SPACETIME WITHOUT REFERENCE FRAMES

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by

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CONTENTS

PART TWO

MATHEMATICAL TOOLS

The star $(*)$ indicates paragraphs or sections which can be skipped at first reading without loss in understanding the essential meaning of the book.

PREFACE

1. Mathematics reached a crisis at the end of the last century when a number of paradoxes came to light. Mathematicians surmounted the difficulties by revealing the origin of the troubles: the obscure notions, the inexact definitions; then the modern mathematical exactness was created and all the earlier notions and results were reappraised. After this great work nowadays mathematics is firmly based upon its exactness.

Theoretical physics — in quantum field theory — reached its own crisis in the last decades. The reason of the troubles is the same. Earlier physics has treated common, visible and palpable phenomena, everything has been obvious. On the other hand, modern physics deals with phenomena of the microworld where nothing is common, nothing is visible, nothing is obvious. Most of the notions applied to describe phenomena of the microworld are the old ones and in the new framework they are necessarily confused.

It is quite evident, that we have to follow a way similar to that followed by mathematicians to create a firm theory based on mathematical exactness; having mathematical exactness as a guiding principle, we must reappraise physics, its most common, most visible and most palpable notions as well. Doing so we can hope we shall be able to overcome the difficulties.

2. According to a new concept, *mathematical physics* should be a *mathematical theory* of the *whole physics,* a mathematical theory based on mathematical exactness, a mathematical theory in which only *mathematically defined notions* appear and in which *all the notions used in physics* are defined in a mathematically exact way.

What does the term “mathematically exact” cover? Since physics is a natural science, its criteria of truth is experiment. As a straightforward consequence, theoretical physics has become a mixture of mathematical notions and mathematically not formulated “tacit agreements”. These agreements are organic parts of theoretical physics; they originate from the period when physics treated palpable phenomena like those in classical mechanics and electrodynamics. Today’s physics deals with phenomena on very small or very large scales.

Unfortunately, since the education of physicists starts with the classical theories which are left more or less as they were at the beginning of this century, the acquired style of thinking is the mixture mentioned above and this is applied further on to describe phenomena in regions where nothing is obvious, resulting in confusion and unclear thinking.

Mathematical exactness means that we formulate all the "tacit agreements" in the language of mathematics starting at the very beginning, with the most natural, most palpable notions. Following this method, we have a good chance of making an important step forward in modern theoretical physics.

At first sight this seems to a physicist like creating unnecessary confusion around obvious things. Such a feeling is quite natural; if one has never driven a car before, the first few occasions are terrible. But after a while it becomes easy and comfortable and much faster than walking on foot; it is worth spending a part of our valuable time on learning to drive.

3. To build up such a mathematical physics, we must start with the simplest, most common notions of physics; we cannot start with quantum field theory but we hope that we can end up with it.

The fundamental notion of mathematical physics is that of *models*. Our aim is to construct mathematical models for physical phenomena. The modelling procedure has two sides of equal importance: the mathematical model and the modelled part of physical reality. We shall sharply distinguish between these two sides. Physical reality is *independent* of our mind, it is such as it is. A mathematical model *depends* on our mind, it is such as it is constructed by us. The confusion of physical reality and its models have led to heavy misunderstandings in connection with quantum mechanics.

A mathematical model is constructed as a result of experiments and theoretical considerations; conclusions based on the model are controlled by experiments. The mathematical model is a *mathematical structure* which is expected to reflect some properties of the modelled part of reality. It lies outside the model to answer what and how it reflects and to decide in what sense it is good or bad. To answer these questions, we have to go *beyond the exact framework* of the model.

4. The whole world is an undivisible unity. However, to treat physics, we are forced by our limited biological, mental etc. capacity to divide it into parts in theory.

Today's physics suggests the arrangement of physical phenomena in three groups; the corresponding three entities can be called Spacetime, Matter and Field.

The phenomena of these three entities interact and determine each other mutually. At present it is impossible to give a good description of the complex situation in which everything interacts with everything, which can be illustrated as follows:

5. Fortunately, a great number of phenomena allows us to neglect some aspects of the interactions. More precisely, we can construct a good theory if we can *replace interaction by action,* i.e. we can consider as if the phenomena of two of the entities above were given, fixed, "stiff" and only the phenomena of the third one were "flexible", unknown and looked for. The stiff phenomena of the two entities are supposed to act upon and even determine the phenomena of the third one which do not react. We obtain different theories according to the entities considered to be fixed.

Mechanics (classical and quantal), if spacetime and field phenomena are given to determine phenomena of matter, can be depicted as:

In some sense *continuum physics* and *thermodynamics,* too, are such theories.

Field theory (classical, i.e. electrodynamics), if spacetime and matter phenomena are given to determine phenomena of field, is:

Gravitation theory, if matter and field are given to determine spacetime, is:

These theories in usual formulation are relatively simple and well applicable to describe a number of phenomena: it is clear, however, that they draw *roughly simplified* pictures of the really existing physical world.

6. Difficulties arise when we want to describe complicated situations in which only one of the three entities can be regarded as known and interactions occur among the phenomena of the other two entities. The following graphically delineated possibilities exist:

The third one is of no physical interest, so far. However, the other two are very important and we are forced to deal with them. They represent qualitatively new problems and they cannot be reduced to the previous well-known theories, except some special cases treated in the next item.

Electromagnetic radiation of microparticles is, for instance, a phenomenon, which needs such a theory. Usual quantum electrodynamics serves as a theory for its description, and in general usual quantum field theory is destined to describe the interaction of field and matter in a given spacetime.

As it is well known, usual quantum field theory has failed to be completely correct and satisfactory. One might suspect the reason of the failure is that usual quantum field theory was created in such a way that the notions and formulae of *mechanics* were mixed with those of *field theory*. This way leads to nowhere: in mechanics the field phenomena are fixed, in field theory the matter phenomena are fixed; the corresponding notions "stiff" on one side cannot be fused correctly to produce notions "flexible" on both sides.

The complicated mathematics of quantum field theory does not allow us to present a simple example to illustrate the foregoings, whereas classical electrodynamics offers an excellent example. The electromagnetic field of a point charge moving on a *prescribed* path is obtained by the Lienard–Wiechert potential which allows us to calculate the force due to electromagnetic radiation acting upon the charge. Then the Newtonian equation is supplemented with this radiation reacting force — which is deduced for a point charge *moving on a given path* — to get the so-called Lorentz–Dirac equation for *giving the motion* of a point charge in an electromagnetic field. No wonder, the result is the nonsense of "runaway solutions".

Electromagnetic radiation is an irreversible process; in fact every process in Nature is irreversible. *The description of interactions must reflect irreversibility.* Mechanics (Newtonian equation, Schrödinger equation) and electrodynamics (Maxwell equations) i.e. the theories dealing with action instead of interaction do not know irreversibility. Evidently, no amalgamation of these theories can describe interaction and irreversibility.

7. There is a special case in which interaction can be reduced to some combination of actions yielding a good approximation. Assume that matter phenomena can be divided into two parts, a "big" one and a "small" one. The big one and field (or spacetime) are considered to be given and supposed to produce spacetime (or field) which in turn acts on the small matter phenomena to determine them. The situations can be illustrated graphically as follows:

An example for the application of this trick is the description of planetary motion in general relativity, more closely, the advance of the perihelion of Mercury. Then field is supposed to be absent, the big Sun produces spacetime and this spacetime determines the motion of the small Mercury. Doing so we neglect that spacetime is influenced by Mercury and the motion of Sun is influenced by spacetime as well, i.e. we neglect interaction.

The second example is similar. A given spacetime and a heavy point charge are supposed to produce an electromagnetic field and this electromagnetic field determines the motion of a light point charge. Doing so we neglect that the electromagnetic field is influenced by the light point charge and the motion of the heavy point charge is influenced by the electromagnetic field, i.e. we neglect interaction.

PART ONE

SPACETIME MODELS

INTRODUCTION

1. The principles of covariance and of relativity

1.1. Today the guiding principle for finding appropriate laws of Nature is the principle of general relativity: any kind of observer should finally conclude the same laws of Nature; the laws are independent of the way we look at them. The usual mathematical method of applying this principle is the following: in a certain reference frame we have an equation that, as we suspect, expresses some law independent of the reference frame. The way to check this is referred to as the principle of covariance: transfer the equation into another reference frame with an appropriate transformation (Galilean, Lorentzian, or a general coordinate transformation), and if the form of the equation remains the same after this procedure, then it can be a law of some phenomenon. This method can be illustrated in the following way:

It seems quite natural to organize the procedure in such a way; this is how Galileo and Newton started it and this is how Einstein finally concluded to the principle of general relativity. What could be the next step? Very simple: since the laws of Nature are the same for all observers, the theoretical description does not need the observer any longer; there should exist a way of describing Nature without observers. In fact, at that time Einstein said this in another way: "the description of Nature should be coordinate-free".

This was some 70 years ago but if we take a glance at some books on theoretical physics today, we stumble upon an enormous amount of indices; thinking starts from reference frames and remains there; the program of coordinate-free description has not yet been accomplished.

The key step (but not the only step) towards being able to describe Nature without observers is the mathematical formulation of the "tacit agreement" behind the non-mathematical notion of observers. This formulation finally lifts the notion of the observer from the mist and starts reorganizing the method of description in a way Einstein suggested. This reorganizing results in describing Nature independently of observers. If we wish to test our theory by experiments, we have to convert absolute quantities into relative ones corresponding to observers and then to turn them into numbers by choosing units of time, distance etc. arriving in this way to indices and transformation rules. Compared with the previous situation, this can be illustrated as follows:

1.2. The most important result of the present book is this reorganizing of the whole method of theoretical description. In this framework the principle of covariance and the principle of relativity sound very simple (encouraging us that this might be the right way).

Principle of covariance: *according to our present knowledge, the description of Nature should be done by first choosing one of the non-relativistic, special relativistic and general relativistic spacetime models and then using the tools of the chosen model.*

Principle of relativity: *there must be a rule in the spacetime models that says how an arbitrary observer derives from the absolute notions its own quantities describing the phenomena.*

2. Units of measurements

2.1. In practice, the magnitudes of a physical quantity (observable) are always related to some unit of measurement i.e. to a chosen and fixed magnitude. We determine, for instance, which distance is called *meter* and then we express all distances as non-negative multiples of meter.

