i\rangle_{i\in I}\right)(\exists! k : \mathfrak{B} \longrightarrow \mathbf{P}_{i\in I}\mathfrak{A}_i)(\forall i \in I)k \circ p_i = h_i\,.$$

See Figure 2.2.7.

$$\begin{array}{ccccc} & & k & & \\ \mathfrak{B} & & & \mathbf{P}_{i\in I}\mathfrak{A}_i & \\ & & & & \\ h_i & & p_i & & \\ \dots & \mathfrak{A}_i & \dots & \mathfrak{A}_j & \dots \end{array}$$

Figure 2.2.7

The **proof** is left to the reader. ∎

When using Lemma 2.2.24, sometimes we say that the cone $\langle \mathfrak{B} \xrightarrow{h_i} \mathfrak{A}_i\rangle_{i\in I}$ *induces* the homomorphism k. Borrowing the expression from category theory, we call the property described by Lemma 2.2.24 the *universal property of direct products*.

LEMMA 2.2.25 (commutativity and associativity of direct product).

(i) $\mathfrak{A}_1 \times \mathfrak{A}_2 \cong \mathfrak{A}_2 \times \mathfrak{A}_1$
(that is, direct product is commutative up to isomorphism).

(ii) $\mathfrak{A}_1 \times (\mathfrak{A}_2 \times \mathfrak{A}_3) \cong (\mathfrak{A}_1 \times \mathfrak{A}_2) \times \mathfrak{A}_3$
(that is, direct product is associative up to isomorphism).

The **proof** is left to the reader. ∎

Definition 2.2.26 (P): Given a class K of similar algebras, we let

$$\mathbf{P}\mathsf{K} \stackrel{\text{def}}{=} \mathbf{I}\left\{\mathbf{P}_{i\in I}\mathfrak{A}_i : I \text{ is a set, and } (\forall i \in I)\mathfrak{A}_i \in \mathsf{K}\right\} .$$

We often write $\mathbf{P}\mathfrak{A}$ instead of $\mathbf{P}\{\mathfrak{A}\}$. ◀

Exercises 2.2.27:

(1) Prove that $\mathbf{SS}\mathsf{K} = \mathbf{S}\mathsf{K}$.
(2) Prove that $\mathbf{PP}\mathsf{K} = \mathbf{P}\mathsf{K}$.
(3) Prove that $\mathbf{SPSP}\mathsf{K} = \mathbf{SP}\mathsf{K}$.
(4) Prove that there is no family $\langle \mathfrak{A}_i : i \in I\rangle$ of non–trivial algebras with $|I| \geqslant \omega$ such that $|\mathbf{P}_{i\in I}\mathfrak{A}_i| = \omega$.
(5) Let $\mathsf{K} = \{\mathfrak{A}\}$, where $|A| = 2$. Prove that $\mathbf{P}\mathsf{K}$ does not contain any countably infinite algebra.
(6) Let K be as in (5) above. Prove that $\mathbf{SP}\mathsf{K}$ does contain countably infinite algebras.

This shows that the operator **SP** is more flexible than **P**! Hence often **SP** is more useful than **P**. Actually, **SP** is one of the most useful operators. ◀

Definition 2.2.28 (directly indecomposable algebras): We say that $\mathfrak{A}$ is *directly indecomposable* iff $\mathfrak{A}$ is not isomorphic to the direct product of two non–trivial algebras. ◀

Clearly, the simple algebras and the algebras of prime cardinality are directly indecomposable.

THEOREM 2.2.29 (direct decomposition of finite algebras). *Every finite algebra is isomorphic to the direct product of some directly indecomposable algebras.*

Proof: By induction. Let $\mathfrak{A}$ be finite.
- If $|A| = 1$, then $\mathfrak{A}$ is directly indecomposable.
- Assume $\mathfrak{A}$ is non–trivial. Our *induction hypothesis* is:

$$\forall \mathfrak{B}(|B| < |A| \Longrightarrow \mathfrak{B} \text{ is isomorphic to the direct product of some directly indecomposable algebras}) .$$

Case 1: $\mathfrak{A}$ is directly indecomposable. Then we are done.
Case 2: $(\exists \mathfrak{A}_1, \mathfrak{A}_2)(\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2 \text{ and } 1 < |A_1|, 1 < |A_2|)$. Then $|A_1| < |A| > |A_2|$. Then, by the induction hypothesis,

$$\mathfrak{A}_1 \cong \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_m \qquad \text{and} \qquad \mathfrak{A}_2 \cong \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_k ,$$

for some directly indecomposable $\mathfrak{B}_i$'s and $\mathfrak{C}_i$'s. Thus, using Lemma 2.2.25 (ii) above,

$$\mathfrak{A} \cong \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_m \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_k .$$ ∎

Exercises 2.2.30: Give such examples for directly indecomposable algebras which are not mentioned above (that is, they should be non–simple and have non–prime cardinalities). ◀

SUBDIRECT DECOMPOSITION

Motivation 2.2.31: As we mentioned before starting to introduce operations on algebras, to study a complex system[1] $\mathfrak{A}$, a standard approach is to simplify $\mathfrak{A}$ first, then study the simplified system, and then try to use the so obtained information for a better understanding of the original system $\mathfrak{A}$ itself. One of our tools for simplifying an algebra $\mathfrak{A}$ is taking homomorphic images of it. If $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ is surjective then we can consider $\mathfrak{B}$ as a *simplified version* of $\mathfrak{A}$, and h a "simplification". Such an h is a *real* (or non–trivial) simplification if $h(a) = h(b)$ for some $a, b \in A$, $a \neq b$. If we have only one real simplification $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then forgetting $\mathfrak{A}$ will cause loss of information, since from $\mathfrak{B}$ we cannot reconstruct $\mathfrak{A}$. The reason for this is that the information which was (deliberately) thrown away when a and b were collapsed, cannot be restored from $\mathfrak{B}$

[1]Here we use the word "system" in an intuitive way. Our motivation for calling an algebra $\mathfrak{A}$ a system in this motivational text comes from applications of algebra to the theory of (complex) systems.

itself. E.g., on Figure 2.2.8, if we look at $\mathfrak{B}$ only (say $\mathfrak{A}$ was forgotten), we will never find out whether "$a \overset{\longleftarrow}{\longrightarrow} b$" or "$\overset{\cap}{a} \qquad \overset{\cap}{b}$" or perhaps "$a \longrightarrow \overset{\cap}{b}$" was the original pattern in $\mathfrak{A}$. However, if we have *another simplification*, say $h_1 : \mathfrak{A} \twoheadrightarrow \mathfrak{B}_1$ such that $h_1(a) \neq h_1(b)$, then the two simplifications h, h_1 *together* might *both* simplify the original picture $\mathfrak{A}$ *and* at the same time, retain all information about $\mathfrak{A}$. Such is the situation, e.g., on Figure 2.2.6, where the algebra $\mathfrak{A} \times \mathfrak{B}$ is simplified into two simple components $\mathfrak{A}$ and $\mathfrak{B}$ without loss of information (the simplifying homomorphisms are the two projections).

$$\begin{tikzcd}
 & & \mathfrak{A} \\
\cdot & b \arrow[d, shift left] & \\
\cdot \arrow[d] & a \arrow[u, shift left] \arrow[d, "h"] & \\
\cdot \arrow[r] & \cdot & \mathfrak{B}
\end{tikzcd}$$

Figure 2.2.8

What is the criterion for not loosing information? Well, we need a sequence $h_i : \mathfrak{A} \twoheadrightarrow \mathfrak{B}_i$ $(i \in I)$ of surjective homomorphisms such that

$$(\forall a, b \in A)[a \neq b \Rightarrow (\exists i \in I) h_i(a) \neq h_i(b)]\,. \tag{2.2.2}$$

This is the case, e.g., when $\mathfrak{A}$ is the direct product of $\langle \mathfrak{B}_i : i \in I \rangle$ and h_i $(i \in I)$ are the projection functions. Another example is on Figure 2.2.9. (here $\mathfrak{A}$ is not the direct product of $\mathfrak{B}_1$ and $\mathfrak{B}_2$):

$$\begin{tikzcd}
\mathfrak{B}_1 & & & \mathfrak{A} \\
\cdot \arrow[d, shift left] & & \cdot \arrow[ll, "h_1"'] \arrow[d, shift left] & \\
\cdot \arrow[u, shift left] & \cdot \arrow[l] \arrow[r] \arrow[d] & \cdot \arrow[u, shift left] \arrow[d, "h_2"] & \\
 & \cdot \arrow[r] & \cdot & \\
 & & & \mathfrak{B}_2
\end{tikzcd}$$

Figure 2.2.9

A sequence of homomorphisms satisfying (2.2.2) is a simplification which certainly does not loose information. In order to make this a *non–trivial* (or real) simplification, we require that $(\forall i \in I)(h_i$ is not injective and $|Rng(h_i)| > 1)$.

Actually, a simplification $\langle h_i : i \in I\rangle$ of $\mathfrak{A}$ as described above can be considered as a *decomposition* of the complex system $\mathfrak{A}$ into simpler systems $\mathfrak{B}_i$, $i \in I$. For certain reasons, we call such decompositions *subdirect decompositions*[2] . The general schema of subdirect decompositions is like this:

$$\begin{array}{ccccccccc}
 & & & & \mathfrak{A} & & & & \\
 & & & h_i & & h_j & & & \\
\dots & & \mathfrak{B}_i & & \dots & & \mathfrak{B}_j & & \dots
\end{array}$$

Figure 2.2.10

◀

Definition 2.2.32 (subdirect decomposition): Let $\mathfrak{A}$ be an algebra.

(i) By a *subdirect decomposition* of $\mathfrak{A}$ we understand a system $\langle h_i : i \in I\rangle$ of surjective homomorphisms $h_i : \mathfrak{A} \twoheadrightarrow \mathfrak{B}_i$ such that

$$(\forall a, b \in A, a \neq b)(\exists i \in I)h_i(a) \neq h_i(b)\,.$$

(ii) A subdirect decomposition is called *non–trivial* if

$$(\forall i \in I)(h_i \text{ is not an isomorphism})\,.$$

We call it *trivial* otherwise.

(iii) If $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i\rangle_{i\in I}$ is a subdirect decomposition of $\mathfrak{A}$, then we call $\mathfrak{B}_i$ the i–th component (of the decomposition). ◀

Exercises 2.2.33 (subdirect decomposition):

(1) Give examples of non–trivial subdirect decompositions. Choose first $\mathfrak{A}$ to have three elements and, say, all operations “trivial”.
(2) Prove that if $|A| = 2$ then $\mathfrak{A}$ has no non–trivial subdirect decomposition.
(3) Prove that if in $\bar{h} = \langle h_i : i \in I\rangle$ we have $|I| = 1$, then $\bar{h}$ cannot be a non–trivial subdirect decomposition.
(4) Give two different (non–isomorphic) subdirect decompositions of some algebra $\mathfrak{A}$. ◀

---

[2]The name “subdirect” comes from a strong connection with the operator **SP**, where **S** is connected with the part “sub” and **P** with the part “direct” of the word.

Exercises 2.2.34 (subdirect decomposition and direct decomposition):

(1) Assume $\langle \mathfrak{A} \xrightarrow{h_i}\!\!\!\!\rightarrow \mathfrak{B}_i\rangle_{i<2}$ is a subdirect decomposition of $\mathfrak{A}$. Prove that $\mathfrak{A}$ is embeddable into $\mathfrak{B}_0 \times \mathfrak{B}_1$.

Hint: First check that the statement is true with $\mathfrak{A}$ a small finite algebra of 3 or 4 elements.

(2) Let $\langle \mathfrak{B}_i : i \in I\rangle$ be a system of similar algebras. Recall the projection functions p_i from subsection 1.5. Prove that $\langle \mathbf{P}_{j\in I}\mathfrak{B}_j \xrightarrow{p_i} \mathfrak{B}_i : i \in I\rangle$ is a subdirect decomposition of $\mathbf{P}_{j\in I}\mathfrak{B}_j$.

(3) Let $\langle \mathfrak{A} \xrightarrow{h_i}\!\!\!\!\rightarrow \mathfrak{B}_i\rangle_{i\in I}$ be a subdirect decomposition. Prove that there is an embedding $\mathfrak{A} \overset{f}{\rightarrowtail} \mathbf{P}_{i\in I}\mathfrak{B}_i$ such that $(\forall i \in I) f \circ p_i = h_i$, that is, the following diagram commutes.

$$\begin{tikzcd}
 & \mathfrak{A} \arrow[rr, "f"] \arrow[d, "h_i"'] & & \mathbf{P}_{i\in I}\mathfrak{B}_i \arrow[dll, "p_i"] & \\
\dots & \mathfrak{B}_i & \dots & \mathfrak{B}_j & \dots
\end{tikzcd}$$

Figure 2.2.13

(4) Let $\mathfrak{A} \subseteq \mathbf{P}_{i\in I}\mathfrak{B}_i$ be an arbitrary subalgebra of an arbitrary direct product. Recall that $p_i \upharpoonright A$ denotes the function p_i restricted to A. Prove that $\langle (p_i \upharpoonright A) : i \in I\rangle$ is a subdirect decomposition of $\mathfrak{A}$. ◀

We will see that subdirect decompositions indeed do not involve loss of information in the following sense. If we are given a subdirect decomposition of an algebra $\mathfrak{A}$ into components $\mathfrak{B}_i$ $(i \in I)$, then $\mathfrak{A}$ can be "built up" from the components $\mathfrak{B}_i$ $(i \in I)$ by the operator **SP**. That is, $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{B}_i : i \in I\}$. (Cf. Exercise 2.2.27 (6).)

THEOREM 2.2.35. *Let $\langle \mathfrak{A} \xrightarrow{h_i}\!\!\!\!\rightarrow \mathfrak{B}_i\rangle_{i\in I}$ be a subdirect decomposition of the algebra $\mathfrak{A}$. Then (i)–(ii) below hold.*

(i) $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{B}_i : i \in I\}$.

(ii) $f : \mathfrak{A} \rightarrowtail \mathbf{P}_{i\in I}\mathfrak{B}_i$, *for some embedding f of $\mathfrak{A}$.*

Proof: (i) clearly follows from (ii), thus it is enough to prove (ii).

Let $\langle \mathfrak{A} \xrightarrow{h_i}\!\!\!\!\rightarrow \mathfrak{B}_i\rangle_{i\in I}$ be a subdirect decomposition of some algebra $\mathfrak{A}$. We define $f : A \longrightarrow \mathbf{P}_{i\in I}B_i$ as follows. For each $a \in A$, $f(a) \overset{\text{def}}{=} \langle h_i(a) : i \in I\rangle$.

Clearly, $f(a) \in \mathbf{P}_{i\in I}B_i$ for every $a \in A$. By the definition of a subdirect decomposition, we have

$$(\forall a, b \in A)[a \neq b \Rightarrow (\exists i \in I) h_i(a) \neq h_i(b)]\,.$$

Therefore f is injective. It remains to check that f is a homomorphism.

