HOMEWORK SET #4 / CO1A / Fall 2009

1. How many subsets does the set $\{1, 2, \dots, n\}$ have that contain no two consecutive integers?
2. Remember that the Fibonacci numbers are defined by $F_n = F_{n-1} + F_{n-2}$ and $F_0 = F_1 = 1$. Find closed formulas for $F_0 + F_2 + F_4 + \cdots + F_{2n}$, $F_1 + F_3 + F_5 + \cdots + F_{2n+1}$ and $F_0 + F_1 + F_2 + \cdots + F_n$ $(n \geq 1)$ (you may express them sums in terms of some — fixed number of — members of the Fibonacci sequence).
3. Determine a recurrence for $f(n)$, the number of regions into which the space is divided by n planes in general position (and for that define the general positions of the planes as well such a way that they will give the highest possible numbers of parts of the plane). Solve the recurrence to obtain the number of parts.
4. Suppose that there are p kinds of objects, each in an infinite supply. Let b_k be the number of ways of choosing k objects (without order) if only an odd number of each kind of object can be taken. Set up a generating function for the sequence (b_k) and solve for b_k.
5. Prove that for general (not necessarily integer) s and integer k we have the equality $\binom{s}{k} + \binom{s}{k-1} = \binom{s+1}{k}$.
6. Suppose that $a_{n+1} = (n+1)b_n$, with $a_0 = b_0 = 1$. Let $A(x)$ denote the exponential generating function of the sequence (a_n) and $B(x)$ denote the exponential generating function of the sequence (b_n). Derive a relation between $A(x)$ and $B(x)$.
7. For each of the following functions, find the sequences for which the function is the ordinary and (for a different sequence) the exponential generating function: $\frac{1}{1-x} + e^{6x}$, $(1+x^2)^n + 1$, $e^{x/2} + x^2 + x^3$

8*. Prove that the number of partitions of an integer k into distinct integers and the number of partitions of the same integer k into (not necessarily distinct) odd integers are equal. (Hint: set up the ordinary generating functions for the two sequences and prove that they are equal. However, setting up the functions is not automatic this time with the help of theorem from class, rather, it requires some genuine new idea).

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