In general the following can be said. Let C be the set of the magnitudes of an observable. Taking an arbitrary element c of C and a non-negative real number α, we can establish which element of C is α times c, denoted by αc. In other words, we give a mapping, called *multiplication by non-negative numbers,*

$$\mathbb{R}_0^+ \times \mathrm{C} \to \mathrm{C}, \quad (\alpha, c) \to \alpha c$$

with the following properties: for all $c \in \mathrm{C}$

(i) $0c$ is the same element, called the *zero* of C and is denoted by 0 as well;

(ii) $1c = c$

(iii) $\beta(\alpha c) = (\beta\alpha)c$ for all $\alpha, \beta \in \mathbb{R}_0^+$ and $c \in \mathrm{C}$;

(iv) if $c \neq 0$ then $J_c : \mathbb{R}_0^+ \to \mathrm{C}$, $\alpha \mapsto \alpha c$ is bijective.

In customary language we can say that C is a one-dimensional cone.

An addition can be defined on this one-dimensional cone. It is easy to see that the mapping, called *addition,*

$$\mathrm{C} \times \mathrm{C} \to \mathrm{C}, \quad (a, b) \mapsto J_c\left({J_c}^{-1}(a) + {J_c}^{-1}(b)\right) =: a + b$$

is independent of c.

Let us introduce the notations $-\mathrm{C} := \{-1\} \times \mathrm{C}$, $-c := (-1, c)$ $(c \in \mathrm{C})$ and $\mathbf{D} := (-\mathrm{C}) \cup \mathrm{C}$. Then we can give a multiplication by real numbers

$$\mathbb{R} \times \mathbf{D} \to \mathbf{D}, \qquad (\alpha, \boldsymbol{d}) \mapsto \alpha\boldsymbol{d}$$

and an addition

$$\mathbf{D} \times \mathbf{D} \to \mathbf{D}, \qquad (\boldsymbol{d}, \boldsymbol{e}) \mapsto \boldsymbol{d} + \boldsymbol{e}$$

that are trivial extensions of the operations given on C, so that $\mathbf{D}$ becomes a one-dimensional real vector space. For instance,

$$\begin{aligned} \alpha c :&= -|\alpha| c \quad \text{for} \quad \alpha < 0, \quad c \in \mathrm{C}, \\ \alpha(-c) :&= -\alpha c \quad \text{for} \quad \alpha > 0, \quad c \in \mathrm{C}, \\ \alpha(-c) :&= |\alpha| c \quad \text{for} \quad \alpha < 0, \quad c \in \mathrm{C}. \end{aligned}$$

Furthermore, the two "halves" of this vector space have different importance: the original cone contains the physically meaningful elements. We express this fact mathematically by orienting $\mathbf{D}$ with the elements of C (see IV.5).

The preceding construction works e.g. for distance, mass, force magnitude, etc. In some cases — e.g. for electric charge — we are given originally a one-dimensional real vector space of observable values.

Thus we accept that the magnitudes of observables are represented by elements of oriented one-dimensional real vector spaces called *measure lines.* Choosing a unit of measurement means that we pick up a positive element of the measure line.

2.2. In practice some units of measurement are deduced from other ones by multiplication and division; for instance, if $\boldsymbol{kg}$, $\boldsymbol{m}$ and $\boldsymbol{s}$ are units of mass, distance and time period, respectively, then $\frac{\boldsymbol{kg}\,\boldsymbol{m}}{\boldsymbol{s}^2}$ is the unit of force. The question arises at once: how can we give a mathematically exact meaning to such a symbol? According to what has been said, $\boldsymbol{kg}$, $\boldsymbol{m}$ and $\boldsymbol{s}$ are elements of one-dimensional vector spaces; how can we take their product and quotient? To give an answer let us list the rules associated usually with these operations; for instance,

$$(\alpha \boldsymbol{kg})(\beta \boldsymbol{m}) = (\alpha\beta)(\boldsymbol{kg}\ \boldsymbol{m}) \qquad (\alpha, \beta \in \mathbb{R}_0^+),$$
$$\frac{\alpha \boldsymbol{m}}{\beta \boldsymbol{s}} = \frac{\alpha}{\beta}\frac{\boldsymbol{m}}{\boldsymbol{s}} \qquad (\alpha \in \mathbb{R}_0^+, \beta \in \mathbb{R}^+).$$

Extending these rules to negative numbers, too, we see that the usual multiplication is a bilinear map on the measure lines and the usual division is a linear-quotient map, with the additional property that the product and quotient of non-zero elements are not zero.

Consequently, we can state that the product and quotient of units of measurements are to be defined by their tensor product and tensor quotient, respectively (see IV.3 and IV.4).

Thus if $\mathbf{D}$, $\mathbf{I}$ and $\mathbf{G}$ denote the measure line of distance, time period and mass, respectively, $\boldsymbol{m} \in \mathbf{D}$, $\boldsymbol{s} \in \mathbf{I}$, $\boldsymbol{kg} \in \mathbf{G}$, then $\frac{\boldsymbol{kg}\,\boldsymbol{m}}{\boldsymbol{s}^2} := \frac{\boldsymbol{kg} \otimes \boldsymbol{m}}{\boldsymbol{s} \otimes \boldsymbol{s}} \in \frac{\mathbf{G} \otimes \mathbf{D}}{\mathbf{I} \otimes \mathbf{I}}$.

3. What is spacetime?

Space and time are fundamental notions in physics: space and time form the general background of phenomena in Nature.

Let us examine these notions more closely.

3.1. Sitting in a room, we conceive that a corner of the room, a spot on the carpet are points and the table is a part of our space. Looking through the window we see trees, chimneys, hills that form other parts of our space. A car travelling on the road is not a part of this space.

On the other hand, the seats, the dashboard, etc. constitute a space for someone sitting in the car. Looking out he sees that the trees, the houses, the hills are running, they are not parts of the space corresponding to the car.

Consequently, the space for us in the room and the space for the one in the car are different. We have ascertained that space itself does not exist, *there is no absolute space, there are only spaces relative to material objects.* A space is constituted by a material object.

3.2. Processes indicate that time passes: we breathe, someone is speaking, a clock is ticking, the Sun proceeds on the sky. In fact this is time: the sequence of processes. *Time, too, is constituted by material objects.*

Immediately the question arises: is time absolute or relative? In other words: is the same time realized in the room and in the car or not? And even: is the same time realized in two different corners of the room?

There are no evident answers to these questions. Our simplest everyday experience suggests that time is absolute. However, some experiments contradict this suggestion.

3.3. To relate the space of the room and that of the car, we must involve time, too. Space and time relative to a material object interweave to express space and time relative to another material object. This reason suggests that a *unique spacetime exists which is observed by material objects as space and time.* We can think that space and time are something like side views of spacetime.

We try to make mathematical models for spacetime on the basis of the properties of space and time observed by material objects.

3.4. Our first abstraction in connection with space is the *point.* The corner of the room, a spot on the carpet stand for points of our space.

Our second abstraction is the *straight line.* A light beam, a spanned thread stand for a segment of a straight line. We discover that one and only one straight line is passing through two different points.

Our third abstraction is the *plane.* A table surface, a window-glass stand for a part of a plane. We find out that one and only one plane passes through three points that are not on a straight line.

The notion of planes offers us the notion of parallelism: two straight lines are parallel if have no common point and there is a plane containing them.

Let x and y be two distinct points of our space. We introduce the vector $\overrightarrow{xy}$ to be the straight line segment between x and y, oriented in such a way that x and y are its initial and final points, respectively. We agree that $\overrightarrow{xy} = \overrightarrow{uv}$ in the case $x \neq u$, $y \neq v$ if and only if the corresponding lines — i.e. the line passing through x and y and the line passing through u and v as well as the line passing through x and u and the line passing through y and v— are parallel.

For a space point x we consider $\overrightarrow{xx}$ to be a "degenerate" segment; if we accept the preceding rule for the equality of $\overrightarrow{xx}$ and $\overrightarrow{uu}$ we find that they are equal for all space points x and u; we call this vector *zero.*

With the aid of parallelism we introduce the sum of two vectors: $\overrightarrow{xu} = \overrightarrow{xy} + \overrightarrow{xz}$ if and only if $\overrightarrow{yu} = \overrightarrow{xz}$.

A fundamental property of this addition is that for arbitrary space points x, y and z

$$\overrightarrow{xy} + \overrightarrow{yz} + \overrightarrow{zx} = \overrightarrow{xx} = \text{zero vector.}$$

We have the well-known Euclidean method of constructing the positive rational multiple of a vector (segment); since we feel the space is "continuous", we are convinced that all positive real multiples of a vector make sense. We accept that a vector multiplied by -1 is the same segment oriented inversely.

3.5. We have given two operations on the vectors: addition and multiplication by real numbers. These operations satisfy the necessary requirement that the vectors be indeed vectors, i.e. the set of vectors endowed with these operations is a *vector space.*

We know that at most three linearly independent space vectors can be found. Moreover, we can compare the vectors with respect to their *length*, and we introduce the *angle* between two vectors. We find that the sum of the lengths of two sides of a triangle is larger than the third side, and the sum of the angles of a triangle is the straight angle.

To sum it up, *the vectors of our space form a three-dimensional Euclidean vector space* (see V.3).

Let us consider three vectors that are not in the same plane (linearly independent vectors). We can order them in two manners: in right-handed way and in left-handed way. It is an interesting question whether the right-handed order and the left-handed order are physically equivalent or not: is there a phenomenon that makes different the two orders, i.e. the phenomenon exists but its reflection does not. Our simplest experience indicates that the two orders are equivalent. However, some more complicated phenomena show that they are not: e.g. the structure of molecules of living organs, or the snail shells. Recently it was demonstrated that the decay of Kmesons exhibits clearly that the right-handed order and the left-handed one are not physically equivalent. We reflect this fact by saying that the vector space in question is *oriented* (see IV.5).

3.6. It is emphasized that the points of our space do not form a vector space; we associate a vector to each ordered pair of space points. This correspondence between space point pairs and vectors has properties which suggest us accepting that *our space is an affine space* (see Chapter VI).

Summarizing what we have found we state that *our space is a three-dimensional oriented Euclidean affine space* (see VI.1.6).

3.7. As concerns time, we are convinced that it passes "uniformly". We can determine the period between two arbitrary time points, and time periods are summed up as time is passing. We give sense to the real multiple of time periods. Time is evidently oriented: past and future are not equivalent.

