MIDTERM / CO1A / Spring 2009 / Sample Midterm, Fall 2009

1.) A codeword from the alphabet $\{a, b, c, d\}$ is legitimate iff a appears even and b and c appear odd number of times (no constrain on d). Find the number of legitimate codewords of length k. **(5 points)**
2.) If a campus telephone extension has four digits, how many different extensions are there with no repeated digits (a) if the first digit cannot be 0? (b) if the first digit cannot be 0 and the second cannot be 1? **(3 points)**
3.) Find the sequence (a_k) such that the function $\frac{1}{1+x^4}$ is the ordinary generating function of (a_k), that is $\frac{1}{1+x^4} = \sum_{n=0}^{\infty} a_n \cdot x^n$. **(4 points)**
4.) Prove that (suppose $n \geq k$)

$$\sum_{r=k}^{n} \binom{r}{k} = \binom{n+1}{k+1}. \qquad \textbf{(5 points)}$$

5.) Suppose that there are $2p$ kinds of objects, each in infinite supply. Let a_k be the number of distinguishable ways of choosing (without order) k objects if only an even number (excluding 0) of each of the first p kinds of object and an odd number of each of the second p kinds of object can be taken. Set up a generating function for the sequence (a_k) and solve for a_k. **(5 points)**
6.) Solve the recurrence relation under the given initial conditions: $h_{k+2} = 2h_{k+1} - h_k + 2^k$, $h_0 = 2$, $h_1 = 1$. **(6 points)**
7.) In how many ways can King Arthur sit n pairs of knights around a round table if no knights from the same pair can be seated next to each other. **(6 points)**
8.) Determine a recurrence for $f(n)$, the number of regions into which the plane is divided by n circles in general position (the circles are in general position if every pair of them have exactly 2 — distinct — intersection points and no three of them go through the same point). Solve the recurrence to obtain the number of parts (be careful with the initial values!) **(7 points)**
9.) Prove that $x^n = \sum_{k=1}^{n} S(n, k)x(x-1)\cdots(x-k+1)$ (Hint: prove that the two polynomials on the two sides of the equation are equal at values $x = 0, 1, 2, \ldots, n$.) **(7 points)**

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