To see this, let g be an n–ary operation of $\mathfrak{A}$, and let $\langle a_1, \dots, a_n\rangle \in {}^nA$. Then

$$\begin{aligned} f\left(g^{\mathfrak{A}}(a_1,\dots,a_n)\right) &= \langle h_i\left(g^{\mathfrak{A}}(a_1,\dots,a_n)\right) : i \in I\rangle && \text{by def. of } f \\ &= \langle g^{\mathfrak{B}_i}\left(h_i(a_1),\dots,h_i(a_n)\right) : i \in I\rangle && h_i \text{ is a homom.} \\ &= g^{\mathbf{P}\mathfrak{B}_i}\left(\langle h_i(a_1)\rangle_{i\in I},\dots,\langle h_i(a_n)\rangle_{i\in I}\right) && \text{by def. of a product} \\ &= g^{\mathbf{P}\mathfrak{B}_i}\left(f(a_1),\dots,f(a_n)\right). \end{aligned}$$

This proves that f is a homomorphism, completing the proof of (ii). ∎

The above theorem shows that subdirect decompositions are useful in the sense that instead of a complicated algebra $\mathfrak{A}$, we may study its "relatively simple" subdirect components $\mathfrak{B}_i$ $(i \in I)$, and after having studied the components, we can reconstruct the original algebra $\mathfrak{A}$ by using $\mathbf{SP}$. As we already said in the starting paragraph of section 2.2, this process of decomposing first, then studying the parts, and then recovering the original by putting the parts together is often called the analysis–synthesis method; namely, we analyze complex system $\mathfrak{A}$ into its "simple" parts $\mathfrak{B}_i$, study the parts, and then synthesize the original $\mathfrak{A}$ from parts $\mathfrak{B}_i$.

This analysis–synthesis method suggests the question whether there are some kind of "smallest" i.e. "atomic" building blocks which cannot be further decomposed subdirectly into even "smaller" (or "simpler") components.[3] This thought motivates the definition of *subdirectly indecomposable* algebras. To follow the "official" terminology, we will say "subdirectly irreducible" instead of "subdirectly indecomposable".

Definition 2.2.36 (subdirectly irreducible algebras):

(i) An algebra $\mathfrak{A}$ is called *subdirectly irreducible* iff $\mathfrak{A}$ has no non–trivial subdirect decomposition (in the sense of Definition 2.2.32 (ii) above). That is, $\mathfrak{A}$ is subdirectly irreducible iff for any subdirect decomposition $\langle h_i : i \in I\rangle$ of $\mathfrak{A}$, if $|I| \neq 0$ then

$$(\exists i \in I)(h_i \text{ is an isomorphism}).$$

Putting it another way, $\mathfrak{A}$ is subdirectly irreducible iff for any subdirect decomposition $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i\rangle_{i\in I}$ of $\mathfrak{A}$, we have $\mathfrak{A} \in \mathbf{I}\{\mathfrak{B}_i : i \in I\}$. That is, for any subdirect decomposition of $\mathfrak{A}$, the original algebra $\mathfrak{A}$ is isomorphic to some of the components $\mathfrak{B}_i$.

(ii) Sir denotes the class of all subdirectly irreducible algebras. If K is a class of algebras then

$$\mathsf{SirK} = \mathsf{Sir}(\mathsf{K}) \stackrel{\text{def}}{=} \mathsf{Sir} \cap \mathsf{K}.$$

That is, SirK is the class of subdirectly irreducible members of K. ◀

[3] Of course, this question motivates the related one, of whether every algebra is built up from such "atomic", that is, "indecomposable" building blocks. We will come to this question soon.

LEMMA 2.2.37 (on the congruences of subdirectly irreducible algebras). *$\mathfrak{A} \in \mathsf{Sir}$ iff $Con(\mathfrak{A}) \smallsetminus \mathsf{Id}_A$ has a smallest element, that is, there is a congruence $\beta \in Con(\mathfrak{A}) \smallsetminus \mathsf{Id}_A$ such that*

$$(\forall \alpha \in Con(\mathfrak{A}))(\beta \subseteq \alpha \Longleftrightarrow \alpha \neq \mathsf{Id}_A)\,. \tag{2.2.3}$$

Proof: Assume $\mathfrak{A} \in \mathsf{Sir}$. Let $\beta \overset{\text{def}}{=} \bigcap (Con(\mathfrak{A}) \smallsetminus \mathsf{Id}_A)$. Then $\beta \in Con(\mathfrak{A})$ by Exercise 2.2.17. Assume $\beta = \mathsf{Id}_A$. Then $\langle h : \ker(h) \in Con(\mathfrak{A}) \smallsetminus \mathsf{Id}_A \rangle$ is a non–trivial subdirect decomposition of $\mathfrak{A}$. By Definition 2.2.36, this contradicts the fact that $\mathfrak{A} \in \mathsf{Sir}$. Thus $\beta \neq \mathsf{Id}_A$, and, clearly, β is the smallest element of $Con(\mathfrak{A}) \smallsetminus \mathsf{Id}_A$.

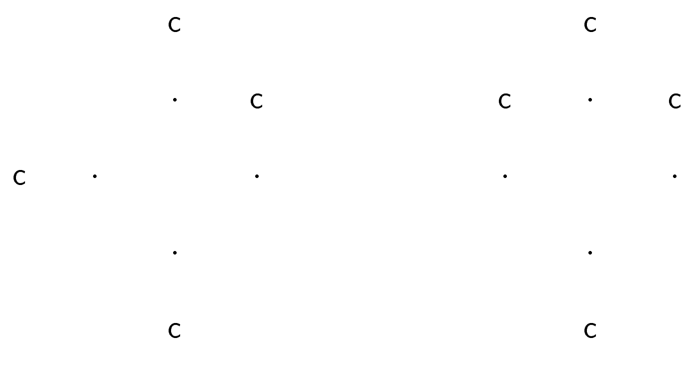
Now assume $\beta \in Con(\mathfrak{A}) \smallsetminus \mathsf{Id}_A$ satisfies (2.2.3). Let $\langle h_i : i \in I \rangle$ be a subdirect decomposition of $\mathfrak{A}$. Let $\langle a, b \rangle \in \beta \smallsetminus \mathsf{Id}_A$. Then $h_i(a) \neq h_i(b)$ for some $i \in I$. Thus, for such an i, $\langle a, b \rangle \notin \ker(h_i) \in Con(\mathfrak{A})$, therefore $\beta \not\subseteq \ker(h_i)$. This implies $\ker(h_i) = \mathsf{Id}_A$ by (2.2.3). Thus h_i is one–one. But then h_i is an isomorphism because it is onto (by subdirect decomposition). This proves $\mathfrak{A} \in \mathsf{Sir}$. ∎

Exercises 2.2.38 (subdirectly irreducible algebras):

(1) Prove that for every algebra $\mathfrak{A}$, $|A| = 2 \Rightarrow \mathfrak{A} \in \mathsf{Sir}$.
(2) Prove that $\underset{\sim}{2} \in \mathsf{Sir}$ and $\mathfrak{P}(2) \notin \mathsf{Sir}$.
(3) Prove that any subalgebra of the algebra $\mathfrak{A}$ given in Example 2.2.20 (3) is subdirectly irreducible.
(4) Prove that every subdirectly irreducible algebra is directly indecomposable. ◀

Exercises 2.2.39 (subdirectly irreducible and simple algebras):

(1) Recall the similarity type ba from Example 2.1.3 **4**. Let $t \overset{\text{def}}{=} ba \cup \{\langle \mathsf{c}, 1 \rangle\}$. Let the algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{Alg}_t$ be isomorphic to $\mathfrak{P}(2)$ on the "ba–part", and c be as indicated on Figure 2.2.11.



$\mathfrak{A}$ $\mathfrak{B}$

Figure 2.2.11

Prove that $\mathfrak{A} \in \mathsf{Sir} \smallsetminus \mathsf{Smp}$ and $\mathfrak{B} \in \mathsf{Sir}(\mathsf{Smp})$.

(2) Prove that if $\mathfrak{A}$ satisfies condition (2.2.4) below then $\mathfrak{A}$ is subdirectly irreducible but not simple.

(2.2.4) Let $a, b \in A$, $a \neq b$. To every $x, y \in A$, if x, y, a, b are all different then there is a fundamental operation $f_{x,y}$ of $\mathfrak{A}$ such that every element of $\mathfrak{A}$ except for x and y are fixed points of f_{xy}, and $f_{xy}(x) = a$ and $f_{xy}(y) = b$. There are no other fundamental operations of $\mathfrak{A}$.

See the illustration on Figure 2.2.12.

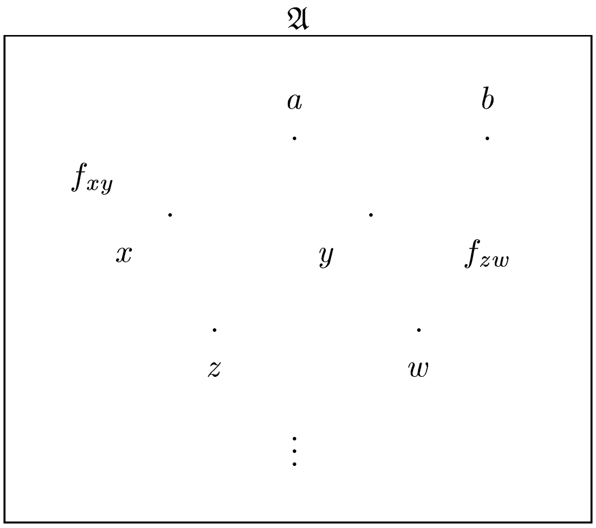


Figure 2.2.12

(3) Prove that $\mathsf{Smp} \subsetneqq \mathsf{Sir}$. ◀

Recall the definitions of Sir and $\mathsf{Sir}(\mathsf{K})$ (for a class K of algebras) from Definition 2.2.36 (ii).

THEOREM 2.2.40 (Birkhoff). *Let $\mathfrak{A}$ be an algebra. Then (i)–(ii) below hold.*

(i) $\mathfrak{A} \in \mathbf{SP}\mathsf{Sir}$.

(ii) $\mathfrak{A} \in \mathbf{SP}\mathsf{Sir}(\mathbf{H}\{\mathfrak{A}\})$.

The **proof** will be discussed below the formulation of the following theorem. ∎

Theorem 2.2.40 above is a powerful tool in simplifying investigations of algebras. Namely, in a sense, it is sufficient to study the subdirectly irreducible algebras only, because the rest is built up from these by $\mathbf{SP}$. (Of course, this is not true for *all* kinds of questions, because building up by $\mathbf{SP}$ might interfere with *certain* kinds of questions. But still, we will experience that it is true for many kinds of questions.) Similarly, if we are investigating a class $\mathsf{K} = \mathbf{H}\mathsf{K}$ of algebras (like Boolean algebras, groups etc.) then it is often sufficient to investigate $\mathsf{Sir}\mathsf{K}$ instead of K, since the rest of K is built up from $\mathsf{Sir}\mathsf{K}$ by $\mathbf{SP}$.

THEOREM 2.2.41 (Birkhoff's Subdirect Decomposition Theorem). *Every algebra $\mathfrak{A}$ admits a subdirect decomposition into subdirectly irreducible components that are quotient algebras of $\mathfrak{A}$. That is:*

Let $\mathfrak{A}$ be an algebra. Then there is a subdirect decomposition $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i \rangle_{i \in I}$ of $\mathfrak{A}$ such that $\{\mathfrak{B}_i : i \in I\} \subseteq \mathsf{Sir}(\mathbf{H}\mathfrak{A})$.

Proof: Since every trivial algebra is subdirectly irreducible, we may assume that $\mathfrak{A}$ is not trivial. Let $I = (A \times A) \smallsetminus \mathsf{Id}_A$. For each $i = \langle a, b\rangle \in I$, choose a maximal member β_i in the set $\{\theta \in Con(\mathfrak{A}) : \langle a, b\rangle \notin \theta\}$. The existence of such a maximal member follows easily by Zorn's lemma.

Let $i = \langle a, b\rangle \in I$ be arbitrary. We claim that $\mathfrak{A}/\beta_i \in \mathsf{Sir}$. This is so because $(\forall\theta \in Con(\mathfrak{A}/\beta_i))a/\beta_i \equiv_\theta b/\beta_i$, by the definition of β_i. Thus the smallest congruence containing $\langle a/\beta_i, b/\beta_i\rangle$ is the smallest non–identity element of $Con(\mathfrak{A}/\beta_i)$. Thus $\mathfrak{A}/\beta_i \in \mathsf{Sir}$ by Lemma 2.2.37. We have seen that $(\forall i \in I)\mathfrak{A}/\beta_i \in \mathsf{Sir}$.

Clearly, $\bigcap\{\beta_i : i \in I\} = \mathsf{Id}_A$. Let q_i be the quotient map associated to β_i, for every $i \in I$. Then $\langle q_i : i \in I\rangle$ is a subdirect decomposition of $\mathfrak{A}$, with components the factor algebras $\mathfrak{A}/\beta_i$ of $\mathfrak{A}$, for every $i \in I$. This completes the proof of Theorem 2.2.41. ∎

Let us return briefly to the analysis–synthesis method outlined above Definition 2.2.36. Theorem 2.2.41 above implies that this method is applicable rather "deterministically" to algebras. Namely, if we want to study a (say, complicated) algebra $\mathfrak{A}$, then we can subdirectly decompose $\mathfrak{A}$ to parts $\mathfrak{B}_i$ $(i \in I)$ which are *not* decomposable subdirectly any further. Since the $\mathfrak{B}_i$'s are not decomposable any further, we have to investigate the $\mathfrak{B}_i$'s, and then "synthesize" the results obtained about the $\mathfrak{B}_i$'s to results about the original $\mathfrak{A}$ along the lines discussed earlier. The important point here is that to any algebra $\mathfrak{A}$, there are "atomic" (i.e. indecomposable) components $\mathfrak{B}_i$ of $\mathfrak{A}$ such that we know when to stop in the "analysis part", that is, in the decomposition process. (Further $\mathfrak{A}$ can be built up from these "atomic" components $\mathfrak{B}_i$ of $\mathfrak{A}$.)

From Theorems 2.2.35 and 2.2.41 we can prove Theorem 2.2.40, as follows.

Proof of Theorem 2.2.40: Let $\mathfrak{A}$ be an algebra. By Theorem 2.2.41, there is a subdirect decomposition $\langle \mathfrak{A} \xrightarrow{h_i} \mathfrak{B}_i \rangle_{i \in I}$ with $(\forall i \in I)\mathfrak{B}_i \in \mathsf{Sir}(\mathbf{H}\mathfrak{A})$. Now, Theorem 2.2.35 says that $\mathfrak{A} \in \mathbf{SP}\{\mathfrak{B}_i : i \in I\} \subseteq \mathbf{SP}\,\mathsf{Sir}(\mathbf{H}\mathfrak{A})$. This completes the proof. ∎

ULTRAPRODUCT, REDUCED PRODUCT

Motivation 2.2.42: Like direct product, *ultraproduct* and *reduced product* are such operations on algebras which "build up bigger, more complex algebras from smaller ones" (cf. the introductory paragraphs of section 2.2). To motivate these new operations to be introduced shortly, observe the following. An equation is valid in the direct product $\mathfrak{A}$ of some system $\langle \mathfrak{B}_i : i \in I \rangle$ of algebras iff it is valid in *each* component $\mathfrak{B}_i$ $(i \in I)$. For example, if the similarity type contains two constant symbols d, e and a unary function symbol f, then $f(d) = e$ is valid in $\mathfrak{A}$ iff it is valid in $\mathfrak{B}_i$ for *every* $i \in I$. Thus $f(d) = e$ is not valid in $\mathfrak{A}$ whenever it is not valid in one of its components, even if $f(d) = e$ is valid in each of the other (possibly infinitely many) components. See Figure 2.2.14.