Thus we state that our *time is a one-dimensional oriented affine space.*

3.8. The affine structure of time(s) and the affine structure of spaces relative to material objects are related to each other.

More closely, if an inertial material object observes that a body moves uniformly on a straight line then another inertial material object observes the same body moving uniformly on a straight line, too. Uniform motion, involving the affine structure of both time and space, is independent of observers.

This indicates that spacetime itself has an affine structure.

3.9. After having gathered the properties of our space and time, and having obtained nice structures, let us hasten to pose the uneasy questions: have we reasoned properly? have we not made some mistakes? have we not left anything out of consideration?

There is a serious objection to our reasoning: *we have extrapolated our experience gained in human size to much larger and much smaller size, too.*

Let us examine first our concept of continuity of space and time. According to our common experience, i.e. from human point of view, water is a continuous material. However, we already know that it is rather coarse: a microbe does not perceive it to be continuous at all. Are perhaps space and time coarse as well? At present no experimental fact supports this possibility but we cannot exclude it in good faith.

Let us accept the continuity of space and time. Our conviction that a vector can be associated with two space points is based on the fact that e.g. we can span a thread between the corner of the room and a spot on the carpet, or we can produce a light beam between them. But how can we determine the vector between two points whose distance is much smaller than the diameter of the thread or the light beam? If we can define vectors for such near points, too, do they obey the customary rules of addition and multiplication by real numbers?

We meet a similar problem if we want to give sense of vectors corresponding to points very far from each other. A thread cannot but a light beam can draw a straight line between Earth and Moon; however, it is not evident at all that addition and multiplication by real numbers of such huge vectors make sense with the customary properties.

Indeed, some experiments show that in astronomical size the vectorial operations cannot be defined for segments defined by light beams. At present we have no similar knowledge regarding minute size.

Evidently, the same problems arise for small and large time periods.

3.10. The part of sea surface seen from a ship seems to be plane though it is a part of a sphere. The domain of space and time observed by us seems to be a part of an affine space. Are space and time affine?

Let us recall that we have gathered the properties of our space and time in order to establish the properties of spacetime. Since we could not settle exactly the properties of our space and time, we cannot do that for spacetime either.

However, this does not matter. We are faced a kind of fundamental problem: on the basis of some experience we have made some abstractions to create mathematical models. Such a model is not the reality itself; it is an image — a necessarily simplified and distorted image — of reality. *Reality and model should not be confused!*

Accepting our experience regarding human size as *global*, i.e. extrapolating it to very small and large size, too, we make models in which *spacetime is a four-dimensional affine space.*

The non-relativistic model and the special relativistic model are of this kind. The difference between the two models is — regarding their physical content and not their mathematical form — that time or light spread are taken to be absolute, respectively.

If we admit that our experience is only *local,* i.e. considering it approximately true even in human size, we give up the affine structure and we make models in which *spacetime is a four-dimensional manifold.* These are the general relativistic spacetime models.

What kind of spacetime models shall we develop? This depends on our intention for what purpose we wish to employ it.

The non-relativistic spacetime model is suitable for the description of "sluggish" mechanical phenomena — when bodies move relative to each other with velocities much smaller than light speed — and of static electromagnetic phenomena.

The special relativistic spacetime model is suitable for the description of all mechanical and electromagnetic phenomena, but it has a more complicated structure than the non-relativistic one, therefore it is suitable for "brisk" mechanical phenomena and non-static electromagnetic phenomena.

To describe cosmic phenomena we have to adopt general relativistic spacetime models.

3.11. At last, let us speak about how we imagine the points of spacetime. We have our notions of space points and time points (instants). Roughly speaking, a point of spacetime is the fusion of a space point and a time point: a spacetime point can be conceived as "here and now" or "there and then". We can say more. A lamp flashes, two billiard balls collide: "there and then" is incarnated. Thus spacetime points can be illustrated by such *occurrences.*

We call attention to the fact that one often says event instead of occurrence which causes a number of misunderstandings. Namely, the notion of an event is well defined in probability theory and in physics as well. An event always happens to the object in question: the flash is an event of the lamp, the collision is an event of the balls, they are not events of spacetime. These events are illustrations only and are not equal to a spacetime point.

That is why we prefer to say occurrence when relating these events of some material objects to spacetime points.

I. NON-RELATIVISTIC SPACETIME MODEL

1. Fundamentals

1.1. Definition of the spacetime model

1.1.1. According to what has been said in the Introduction, now we model spacetime by a four-dimensional oriented affine space, denoted by M; let $\mathbf{M}$ be the corresponding vector space.

The affine structure does not reflect completely our fundamental knowledge of spacetime.

Let us accept that absolute time exists. This will be the most important feature of a non-relativistic spacetime model. Time is modelled by a one-dimensional oriented affine space, denoted by I; let $\mathbf{I}$ denote the underlying vector space.

Spacetime and time are not independent; the phrase "time is absolute" means that the time point corresponding to each spacetime point is determined uniquely; in other words, there is a mapping $\tau : \mathrm{M} \to \mathrm{I}$. The affine structures of spacetime and time are evidently related somehow. We express this relation by supposing that τ is an affine map (over the linear map $\boldsymbol{\tau} : \mathbf{M} \to \mathbf{I}$).

Now we have to put the Euclidean structure of "space" into the model (quotation marks are used because we well know that absolute space does not exist, we do not want to put space into the model). To go on the right way, let us observe that the Euclidean structure of our space is established on the basis of simultaneity: the vector between the corner of the room and the spot on the carpet is not defined by the corner yesterday and the spot today. Without introducing space, we can introduce the Euclidean structure with the aid of simultaneity as follows.

Let t be an instant, i.e. an element of I. Then

$$\overset{-1}{\tau}(\{t\}) = \{x \in \mathrm{M} |\ \tau(x) = t\}$$

is the set of simultaneous spacetime points to which the same instant t corresponds. It is a three-dimensional affine subspace of M over the vector space (see VI.2.2)

$$\mathbf{E} := \mathrm{Ker}\, \boldsymbol{\tau} = \{\boldsymbol{x} \in \mathbf{M}\} |\ \boldsymbol{\tau} \cdot \boldsymbol{x} = \mathbf{0}\}.$$

We accept that there is a Euclidean structure on $\mathbf{E}$: we introduce the measure line of distances, $\mathbf{D}$, and a positive definite symmetric bilinear map $\mathbf{b} : \mathbf{E} \times \mathbf{E} \to \mathbf{D} \otimes \mathbf{D}$; so $(\mathbf{E}, \mathbf{D}, \mathbf{b})$ is a three-dimensional Euclidean vector space.

1.1.2. Now we are ready to formulate a correct definition.

Definition. A *non-relativistic spacetime model* is a quintuplet $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$ where

— M is an oriented four-dimensional real affine space (over the vector space $\mathbf{M}$),

— I is an oriented one-dimensional real affine space (over the vector space $\mathbf{I}$),

— $\tau : \mathrm{M} \to \mathrm{I}$ is an affine surjection (over the linear surjection $\boldsymbol{\tau} : \mathbf{M} \to \mathbf{I}$),

— $\mathbf{D}$ is an oriented one-dimensional real vector space,

— $\mathbf{b} : \mathbf{E} \times \mathbf{E} \to \mathbf{D} \otimes \mathbf{D}$ is a positive definite symmetric bilinear map where $\mathbf{E} := \mathrm{Ker}\, \boldsymbol{\tau}$. ■

We shall use the following names:

M is *spacetime* or *world*,

I is *time,* $\mathbf{I}$ is *the measure line of time periods,*

τ is *the time evaluation,*

$\mathbf{D}$ is *the measure line of distances,*

$\mathbf{b}$ is *the Euclidean structure on simultaneity.*

Elements of M and I are called *world points* and *instants,* respectively. The world points x and y are *simultaneous* if $\tau(x) = \tau(y)$.

1.1.3. A *world vector,* i.e. an element $\boldsymbol{x}$ of $\mathbf{M}$ is called *spacelike* or *timelike* if $\boldsymbol{x} \in \mathbf{E}$ or $\boldsymbol{x} \notin \mathbf{E}$, respectively. Evidently, $\boldsymbol{x}$ is spacelike if and only if $\boldsymbol{\tau} \cdot \boldsymbol{x} = \mathbf{0}$. The set of timelike elements consists of two disjoint open subsets:

$$\mathrm{T}^{\to} := \{\boldsymbol{x} \in \mathbf{M} |\ \boldsymbol{\tau} \cdot \boldsymbol{x} > \mathbf{0}\}, \qquad \mathrm{T}^{\leftarrow} := \{\boldsymbol{x} \in \mathbf{M} |\ \boldsymbol{\tau} \cdot \boldsymbol{x} < \mathbf{0}\}.$$

(Recall that $\mathbf{I}$ is oriented, thus it makes sense to speak about its positive and negative elements, see IV.5.3.) The vectors in $\mathrm{T}^{\to}$ and in $\mathrm{T}^{\leftarrow}$ are called *future-directed* and *past-directed,* respectively.

We often illustrate the world vectors in the plane of the page:

1.1.4. If $t \in \mathrm{I}$ then $\overset{-1}{\tau}(\{t\})$ is an affine hyperplane in M, directed by $\mathbf{E}$. The correspondence $t \mapsto \overset{-1}{\tau}(\{t\})$ is a bijection between I and the affine hyperplanes directed by $\mathbf{E}$. We make use of this correspondence *to identify the two sets,* i.e. *to regard instants as affine hyperplanes directed by* $\mathbf{E}$. In this way an instant equals the set of corresponding simultaneous world points. This trick makes thinking simpler and creates the possibility of comparison with relativistic models where time can be defined only by hypersurfaces.

Since I is oriented and one-dimensional, a total ordering is given on it: we say that $t \in \mathrm{I}$ is *later* than $s \in \mathrm{I}$ (or s is *earlier* than t) and we write $s < t$ if $t - s$ is a positive element of **I**.

Spacetime, too, will be illustrated in the plane of the page. Then vertical lines stand for the instants (hyperplanes of simultaneous world points). A line standing to the right of another is taken to be later.

If x is a world point, $x + \mathrm{T}^{\rightarrow}$ and $x + \mathrm{T}^{\leftarrow}$ are called the *future-like* and *past-like part* of M, with respect to x.