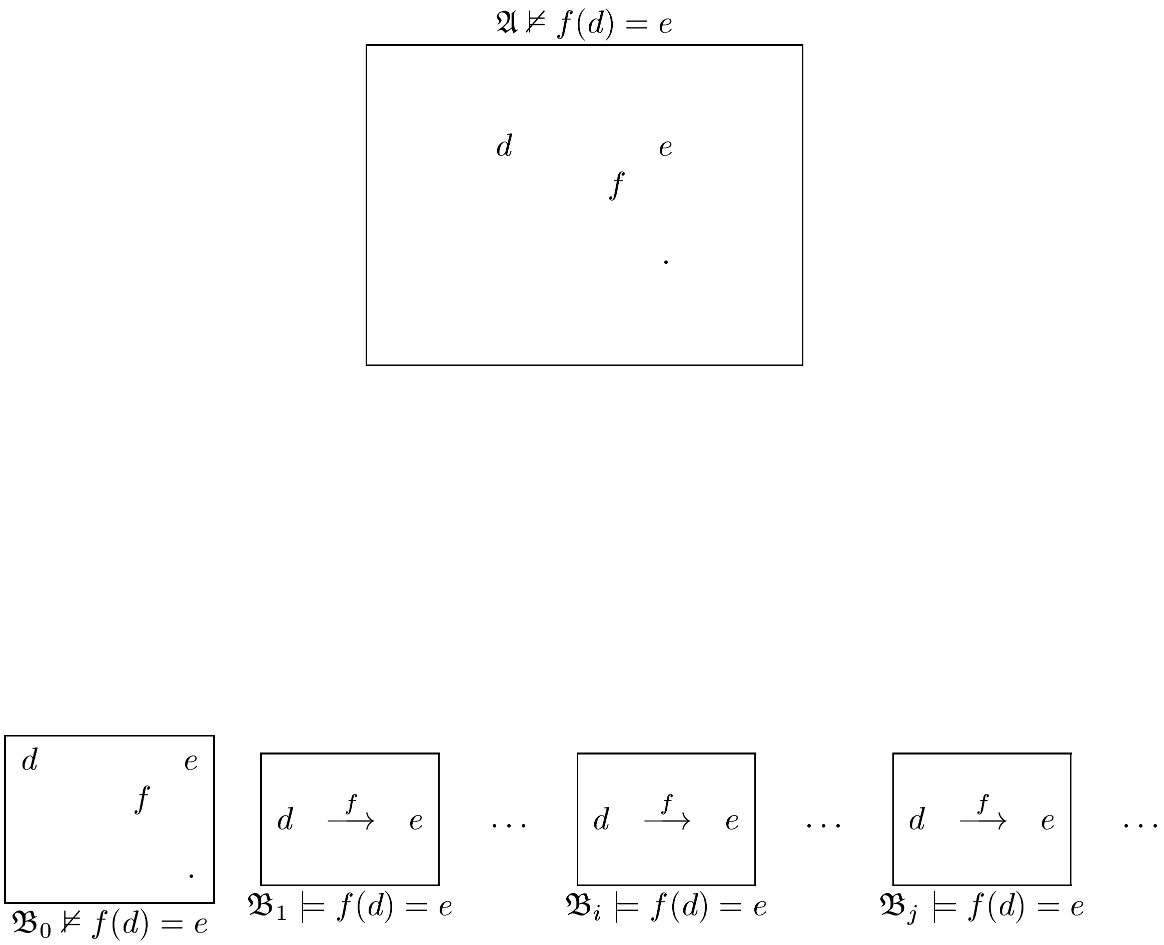


Figure 2.2.14

The property of direct product illustrated above shows that this operation is rigid in that it does not "tolerate" any "deviation": an equation cannot be valid in the value of the operation even when it is valid in "almost every" argument (component) of the operation. Now we are aiming at defining operations on algebras which are not so rigid in this sense. We would like to define a "product–like" operation in the value of which a property holds iff it holds in almost every argument (almost everywhere). To have such a concept, we need to clarify first what we want to mean by "almost every" or "almost everywhere".

From a (mathematical) concept of "almost everywhere", we require the following, intuitively natural things.

(i) If something is true everywhere, then it should be true almost everywhere.
(ii) If two properties φ and ψ hold almost everywhere then their conjunction $(\varphi \wedge \psi)$ should hold almost everywhere, too.
(iii) If φ holds almost everywhere and φ (always) implies ψ then ψ should hold almost everywhere, too.

We may also add:

(iv) A property φ holds almost everywhere iff its negation $\neg\varphi$ does not hold almost everywhere.

In the following definition we give two (mathematical) concepts corresponding to the above described intuition of "almost everywhere". These concepts are called *filter* and *ultrafilter*. ◀

Definition 2.2.43 (filter, ultrafilter, special filters): Let I be a nonempty set. Recall that $\mathcal{P}(I)$ denotes the powerset (set of all subsets) of I. Let $\mathcal{F} \subseteq \mathcal{P}(I)$.

1. $\mathcal{F}$ is called a *filter over I* iff conditions (i)–(iii) below are satisfied.

(i) $I \in \mathcal{F}$
(ii) if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$
(iii) if $X \in \mathcal{F}$ and $X \subseteq Z \subseteq I$ then $Z \in \mathcal{F}$.

2. A filter $\mathcal{F}$ over I is called and *ultrafilter* over I iff for every $X \in \mathcal{P}(I)$, (iv) below holds.

(iv) $X \in \mathcal{F} \iff (I \smallsetminus X \notin \mathcal{F})$.

If we simply say that $\mathcal{F}$ is an ultrafilter then we tacitly assume that $\mathcal{F}$ is an ultrafilter over $\bigcup \mathcal{F}$.

3. $\{I\}$ is called the *trivial filter* (over I), $\mathcal{P}(I)$ the *improper filter*. A filter $\mathcal{F}$ over I is said to be a *proper filter* iff it is not the improper filter $\mathcal{P}(I)$. For each $Y \subseteq I$, the filter $\{X \subseteq I : Y \subseteq X\}$ is called the *principal filter* generated by Y. A filter (ultrafilter) is called *non–principal* iff it is not a principal filter (ultrafilter). If $|I| \geqslant \omega$ then $\mathcal{F}\nabla(I) \stackrel{\text{def}}{=} \{X \in \mathcal{P}(I) : |I \smallsetminus X| < \omega\}$ is called the *Fréchet filter* (over I). ◀

Exercise 2.2.44: Prove that, for any nonempty set I, the trivial filter over I, any principal filter, and the Fréchet filter over I are all filters. ◀

If $\mathcal{F}$ is a filter over I then we sometimes refer to the elements of $\mathcal{F}$ as to "*big*" or "$\mathcal{F}$*–big*" sets, while we call the rest of the subsets of I "*small*" sets.

Next we investigate filters and ultrafilters.

Definition 2.2.45 (generated filter, finite intersection property): Let E be a subset of $\mathcal{P}(I)$.

(i) By a *filter generated by* E we mean the intersection $\mathcal{F}$ of all filters over I which include E, that is,

$$\mathcal{F} = \bigcap\{\mathcal{D} : E \subseteq \mathcal{D} \text{ and } \mathcal{D} \text{ is a filter over } I\}.$$

(ii) E is said to have the *finite intersection property* iff the intersection of any finite number of elements of E is nonempty. ◄

PROPOSITION 2.2.46. *Let E be any subset of $\mathcal{P}(I)$ for some nonempty set I, and let $\mathcal{F}$ be the filter generated by E. Then (i)–(iii) below hold.*

(i) *$\mathcal{F}$ is a filter over I.*

(ii) *$\mathcal{F}$ is the set of all $X \subseteq \mathcal{P}(I)$ such that either $X = I$ or for some $Y_1, \dots, Y_n \in E$,*

$$Y_1 \cap \dots \cap Y_n \subseteq X.$$

(iii) *$\mathcal{F}$ is a proper filter iff E has the finite intersection property.*

Proof: (i) is easy to prove. (iii) follows easily from (ii). To prove (ii), let

$$\mathcal{F}' \stackrel{\text{def}}{=} \{X \subseteq I : X = I \text{ or } Y_1 \cap \dots \cap Y_n \subseteq X \text{ for some } Y_1, \dots, Y_n \in E\}.$$

We prove that $\mathcal{F} = \mathcal{F}'$. First we check that $\mathcal{F}'$ is a filter, as follows. $I \in \mathcal{F}'$ by the definition of $\mathcal{F}'$. Let $X, X' \in \mathcal{F}'$, and let $Y_i, Y'_j \in E$ be such that

$$Y_1 \cap \dots \cap Y_n \subseteq X \quad \text{and} \quad Y'_1 \cap \dots \cap Y'_k \subseteq X'.$$

If $X \subseteq Z \subseteq I$, then $Y_1 \cap \dots \cap Y_n \subseteq Z$, thus $Z \in \mathcal{F}'$. Further,

$$Y_1 \cap \dots \cap Y_n \cap Y'_1 \cap \dots \cap Y'_k \subseteq X \cap X',$$

thus $X \cap X' \in \mathcal{F}'$. Therefore $\mathcal{F}'$ is a filter over I. Obviously $E \subseteq \mathcal{F}'$. Thus $\mathcal{F} \subseteq \mathcal{F}'$, by the definition of $\mathcal{F}$.

To see $\mathcal{F}' \subseteq \mathcal{F}$, consider any filter $\mathcal{D}$ over I which includes E. Then $\mathcal{F}' \subseteq \mathcal{D}$ can be seen as follows. Clearly $I \in \mathcal{D}$. For any $Y_1, \dots, Y_n \in E$, we have $Y_1 \cap \dots \cap Y_n \in \mathcal{D}$, and hence any $X \subseteq I$ which includes $Y_1 \cap \dots \cap Y_n$ belongs to $\mathcal{D}$. Thus $\mathcal{F}' \subseteq \mathcal{D}$. This shows that $\mathcal{F}' \subseteq \mathcal{F}$. Thus $\mathcal{F} = \mathcal{F}'$. ∎

Exercise 2.2.47:

(1) Prove (i) and (iii) of Proposition 2.2.46 above.

(2) Prove that for any set I, if C is a nonempty chain of proper filters over I, then $\bigcup C$ is a proper filter over I.

(3) Prove that every proper filter has the finite intersection property. ◄

PROPOSITION 2.2.48. *The following (i) and (ii) are equivalent.*

(i) *$\mathcal{F}$ is an ultrafilter over I.*
(ii) *$\mathcal{F}$ is a maximal proper filter over I. That is, $\mathcal{F}$ is a proper filter over I, and the only proper filter over I which includes $\mathcal{F}$ is $\mathcal{F}$ itself.*

Proof: (i)$\Rightarrow$(ii). Assume (i). Then $0 \notin \mathcal{F}$, because $I \in \mathcal{F}$ and $0 = I \smallsetminus I$. Hence $\mathcal{F}$ is a proper filter. Let $\mathcal{D}$ be any proper filter over I which includes $\mathcal{F}$. If $X \in \mathcal{D} \smallsetminus \mathcal{F}$, then $I \smallsetminus X \in \mathcal{F}$ (because $\mathcal{F}$ is an *ultra*filter), thus $I \smallsetminus X \in \mathcal{D}$, and $0 = X \cap (I \smallsetminus X) \in \mathcal{D}$. This contradicts the assumption that $\mathcal{D}$ is proper. Thus $\mathcal{D} \subseteq \mathcal{F}$, so $\mathcal{D} = \mathcal{F}$, and (ii) holds.

(ii)$\Rightarrow$(i). Assume (ii). Consider any $X \subseteq I$. We cannot have both $X \in \mathcal{F}$ and $I \smallsetminus X \in \mathcal{F}$, because then $0 \in \mathcal{F}$, thus $(\forall Y \subseteq I) Y \in \mathcal{F}$, and $\mathcal{F}$ is not proper. It is enough to prove that $I \smallsetminus X \notin \mathcal{F} \Rightarrow X \in \mathcal{F}$. Suppose $I \smallsetminus X \notin \mathcal{F}$. Let $E = \mathcal{F} \cup \{X\}$, and let $\mathcal{D}$ be the filter generated by E. We show that E has the finite intersection property. Consider $Y_1, \ldots, Y_n \in E$, and let $Z = Y_1 \cap \cdots \cap Y_n$. Since $\mathcal{F}$ is closed under finite intersections, we either have $Z \in \mathcal{F}$ or $Z = Y \cap X$ for some $Y \in \mathcal{F}$. In the first case, $Z \neq 0$, because $0 \notin \mathcal{F}$. In the second case, we also have $Z \neq 0$, for otherwise we would have $Y \cap X = 0$, $Y \subseteq I \smallsetminus X$, whence $I \smallsetminus X \in \mathcal{F}$. Thus in any case, $Z \neq 0$. By Proposition 2.2.46, $0 \notin \mathcal{D}$. Thus $\mathcal{D}$ is a proper filter including $\mathcal{F}$, thus by (ii), $\mathcal{D} = \mathcal{F}$. Therefore $E \subseteq \mathcal{F}$ and $X \in \mathcal{F}$. This proves (i). ∎

PROPOSITION 2.2.49. *If $E \subseteq \mathcal{P}(I)$ and E has the finite intersection property, then there exists an ultrafilter $\mathcal{F}$ over I such that $E \subseteq \mathcal{F}$.*

Proof: By Proposition 2.2.46, the filter $\mathcal{D}$ generated by E does not contain the empty set, thus $\mathcal{D}$ is proper. Moreover, if C is any nonempty chain of proper filters over I, then $\bigcup C$ is a proper filter over I, by Exercise 2.2.47 (2). Furthermore, if $\mathcal{F} \in C$ includes E, then $\bigcup C$ includes E. Then, by Zorn's lemma, the class H of all proper filters over I including E has a maximal element, say $\mathcal{F}$. Thus $E \subseteq \mathcal{F}$. $\mathcal{F}$ is a maximal proper filter over I, because if $\mathcal{F}'$ is a proper filter including $\mathcal{F}$, then $E \subseteq \mathcal{F}'$, and so $\mathcal{F}'$ belongs to H and $\mathcal{F}' = \mathcal{F}$. Thus, by Proposition 2.2.48, $\mathcal{F}$ is an ultrafilter over I. ∎

COROLLARY 2.2.50. *Any proper filter over I can be extended to an ultrafilter over I.*

Proof: Every proper filter has the finite intersection property by Exercise 2.2.47 (3). ∎

Exercises 2.2.51: Below, I is a non–empty set, $\mathcal{U} \subseteq \mathcal{P}(I)$ is an ultrafilter, and $\mathcal{F}r(I)$ denotes the Fréchet filter over I.

(1) (a) Let $X \in \mathcal{U}$, $X = Y_0 \cup \cdots \cup Y_n$ for some $n \in \omega$. Prove that
$$Y_i \in \mathcal{U} \text{ for some } i \leqslant n\,.$$
 (b) Prove that $\mathcal{U}$ is principal iff $\mathcal{U} = \{Y \subseteq I : \{x\} \subseteq Y\}$ for some $x \in I$.
(2) (a) Prove that $\mathcal{F}r(I)$ is not an ultrafilter (but is a filter by Exercises 2.2.44).
 (b) Prove that $\mathcal{U}$ is non–principal iff $\mathcal{F}r(I) \subseteq \mathcal{U}$.
(3) Let $E \subseteq \mathcal{P}(I)$ be such that $(\forall \text{ finite } H \subseteq E) |\bigcap H| \geqslant \omega$. Prove that there exists a non–principal ultrafilter containing E. ◀

Definition 2.2.52 (reduced product, ultraproduct): Let I be a nonempty set and $\mathcal{F}$ a proper filter over I. Let $\langle A_i : i \in I\rangle$ be a system of nonempty sets.