1.2. Structure of world vectors and covectors

1.2.1. There are spacetime and time in our non-relativistic spacetime model and there is no space. However, there is something spacelike: the linear subspace $\mathbf{E}$ of $\mathbf{M}$. Later we see what the spacelike feature of $\mathbf{E}$ consists in. We find an important "complementary" connection between $\mathbf{E}$ and $\mathbf{I}$. Let

$$\mathbf{i} : \mathbf{E} \to \mathbf{M}$$

denote the canonical injection (embedding; if $\boldsymbol{q} \in \mathbf{E}$, then $\mathbf{i} \cdot \boldsymbol{q}$ equals $\boldsymbol{q}$ regarded as an element of $\mathbf{M}$; evidently, $\mathbf{i}$ is linear). Then we can draw the diagram

$$\mathbf{E} \xrightarrow{\mathbf{i}} \mathbf{M} \xrightarrow{\boldsymbol{\tau}} \mathbf{I} \ ;$$

$\mathbf{i}$ is injective, τ is surjective, and Ran $\mathbf{i}$ = Ker $\boldsymbol{\tau}$, thus $\boldsymbol{\tau} \cdot \mathbf{i} = \mathbf{0}$.

$\mathbf{M}^*$, the dual of $\mathbf{M}$ will play an important role. Though it is also a four-dimensional oriented vector space, there is no canonical isomorphism between $\mathbf{M}$ and $\mathbf{M}^*$; these vector spaces are different.

A diagram similar to the previous one is drawn for the transposed maps:

$$\mathbf{I}^* \xrightarrow{\boldsymbol{\tau}^*} \mathbf{M}^* \xrightarrow{\mathbf{i}^*} \mathbf{E}^* \ .$$

$\boldsymbol{\tau}^*$ is injective, $\mathbf{i}^*$ is surjective (see IV.1.4) and Ran $\boldsymbol{\tau}^*$ = Ker $\mathbf{i}^*$, thus $\mathbf{i}^* \cdot \boldsymbol{\tau}^* = \mathbf{0}$.

It is worth mentioning that for $\boldsymbol{k} \in \mathbf{M}^*$, $\mathbf{i}^* \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \mathbf{i}$ is the restriction of $\boldsymbol{k}$ onto $\mathbf{E}$.

1.2.2. Since $\boldsymbol{\tau}^*$ is injective, its range is a one-dimensional linear subspace of $\mathbf{M}^*$ which will play an important role:

$$\text{Ran}\ \boldsymbol{\tau}^* = \{\boldsymbol{\tau}^* \cdot \boldsymbol{e} |\ \boldsymbol{e} \in \mathbf{I}^*\} = \{\boldsymbol{e} \cdot \boldsymbol{\tau} |\ \boldsymbol{e} \in \mathbf{I}^*\} = \mathbf{I}^* \cdot \boldsymbol{\tau}.$$

Observe that $\boldsymbol{k} \in \mathbf{M}^*$ is in $\mathbf{I}^* \cdot \boldsymbol{\tau}$ if and only if $\mathbf{i}^* \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \mathbf{i} = \mathbf{0}$, thus

$$\mathbf{I}^* \cdot \boldsymbol{\tau} = \{\boldsymbol{k} \in \mathbf{M}^* |\ \boldsymbol{k} \cdot \boldsymbol{q} = \mathbf{0} \quad \text{for all} \quad \boldsymbol{q} \in \mathbf{E}\}.$$

If the dot denoted an inner product on some vector space then this would mean that $\mathbf{I}^* \cdot \boldsymbol{\tau}$ is orthogonal to $\mathbf{E}$; please, note, *now* $\mathbf{I}^* \cdot \boldsymbol{\tau}$ *and* $\mathbf{E}$ *are in different vector spaces,* they cannot be orthogonal to each other. We say that $\mathbf{I}^* \cdot \boldsymbol{\tau}$ is *the annullator* of $\mathbf{E}$.

Illustrating $\mathbf{M}^*$ on the plane of the page, we draw a horizontal line for the one-dimensional linear subspace $\mathbf{I}^* \cdot \boldsymbol{\tau}$.

As usual, the elements of $\mathbf{M}^*$ are called *covectors.* The covectors in the linear subspace $\mathbf{I}^* \cdot \boldsymbol{\tau}$ are *timelike,* and the other ones are *spacelike.*

1.2.3. It will be often convenient to use tensorial forms of the above linear maps. According to IV.3.4 and IV.1.2 we have

$$\begin{aligned} \boldsymbol{\tau} &\in \mathbf{I} \otimes \mathbf{M}^* , \qquad & \mathbf{i} &\in \mathbf{M} \otimes \mathbf{E}^* , \\ \boldsymbol{\tau}^* &\in \mathbf{M}^* \otimes \mathbf{I} , \qquad & \mathbf{i}^* &\in \mathbf{E}^* \otimes \mathbf{M} . \end{aligned}$$

1.2.4. With the aid of $\boldsymbol{\tau}$, the orientations of $\mathbf{M}$ and of $\mathbf{I}$ determine a unique orientation of $\mathbf{E}$.

Proposition. If $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is an ordered basis of $\mathbf{E}$, then $(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ and $(\boldsymbol{y}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ are equally oriented for all $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{T}^{\rightarrow}$.

Proof. Evidently, $(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ and $(\alpha\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ are equally oriented if $\alpha \in \mathbb{R}^+$, hence we can suppose that $\boldsymbol{\tau} \cdot \boldsymbol{y} = \boldsymbol{\tau} \cdot \boldsymbol{x}$, i.e. $\boldsymbol{q} := \boldsymbol{y} - \boldsymbol{x} \in \mathbf{E}$. Then

$$\boldsymbol{y} \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 = (\boldsymbol{x} + \boldsymbol{q}) \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 = \boldsymbol{x} \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 ,$$

hence the statement is true by IV.5.1.

Definition. An ordered basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ of $\mathbf{E}$ is called *positively oriented* if $(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a positively oriented ordered basis of $\mathbf{M}$ for some (hence for all) $\boldsymbol{x} \in \mathrm{T}^{\rightarrow}$. ■

1.2.5. $(\mathbf{E}, \mathbf{D}, \mathbf{b})$ is a three-dimensional Euclidean vector space, $\mathbf{E}$ and $\mathbf{D}$ are oriented. An important relation is the identification

$$\frac{\mathbf{E}}{\mathbf{D} \otimes \mathbf{D}} \equiv \mathbf{E}^* .$$

We shall use the notation

$$\mathbf{N} := \frac{\mathbf{E}}{\mathbf{D}}$$

and all the results of section V.3.

In particular, we use a dot product notation instead of $\mathbf{b}$:

$$\boldsymbol{q} \cdot \boldsymbol{q}' := \mathbf{b}(\boldsymbol{q}, \boldsymbol{q}') \in \mathbf{D} \otimes \mathbf{D} \qquad\qquad (\boldsymbol{q}, \boldsymbol{q}' \in \mathbf{E}).$$

The *length* of $\boldsymbol{q} \in \mathbf{E}$ is

$$|\boldsymbol{q}| := \sqrt{\boldsymbol{q} \cdot \boldsymbol{q}} \in \mathbf{D}_0^+ ,$$

and the *angle between* the non-zero elements $\boldsymbol{q}$ and $\boldsymbol{q}'$ of $\mathbf{E}$ is

$$\arg(\boldsymbol{q}, \boldsymbol{q}') := \arccos \frac{\boldsymbol{q} \cdot \boldsymbol{q}'}{|\boldsymbol{q}||\boldsymbol{q}'|}.$$

The dot product can be defined between spacelike vectors of different types (see later, Section 1.4) as well; e.g. if $\mathbf{A}$ and $\mathbf{B}$ are measure lines, for $\boldsymbol{w} \in \frac{\mathbf{E}}{\mathbf{A}}$ and $\boldsymbol{z} \in \frac{\mathbf{E}}{\mathbf{B}}$ we have

$$\boldsymbol{w} \cdot \boldsymbol{z} \in \frac{\mathbf{D} \otimes \mathbf{D}}{\mathbf{A} \otimes \mathbf{B}}, \quad \arg(\boldsymbol{w}, \boldsymbol{z}) := \arccos \frac{\boldsymbol{w} \cdot \boldsymbol{z}}{|\boldsymbol{w}||\boldsymbol{z}|},$$

$$|\boldsymbol{w}| := \sqrt{\boldsymbol{w} \cdot \boldsymbol{w}} \in \frac{\mathbf{D}}{\mathbf{A}}.$$

1.2.6. Do not forget that timelike vectors (elements of $\mathbf{M}$ outside $\mathbf{E}$) have no length, no angles between them.

$\mathbf{I}^* \cdot \boldsymbol{\tau}$ is an oriented one-dimensional vector space, hence the absolute value of its elements makes sense; thus a length (absolute value) can be assigned to a timelike covector. However, the length of spacelike covectors (elements of $\mathbf{M}^*$ outside $\mathbf{I}^* \cdot \boldsymbol{\tau}$) and the angle between two covectors are not meaningful.

1.2.7. The Euclidean structure of our space is deeply fixed in our mind, therefore we must be careful when dealing with $\mathbf{M}$ which has not a Euclidean structure; especially when illustrating it in the Euclidean plane of the page. Keep in mind that vectors out of $\mathbf{E}$ have no length, do not form angles. The following considerations help us to take in the situation.

Recall that the linear map $\boldsymbol{\tau} : \mathbf{M} \to \mathbf{I}$ can be applied to element of $\frac{\mathbf{M}}{\mathbf{I}}$ and then has values in $\frac{\mathbf{I}}{\mathbf{I}} \equiv \mathbb{R}$ (see V.2.1). Put

$$\mathrm{V}(1) := \left\{ \boldsymbol{u} \in \frac{\mathbf{M}}{\mathbf{I}} \mid \boldsymbol{\tau} \cdot \boldsymbol{u} = 1 \right\}.$$

According to VI.2.2, V(1) is an affine subspace of $\frac{\mathbf{M}}{\mathbf{I}}$ over $\frac{\mathbf{E}}{\mathbf{I}}$. It is illustrated as follows:

Three elements of V(1) appear in the figure. Observe that it makes no sense that

— $\boldsymbol{u}_1$ is orthogonal to $\frac{\mathbf{E}}{\mathbf{I}}$ (there are no vectors orthogonal to $\frac{\mathbf{E}}{\mathbf{I}}$),

— the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is less than the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_3$ (there is no angle between the elements of V(1),)

— $\boldsymbol{u}_2$ is longer than $\boldsymbol{u}_1$ (the elements of V(1) have no length).

We shall see in 2.1.2 that the elements of V(1) can be interpreted as *velocity values*.

1.2.8. Since there is no vector orthogonal to $\mathbf{E}$, the orthogonal projection of vectors onto $\mathbf{E}$ makes no sense. Of course, we can project onto $\mathbf{E}$ in many equivalent ways; the following projections will play an important role.

Let $\boldsymbol{u}$ be an element of V(1). Then $\boldsymbol{u} \otimes \mathbf{I} := \{\boldsymbol{u}\boldsymbol{t} \mid \boldsymbol{t} \in \mathbf{I}\}$ is a one- dimensional linear subspace of $\mathbf{M}$; $\boldsymbol{u} \otimes \mathbf{I}$ and $\mathbf{E}$ are complementary subspaces, thus every vector $\boldsymbol{x}$ can be uniquely decomposed into the sum of components in $\boldsymbol{u} \otimes \mathbf{I}$ and in $\mathbf{E}$, respectively:

$$\boldsymbol{x} = \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{x}) + (\boldsymbol{x} - \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{x}))\,.$$

The linear map

$$\boldsymbol{\pi}_{\boldsymbol{u}} : \mathbf{M} \to \mathbf{E}\,, \quad \boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{x})$$

is the *projection onto* $\mathbf{E}$ *along* $\boldsymbol{u}$. It is illustrated as follows:

V(1) is represented by a dashed line expressing that V(1) is in fact a subset of $\frac{\mathbf{M}}{\mathbf{I}}$.

Observe that

$$\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \mathbf{i} = \mathrm{id}_{\mathbf{E}}$$

and in a tensorial form $\boldsymbol{\pi_u} \in \mathbf{E} \otimes \mathbf{M}^*$.

1.2.9. Proposition. Let $\boldsymbol{u} \in \mathrm{V}(1)$. Then

$$\boldsymbol{h_u} := (\boldsymbol{\tau}, \boldsymbol{\pi_u}) : \mathbf{M} \to \mathbf{I} \times \mathbf{E} \qquad \boldsymbol{x} \mapsto (\boldsymbol{\tau} \cdot \boldsymbol{x},\ \boldsymbol{\pi_u} \cdot \boldsymbol{x})$$

is an orientation-preserving linear bijection, and

$$\boldsymbol{h_u}^{-1}(\boldsymbol{t}, \boldsymbol{q}) = \boldsymbol{u}\boldsymbol{t} + \boldsymbol{q} \qquad\qquad (\boldsymbol{t} \in \mathbf{I}, \boldsymbol{q} \in \mathbf{E}).$$

1.3. The arithmetic spacetime model

1.3.1. Let us number the coordinates of elements of $\mathbb{R}^4$ from 0 to 3 : $(\xi^0, \xi^1, \xi^2, \xi^3) \in \mathbb{R}^4$. The canonical projection onto the zeroth coordinate,

$$\mathrm{pr}^0 \ : \mathbb{R}^4 \to \mathbb{R}, \quad (\xi^0, \xi^1, \xi^2, \xi^3) \mapsto \xi^0$$

is a linear map whose kernel is $\{0\} \times \mathbb{R}^3$ which we *identify* with $\mathbb{R}^3$. Let B denote the usual inner product on $\mathbb{R}^3$: $\mathrm{B}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{3} x^i y^i$. Endow $\mathbb{R}$ and $\mathbb{R}^4$ with the standard orientation.

It is quite evident that $\left(\mathbb{R}^4, \mathbb{R}, \mathrm{pr}^0, \mathbb{R}, \mathrm{B}\right)$ is a non-relativistic spacetime model which we call the *arithmetic non-relativistic spacetime model.*

In the arithmetic spacetime model we have:

$$\mathrm{M} = \mathbf{M} = \mathbb{R}^4, \qquad \mathrm{I} = \mathbf{I} = \mathbb{R}, \qquad \mathbf{D} = \mathbb{R},$$

$$\tau = \boldsymbol{\tau} = \mathrm{pr}^0,$$

$$\mathbf{E} = \{0\} \times \mathbb{R}^3 \equiv \mathbb{R}^3, \qquad \mathbf{b} = \mathrm{B}.$$

Then

$$\mathbf{i} : \mathbf{E} \to \mathbf{M} \quad \text{equals} \quad \mathbb{R}^3 \to \mathbb{R}^4, \quad (x^1, x^2, x^3) \mapsto (0, x^1, x^2, x^3).$$

The usual identification yields $\mathbf{M}^* = (\mathbb{R}^4)^* \equiv \mathbb{R}^4$; the covectors are indexed in subscripts: $(k_0, k_1, k_2, k_3) \in (\mathbb{R}^4)^*$ (see IV.1.4).

In the same way, $\mathbf{I}^* = \mathbb{R}^* \equiv \mathbb{R}$, but here we cannot make distinction with the aid of indices.

Then

$$\mathbf{i}^* : \mathbf{M}^* \mapsto \mathbf{E}^* \quad \text{equals} \quad (\mathbb{R}^4)^* \to (\mathbb{R}^3)^*, \quad (k_0, k_1, k_2, k_3) \mapsto (k_1, k_2, k_3)$$

and

$$\boldsymbol{\tau}^* : \mathbf{I}^* \to \mathbf{M}^* \quad \text{equals} \quad \mathbb{R}^* \to (\mathbb{R}^4)^*, \quad e \mapsto (e, 0, 0, 0).$$

1.3.2. It is an unpleasant feature of the arithmetic spacetime model that the same object, $\mathbb{R}^4$, represents the affine space of world points and the vector space of world vectors (and even the vector space of covectors). For a clear distinction we shall write Greek letters indicating world points (affine space elements) and Latin letters indicating world vectors or covectors.

Moreover, the notations will be much simpler if we consider $\mathbb{R}^4 \equiv \mathbb{R} \times \mathbb{R}^3$, and we write $(\alpha, \boldsymbol{\xi})$ or $(\xi^0, \boldsymbol{\xi})$ and $(t, \boldsymbol{q})$ for its elements; similarly, $(e, \boldsymbol{p})$ denotes an element of $(\mathbb{R} \times \mathbb{R}^3)^* \equiv \mathbb{R}^* (\mathbb{R}^3)^*$. Then

$$\begin{aligned}
\tau : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, &\qquad (\alpha, \boldsymbol{\xi}) \mapsto \alpha, \\
\boldsymbol{\tau} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, &\qquad (t, \boldsymbol{q}) \mapsto t, \\
\mathbf{i} : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3, &\qquad \boldsymbol{q} \mapsto (0, \boldsymbol{q}), \\
\mathbf{i}^* : (\mathbb{R} \times \mathbb{R}^3)^* \to (\mathbb{R}^3)^*, &\qquad (e, \boldsymbol{p}) \mapsto \boldsymbol{p}, \\
\boldsymbol{\tau}^* : \mathbb{R}^* \to (\mathbb{R} \times \mathbb{R}^3)^*, &\qquad e \mapsto (e, \mathbf{0}).
\end{aligned}$$

The last formula means that $\mathbf{I}^* \cdot \boldsymbol{\tau}$ now equals $\mathbb{R} \times \{\mathbf{0}\}$.

Of course, τ and $\boldsymbol{\tau}$ are equal though we have written the same formula in different symbols. This is a trick similar to that of subscripts and superscripts: we wish to distinguish between different objects that appear in the same form.

1.3.3. Now we have $\frac{\mathbf{M}}{\mathbf{I}} = \frac{\mathbb{R}^4}{\mathbb{R}} \equiv \mathbb{R}^4$, and

$$\mathrm{V}(1) = \{(v^0, \boldsymbol{v}) \in \mathbb{R} \times \mathbb{R}^3 \mid v^0 = 1\} = \{1\} \times \mathbb{R}^3.$$

$\mathrm{V}(1)$ has a simplest element: $(1, \mathbf{0})$ which is called the *basic velocity value.*

For $(1, \boldsymbol{v}) \in \mathrm{V}(1)$ we easily derive that

$$\boldsymbol{\pi}_{(1,\boldsymbol{v})} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \qquad (t, \boldsymbol{q}) \mapsto \boldsymbol{q} - \boldsymbol{v}t.$$

In particular, $\boldsymbol{\pi}_{(1,\mathbf{0})}$ is the canonical projection from $\mathbb{R} \times \mathbb{R}^3$ onto $\mathbb{R}^3$.

1.4. Classification of physical quantities

1.4.1. In physics one usually says e.g. that (relative) velocity and acceleration are three-dimensional vectors and are considered as triplets of real numbers. Although both are taken as elements of $\mathbb{R}^3$, they cannot be added because they have "different physical dimensions". The framework of our spacetime model assures a precise meaning of these notions.

A physical dimension is represented by a measure line. Let $\mathbf{A}$ be a measure line. Then the elements of

$$\begin{array}{rll} \mathbf{A} & \text{are called} & \textit{scalars of type} \quad \mathbf{A}, \\ \mathbf{A}\otimes\mathbf{M} & \text{are called} & \textit{vectors of type} \quad \mathbf{A}, \\ \dfrac{\mathbf{M}}{\mathbf{A}} & \text{are called} & \textit{vectors of cotype} \quad \mathbf{A}, \\ \mathbf{A}\otimes(\mathbf{M}\otimes\mathbf{M}) & \text{are called} & \textit{tensors of type} \quad \mathbf{A}, \\ \dfrac{\mathbf{M}\otimes\mathbf{M}}{\mathbf{A}} & \text{are called} & \textit{tensors of cotype} \quad \mathbf{A}. \end{array}$$

Covectors of type $\mathbf{A}$, etc. are defined similarly with $\mathbf{M}^*$ instead of $\mathbf{M}$.

In the case $\mathbf{A}=\mathbb{R}$ we omit the term "of type $\mathbb{R}$". In particular, the elements of $\mathbf{M}\otimes\mathbf{M}$ and $\mathbf{M}^*\otimes\mathbf{M}^*$ are called *tensors* and *cotensors,* respectively; the elements of $\mathbf{M}^*\otimes\mathbf{M}$ and $\mathbf{M}\otimes\mathbf{M}^*$ are *mixed tensors.*

Recall the identifications $\mathbf{A}\otimes\mathbf{M}\equiv\mathbf{M}\otimes\mathbf{A}$ etc. (see IV.3.6).