1. Consider the Cartesian product $\mathbf{P}_{i\in I}A_i$ of the sets A_i. Let $a, b \in \mathbf{P}_{i\in I}A_i$. We say that a and b are $\mathcal{F}$–equivalent, in symbols $a \equiv_{\mathcal{F}} b$, iff

$$\{i \in I : a(i) = b(i)\} \in \mathcal{F}\,.$$

Then $\equiv_{\mathcal{F}}$ is an equivalence relation over $\mathbf{P}_{i\in I}A_i$. (Exercise: Prove this!) Let $a/\mathcal{F}$ denote the equivalence class of a:

$$a/\mathcal{F} \overset{\text{def}}{=} \{b \in \mathbf{P}_{i\in I}A_i : a \equiv_{\mathcal{F}} b\}\,.$$

We define the *reduced product of* $\langle A_i : i \in I\rangle$ *modulo* $\mathcal{F}$, in symbols $\mathbf{P}\langle A_i : i \in I\rangle/\mathcal{F}$ or $\mathbf{P}_{i\in I}A_i/\mathcal{F}$, to be the set of all equivalence classes of $\equiv_{\mathcal{F}}$. That is,

$$\mathbf{P}_{i\in I}A_i/\mathcal{F} \overset{\text{def}}{=} \{a/\mathcal{F} : a \in \mathbf{P}_{i\in I}A_i\}\,.$$

We call the set I the *index set* for $\mathbf{P}_{i\in I}A_i/\mathcal{F}$. In the special case when $\mathcal{F}$ is an ultrafilter, the reduced product $\mathbf{P}_{i\in I}A_i/\mathcal{F}$ is called an *ultraproduct*.

2. Now let $\langle \mathfrak{A}_i : i \in I\rangle$ be a system of similar algebras. We define the *reduced product* $\mathbf{P}\langle \mathfrak{A}_i : i \in I\rangle/\mathcal{F} = \mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F}$ of this system as follows. $\mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F}$ is defined to be an algebra with universe $\mathbf{P}_{i\in I}A_i/\mathcal{F}$ (the reduced product of the universes), and if f is an n–ary function symbol of the similarity type of the $\mathfrak{A}_i$'s then f is interpreted on $\mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F}$ as follows.

$$f^{\mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F}}(a_1/\mathcal{F},\ldots,a_n/\mathcal{F}) \overset{\text{def}}{=} \langle f^{\mathfrak{A}_i}(a_1(i),\ldots,a_n(i)) : i \in I\rangle/\mathcal{F} \tag{2.2.5}$$

for every $a_1,\ldots,a_n \in \mathbf{P}_{i\in I}A_i$. ◀

Illustration: Assume that the similarity type of $\langle \mathfrak{A}_i : i \in I\rangle$ contains a unary function symbol f. Then equation (2.2.5) above is reduced to

$$f^{\mathbf{P}_{i\in I}\mathfrak{A}_i}(a/\mathcal{F}) = \langle f^{\mathfrak{A}_i}(a(i))\rangle_{i\in I}/\mathcal{F}\,.$$

On Figure 2.2.15, the equivalence class $a/\mathcal{F}$ is drawn as a long "sausage" (containing "line" a).

A_i A_j $\langle f^{\mathfrak{A}_i}(a(i))\rangle_{i\in I}/\mathcal{F}$

– $f^{\mathfrak{A}_i}(a(i))$ $\langle f^{\mathfrak{A}_i}(a(i))\rangle_{i\in I}$

$f^{\mathfrak{A}_i}$ $f^{\mathbf{P}_{i\in I}\mathfrak{A}_i}$

– $a(i)$ a

$a/\mathcal{F}$

… …

i j I

Figure 2.2.15

The following proposition states that the definition of an ultraproduct given above is correct.

PROPOSITION 2.2.53. *Let I, $\mathcal{F}$, $\mathfrak{A}_i$ be as in Definition 2.2.52 above. Suppose that $a_1 \equiv_{\mathcal{F}} b_1, \ldots, a_n \equiv_{\mathcal{F}} b_n$. Then for every function symbol f of the similarity type of the $\mathfrak{A}_i$'s,*

$$\langle f^{\mathfrak{A}_i}((a_1)_i, \ldots, (a_n)_i) : i \in I\rangle \equiv_{\mathcal{F}} \langle f^{\mathfrak{A}_i}((b_1)_i, \ldots, (b_n)_i) : i \in I\rangle \,. \qquad \blacksquare$$

Exercise 2.2.54: Prove Proposition 2.2.53. ◀

Convention 2.2.55: We sometimes omit the subscript $i \in I$ from $\mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F}$ when it causes no misunderstanding.

We call the ultraproduct of a system $\langle \mathfrak{A} : i \in I\rangle$ (all the models are the same) an *ultrapower*. An ultrapower is denoted as ${}^I\mathfrak{A}/\mathcal{F}$. ◀

Exercise 2.2.56: Generalize the concept of a reduced product (ultraproduct) of algebras to that of *models*. ◀

The most important theorem we need concerning ultraproducts is the following. Recall our notation for the truth of a formula in an algebra at a valuation from the end of Definition 1.8 (4) (in subsection 1.7).

THEOREM 2.2.57 (Łos ultraproduct theorem).
Let $\mathfrak{B}$ be the ultraproduct $\mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F}$. Then (i)–(iii) below hold.

(i) *For any term $\tau(x_1, \ldots, x_n)$ in the language of $\mathfrak{B}$ and elements $a_1/\mathcal{F}, \ldots, a_n/\mathcal{F} \in B$, we have*

$$\tau^{\mathfrak{B}}(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}) = \langle \tau^{\mathfrak{A}_i}(a_1(i), \ldots, a_n(i)) : i \in I\rangle/\mathcal{F} \,.$$

(ii) *Given any formula $\varphi(x_1, \ldots, x_n)$ of the language of $\mathfrak{B}$ and elements $a_1/\mathcal{F}, \ldots, a_n/\mathcal{F} \in B$, we have*

$$\mathfrak{B} \models \varphi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] \qquad \textit{iff} \qquad \{i \in I : \mathfrak{A}_i \models \varphi[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F} \,.$$

(iii) *For any sentence φ of the language of $\mathfrak{B}$,*

$$\mathfrak{B} \models \varphi \qquad \textit{iff} \qquad \{i \in I : \mathfrak{A}_i \models \varphi\} \in \mathcal{F} \,.$$

Proof: (iii) is an immediate consequence of (i) and (ii). The proofs of (i) and (ii) are by induction on the complexity of terms and formulas, respectively.

Proof of (i): (i) holds whenever $\tau(x_1, \ldots, x_n)$ is a variable, a constant, or is of the form $f(x_1, \ldots, x_n)$, by the definition of reduced product. Suppose that

$$\tau(x_1, \ldots, x_n) = f(\tau_1(x_1, \ldots, x_n), \ldots, \tau_k(x_1, \ldots, x_n)) \text{ and } \tau_1, \ldots, \tau_k \text{ satisfy (i).}$$

Then for any elements $a_1/\mathcal{F}, \ldots, a_n/\mathcal{F} \in B$,

$$\tau^{\mathfrak{B}}(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}) = f^{\mathfrak{B}}(\tau_1^{\mathfrak{B}}(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}), \ldots, \tau_k^{\mathfrak{B}}(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}))\,. \tag{2.2.6}$$

If $j \in \{1, \ldots, k\}$ then $\tau_j^{\mathfrak{B}}(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}) = \langle \tau_j^{\mathfrak{A}_j}(a_1(i), \ldots, a_n(i)) : i \in I\rangle/\mathcal{F}$ by the induction hypothesis. By the definition of reduced product, we have:

$$f^{\mathfrak{B}}(b_1/\mathcal{F}, \ldots, b_n/\mathcal{F}) = \langle f^{\mathfrak{A}_i}(b_1(i), \ldots, b_n(i)) : i \in I\rangle/\mathcal{F}\,, \tag{2.2.7}$$

for any $b_1, \ldots, b_n \in \mathbf{P}_{i\in I}\mathfrak{A}_i$. For every $i \in I$, there are $b_1, \ldots, b_n$ such that

$$\tau^{\mathfrak{A}_i}(a_1(i), \ldots, a_n(i)) = f^{\mathfrak{A}_i}(b_1(i), \ldots, b_n(i))\,. \tag{2.2.8}$$

Now

$$\begin{aligned}
\tau^{\mathfrak{B}}(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}) &= f^{\mathfrak{B}}\left(\tau_1^{\mathfrak{B}}(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}), \ldots, \tau_k^{\mathfrak{B}}(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F})\right) && \text{by (2.2.6)}\\
&= f^{\mathfrak{B}}(b_1/\mathcal{F}, \ldots, b_n/\mathcal{F}) && \text{for some } b_1, \ldots, b_n \in \mathbf{P}_{i\in I}\mathfrak{A}_i\\
&= \langle f^{\mathfrak{A}_i}(b_1(i), \ldots, b_n(i)) : i \in I\rangle/\mathcal{F} && \text{by (2.2.7)}\\
&= \langle \tau^{\mathfrak{A}_i}(a_1(i), \ldots, a_n(i)) : i \in I\rangle/\mathcal{F} && \text{by (2.2.8).}
\end{aligned}$$

Proof of (ii): The proof of (ii) for atomic formula (equations) is similar to the above proof of (i); we note that (i) is used in the proof of (ii).

Suppose $\varphi = \neg\psi(x_1, \ldots, x_n)$ and (ii) holds for $\psi(x_1, \ldots, x_n)$. Let $a_1, \ldots, a_n \in \mathbf{P}_{i\in I}\mathfrak{A}_i$. Then

$$\begin{aligned}
\mathfrak{B} \models \varphi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] &\iff \mathfrak{B} \not\models \psi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}]\\
&\iff \{i \in I : \mathfrak{A}_i \models \psi[a_1(i), \ldots, a_n(i)]\} \notin \mathcal{F}\\
&\iff \{i \in I : \mathfrak{A}_i \not\models \psi[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F}\\
&\iff \{i \in I : \mathfrak{A}_i \models \varphi[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F}\,.
\end{aligned}$$

The fact that $\mathcal{F}$ is an ultrafilter is used to show that the second "$\iff$" holds. Indeed, this is the only point in the entire proof of the theorem where we need the fact that $\mathcal{F}$ is an ultrafilter, and not merely a proper filter.

The next step is to prove that if ψ and χ satisfy (ii), then so does $(\psi \wedge \chi)$. This is done by writing a string of equivalences like the one we used for $\neg\psi$. This time the crucial fact about $\mathcal{F}$ which we need is that $X \cap Y \in \mathcal{F} \iff X \in \mathcal{F}$ and $Y \in \mathcal{F}$. Every filter has this property. The details of this step in the proof are straightforward, and are left to the reader.

Now suppose that $\varphi(x_1, \ldots, x_n) = \exists x\psi(x, x_1, \ldots, x_n)$, and that (ii) holds for ψ. Let $a_1/\mathcal{F}, \ldots, a_n/\mathcal{F} \in B$. Then

$$\begin{aligned}
\mathfrak{B} \models \varphi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] &\iff \mathfrak{B} \models \exists x\psi(x, x_1, \ldots, x_n)[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] \\
&\iff \mathfrak{B} \models \psi[b/\mathcal{F}, a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] \text{ for some } b \in \mathbf{P}_{i\in I}\mathfrak{A}_i \\
&\iff \{i \in I : \mathfrak{A}_i \models \psi[b(i), a_1(i), \ldots, a_n(i)]\} \in \mathcal{F} \\
&\qquad\qquad\qquad \text{for the above } b, \text{ by ind. hyp.} \\
&\implies \{i \in I : \mathfrak{A}_i \models \exists x\psi(x, x_1, \ldots, x_n)[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F} \\
&\iff \{i \in I : \mathfrak{A}_i \models \varphi[a_1(i), \ldots, a_n(i)]\} \in \mathcal{F}\,.
\end{aligned}$$

To see the other direction, assume $\{i \in I : \mathfrak{A}_i \models \varphi[a_1, \ldots, a_n]\} \in \mathcal{F}$. Then

$$\{i \in I : \mathfrak{A}_i \models \psi[d(i), a_1(i), \ldots, a_n(i)]\} \in \mathcal{F} \text{ for some } d \in \mathbf{P}_{i\in I}\mathfrak{A}_i$$

$$\begin{aligned}
&\iff \mathfrak{B} \models \psi[d/\mathcal{F}, a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] \\
&\iff \mathfrak{B} \models \varphi[a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}]\,.
\end{aligned}$$

By this, our induction for step (ii) is completed. ∎

Exercise 2.2.58: Prove Theorem 2.2.57 for the case of *models*. That is, consider the theorem we get from Theorem 2.2.57 via substituting models for algebras in it, and using ultraproduct of models (cf. Exercise 2.2.56) instead of that of algebras. Prove this theorem. ◀

Definition 2.2.59 ($\mathbf{P^r}$, Up): Given a class K of similar algebras, we let

$$\mathbf{P^r}\mathsf{K} \stackrel{\text{def}}{=} \mathbf{I}\,\{\mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F} : I \text{ is a set, } \mathcal{F} \text{ is a filter over the set } I, \text{ and } (\forall i \in I)\mathfrak{A}_i \in \mathsf{K}\}$$

$$\mathbf{Up}\mathsf{K} \stackrel{\text{def}}{=}$$

$$\mathbf{I}\,\{\mathbf{P}_{i\in I}\mathfrak{A}_i/\mathcal{F} : I \text{ is a set, } \mathcal{F} \text{ is an ultrafilter over the set } I, \text{ and } (\forall i \in I)\mathfrak{A}_i \in \mathsf{K}\}\,.$$

◀

LEMMA 2.2.60. $\mathbf{P^r}\mathbf{P^r}\mathsf{K} = \mathbf{P^r}\mathsf{K}$.

Proof: [4] Let K be an arbitrary class of algebras. Then $\mathbf{P}^r\mathsf{K} \subseteq \mathbf{P}^r\mathbf{P}^r\mathsf{K}$ follows from the definition of reduced product. It remains to prove $\mathbf{P}^r\mathbf{P}^r\mathsf{K} \subseteq \mathbf{P}^r\mathsf{K}$.

Let J be a set and $\langle I_j : j \in J\rangle$ a system of pairwise disjoint sets. For every $j \in J$, let $\langle \mathfrak{A}_i : i \in I_j\rangle$ be a system of algebras. Let $\mathcal{F}$ be a filter over J, and for every $j \in J$, let $\mathcal{F}_j$ be a filter over I_j. We define

$$I \stackrel{\text{def}}{=} \bigcup_{j\in J} I_j \quad \text{and} \quad \hat{\mathcal{F}} \stackrel{\text{def}}{=} \{L \subseteq I : \{j \in J : L \cap I_j \in \mathcal{F}_j\} \in \mathcal{F}\}\,.$$

[4] In Andréka–Németi [AN80] as well as in Andréka–Burmeister–Németi [ABN81] it is shown that Lemma 2.2.60 cannot be proved without the Axiom of Choice.