Because of the identification $\frac{\mathbf{M}}{\mathbf{A}}\equiv\mathbf{M}\otimes\mathbf{A}^*$ the vectors of cotype $\mathbf{A}$ coincide with the vectors of type $\mathbf{A}^*$.

1.4.2. The vectors and tensors of type $\mathbf{A}$ in the subspaces $\mathbf{A}\otimes\mathbf{E}$ and $\mathbf{A}\otimes(\mathbf{E}\otimes\mathbf{E})$, respectively, are called *spacelike.*

The covectors of type $\mathbf{A}$ in the subspace $\mathbf{A}\otimes(\mathbf{I}^*\cdot\boldsymbol{\tau})$ are called *timelike.*

According to our convention (V.2.1 and V.2.2), the dot product of covectors and vectors of different types makes sense; e.g.

$$\text{for} \quad \boldsymbol{k}\in\mathbf{B}\otimes\mathbf{M}^* \quad \text{and} \quad \boldsymbol{z}\in\mathbf{A}\otimes\mathbf{M}\equiv\mathbf{M}\otimes\mathbf{A} \quad \text{we have} \quad \boldsymbol{k}\cdot\boldsymbol{z}\in\mathbf{B}\otimes\mathbf{A}.$$

In particular,

$$\text{for} \quad \boldsymbol{\tau}\in\mathbf{I}\otimes\mathbf{M}^* \quad \text{and} \quad \boldsymbol{z}\in\mathbf{A}\otimes\mathbf{M} \quad \text{we have} \quad \boldsymbol{\tau}\cdot\boldsymbol{z}\in\mathbf{I}\otimes\mathbf{A};$$

similarly,

$$\begin{array}{rl} \text{for} \quad \boldsymbol{w}\in\dfrac{\mathbf{M}}{\mathbf{A}} & \text{we have} \quad \boldsymbol{\tau}\cdot\boldsymbol{w}\in\dfrac{\mathbf{I}}{\mathbf{A}}; \\ \text{for} \quad \boldsymbol{T}\in\mathbf{A}\otimes(\mathbf{M}\otimes\mathbf{M}) & \text{we have} \quad \boldsymbol{\tau}\cdot\boldsymbol{T}\in\mathbf{I}\otimes\mathbf{A}\otimes\mathbf{M}. \end{array}$$

Evidently, $\boldsymbol{z}\in\mathbf{A}\otimes\mathbf{M}$ is spacelike if and only if $\boldsymbol{\tau}\cdot\boldsymbol{z}=\mathbf{0}$.

In the same way, $\mathbf{i}^*:\mathbf{M}^*\to\mathbf{E}^*$ is lifted to covectors of type $\mathbf{A}$, etc. i.e.

$$\text{for} \quad \mathbf{i}^*\in\mathbf{E}^*\otimes\mathbf{E} \quad \text{and} \quad \boldsymbol{h}\in\mathbf{A}\otimes\mathbf{M}^* \quad \text{we have} \quad \mathbf{i}^*\cdot\boldsymbol{h}\in\mathbf{A}\otimes\mathbf{E}^* \quad \text{etc.}$$

Evidently, $\boldsymbol{h}\in\mathbf{A}\otimes\mathbf{M}^*$ is timelike if and only if $\mathbf{i}^*\cdot\boldsymbol{h}=\mathbf{0}$.

1.4.3. In non-relativistic physics one usually introduces the notion of scalars, three-dimensional vectors, three-dimensional pseudo-vectors and pseudo-scalars as quantities having some prescribed transformation properties. One is forced to adapt such a definition because only coordinates are considered, only numbers and triplets of numbers are used, and one must know whether a triplet of numbers is the set of coordinates of a vector, or not. Of course, vectors can have different "physical dimensions".

Now we formulate the corresponding notion in the framework of our non-relativistic spacetime model. The elements of

$$\begin{array}{rl} \mathbb{R} & \text{are the scalars,} \\ \mathbf{E} & \text{are the spacelike vectors,} \\ \mathbf{E}\wedge\mathbf{E} & \text{are the spacelike pseudo-vectors of type} \quad \mathbf{D}, \\ \mathbf{E}\wedge\mathbf{E}\wedge\mathbf{E} & \text{are the pseudo-scalars of type} \quad \overset{3}{\otimes}\mathbf{D}. \end{array}$$

The first and the second names do not require explanation. The third and fourth names are based on the fact that we have canonical linear bijections $\mathbf{E}\wedge\mathbf{E}\to\mathbf{E}\otimes\mathbf{D}$ and $\mathbf{E}\wedge\mathbf{E}\wedge\mathbf{E}\to\mathbf{D}\otimes\mathbf{D}\otimes\mathbf{D}$ (see V.3.17); the pseudo-vectors are "similar" to spacelike vectors of type $\mathbf{D}$, and the pseudo-scalars are "similar" to scalars of type $\overset{3}{\otimes}\mathbf{D}$.

Having the notion of vectors of type $\mathbf{A}$, it is evident, how we shall define spacelike pseudo-vectors and pseudo-scalars of diverse types. For the sake of simplicity, we consider now "physically dimensionless" quantities: $\mathbb{R}$, $\mathbf{N}$, $\mathbf{N}\wedge\mathbf{N}$, $\mathbf{N}\wedge\mathbf{N}\wedge\mathbf{N}$. Then we have the linear bijections $\mathbf{j}:\mathbf{N}\wedge\mathbf{N}\to\mathbf{N}$ and $\mathrm{j}_o:\mathbf{N}\wedge\mathbf{N}\wedge\mathbf{N}\to\mathbb{R}$.

Let $\boldsymbol{R}:\mathbf{E}\to\mathbf{E}$ be an orthogonal map which is considered to be an orthogonal map $\mathbf{N}\to\mathbf{N}$ as well. We say that $\boldsymbol{R}$ is a rotation if it has positive determinant. The determinant of the inversion $\boldsymbol{S}:=-\mathrm{id}_{\mathbf{E}}$ is negative.

By definition, $\overset{0}{\otimes}\boldsymbol{R}:=\overset{0}{\otimes}\boldsymbol{S}:=\mathrm{id}_{\mathbb{R}}$; the scalars are not transformed.

Vectors are transformed under $\boldsymbol{R}$ and $\boldsymbol{S}$ according to the definition of these operations.

Pseudo-vectors are transformed by $\boldsymbol{R}\wedge\boldsymbol{R}$ and $\boldsymbol{S}\wedge\boldsymbol{S}$ (IV.3.2.1); formulae in V.3.16 say that

$$\mathbf{j}\circ(\boldsymbol{R}\wedge\boldsymbol{R})=\boldsymbol{R}\circ\mathbf{j},\qquad \mathbf{j}\circ(\boldsymbol{S}\wedge\boldsymbol{S})=-\boldsymbol{S}\circ\mathbf{j}=\mathbf{j}$$

which means that the pseudo-vectors are transformed by rotations like vectors but they are not transformed by the inversion.

Similarly we have that

$$\mathrm{j}_o\circ(\boldsymbol{R}\wedge\boldsymbol{R}\wedge\boldsymbol{R})=\mathrm{j}_o,\qquad \mathrm{j}_o\circ(\boldsymbol{S}\wedge\boldsymbol{S}\wedge\boldsymbol{S})=-\mathrm{j}_o,$$

the pseudo-scalars are not transformed by rotations and they change sign by the inversion.

1.5. Comparison of spacetime models

1.5.1. The spacetime model is defined as a mathematical structure. It is an interesting question both from mathematical and from physical points of view: how many "different" non-relativistic spacetime models exist?

To answer, first we must define what the "difference" and the "similarity" between two spacetime models mean. We proceed as it is usual in mathematics; for instance, one defines the linear structure (vector space) and then the linear maps as the tool of comparison between linear structures; two vector spaces are of the same kind if there is a linear bijection between them, in other words, if they are isomorphic.

Definition. The non-relativistic spacetime model $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$ is *isomorphic* to the non-relativistic spacetime model $(\mathrm{M}', \mathrm{I}', \tau', \mathbf{D}', \mathbf{b}')$ if there are

(i) an orientation-preserving affine bijection $F : \mathrm{M} \to \mathrm{M}'$,

(ii) an orientation-preserving affine bijection $B : \mathrm{I} \to \mathrm{I}'$,

(iii) an orientation-preserving linear bijection $\boldsymbol{Z} : \mathbf{D} \to \mathbf{D}'$ such that

(I) $\tau' \circ F = B \circ \tau$,

(II) $\mathbf{b}' \circ (\boldsymbol{F} \times \boldsymbol{F}) = (\boldsymbol{Z} \otimes \boldsymbol{Z}) \circ \mathbf{b}$.

The triplet $(F, B, \boldsymbol{Z})$ is an *isomorphism* between the two spacetime models.

If the two models coincide, isomorphism is called *automorphism.* An automorphism $(F, B, \boldsymbol{Z})$ of $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$ is *strict* if $\boldsymbol{B} = \mathrm{id}_{\mathbf{I}}$ and $\boldsymbol{Z} = \mathrm{id}_{\mathbf{D}}$. ■

In the definition and later on, $\boldsymbol{B}$ and $\boldsymbol{F}$ are the linear maps under B and F, respectively.

Two commutative diagrams illustrate the isomorphism:

$$\begin{array}{ccc} \mathrm{M} & \xrightarrow{\tau} & \mathrm{I} \\ F\big\downarrow & & \big\downarrow B \\ \mathrm{M}' & \xrightarrow[\tau']{} & \mathrm{I}' \end{array} \qquad \begin{array}{ccc} \mathbf{E} \times \mathbf{E} & \xrightarrow{\mathbf{b}} & \mathbf{D} \otimes \mathbf{D} \\ \boldsymbol{F} \times \boldsymbol{F}\big\downarrow & & \big\downarrow \boldsymbol{Z} \otimes \boldsymbol{Z}\,. \\ \mathbf{E}' \times \mathbf{E}' & \xrightarrow[\mathbf{b}']{} & \mathbf{D}' \otimes \mathbf{D}' \end{array}$$

The definition is quite natural and simple. It is worth mentioning that (I) implies

$$\boldsymbol{\tau}' \circ \boldsymbol{F} = \boldsymbol{B} \circ \boldsymbol{\tau},$$

thus for $\boldsymbol{q} \in \mathbf{E}$ we have $\boldsymbol{\tau}' \cdot \boldsymbol{F} \cdot \boldsymbol{q} = \boldsymbol{B} \cdot \boldsymbol{\tau} \cdot \boldsymbol{q} = \mathbf{0}$ which means that $\boldsymbol{F}$ maps $\mathbf{E}$ into (and even onto) $\mathbf{E}'$; hence the requirement in (II) is meaningful.