($L \in \hat{\mathcal{F}}$ iff L is an "$\mathcal{F}$–big union of $\mathcal{F}_j$–big sets".) It is easy to check that $\hat{\mathcal{F}}$ is a filter over I. It is enough to prove that

$$\mathop{\mathbf{P}}\limits_{j\in J}\left(\mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i \Big/ \mathcal{F}_j\right)\Big/ \mathcal{F} \cong \mathop{\mathbf{P}}\limits_{i\in I}\mathfrak{A}_i/\hat{\mathcal{F}}\,. \tag{2.2.9}$$

For every $j \in J$, let $f_j : \mathop{\mathbf{P}}\limits_{i\in I}\mathfrak{A}_i \longrightarrow \mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i$ be defined as:

$$\left(\forall a \in \mathop{\mathbf{P}}\limits_{i\in I} A_i\right) f_j(a) \stackrel{\text{def}}{=} a \restriction I_j\,.$$

Clearly f_j is a surjective homomorphism, for every $j \in J$. Let $q_j : \mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i \longrightarrow \mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j$ be the quotient map for every $j \in J$, thus $(\forall b \in \mathop{\mathbf{P}}\limits_{i\in I_j} A_i) q_j(b) = b/\mathcal{F}_j$. Follow the proof on Figure 2.2.16.

$$\mathop{\mathbf{P}}\limits_{i\in I}\mathfrak{A}_i \xrightarrow{q} \mathop{\mathbf{P}}\limits_{i\in I}\mathfrak{A}_i/\hat{\mathcal{F}}$$

$f_{j'}$ $\qquad\qquad\qquad\qquad$ f_j

$$\mathbf{P}\ldots \xrightarrow{q'_j} \bullet \qquad \cdots \qquad \mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i \xrightarrow{q_j} \mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j$$

h $\qquad\qquad$ g

$p_{j'}$ $\qquad\qquad\qquad\qquad$ p_j

$$\mathfrak{P} \xrightarrow[k]{} \mathfrak{P}/\mathcal{F}$$

Figure 2.2.16

Let $\mathfrak{P} \stackrel{\text{def}}{=} \mathop{\mathbf{P}}\limits_{j\in J}\left(\mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j\right)$. Consider the cone

$$\langle \mathop{\mathbf{P}}\limits_{i\in I}\mathfrak{A}_i \overset{f_j}{\twoheadrightarrow} \mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i \xrightarrow{q_i} \mathop{\mathbf{P}}\limits_{i\in I_j}\mathfrak{A}_i/\mathcal{F}_j : j \in J\rangle\,.$$

By Lemma 2.2.23, this cone induces a unique homomorphism

$$h : \mathop{\mathbf{P}}\limits_{i\in I}\mathfrak{A}_i \longrightarrow \mathfrak{P}\,,$$

thus, denoting the projection homomorphisms of $\mathfrak{P}$ by p_j $(j \in J)$, we have

(2.2.10) $h \circ p_j = f_j \circ q_j$, i.e.,
$$(\forall a \in \mathop{\mathbf{P}}_{i\in I} A_i) h(a)(j) = q_j(f_j(a)) = q_j(a \restriction I_j) = (a \restriction I_j)/\mathcal{F}_j\,,$$

for every $j \in J$. Let k denote the quotient map $k : \mathfrak{P} \longrightarrow \mathfrak{P}/\mathcal{F}$. Notice that

(2.2.11) $$h \circ k \ \text{ is surjective.}$$

The following sequence proves that

(2.2.12) $$\theta \stackrel{\text{def}}{=} \ker(h \circ k) = \equiv_{\hat{\mathcal{F}}}\ .$$

$$\begin{aligned}
\langle a, b\rangle \in \theta &\iff \langle h(a), h(b)\rangle \in \ker(k) = \equiv_{\mathcal{F}} \\
&\iff \{j \in J : h(a)(j) = h(b)(j)\} \in \mathcal{F} && \text{by def's of } k, \equiv_{\mathcal{F}} \\
&\iff \{j \in J : (a \restriction I_j)/\mathcal{F}_j = (b \restriction I_j)/\mathcal{F}_j\} \in \mathcal{F} && \text{by (2.2.10)} \\
&\iff \{j \in J : \{i \in I_j : (a \restriction I_j)(i) = (b \restriction I_j)(i)\} \in \mathcal{F}_j\} \in \mathcal{F} \\
&\iff \{j \in J : (\{i \in I : a(i) = b(i)\} \cap I_j) \in \mathcal{F}_j\} \in \mathcal{F} \\
&\iff \{i \in I : a(i) = b(i)\} \in \hat{\mathcal{F}} \\
&\iff \langle a, b\rangle \in \equiv_{\hat{\mathcal{F}}}\ .
\end{aligned}$$

We proved (2.2.12).

Let q denote the quotient map $q : \mathop{\mathbf{P}}_{i\in I} \mathfrak{A}_i \longrightarrow \mathop{\mathbf{P}}_{i\in I} \mathfrak{A}_i/\hat{\mathcal{F}}$. Applying the homomorphism theorem (Theorem 2.2.18), (2.2.11) and (2.2.12) together imply the existence of an isomorphism $g : \mathop{\mathbf{P}}_{i\in I} \mathfrak{A}_i/\hat{\mathcal{F}} \rightarrowtail\!\!\!\!\rightarrow \mathfrak{P}/\mathcal{F}$ such that $q \circ g = h \circ k$. This proves (2.2.9). ∎

Exercise 2.2.61: Prove Lemma 2.2.60 for models instead of algebras. ◀

Exercise 2.2.62: Prove that for any class K of similar algebras, (1)–(6) below hold.

(1) $\mathbf{Up}\mathsf{K} \subseteq \mathbf{HP}\mathsf{K}$
(2) $\mathbf{PH}\mathsf{K} \subseteq \mathbf{HP}\mathsf{K}$
(3) $\mathbf{SH}\mathsf{K} \subseteq \mathbf{HS}\mathsf{K}$
(4) $\mathbf{PP}\mathsf{K} = \mathbf{P}\mathsf{K}$
(5) $\mathbf{UpUp}\mathsf{K} = \mathbf{Up}\mathsf{K}$
(6) $\mathbf{UpS}\mathsf{K} \subseteq \mathbf{SUp}\mathsf{K}$, $\mathbf{P}^r\mathbf{S}\mathsf{K} \subseteq \mathbf{SP}^r\mathsf{K}$. ◀

æ

2.3. Variety characterization, quasi–variety characterization.

Definition 2.3.1 (term-algebra): Let t be a similarity type and let X be an arbitrary set with $X \cap Dom(t) = \emptyset$. Recall the set $Trm_t(X)$ of terms of similarity type t with variables from X from Def.1.8. The *term-algebra* $\mathfrak{F}_t(X)$ *of similarity type* t *over* X is defined as follows.

(i) The universe of $\mathfrak{F}_t(X)$ is $Trm_t(X)$;
(ii) if $f \in Fns_t,\ t(f) = n$ and $\tau_1, \ldots, \tau_n \in Trm_t(X)$ then

$$f^{\mathfrak{F}_t(X)}(\tau_1, \ldots, \tau_n) \stackrel{\text{def}}{=} f(\tau_1, \ldots, \tau_n).$$ ◀

Exercise 2.3.2: Prove that if $|A| = |B|$ for some sets A, B then $\mathfrak{F}_t(A) \cong \mathfrak{F}_t(B)$. ◀

LEMMA 2.3.3. *For every algebra $\mathfrak{A} \in \mathsf{Alg}_t$ and function $g : X \longrightarrow A$ there exists a homomorphism $h : \mathfrak{F}_t(X) \longrightarrow \mathfrak{A}$ such that $h \upharpoonright X = g$.*

Proof: We define h by the following recursion:

- For any $x \in X$, $h(x) \stackrel{\text{def}}{=} g(x)$;
- for any $f \in Fns_t,\ t(f) = n$ and $\tau_1, \ldots, \tau_n \in Trm_t(X)$,

$$h\big(f(\tau_1, \ldots, \tau_n)\big) \stackrel{\text{def}}{=} f^{\mathfrak{A}}\big(h(\tau_1), \ldots, h(\tau_n)\big).$$

Then h is a homomorphism with the required properties. ∎

Definition 2.3.4: We say that a formula φ (φ may be, e.g. , an equation or a quasi–equation) is *preserved* under an operator like, e.g., $\mathbf{P}$ iff for any class K of (similar) algebras,

$$\mathsf{K} \models \varphi \Longrightarrow \mathbf{P}\mathsf{K} \models \varphi\,.$$

Similarly, φ is preserved under $\mathbf{SP}$ iff

$$\mathsf{K} \models \varphi \Longrightarrow \mathbf{SP}\mathsf{K} \models \varphi\,.$$

The definition for other operations like $\mathbf{H}$, $\mathbf{P}^r$ etc. is similar. ◀

By the Łos ultraproduct theorem (Theorem 2.2.57), every first–order formula is preserved under $\mathbf{Up}$.

Exercises 2.3.5:

(1) Let t be a similarity type containing two constant symbols: 0 and 1. Prove that the formula $(x = 0 \vee x = 1)$ (written in this language) is preserved under $\mathbf{S}$ but not under $\mathbf{P}$.

(2) Prove that equations are preserved under $\mathbf{S}$ and under $\mathbf{H}$.

(3) Prove that equations are preserved under $\mathbf{P}$. Use this and (2) above to prove that equations are preserved under $\mathbf{HSP}$.

(4) Prove that $\mathbf{S}$, $\mathbf{P}$, $\mathbf{Up}$ preserve quasi–equations. ◀

Definition 2.3.6: Let $\mathsf{K} \subseteq \mathsf{Alg}_t$. Then both $Eq_t(\mathsf{K})$ and $Eq_t\mathsf{K}$ denote the set of all equations of similarity type t which are valid in K, that is,

$$Eq_t\mathsf{K} = Eq_t(\mathsf{K}) \stackrel{\text{def}}{=} \{e : \mathsf{K} \models e \text{ and } e \text{ is an equation of similarity type } t\}\,.$$

$Qeq_t(\mathsf{K})$ and $Qeq_t\mathsf{K}$ denote the set of all quasi–equations valid in K, that is,

$$Qeq_t\mathsf{K} = Qeq_t(\mathsf{K}) \stackrel{\text{def}}{=} \{q : \mathsf{K} \models q \text{ and } q \text{ is a quasi–equation of similarity type } t\}\,. \quad ◀$$

Convention 2.3.7: From now on, by a class K of algebras we always mean a class of *similar* algebras. ◀

Definition 2.3.8 (variety, quasi–variety): A class of algebras of similarity type t is called a *variety* (*quasi–variety*) iff it can be defined by a set of equations (quasi–equations) of type t. That is, K is a variety iff

$$(\exists \text{ set } \Sigma \text{ of equations })(\mathfrak{A} \in \mathsf{K} \Longleftrightarrow \mathfrak{A} \models \Sigma)\,;$$

and K is a quasi–variety iff

$$(\exists \text{ set } \Gamma \text{ of quasi–equations })(\mathfrak{A} \in \mathsf{K} \Longleftrightarrow \mathfrak{A} \models \Gamma)\,. \quad ◀$$

We will also use the expressions *equational class* and *quasi–equational class* as synonyms of "variety" and "quasi–variety", respectively.

$\mathbf{Mod}(Eq_t\mathsf{K})$ ($\mathbf{Mod}(Qeq_t\mathsf{K})$) denotes the class of all models of the equational theory $Eq_t\mathsf{K}$ (quasi–equational theory $Qeq_t\mathsf{K}$, respectively). $\mathbf{Mod}(Eq_t\mathsf{K})$ ($\mathbf{Mod}(Qeq_t\mathsf{K})$) is called the variety (quasi–variety) *generated* by K.

Example 2.3.9: It can be proven that the class BA of all Boolean algebras can be defined by finitely many equations, thus it is a variety. Such an equational axiomatization will be given in section 3. ◀

THEOREM 2.3.10 (Variety characterization). *Let* $\mathsf{K} \subseteq \mathsf{Alg}_t$. *Then*

$$\mathbf{Mod}(Eq_t\mathsf{K}) = \mathbf{HSP}\mathsf{K}\,.$$

Thus, V *is a variety iff*

$$\mathsf{V} = \mathbf{HSP}\mathsf{V}\,.$$

Proof: $\mathbf{HSP}\mathsf{K} \subseteq \mathbf{Mod}(Eq_t\mathsf{K})$, because equations are preserved under $\mathbf{HSP}$ (cf. Exercise 2.3.5 (3) above).

To prove $\mathbf{Mod}(Eq_t\mathsf{K}) \subseteq \mathbf{HSP}\mathsf{K}$, let $\mathfrak{A} \in \mathbf{Mod}(Eq_t\mathsf{K})$. Consider the term-algebra $\mathfrak{F}_t(A)$ over the universe A of $\mathfrak{A}$. Then, by Lemma 2.3.3, there is a surjective homomorphism $f : \mathfrak{F}_t(A) \twoheadrightarrow \mathfrak{A}$ which is an extension of the identity map on A.

Now define for all $\sigma, \tau \in Trm_t(A)$

$$\sigma \equiv \tau \quad \stackrel{\text{def}}{\iff} \quad \mathsf{K} \models (\sigma = \tau).$$

Clearly, $\equiv$ is a congruence relation on $\mathfrak{F}_t(A)$ (check it!). We define a function $h : Trm_t(A)/\equiv \longrightarrow A$ by

$$h(\tau/\equiv) \stackrel{\text{def}}{=} f(\tau) \quad \text{for every } \tau \in Trm_t(A).$$

This definition is sound, since $\equiv \subseteq ker(f)$ and h is a homomorphism from $\mathfrak{F}_t(A)/\equiv$ to $\mathfrak{A}$ (check it: cf. the proof of the Homomorphism theorem Thm.2.2.18). h is surjective, because f is surjective, thus $\mathfrak{A}$ is a homomorphic image of $\mathfrak{F}_t(A)/\equiv$.

Next we prove that $\mathfrak{F}_t(A) \in \mathbf{SP}\mathsf{K}$. First, there exists a *set* $\mathsf{K}_0 \subseteq \mathsf{K}$ such that $Eq_t\mathsf{K}_0 = Eq_t\mathsf{K}$, because there are only "set many" equations of similarity type t. If e is an equation of similarity type t such that $\mathsf{K} \not\models e$ then let $\mathfrak{A}_e \in \mathsf{K}$ and let v_e be a valuation of the variables in $\mathfrak{A}_e$ such that $\mathfrak{A}_e \not\models e[v_e]$. Let

$$\mathsf{K}_0 \stackrel{\text{def}}{=} \{\mathfrak{A}_e : e \text{ is an equation of similarity type } t \text{ and } \mathsf{K} \not\models e\}\,.$$

Clearly $Eq_t\mathsf{K}_0 = Eq_t\mathsf{K}$ and $\mathsf{K}_0 \subseteq \mathsf{K}$ is a set.

$$\begin{tikzcd}
\mathfrak{F}_t(A) \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathfrak{A} \\
& \mathfrak{F}_t(A)/\equiv \arrow[ur, "h"'] \arrow[r, "g"] & \mathbf{P}_{\mathfrak{A}_e \in \mathsf{K}_0}\mathfrak{A}_e \\
& & \mathsf{K}_0 \\
& & \mathfrak{A}_e
\end{tikzcd}$$

Figure 2.3.1

Define the function $g : Trm_t(A)/\equiv \longrightarrow \mathbf{P}_{\mathfrak{A}_e \in \mathsf{K}_0} A_e$ as follows.

$$g(\tau/\equiv) \stackrel{\text{def}}{=} \langle \ldots, \tau^{\mathfrak{A}_e}[v_e], \ldots \rangle_{\mathfrak{A}_e \in \mathsf{K}_0}.$$

This definition is sound, since if $\tau \equiv \sigma$ then $\mathsf{K} \models (\tau = \sigma)$ holds, therefore for all $\mathfrak{A}_e \in \mathsf{K}_0$ we have $\mathfrak{A}_e \models (\tau = \sigma)[v_e]$. We show that g is an injective homomorphism (an embedding) from $\mathfrak{F}_t(A)/\equiv$ into $\mathbf{P}_{\mathfrak{A}_e \in \mathsf{K}_0} \mathfrak{A}_e$.

g is a homomorphism: If $k \in Fns_t$, $t(k) = n$ and $\tau_1, \ldots, \tau_n \in Trm_t(A)$ then

$$g\big(k(\tau_1/\equiv, \ldots, \tau_n/\equiv)\big)_{\mathfrak{A}_e} = g\big(k(\tau_1, \ldots, \tau_n)/\equiv\big)_{\mathfrak{A}_e} = k(\tau_1, \ldots, \tau_n)^{\mathfrak{A}_e}[v_e] =$$
$$k^{\mathfrak{A}_e}(\tau_1^{\mathfrak{A}_e}[v_e], \ldots, \tau_n^{\mathfrak{A}_e}[v_e]) = k^{\mathfrak{A}_e}\big(g(\tau_1/\equiv)_{\mathfrak{A}_e}, \ldots, g(\tau_n/\equiv)_{\mathfrak{A}_e}\big).$$

g is injective: Assume $\sigma \not\equiv \tau$. Then $\mathfrak{A}_{(\sigma=\tau)} \not\models (\sigma = \tau)[v_{(\sigma=\tau)}]$ for some algebra $\mathfrak{A}_{(\sigma=\tau)} \in \mathsf{K}_0$, that is, the $\mathfrak{A}_{(\sigma=\tau)}$ coordinates of $g(\tau/\equiv)$ and $g(\sigma/\equiv)$ are different.

Thus $\mathfrak{F}_t(A)/\equiv \in \mathbf{ISP}\mathsf{K}$ that is, $\mathfrak{A} \in \mathbf{HSP}\mathsf{K}$. ∎

THEOREM 2.3.11 (Quasi–variety characterization). *For any class* K *of similar algebras*

$$\mathbf{Mod}\, Qeq(\mathsf{K}) = \mathbf{SPUp}\mathsf{K} = \mathbf{SP}^r\mathsf{K}\,.$$

For proving Theorem 2.3.11, we need Lemmas 2.3.12, 2.3.13 below.

Let t be an arbitrary similarity type. Let I be an arbitrary set, and let e, e_i $(i \in I)$ be equations of type t. Then we call the formula $\bigwedge_{i\in I} e_i \to e$ an *infinitary quasi–equation*. If $\mathfrak{A} \in \mathbf{Mod}_t$, k is a valuation of the variables into A, then

$$\mathfrak{A} \models \bigwedge_{i\in I} e_i \to e[k] \quad \stackrel{\text{def}}{\iff} \quad \mathfrak{A} \not\models e_i\ [k] \ \text{ for some } i \in I, \text{ or } \mathfrak{A} \models e[k]\,.$$

Notice that if I is finite then $\bigwedge_{i\in I} e_i \to e$ is a quasi–equation. Also notice that if $I_0 \subseteq I$ then $\bigwedge_{i\in I_0} e_i \to e \models \bigwedge_{i\in I} e_i \to e$.

For any class K of similar algebras we define

$$Qeq_\infty(\mathsf{K}) \stackrel{\text{def}}{=} \{\varphi : \varphi \text{ is an infinitary quasi–equation and } \mathsf{K} \models \varphi\}\,.$$

LEMMA 2.3.12. *For any class* K *of similar algebras,*

$$\mathbf{Mod}\, Qeq(\mathsf{K}) \subseteq \mathbf{Mod}\, Qeq_\infty(\mathbf{Up}\mathsf{K})\,.$$

Proof of Lemma 2.3.12: Let $\bigwedge_{i\in I} e_i \to e$ be an arbitrary infinitary quasi–equation. We will show that

(2.3.2) if $\mathbf{Up}\mathsf{K} \models \bigwedge_{i\in I} e_i \to e$ then there is a finite $I_0 \subseteq I$ such that $\mathsf{K} \models \bigwedge_{i\in I_0} e_i \to e\,.$

Assume that (2.3.2) holds, and let $\mathfrak{A} \in \mathbf{Mod}\, Qeq(\mathsf{K})$, $qeqiI \in Qeq_\infty(\mathbf{Up}\mathsf{K})$. Then, by (2.3.2), there is a finite $I_0 \subseteq I$ such that $\mathsf{K} \models \bigwedge_{i\in I_0} e_i \to e$. Then $\mathfrak{A} \models \bigwedge_{i\in I_0} e_i \to e$, and thus $\mathfrak{A} \models \bigwedge_{i\in I} e_i \to e$ by $\bigwedge_{i\in I_0} e_i \to e \models \bigwedge_{i\in I} e_i \to e$. Therefore $\mathfrak{A} \in \mathbf{Mod}\, Qeq_\infty(\mathbf{Up}\mathsf{K})$, since we choose $\bigwedge_{i\in I} e_i \to e$ arbitrarily. This proves Lemma 2.3.12. It remains to prove (2.3.2).

To see (2.3.2), let $J = \{j \subseteq I : |j| < \omega\}$, and assume that $\mathsf{K} \not\models \bigwedge_{i\in j} e_i \to e$ for every $j \in J$. Then

$$(\forall j \in J)(\exists \mathfrak{A}_j \in \mathsf{K})(\exists \text{ valuation } k_j \text{ of the variables into } A_j)$$
$$\left(\mathfrak{A}_j \models \bigwedge_{i\in j} e_i[k_j] \quad \text{but} \quad \mathfrak{A}_j \not\models e[k_j]\right).$$

Let $\mathcal{F}$ be an ultrafilter over J, $\mathfrak{B} \stackrel{\text{def}}{=} \mathbf{P}\langle \mathfrak{A}_j : j \in J\rangle/\mathcal{F}$, and $k(x) \stackrel{\text{def}}{=} \langle k_j(x) : j \in J\rangle/\mathcal{F}$ for every variable x. Then $\mathfrak{B} \models \bigwedge_{i\in I} e_i[k]$ but $\mathfrak{B} \not\models e[k]$ by Theorem 2.2.57 (ii). Since $\mathfrak{B} \in \mathbf{Up}\mathsf{K}$, this implies that $\mathbf{Up}\mathsf{K} \models \bigwedge_{i\in I} e_i \to e$ does not hold. This proves (2.3.2). ∎

LEMMA 2.3.13. *For any class of similar algebras* K,

$$\mathbf{Mod}\, Qeq_\infty(\mathsf{K}) \subseteq \mathbf{SP}\mathsf{K}\,.$$

Proof of Lemma 2.3.13: Let $\mathfrak{A} \in \mathbf{Mod}\, Qeq_\infty(\mathsf{K})$. Let

$$H \stackrel{\text{def}}{=} \{h : h \text{ is a homomorphism with } Dom(h) = A \text{ and } Rng(h) \subseteq C \text{ for some } \mathfrak{C} \in \mathsf{K}\}\,.$$

For arbitrary $h_1, h_2 \in H$, we say that h_1 and h_2 are equivalent if $\ker(h_1) = \ker(h_2)$. There are only set many non–equivalent homomorphisms in H, because $Con(\mathfrak{A})$ is a set and $(\forall h \in H) \ker(h) \in Con(\mathfrak{A})$. Thus we can take a system $\langle h_i : i \in I\rangle$ of homomorphisms from the class H indexed by some *set* I such that $\{\ker(h_i) : i \in I\} \subseteq \{\ker(h) : h \in H\}$. For every $i \in I$, let $\mathfrak{A}_i$ be an element of K such that $Rng(h_i) \subseteq A_i$. Let $\mathfrak{B} \stackrel{\text{def}}{=} \mathbf{P}_{i\in I}\mathfrak{A}_i$. Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be the homomorphism induced by the product $\mathfrak{B}$. (Cf. Lemma 2.2.23 (iv).) Thus

$$(\forall i \in I) f \circ p_i = h_i\,, \tag{2.3.4}$$

where p_i is the i–th projection. It is enough to prove that f is injective.

Let V denote the set of all variables. Let $g : V \rightarrowtail A$ be a bijection. Let $Diag^+(\mathfrak{A})$ denote the set of all atomic formulas true in $\mathfrak{A}$ at the valuation g of the variables. (We call $Diag^+(\mathfrak{A})$ the *positive diagram* of $\mathfrak{A}$.)

Claim: For any $x, y \in Dom(g)$, if $f(gx) = f(gy)$ then

$$\mathsf{K} \models Diag^+(\mathfrak{A}) \to x = y\,.$$

Proof of Claim: The proof is illustrated on Figure 2.3.3.

$$\begin{array}{ccccc} & & V & & \\ & & \Big| \quad g & & \\ k & & \mathfrak{A} & \xrightarrow{f} & \mathfrak{B} \\ & k' & h_i & & \\ & & p_i & & \\ \mathfrak{C} & & & & \mathsf{K} \end{array}$$

Figure 2.3.3

Let $x, y \in V$ be arbitrary but fixed. Assume

$$f(gx) = f(gy)\,. \tag{2.3.5}$$

Let $\mathfrak{C} \in \mathsf{K}$ be such that

$$Diag^{+}(\mathfrak{A})[k] \tag{2.3.6}$$

for some valuation k of the variables. Let $k' : A \longrightarrow C$ be the function defined as follows:

$$(\forall a \in A)k'(a) \stackrel{\text{def}}{=} f(g^{-1}a)\,.$$

Because of (2.3.6), k' is a homomorphism. Then $k' \in H$, and

$$\ker(k') = \ker(h_i) \tag{2.3.7}$$

for some $i \in I$. Then $h_i(gx) = h_i(gy)$ by (2.3.5) and (2.3.4), therefore $k'(gx) = k'(gy)$ by (2.3.7). Then $k(x) = k(y)$ be the definition of k'. Thus $\mathfrak{C} \models Diag^{+}(\mathfrak{A}) \to x = y[k]$, which proves the claim, since $\mathfrak{C}$ and k were chosen arbitrarily. ∎

Now assume that $a, b \in A$, $f(a) = f(b)$. Then there are variables $x, y \in V$ such that $g(x) = a$ and $g(y) = b$, and by our assumption we have $f(g(x)) = f(g(y))$. Then, by the above Claim, $\mathsf{K} \models Diag^{+}(\mathfrak{A}) \to x = y$. Notice that $Diag^{+}(\mathfrak{A}) \to x = y$ is an infinitary quasi–equation. Thus, since $\mathfrak{A} \in \mathbf{Mod}\,Qeq_{\infty}\mathsf{K}$, $\mathfrak{A} \models Diag^{+}(\mathfrak{A}) \to x = y$ as well. Therefore, in particular, $\mathfrak{A} \models Diag^{+}(\mathfrak{A}) \to x = y\ [q]$ for any valuation q such that $q(x) = g(x)$ and $q(y) = g(y)$. Then $g(x) = g(y)$, since $\mathfrak{A} \models Diag^{+}(\mathfrak{A})\ [q]$. Thus $a = b$, proving that f is a injection. This completes the proof of Lemma 2.3.13. ∎

Proof of Theorem 2.3.11: $\mathbf{SPUp}\mathsf{K} \subseteq \mathbf{Mod}\,Qeq(\mathsf{K})$ holds, because $\mathbf{S}, \mathbf{P}, \mathbf{Up}$ preserve quasi–equations (cf. Exercise 2.3.2 (4)).

To see $\mathbf{Mod}\,Qeq(\mathsf{K}) \subseteq \mathbf{SPUp}\mathsf{K}$, let $\mathfrak{A} \in \mathbf{Mod}\,Qeq(\mathsf{K})$. By Lemmas 2.3.12 and 2.3.13, $\mathfrak{A} \in \mathbf{Mod}\,Qeq_{\infty}(\mathbf{Up}\mathsf{K}) \subseteq \mathbf{SPUp}\mathsf{K}$. ∎

COROLLARY 2.3.14. *For any class of similar algebras,*

$$\mathbf{Mod}\,Qeq(\mathsf{K}) = \mathbf{SP^{r}}\mathsf{K}\,.$$

Proof: This follows directly from Theorem 2.3.11 and Lemma 2.2.60. ∎

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2.4. Discriminator Varieties.

Definition 2.4.1 (discriminator term, discriminator variety):

(i) A class K of algebras is said to *have a discriminator term* iff there is a term $\tau(x, y, z, u)$ in the language of K such that in every member of K we have

$$\tau(x, y, z, u) = \begin{cases} z, & \text{if } x = y, \\ u, & \text{otherwise.} \end{cases}$$

(ii) A variety V is called a *discriminator variety* if the class $\mathsf{Sir}(\mathsf{V})$ of subdirectly irreducible members of V has a discriminator term. ◀

Warning: Sometimes instead of the 4–ary τ, the ternary discriminator term $t(x, y, z) = \tau(x, y, z, x)$ is used. They are interdefinable, since $\tau(x, y, z, u) = t(t(x, y, z), t(x, y, u), z)$. Therefore, it does not matter which one is used.

Exercises 2.4.2:

(1) Show that if K has a discriminator term then K consists of simple algebras.
(2) Assume that the Boolean operations $-, \wedge, 0, 1$ are available in K and that they satisfy the Boolean axioms (i.e. every element of K is a Boolean algebra possibly with some further operations). This property will be referred to as 'K *has a Boolean reduct*' or that the elements of K are *Boolean ordered algebras*. Prove that K has a discriminator term iff there is a term $c(x)$ in the language of K such that

$$c(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise} \end{cases}$$

in every member of K.
(*Hint:* $\tau(x, y, z, u) = \big[c(x \oplus y) \wedge u\big] \vee \big[z \wedge - c(x \oplus y)\big]$, where $\oplus$ denotes symmetric difference.) ◀

THEOREM 2.4.3. *Let K be a class of similar algebras. Assume that K has a discriminator term. Then*

$$\mathbf{HSP}\mathsf{K} = \mathbf{SPUp}\mathsf{K}.$$

The proof we give here is a model theory oriented, more intuitive proof as opposed to the more "computational" proofs in the standard universal algebra books. The idea of giving here such a more intuitive proof was suggested us by Johan van Benthem.

In the proof we will use Proposition 2.4.4 below.

PROPOSITION 2.4.4: Assume K has a discriminator term. Then to every quasi–equation q in the language of K there is an equation $e(q)$ such that (i)–(ii) below hold:

(i) $\mathsf{K} \models e(q) \leftrightarrow q$,
(ii) $\mathbf{HSP}\mathsf{K} \models e(q) \to q$.

Proof of Proposition 2.4.4: By the *length of a quasi–equation* we mean the number of the equations on the left hand side of the implication of the quasi–equation. The proof goes by induction on the length of quasi–equations. For any quasi–equation q, $length(q)$ denotes its length.

If $length(q) = 0$ for a quasi–equation q, then it is an equation, and then statements (i) and (ii) are obvious.

The induction step: Assume that for any quasi–equation q' we have:

(IH) $\quad length(q') < n \Longrightarrow (\exists \text{ equation } e(q'))\,(\mathsf{K} \models e(q') \leftrightarrow q' \;\&\; \mathbf{HSP}\mathsf{K} \models e(q') \to q')$.

Let q be a quasi–equation of length n, say q is

$$(\sigma_1 = \rho_1 \wedge \cdots \wedge \sigma_n = \rho_n) \to \sigma_0 = \rho_0 .$$

Then q is equivalent to

$$\sigma_1 = \rho_1 \to q' ,$$

where q' is the quasi–equation $(\sigma_2 = \rho_2 \wedge \cdots \wedge \sigma_n = \rho_n) \to \sigma_0 = \rho_0$. Since $length(q') < n$, by the induction hypothesis (IH) above we have

(2.4.1) $$\mathsf{K} \models \sigma = \rho \leftrightarrow q' \qquad \text{and} \qquad \mathbf{HSP}\mathsf{K} \models \sigma = \rho \to q'$$

for some equation $\sigma = \rho$.