It is evident that $(F^{-1}, B^{-1}, \boldsymbol{Z}^{-1})$, the inverse of $(F, B, \boldsymbol{Z})$, is an isomorphism as well. Moreover, if $(F', B', \boldsymbol{Z}')$ is an isomorphism between $(\mathrm{M}', \mathrm{I}', \tau' \mathbf{D}', \mathbf{b}')$ and $(\mathrm{M}'', \mathrm{I}'', \tau'', \mathbf{D}'', \mathbf{b}'')$, then $(F' \circ F, B' \circ B, \boldsymbol{Z}' \circ \boldsymbol{Z})$ is an isomorphism, too.

1.5.2. Proposition. The non-relativistic spacetime model $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$ is isomorphic to the arithmetic spacetime model.

Proof. Take

(i) a positive element $\boldsymbol{s}$ of $\mathbf{I}$,
(ii) a positive element $\boldsymbol{m}$ of $\mathbf{D}$,
(iii) an element $\boldsymbol{e}_0$ of $T^{\rightarrow}$ such that $\boldsymbol{\tau} \cdot \boldsymbol{e}_0 = \boldsymbol{s}$,
(iv) a positively oriented orthogonal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, normed to $\boldsymbol{m}$, of $\mathbf{E}$,
(v) an element o of M.

Then $\boldsymbol{u} := \frac{\boldsymbol{e}_0}{\boldsymbol{s}}$ is in V(1) and it is not hard to see that

$$
\begin{aligned}
F : \mathrm{M} \to \mathbb{R}^4, &\quad x \mapsto \left(\frac{\boldsymbol{\tau} \cdot (x - o)}{\boldsymbol{s}}, \left(\frac{\boldsymbol{e}_\alpha \cdot \pi_{\boldsymbol{u}} \cdot (x - o)}{\boldsymbol{m}^2} \right)_{\alpha = 1,2,3} \right), \\
B : \mathrm{I} \to \mathbb{R}, &\quad t \mapsto \frac{t - \tau(o)}{\boldsymbol{s}}, \\
\boldsymbol{Z} : \mathbf{D} \to \mathbb{R}, &\quad \boldsymbol{d} \mapsto \frac{\boldsymbol{d}}{\boldsymbol{m}}
\end{aligned}
$$

is an isomorphism. ■

Observe that $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a positively oriented basis in $\mathbf{M}$, and F is the affine coordinatization of M corresponding to o and that basis.

The isomorphism above has the inverse

$$
\begin{aligned}
\mathbb{R}^4 \to \mathrm{M}, &\quad (\xi^0, \xi^1, \xi^2, \xi^3) \mapsto o + \sum_{i=o}^{3} \xi^i \boldsymbol{e}_i, \\
\mathbb{R} \to \mathrm{I}, &\quad \alpha \mapsto \alpha \boldsymbol{s}, \\
\mathbb{R} \to \mathbf{D}, &\quad \delta \mapsto \delta \boldsymbol{m}.
\end{aligned}
$$

1.5.3. An important consequence of the previous result is that *two arbitrary non-relativistic spacetime models are isomorphic,* i.e. are of the same kind. The non-relativistic spacetime model as a mathematical structure is unique. This means that there is a unique "non-relativistic physics".

Please, note: the non-relativistic spacetime models are of the same kind, but, in general, are not identical. They are isomorphic, but, in general, there is no "canonical" isomorphism between them, we cannot identify them by a distinguished isomorphism. It is a situation similar to that well known in the theory of vector spaces: all N-dimensional vector spaces are isomorphic to $\mathbb{K}^N$ but, in general, there is no canonical isomorphism between them.

Since all non-relativistic spacetime models are isomorphic, we can use an arbitrary one for investigation and application. However, an actual model can have additional structures. For instance, in the arithmetic model, spacetime and time are vector spaces, time is canonically embedded into spacetime as $\mathbb{R} \times \{\mathbf{0}\}$, $\mathrm{V}(1)$ has a distinguished element, $(1, \mathbf{0})$. This model tempts us to multiply world points by real numbers (although this has no physical meaning and that is why it is not meaningful in the abstract spacetime structure), to consider spacetime to be the Cartesian product of time and space (but space does not exist!), to say that the distinguished element of $\mathrm{V}(1)$ is orthogonal to the space (such an orthogonality makes no sense in the abstract spacetime structure), etc.

To avoid such confusions, we should keep away from similar specially constructed models for investigation and general application of the non-relativistic spacetime model. However, for solving special problems, for executing some particular calculations, we can choose a convenient actual model. In the same way as in the theory of vector spaces where a coordinatization — i.e. the use of $\mathbb{K}^N$ — may help us to perform our task.

1.5.4. In present day physics one uses tacitly the arithmetic spacetime model. One represents time points by real numbers, space points by triplets of real numbers. To arrive at such representations, one chooses a unit of measurement for time and an initial time point, a unit of measurement for distance and an initial space point (origin) and an orthogonal spatial basis whose elements have unit length.

However, all the previous notions have merely a heuristic sense. Take a glance at the isomorphism established in 1.5.2 to recognize that the non-relativistic spacetime model will give these notions a mathematically precise meaning. Evidently, $\boldsymbol{s}$ and $\boldsymbol{m}$ are the units of time period and distance, respectively, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the orthogonal spatial basis whose elements have unit length; $\tau(o)$ is the initial time point and o includes somehow the origin of space as well. At present only the sense of $\boldsymbol{e}_0$ is not clear; later we shall see that it determines the space in question, because we know that absolute space does not exist; $\boldsymbol{e}_0$ characterizes an observer which realizes a space.

1.6. The split spacetime model

1.6.1. As we have said, the arithmetic spacetime model is useful for solving particular problems, for executing practical calculations. Moreover, at present, one usually expounds theories, too, in the frame of the arithmetic spacetime model, so we ought to "translate" every notion in the arithmetic language. However, the arithmetic spacetime model is a little ponderous; that is why we introduce an "intermediate" spacetime model between the abstract and the arithmetic ones, a more terse model which has all the essential features of the arithmetic spacetime model.

1.6.2. Let $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$ be a non-relativistic spacetime model, and use the notations introduced in this chapter. Let $\mathrm{pr}_{\mathbf{I}} : \mathbf{I} \times \mathbf{E} \to \mathbf{I}$ be the canonical projection $(\boldsymbol{t}, \boldsymbol{q}) \mapsto \boldsymbol{t}$.

Then $(\mathbf{I} \times \mathbf{E}, \mathbf{I}, \mathrm{pr}_{\mathbf{I}}, \mathbf{D}, \mathbf{b})$ is a non-relativistic spacetime model, called the *split non-relativistic spacetime model* corresponding to $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$.

It is quite obvious that for all $o \in \mathrm{M}$ and $\boldsymbol{u} \in \mathrm{V}(1)$,

$$\begin{aligned} \mathrm{M} \to \mathbf{I} \times \mathbf{E}, &\qquad x \mapsto \boldsymbol{h_u} \cdot (x - o) \\ \mathrm{I} \to \mathbf{I}, &\qquad t \to t - \tau(o) \\ \mathbf{D} \to \mathbf{D}, &\qquad \boldsymbol{d} \to \boldsymbol{d} \end{aligned}$$

is an isomorphism of the two non-relativistic spacetime models where $\boldsymbol{h_u}$ is defined in 1.2.9.

1.6.3. In the split spacetime model

$$\begin{aligned} \boldsymbol{\tau} : \mathbf{I} \times \mathbf{E} \to \mathbf{I}, &\qquad (\boldsymbol{t}, \boldsymbol{q}) \mapsto \boldsymbol{t}, \\ \mathbf{i} : \mathbf{E} \to \mathbf{I} \times \mathbf{E}, &\qquad \boldsymbol{q} \mapsto (\mathbf{0}, \boldsymbol{q}). \end{aligned}$$

With the usual identification (see IV.1.3) we have that in the split spacetime model the covectors are elements of $\mathbf{I}^* \times \mathbf{E}^*$, correspondingly,

$$\begin{aligned} \boldsymbol{\tau}^* : \mathbf{I}^* \to \mathbf{I}^* \times \mathbf{E}^*, &\qquad \boldsymbol{e} \mapsto (\boldsymbol{e}, \mathbf{0}), \\ \mathbf{i}^* : \mathbf{I}^* \times \mathbf{E}^* \to \mathbf{E}^*, &\qquad (\boldsymbol{e}, \boldsymbol{p}) \mapsto \boldsymbol{p}. \end{aligned}$$

As a consequence, $\mathbf{I}^* \cdot \boldsymbol{\tau} = \mathbf{I}^* \times \{\mathbf{0}\}$.

1.6.4. In the split spacetime model

$$\mathrm{V}(1) = \{1\} \times \frac{\mathbf{E}}{\mathbf{I}},$$

so there is a simplest element (the *basic velocity value*) in it: $(1, \mathbf{0})$.

We easily derive for $(1, \boldsymbol{v}) \in \mathrm{V}(1)$:

$$\boldsymbol{\pi}_{(1,\boldsymbol{v})} : \mathbf{I} \times \mathbf{E} \to \mathbf{E}, \qquad (\boldsymbol{t}, \boldsymbol{q}) \mapsto \boldsymbol{q} - \boldsymbol{v}\boldsymbol{t}.$$

1.7. Exercises

1. Let $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a basis in $\mathbf{M}$ such that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an orthogonal basis in $\mathbf{E}$, normed to $\boldsymbol{m} \in \mathbf{D}^+$. Put $\boldsymbol{s} := \boldsymbol{\tau} \cdot \boldsymbol{e}_0$, $\boldsymbol{u} := \frac{\boldsymbol{e}_0}{\boldsymbol{s}}$. Then $\left\{ \frac{\boldsymbol{\tau}}{\boldsymbol{s}}, \left(\frac{\boldsymbol{\pi}_{\boldsymbol{u}}^* \cdot \boldsymbol{e}_i}{\boldsymbol{m}^2} \right)_{i=1,2,3} \right\}$ is the dual of the basis in question.