Let τ denote the discriminator term of K, and we define $e(q)$ to be

$$\tau(\sigma_1, \rho_1, \sigma, \rho) = \rho .$$

First we prove that

(2.4.2) $$\mathsf{K} \models q \to e(q) .$$

To see this, let $\mathfrak{A} \in \mathsf{K}$ and the valuation $\bar{a} \in {}^{\omega}A$ of the variables into A be arbitrary. It is enough to prove that

(2.4.3) $$\mathfrak{A} \models \neg e(q) \to \neg q[\bar{a}] .$$

Assume $\mathfrak{A} \models \neg e(q)[\bar{a}]$, that is, $\mathfrak{A} \models \tau(\sigma_1, \rho_1, \sigma, \rho) \neq \rho[\bar{a}]$. Since τ is a discriminator term, this implies

(2.4.4) $$\mathfrak{A} \models \sigma_1 = \rho_1[\bar{a}]$$

and

$$\mathfrak{A} \models \sigma \neq \rho[\bar{a}] .$$

By the latter and (2.4.1),

$$\mathfrak{A} \not\models q'[\bar{a}]\,. \tag{2.4.5}$$

By (2.4.4) and (2.4.5), $\mathfrak{A} \not\models q[\bar{a}]$, proving (2.4.3). This proves (2.4.2).

Next we are proving

$$\mathbf{HSP}\mathsf{K} \models \neg q \to \neg e(q)\,. \tag{2.4.6}$$

To do this, let $\mathfrak{A} \in \mathbf{HSP}\mathsf{K}$ and $\bar{a} \in {}^{\omega}A$, and assume that $\mathfrak{A} \not\models q[\bar{a}]$. Then

$$\mathfrak{A} \models \sigma_1 = \rho_1[\bar{a}] \tag{2.4.7}$$

and $\mathfrak{A} \not\models q'[\bar{a}]$, thus

$$\mathfrak{A} \models \sigma \neq \rho[\bar{a}] \tag{2.4.8}$$

by (the second part of) (2.4.1). Now $\mathfrak{A} \models \tau(\sigma_1, \rho_1, \sigma, \rho) = \sigma \neq \rho[\bar{a}]$ by (2.4.7) and (2.4.8). This proves $\mathfrak{A} \not\models e(q)[\bar{a}]$, proving (2.4.6), since $\mathfrak{A}$ and $\bar{a}$ were chosen arbitrarily. We have proved (2.4.6).

(2.4.2) and (2.4.6) together prove Proposition 2.4.4. ∎

Proof of Theorem 2.4.3: First we prove that $\mathbf{HSP}\mathsf{K} \models Qeq(\mathsf{K})$. Indeed, let $q \in Qeq(\mathsf{K})$. Then, by Proposition 2.4.4 (i), there is an equation $e(q)$ with $\mathsf{K} \models \big(e(q) \leftrightarrow q\big)$. Since equations are preserved under **HSP**, $\mathbf{HSP}\ \mathsf{K} \models e(q)$ also holds. Now by Proposition 2.4.4 (ii), $\mathbf{HSP}\mathsf{K} \models q$.

Now

$$\mathbf{HSP}\mathsf{K} \subseteq \mathbf{Mod}\,Qeq(\mathsf{K}) \overset{\text{Thm.2.3.11}}{=} \mathbf{SPUp}\mathsf{K},$$

which proves one direction of Theorem 2.4.3.

The other direction, $\mathbf{SPUp}\mathsf{K} \subseteq \mathbf{HSP}\mathsf{K}$ is straightforward, since, by Exercise 2.2.62, $\mathbf{Up} \subseteq \mathbf{HP}$, $\mathbf{PH} \subseteq \mathbf{HP}$, $\mathbf{SH} \subseteq \mathbf{HS}$, and $\mathbf{PP} = \mathbf{P}$. ∎

Exercise 2.4.5: Check how much one can simplify the proof of Theorem 2.4.3 above if we assume that K has a Boolean reduct (cf. item (2) of Exercises 2.4.2 above). ◀

COROLLARY 2.4.6: Assume K has a discriminator term. Then

(i) K is contained in some discriminator variety.
(ii) The subdirectly irreducible members of $\mathbf{HSP}\ \mathsf{K}$ are exactly the subdirectly irreducibles of $\mathbf{SUp}\mathsf{K}$.

Proof: (ii): Let $\mathfrak{A}$ be a subdirectly irreducible member of $\mathbf{HSP}\mathsf{K}$. By Theorem 2.4.3, $\mathfrak{A} \in \mathbf{SP}(\mathbf{Up}\mathsf{K})$. Then $\mathfrak{A}$ is a subdirect product of algebras from $\mathbf{SUp}\mathsf{K}$. By irreducibility, then $\mathfrak{A} \in \mathbf{SUp}\mathsf{K}$. This proves (ii).

(i): The discriminator term τ which works for K also works for $\mathbf{SUp}\mathsf{K}$, since the discriminator property

$$\forall x, y, z, u \ \big([x \neq y \Rightarrow \tau(x,y,z,u) = u] \wedge [x = y \Rightarrow \tau(x,y,z,u) = z]\big)$$

is defined by a universal formula, thus is preserved under $\mathbf{SUp}$. Thus $\mathbf{SUp}\mathsf{K}$ has a discriminator term. But by (ii) the class $\mathsf{Sir}(\mathbf{HSP}\mathsf{K})$ of subdirectly irreducibles of $\mathbf{HSP}\mathsf{K}$ is in $\mathbf{SUp}\mathsf{K}$. Then by definition, $\mathbf{HSP}\mathsf{K}$ is a discriminator variety. ∎

Exercise 2.4.7*:

(i) Prove that $(d_1 - d_2)$ below are equivalent with saying that τ is a discriminator term for K.

$$(d_1) \qquad \tau(x,x,z,u) = z$$

$$(d_2) \qquad \tau(x,y,z,u) = u \vee x = y\,.$$

(ii) Prove that if τ is a discriminator term then also $(d_3 - d_5)$ below are true.

$$(d_3) \qquad \tau(x,y,z,u) = z \to (x = y \vee z = u$$

$$(d_4) \qquad \tau(x,y,z,u) = z \vee \tau(x,y,z,u) = u$$

$$(d_5) \qquad \tau(x,y,z,z) = z\,.$$

Exercise 2.4.8*: Assume τ is a discriminator term for K.

(i) Prove that $\mathbf{HSP}\mathsf{K} \models (d_1 \wedge d_5)$ (for (d_1) and (d_5) see previous exercise).
(ii) Prove that $\mathbf{HS}\mathsf{K} \models (d_1, d_2)$.

Hint: Basic universal algebra:
(i) Equations are preserved under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$.
(ii) Quasi–equations (implications between equations) are preserved under $\mathbf{S}$ and $\mathbf{P}$.
(iii) Disjunctions of equations are preserved under $\mathbf{H}$ and $\mathbf{S}$. ◀

Exercise 2.4.9: Assume τ is a discriminator term for K. Prove that in $\mathbf{HSP}\mathsf{K}$ (the variety generated by K) d_1 and d_3 are valid.

Hint: Use Theorem 2.3.11 above saying that $\mathbf{SPUp}\mathsf{K} = \mathbf{HSPUp}\mathsf{K}$ if K has a discriminator. Then apply Exercise 2.4.8 (ii) above together with the fact that $\mathbf{Up}$ preserves all formulas, hence $\mathbf{Up}\mathsf{K} \models (d_1 - d_3)$. ◀

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3. Boolean Algebras (BA's)
(Fragments)

The basic theory of Boolean algebras (BA's for short) can be found in any textbook on universal algebra, e.g. in Burris et al. [BS81] or in Halmos's book [Ha70] or Sikorski's one [Si60] or in the handbook [Handba]. For the reader not familiar with BA's, we would like to point out that Halmos's little book [Ha70] is a very easily readable introduction. Since the literature is so rich, we do not reproduce it here, instead we establish only notation and terminology to be used later, and give some basic theorems with outlines of proofs only.

As we already wrote in previous sections, the similarity type of BA$'s$ is $ba = \{\langle\vee, 2\rangle, \langle -, 1\rangle\}$. We defined BA's in the following steps: the class of *powerset* BA's was defined to be

$$\{\mathfrak{P}(U) : \mathfrak{P}(U) = \langle \mathcal{P}(U), \cup, - \rangle \text{ for some set } U\},$$

where set theoretic union $\cup$ is the interpretation of $\vee$, and the set theoretic complementation $-$ is the interpretation of the symbol $-$ (denoted both the same way, ambiguously). (Cf. Examples 2.1.3 **4.**) Then we defined *Boolean set algebras* in Definition 2.2.3 as

$$\mathsf{SetBA} = \mathbf{S}\{\mathfrak{P}(U) : U \text{ is a set}\}.$$

Finally, in Definition 2.2.11, BA's were defined as

$$\mathsf{BA} = \mathbf{I}\mathsf{SetBA}.$$

We use the following derived operations:

$$\begin{aligned} x \wedge y &= -(-x \vee -y) \\ 0 &= x \wedge -x \\ 1 &= x \vee -x \\ x - y &= x \wedge -y \\ x \oplus y &= (x - y) \vee (y - x). \end{aligned}$$

As can be seen, we use the complementation symbol $-$ in two different senses, both as a unary operation (this is the basic one), and sometimes also as a binary operation abbreviating $x \wedge -y$. We hope context will help to avoid misunderstanding.

As it is usual, we define the ordering $\leqslant$ of BA's as:

$$x \leqslant y \overset{\text{def}}{\iff} x \vee y = y.$$

The "*generic*" BA is the following:

$$\underset{\sim}{2} = \mathfrak{P}(\{0\}) .$$

Exercises 3.1: Recall that $1 = \{0\}$.

(1) Prove that $\underset{\sim}{2} = \mathfrak{P}(1)$ has exactly 2 elements.
(2) Prove that $\underset{\sim}{2}$ has no subalgebras (except itself).
(3) Prove that $\underset{\sim}{2}$ has no nontrivial homomorphic images.
(4) Prove that there exists a one–element BA.
(5) Prove that there is no 3–element BA.
(6) Prove that in any Alg_t there is only one one–element algebra up to isomorphism. (The expression "up to isomorphism" will be used quite often, and its above use makes the above sentence mean that any two one–element algebras are isomorphic.)
(7) Prove that there is only one two–element BA (up to isomorphism).
(8)* Prove that there is at most one 4–element BA (up to isomorphism). ◀

Definition 3.2: Let $\mathfrak{A} \in \mathsf{BA}$. By a *filter* of $\mathfrak{A}$ we understand a congruence class of $1^{\mathfrak{A}}$. By an *ultrafilter* of $\mathfrak{A}$ we understand a maximal filter, that is, the kernel of a homomorphisms of $\mathfrak{A}$ onto $\underset{\sim}{2}$. ◀

LEMMA 3.3. *Let $\mathfrak{A}$ be an arbitrary* BA.

(i) *$\mathcal{F}$ is a filter of $\mathfrak{A}$ iff the following conditions hold for all $a, b \in A$.*

$$\begin{gathered} 1 \in \mathcal{F} \\ a \in \mathcal{F}, a \leqslant b \Rightarrow b \in \mathcal{F} \\ a, b \in \mathcal{F} \Rightarrow a \wedge b \in \mathcal{F} . \end{gathered}$$

(ii) *$\mathcal{F}$ is an ultrafilter of $\mathfrak{A}$ iff it is a filter of $\mathfrak{A}$ and for all $a \in A$,*

$$a \in \mathcal{F} \Longleftrightarrow -a \in A \smallsetminus \mathcal{F} .$$

The **proof** is left to the reader. ∎

Definition 3.4: Let *Bax* be the following set of axioms:

(b0) $$x \wedge y = -(-x \vee -y)$$
(b1) $$x \wedge y = y \wedge x$$
(b2) $$x - (-y - z) = -(-(x \wedge y) - (x \wedge z))$$
(b3) $$x - (y - y) = x .$$

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THEOREM 3.5 (Stone). BA *is a finitely axiomatizable variety. Moreover,*

$$\mathsf{BA} = \mathbf{Mod}(Bax)\,,$$

where Bax is the set of axioms given in Definition 3.4 above.

Outline of proof: $\mathsf{BA} \subseteq \mathbf{Mod}(Bax)$ is easy to check. To see $\mathbf{Mod}(Bax) \subseteq \mathsf{BA}$, let $\mathfrak{A} \in \mathbf{Mod}(Bax)$ be arbitrary, and we define the following function rep: for every $a \in A$,

$$\mathrm{rep}(a) \stackrel{\mathrm{def}}{=} \{U : U \text{ is an ultrafilter of } \mathfrak{A} \text{ and } a \in U\}\,.$$

Using that $\mathfrak{A} \models Bax$, it is not hard to check that rep is a Boolean homomorphism into the Boolean set algebra $\langle\{\mathrm{rep}(a) : a \in A\}, \cup, -\rangle$. Thus

$$\mathbf{Mod}(Bax) \subseteq \mathbf{HMod}(Bax) \subseteq \mathsf{SetBA} \subseteq \mathsf{BA} \subseteq \mathbf{Mod}(Bax)$$

completes the proof. ∎

LEMMA 3.6. $\mathsf{Sir} \cap \mathsf{BA} = \mathbf{I}\{\underset{\sim}{2}\}\,.$

Outline of proof: First one proves that $\underset{\sim}{2}$ is subdirectly irreducible, i.e. $\underset{\sim}{2} \in \mathsf{SirBA}$. This was done in Exercise 2.2.33 (2) and also in Exercise (2.2.38 (2)); but for completeness we include a proof here. This proof is based on the fact that, by Exercise 3.1 (3) above, $\underset{\sim}{2}$ has no nontrivial homomorphic image, but any subdirect decomposition of an algebra $\mathfrak{A}$ consists of homomorphic images of $\mathfrak{A}$. Therefore any subdirect decomposition of $\underset{\sim}{2}$ consists of the one element BA and the two element BA and nothing else. But the one element BA is trivial, so should be omitted from the decomposition, while any two element BA is isomorphic to $\underset{\sim}{2}$, hence any decomposition of $\underset{\sim}{2}$ consists of the original $\underset{\sim}{2}$ itself as factors. This means that $\underset{\sim}{2}$ is subdirectly indecomposable.

Next one proves that any subdirectly indecomposable BA is isomorphic to $\underset{\sim}{2}$. This goes as follows. Assume $\mathfrak{A} \in \mathsf{SirBA}$. Assume $x \in A$ and $x \neq 0$, $x \neq 1$. Then relativising with x as well as relativising with $-x$ are homomorphisms, and they provide a subdirect decomposition of $\mathfrak{A}$; where relativising with x is the function rl_x sending y to $x \cap y$, that is, $\mathrm{rl}_x \stackrel{\mathrm{def}}{=} \langle x \cap y : y \in A\rangle$. ∎

THEOREM 3.7. $\mathsf{BA} = \mathbf{SP}\{\underset{\sim}{2}\}$.

Proof: By Lemma 3.6, $\mathsf{SirBA} = \mathbf{I}\{\underset{\sim}{2}\}$. Now, by Birkhoff's "subdirect decomposability" theorem (Theorem 2.2.40), we have that $\mathsf{BA} = \mathbf{SP}\mathsf{SirBA} = \mathbf{SPI}\{\underset{\sim}{2}\} = \mathbf{SP}\{\underset{\sim}{2}\}$. ∎

Exercise 3.8: Let $\mathfrak{B} \in \mathsf{BA}$. Represent $\mathfrak{B}$ as a set BA $\mathfrak{C}$ such that $\mathfrak{C} \subseteq \mathfrak{P}(U \times U)$ for some set U.

Hint: To any sets U and W, if $|U \times U| \geqslant |W|$ then there is a partition P of $U \times U$ with $|P| = |W|$. ◀

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4. Boolean Ordered Algebras (Boa's) and Boolean Algebras with Operators (BAO's)

By a *Boolean ordered algebra* (Boa) we understand an (arbitrary) universal algebra in which the BA–operations are term definable. A Boolean algebra with operators (BAO) is a special Boa. (A more precise definition follows below.)

More concretely, recall from section 2.1 that by a similarity type we understand a function $t : I \longrightarrow \omega$ for some set I. By a Boa of similarity type t we understand an algebra $\mathfrak{A} = \langle \mathfrak{B}, f_i \rangle_{i \in I}$, where $\mathfrak{B} \in \mathsf{BA}$ and $(\forall i \in I)\, f_i : {}^{t(i)}B \to B$.

By a Boa we understand a Boa of similarity type t for some t. Notice that a Boa of type t is a (special) universal algebra of type $t +$ (Boolean operations).[5]

Convention 4.1: Throughout t is fixed to denote the similarity type of the algebra in question. So we might be discussing some algebra $\langle \mathfrak{B}, f_i \rangle_{i \in I}$ without mentioning any "t." Then suddenly we start talking about $t(i)$ (or $t(f_i)$ sometimes). Then $t(i)$ defines (by the force of the present convention) the rank of f_i. ◀

Examples 4.2:

1. $\mathsf{BA} \subseteq \mathsf{Boa}$.
2. Let $\mathfrak{B} \in \mathsf{BA}$, $\mathsf{c} : B \longrightarrow B$. We define

$$\mathsf{Cs}_1 \stackrel{\text{def}}{=} \left\{ \quad \langle \mathfrak{B}, \mathsf{c} \rangle \quad : \quad \mathfrak{B} \in \mathsf{BA} \quad \text{and} \quad \mathsf{c}(x) = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases} \quad \right\}.$$

   We call the class Cs_1 the class of *cylindric set algebras of dimension* 1. Clearly $\mathsf{Cs}_1 \subseteq \mathsf{Boa}$.

   Claim 4.2.1: $\mathsf{Cs}_1 \subseteq \mathsf{Smp}$.
3. Let $\mathfrak{B} \in \mathsf{BA}$, $\mathsf{c} : B \longrightarrow B$. We call c a *closure operator* iff $x \leqslant \mathsf{c}x \leqslant \mathsf{cc}x$ and $x \leqslant y \Rightarrow \mathsf{c}x \leqslant \mathsf{c}y$ for every $x \in B$. We call c a *complemented closure operator* iff c is a closure operator and, in addition, $\mathsf{c}(-\mathsf{c}x) = -\mathsf{c}x$ for every $x \in B$. c is called *additive* iff $\mathsf{c}(x \vee y) = \mathsf{c}x \vee \mathsf{c}y$. Now we define

$$\mathsf{CA}_1 \stackrel{\text{def}}{=} \{\langle \mathfrak{B}, \mathsf{c} \rangle : \mathfrak{B} \in \mathsf{BA} \text{ and } \mathsf{c} \text{ is an additive complemented closure operator}\}.$$

   CA_1 is called the class of *cylindric algebras of dimension* 1. Clearly $\mathsf{CA}_1 \subseteq \mathsf{Boa}$.

   Claim 4.2.2: $\mathsf{Sir}(\mathsf{CA}_1) = \mathsf{Cs}_1$.

   The **proof** of this can be found in the Andréka–Németi paper in this volume.

   Corollary 4.2.3: CA_1 *is a discriminator variety.*

   Proof: By Claim 4.2.2, $\mathsf{Sir}(\mathsf{CA}_1) = \mathsf{Cs}_1$. As a consequence of Exercise 2.4.2, Cs_1 has a discriminator term. ∎

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[5] The careful reader may read "Boa of type t" as an abbreviation for "Boa of extra–Boolean type t". Here we use the former expression for keeping things short.

LEMMA 4.3. *Let* $\mathfrak{A} \in \mathsf{Boa}$, $\theta \in Con(\mathfrak{A})$, $a, b \in A$. *Then*

$$a\,\theta\,b \Longleftrightarrow (a \oplus b)\,\theta\,0\,.$$

Proof: Recall that $x = y \Longleftrightarrow x \oplus y = 0$. Now

$$\begin{aligned} a\,\theta\,b &\Longleftrightarrow a/\theta = b/\theta \\ &\Longleftrightarrow a/\theta \oplus b/\theta = 0/\theta \\ &\Longleftrightarrow (a \oplus b)/\theta = 0/\theta \\ &\Longleftrightarrow (a \oplus b)\,\theta\,0\,. \end{aligned}$$

∎

Definition 4.4: By a BAO *(Boolean Algebra with operators) of type* t (a BAO_t) we mean a Boa $\mathfrak{A} = \langle \mathfrak{B}, f_i \rangle_{i \in I}$ of type t such that the set (Dst) of distributivity equations below is valid in $\mathfrak{A}$.

(Dst) each f_i distributes over Boolean "$\vee$", i.e.

$$f_i(x_1, \ldots, x_{n-1}, (x_n \vee y_n), x_{n+1}, \ldots, x_{t(i)}) = \\ f_i(x_1, \ldots, x_{t(i)}) \vee f_i(x_1, \ldots, x_{n-1}, y_n, x_{n+1}, \ldots, x_{t(i)})$$

for each $i \in I$ and each $n \leqslant t(i)$.

By a *normal* BAO we understand a BAO $\langle \mathfrak{B}, f_i \rangle_{i \in I}$ in which the equations (Nrm) below are valid.

(Nrm) $f_i(x_1, \ldots, x_{n-1}, 0, x_{n+1}, \ldots, x_{t(i)}) = 0$ for each $n \leqslant t(i)$ and $i \in I$.

BAO_t and $\mathsf{normal\,BAO}_t$ denote the classes of all BAO_t's and all normal BAO's of similarity type t, respectively. ◀

Exercises 4.5:

(1) Recall the definitions of the classes Cs_1 and CA_1 of algebras from Examples 4.2. Prove that $\mathsf{Cs}_1 \subseteq \mathsf{BAO}$ and $\mathsf{CA}_1 \subseteq \mathsf{BAO}$.
(2) Give an example for an algebra $\mathfrak{A}$ such that $\mathfrak{A} \in \mathsf{Boa} \smallsetminus \mathsf{BAO}$.
(3) Prove that BAO_t and $\mathsf{normal\,BAO}_t$ are varieties.
(4) When are these two varieties finitely axiomatizable?[6] ◀

Notation 4.6: For a BAO $\mathfrak{A} = \langle \mathfrak{B}, f_i \rangle_{i \in I}$ we define $\mathfrak{A}^\circ \stackrel{\text{def}}{=} \mathfrak{B}$. That is, $\mathfrak{A}^\circ$ is the BA–reduct of $\mathfrak{A}$. ◀

Let $\mathfrak{A} \in \mathsf{Boa}$ and $a \in A$. We call a an *atom* iff $0 < a$ and $(\forall b \in A)(b \leqslant a \Rightarrow b = a)$. $\mathfrak{A}$ is called *atomic* iff every element of A is the union of some of the atoms of $\mathfrak{A}$. Notice that every finite Boolean algebra is atomic. A Boa is called *complete* iff its lattice reduct is a complete lattice.

[6]Note that t is a set of pairs hence if $t : I \longrightarrow \omega$ then $|t| = |I|$.

THEOREM 4.5 (Jónsson–Tarski [JT48][7]).

(i) *Every* BAO $\mathfrak{A}$ *is contained (as a subalgebra) in a complete and atomic* BAO $\mathfrak{A}^+$.

(ii) *If* $\mathfrak{A}$ *is normal, then so is* $\mathfrak{A}^+$.

(iii) $\mathfrak{A}^+$ *is completely distributive, i.e. each* f_i *commutes with arbitrary suprema in* $\mathfrak{A}^+$.

(iv) *For every proper filter*[8] $\mathcal{F}$ *of* $\mathfrak{A}^\circ$, *the infimum of* $\mathcal{F}$ *taken in* $\mathfrak{A}^+$ *is nonzero. Formally,* $\inf^{(\mathfrak{A}^+)} \mathcal{F} \neq 0$.

For proving Theorem 4.5 we will need the following proposition of BA–theory.

PROPOSITION 4.8.

(i) *Every* BA $\mathfrak{A}$ *is contained (as a subalgebra) in a complete and atomic* BA $\mathfrak{A}^+$.

(ii) *For every proper filter* $\mathcal{F}$ *of* $\mathfrak{A}^\circ$, $\inf^{(\mathfrak{A}^+)} \mathcal{F} \neq 0$.

Proof: This is an immediate consequence of the well known (Stone) representation theorem of BA's. It can be found in any textbook on BA's (and most ones on universal algebra).

Hint: let $X_{\mathfrak{A}}$ be the set of ultrafilters of $\mathfrak{A}$, and let $\mathfrak{B}$ be the BA of all subsets of $X_{\mathfrak{A}}$. The embedding of $\mathfrak{A}$ into $\mathfrak{B}$ sends each element of $\mathfrak{A}$ to the set of ultrafilters containing it, i.e. $f(a) = \{x \in X_{\mathfrak{A}} : a \in x\}$ for all $a \in A$. ∎

Proof of Theorem 4.5: Let $\mathfrak{A} \in$ BAO. Consider $\mathfrak{A}^\circ \in$ BA. By Proposition 4.8, there is a complete and atomic $\mathfrak{B} \in$ BA with $\mathfrak{B} \supseteq \mathfrak{A}^\circ$ satisfying (iv). Let this $\mathfrak{B}$ be fixed, see Figure 4.1 below.

Figure 4.1

[8] A filter $\mathcal{F}$ is proper if $0 \notin \mathcal{F}$.

It is enough to define operations $f_i^+ : {}^{t(i)}B \to B$ (for each i) such that f_i^+ commutes with suprema and agrees with f_i on A. First we define f_i^+ on the atoms $\mathrm{At}\mathfrak{B}$ of $\mathfrak{B}$. To make the idea of the proof more visible we write out the rest of the proof for the special case when each f_i is normal and unary. The generalization to n–ary f_i is straightforward.

Definition 4.8.1: We define

(i) $f_i^+(b) \stackrel{\text{def}}{=} \inf^{\mathfrak{B}}\{f_i(x) : b \leqslant x \in A\}]$ for each $b \in \mathrm{At}\mathfrak{B}$, and

(ii) $f_i^+(y) \stackrel{\text{def}}{=} \sup^{\mathfrak{B}}\{f_i^+(b) : b \in \mathrm{At}\mathfrak{B} \text{ and } b \leqslant y\}$. ◀

Figure 4.2 represents the definition of $f_i^+(b)$ with b an atom.

Figure 4.2

Both parts of the definition make sense since $\mathfrak{B}$ is complete, and they define a function $f_i^+ : B \to B$. It is useful to notice here that

$$y = \sup^{\mathfrak{B}}\{b \in \mathrm{At}\mathfrak{B} : b \leqslant y\} \tag{4.1}$$

for any $y \in B$. (This explains how $f_i^+(y)$ is determined by part (i) of the definition.) Clearly, f_i^+ is completely distributive by a bit of Boolean reasoning, using part (ii) of the definition. We let $\mathfrak{A}^+ \stackrel{\text{def}}{=} \langle \mathfrak{B}, f_i^+ \rangle_{i \in I}$. Clearly $\mathfrak{A}^+ \in \mathsf{BAO}$ is complete and atomic, etc. The question is how $\mathfrak{A}^+$ is related to $\mathfrak{A}$.

Claim 4.8.2. *Let $y \in A$. Then $f_i^+(y) \leqslant f_i(y)$.*

Proof of Claim 4.8.2: Assume $f_i^+(y) \geqslant a \in \mathrm{At}\mathfrak{B}$. Then by (ii) $(\exists b \in \mathrm{At}\mathfrak{B})$ $[b \leqslant y$ and $f_i^+(b) \geqslant a]$. Fix this b. By (i) then a is a lower bound of $\{f_i(x) : b \leqslant x \in A\}$. Since $b \leqslant y \in A$, this implies $a \leqslant f_i(y)$. We proved that $f_i(y)$ is an upper bound of $\{a \in \mathrm{At}\mathfrak{B} : a \leqslant f_i^+(y)\}$. By (4.1) then $f_i(y) \geqslant \sup\{a \in \mathrm{At}\mathfrak{B} : a \leqslant f_i^+(y)\} = f_i^+(y)$. ∎

From now on we write f and f^+ for f_i and f_i^+ respectively. Similarly, we write At for At$\mathfrak{B}$.

Claim 4.8.3. *Let* $y \in A$, $f(y) \geqslant b \in \mathrm{At}$. *Then there is* $a \in \mathrm{At}$ *such that* $a \leqslant y$ *and* $b \leqslant f^+(a)$.

Proof of Claim 4.8.3: Let $N \stackrel{\text{def}}{=} \{x \in A : b \nleqq f(x)\}$. Note that $N \neq \emptyset$ by normality. Let $x_0, \ldots, x_n \in N$. We show that

$$-y \vee x_0 \vee \ldots \vee x_n \neq 1\,. \tag{4.2}$$

Assume the contrary. Then $y \leqslant x_0 \vee \ldots \vee x_n$, thus $b \leqslant f(y) \leqslant f(x_0 \vee \ldots \vee x_n) = f(x_0) \vee \ldots \vee f(x_n)$. Since b is an atom, this means $(\exists i \leqslant n)\ b \leqslant f(x_i)$ contradicting the definition of N. This proves (4.2).

Let $N^- = \{-x : x \in N\} \cup \{y\}$. Then by (4.2) for any finite $H \subseteq N^-$, $\inf(H) \neq 0$. Hence N^- is contained in a proper filter of $\mathfrak{A}$ whence, by (iv), there is an atom $a \leqslant \inf(N^-)$. Assume now $a \leqslant x \in A$, for some x. Then

$$a \leqslant x \underset{\substack{\uparrow \\ a \neq 0}}{\Rightarrow} a \nleqq -x \quad \Rightarrow \quad -x \notin N^- \Rightarrow x \notin N \Rightarrow b \leqslant f(x).$$

This proves $b \leqslant \inf\{f(x) : a \leqslant x \in A\} = f^+(a)$. By $a \leqslant \inf(N^-) \leqslant y$ we are done. ∎

Let $y \in A$ and $f(y) \geqslant b \in \mathrm{At}$ be arbitrary. By Claim 4.8.3, there is an atom $a \leqslant y$ with $f^+(a) \geqslant b$. Now,

$$f^+(y) = \sup\{f^+(a) : y \geqslant a \in \mathrm{At}\} \geqslant f^+(a) \geqslant b.$$

So for any atom $b \leqslant f(y)$ we proved $f^+(y) \geqslant b$. Then $f(y) = \sup\{b \in \mathrm{At} : f(y) \geqslant b\} \leqslant f^+(y)$. By the choice of y we proved $(\forall y \in A)\ f^+(y) \geqslant f(y)$. Together with Claim 4.8.2 this implies $(\forall y \in A)\ f^+(y) = f(y)$. That is,

$$f_i^+ \restriction A = f_i \quad \text{for all } i \in I.$$

But this means exactly that $\mathfrak{A} \subseteq \mathfrak{A}^+$. The rest is immediate from the properties of $\mathfrak{B}$ and its relationship with $\mathfrak{A}^\circ$ both of which we inherited from Proposition 4.8.

∎**(Theorem 4.5).**

Definition 4.9: An equation e in the language of BAO's is called *positive* if "$-$" does not occur in e except perhaps in front of constant symbols. I.e. if $t(i) = 0$ then $-f_i$ may occur in e. ◀

THEOREM 4.10 (Jónsson–Tarski). *Let* $\mathfrak{A} \in \mathsf{BAO}$. *Then there is a complete and atomic* BAO $\mathfrak{A}^+ \supseteq \mathfrak{A}$ *as in Theorem 4.5, such that the following holds. For any positive equation* e, *we have* $(\mathfrak{A} \models e) \Rightarrow (\mathfrak{A}^+ \models e)$.

The **proof** can be found, e.g. in Jónsson–Tarski [JT52], [HMII, §2.7] (Theorems 2.7.13, 2.7.14, 1.7.16). ∎

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