2. (i) Let $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be a positively oriented basis in $\mathbf{M}$ such that $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a positively oriented basis in $\mathbf{E}$, normed to $\boldsymbol{m} \in \mathbf{D}^+$. Put $\boldsymbol{s} := \boldsymbol{\tau} \cdot \boldsymbol{e}_0$. Take another "primed" basis with the same properties. Then

$$\varepsilon := \frac{\overset{3}{\underset{i=0}{\wedge}} \boldsymbol{e}_i}{\boldsymbol{s}\boldsymbol{m}^3} = \frac{\overset{3}{\underset{i=0}{\wedge}} \boldsymbol{e}'_i}{\boldsymbol{s}'\boldsymbol{m}'^3} \in \frac{\overset{4}{\wedge} \mathbf{M}}{\mathbf{I} \otimes \overset{3}{\otimes} \mathbf{D}},$$

which is called the *Levi–Civita tensor* of the non-relativistic spacetime model.

In other words, if $\boldsymbol{u} \in \mathrm{V}(1)$ and $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ is a positively oriented orthonormal basis in $\mathbf{N} = \frac{\mathbf{E}}{\mathbf{D}}$, then

$$\varepsilon = \boldsymbol{u} \wedge \overset{3}{\underset{\alpha=1}{\wedge}} \boldsymbol{n}_\alpha.$$

(ii) Let $(k^0, \boldsymbol{k}^1, \boldsymbol{k}^2, \boldsymbol{k}^3))$ and $(\boldsymbol{k}'^0, \boldsymbol{k}'^1, \boldsymbol{k}'^2, \boldsymbol{k}'^3)$ be the dual of the bases in question (see the previous exercise). Then

$$\overline{\varepsilon} := \boldsymbol{s}\boldsymbol{m}^3 \overset{3}{\underset{i=0}{\wedge}} \boldsymbol{k}^i = \boldsymbol{s}'\boldsymbol{m}'^3 \overset{3}{\underset{i=0}{\wedge}} \boldsymbol{k}'^i \in \mathbf{I} \otimes \overset{3}{\otimes} \mathbf{D} \otimes \overset{4}{\wedge} \mathbf{M}^*,$$

which is called the *Levi–Civita cotensor* of the non-relativistic spacetime model.

In other words, if $\boldsymbol{r}^\alpha \in \mathbf{D} \otimes \mathbf{M}^*$ and $\mathbf{i}^* \cdot \boldsymbol{r}^\alpha$ $(\alpha = 1,2,3)$ form a positively oriented orthonormal basis in $\mathbf{N} = \frac{\mathbf{E}}{\mathbf{D}}$, then

$$\overline{\varepsilon} = \boldsymbol{\tau} \wedge \overset{3}{\underset{\alpha=1}{\wedge}} \boldsymbol{r}^\alpha.$$

3. ε and $\overline{\varepsilon}$ can be regarded as linear maps from $\mathbf{I} \otimes \overset{3}{\otimes} \mathbf{D}$ into $\overset{4}{\wedge} \mathbf{M}$ and from $\overset{4}{\wedge} \mathbf{M}$ into $\mathbf{I} \otimes \overset{3}{\otimes} \mathbf{D}$ (recall that $\overset{4}{\wedge} \mathbf{M}^* \equiv \left[\overset{4}{\wedge} \mathbf{M} \right]^*$). Prove that $\overline{\varepsilon}$ is the inverse of ε.

4. Take the arithmetic spacetime model and the usual matrix form of linear maps $\mathbb{R}^M \to \mathbb{R}^N$. Then

$$\boldsymbol{\tau} = (1\ 0\ 0\ 0),$$

$$\mathbf{i} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \boldsymbol{\pi}_{(1,\boldsymbol{v})} = \begin{pmatrix} -v^1 & 1 & 0 & 0 \\ -v^2 & 0 & 1 & 0 \\ -v^3 & 0 & 0 & 1 \end{pmatrix}.$$

2. World lines

2.1. History of a masspoint: world line

2.1.1. Let us consider a material body which is much smaller than the other ones around it. It can be considered point-like in our usual space, and its motion is described by a function that assigns space points (the instantaneous positions of the body) to time points. A larger body can be considered point-like, too, if we are interested only in some aspects of its motion; e.g. we neglect that a ball twirls when flying, and is compressed when bouncing over a wall.

In most of the textbooks it is emphasized, rightly, that motion is a relative notion. The motion of a material body makes sense only relative to another material object and the same body moves differently relative to different material objects. However, this does not imply that a body can be described only with respect to a chosen material object (in a "reference frame"). Our spacetime model allows an absolute description (independent of "reference frames"). We have to recognize only that the *existence* (which is usually called the *history*) of the body is an absolute notion and this history seems to be a motion to another material object.

The history of a material point is modelled in the spacetime model by a function that assigns world points to instants; the world point assigned to an instant gives the instantaneous spacetime position of the existence of the material point. Of course, the instant of the assigned world point must coincide with the instant itself.

Definition. A function $r : \mathrm{I} \rightarrowtail \mathrm{M}$ is called a *world line function* if
(i) $\mathrm{Dom}\, r$ is an interval,
(ii) r is piecewise twice continuously differentiable,
(iii) $\tau(r(t)) = t$ for all $t \in \mathrm{Dom}\, r$.

A subset C of M is a *world line* if it is the range of a world line function.

The world line function r and the world line $\mathrm{Ran}\, r$ is *global* if $\mathrm{Dom}\, r = \mathrm{I}$. ■

It can be shown easily that a world line C uniquely determines the world line function r such that $\mathrm{C} = \mathrm{Ran}\, r$.

2.1.2. Let the world line function r be twice differentiable at t. Then $\dot{r}(t) \in \frac{\mathbf{M}}{\mathbf{I}}$ and $\ddot{r}(t) \in \frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$ (see VI.3.9); moreover,

$$\boldsymbol{\tau} \cdot \dot{r}(t) = \lim_{s\to t} \frac{\boldsymbol{\tau} \cdot (r(s) - r(t))}{s - t} = \lim_{s\to t} \frac{\tau(r(s)) - \tau(r(t))}{s - t} = \lim_{s\to t} \frac{s - t}{s - t} = 1$$

and similarly we deduce $\boldsymbol{\tau} \cdot \ddot{r}(t) = \mathbf{0}$; in other words,

$$\dot{r}(t) \in \mathrm{V}(1), \qquad \ddot{r}(t) \in \frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{I}}.$$

The same is true for the right and left derivatives at instants t where r is not twice differentiable.

The functions $\dot{r} : \mathrm{I} \rightarrowtail \mathrm{V}(1)$ and $\ddot{r} : \mathrm{I} \rightarrowtail -\frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{I}}$ can be interpreted as the (absolute) *velocity* and the (absolute) *acceleration* of the material point whose history is described by r.

That is why we call the elements of $V(1)$ *velocity values* and the elements of $\frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{I}}$ *acceleration values.*

2.1.3. Recall that $\mathrm{V}(1)$ is a three-dimensional affine space over $\frac{\mathbf{E}}{\mathbf{I}}$. The elements of $\frac{\mathbf{E}}{\mathbf{I}}$ will be called *relative velocity values;* later we shall see the motivation of this name.

We know that the Euclidean structure of $\mathbf{E}$ induces Euclidean structures on $\frac{\mathbf{E}}{\mathbf{I}}$ and on $\frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{I}}$ (see 1.2.5). The magnitude of a relative velocity value is a positive element of $\frac{\mathbf{D}}{\mathbf{I}}$; the magnitude of an acceleration value is a positive element of $\frac{\mathbf{D}}{\mathbf{I}\otimes\mathbf{I}}$.

$\mathbf{D}$ and $\mathbf{I}$ are the measure lines of distances and time periods, respectively. Choosing a positive element in $\mathbf{D}$ and in $\mathbf{I}$ we fix the unit of distances and the unit of time periods; for instance, (meter=)$\boldsymbol{m} \in \mathbf{D}$ and (secundum=)$\boldsymbol{s} \in \mathbf{I}$. Then the units of measurements of the relative velocity and the acceleration are $\frac{\boldsymbol{m}}{\boldsymbol{s}} \in \frac{\mathbf{D}}{\mathbf{I}}$ and $\frac{\boldsymbol{m}}{\boldsymbol{s}^2} := \frac{\boldsymbol{m}}{\boldsymbol{s}\otimes\boldsymbol{s}} \in \frac{\mathbf{D}}{\mathbf{I}\otimes\mathbf{I}}$, respectively.

We emphasize the following important facts.

(i) The velocity values are timelike vectors of cotype $\mathbf{I}$, in particular they are future-directed. They form a three-dimensional affine space which is not a vector space; in particular, there is no zero velocity value. A velocity value has no magnitude, velocity values have no angles between themselves.

(ii) The relative velocity values are spacelike vectors of cotype $\mathbf{I}$, they form a three-dimensional Euclidean vector space; there is a zero relative velocity value. Magnitudes and angles make sense for relative velocity values.

(iii) The acceleration values are spacelike vectors of cotype $\mathbf{I}\otimes\mathbf{I}$, they form a three-dimensional Euclidean vector space; the acceleration values have magnitudes and angles between themselves.

The absence of magnitudes of velocity values means that “quickness” makes no absolute sense; it is not meaningful that a material object exists more quickly than another. A velocity value characterizes somehow the *tendency* of the history of a material point. Masspoints can move slowly or quickly *relative to each other.*

2.1.4. A world line function in the arithmetic spacetime model is $r = (r^0, \boldsymbol{r}) : \mathbb{R} \rightarrowtail \mathbb{R} \times \mathbb{R}^3$ such that $r^0(t) = t$ for all $t \in \mathrm{Dom}\, r$. In other words, a world line function is given by a function $\boldsymbol{r} : \mathbb{R} \rightarrowtail \mathbb{R}^3$ in the form $t \mapsto (t, \boldsymbol{r}(t))$.

The first and the second derivative of the world line function (i.e. velocity and acceleration) are $t \mapsto (1, \dot{\boldsymbol{r}}(t))$ and $t \mapsto (0, \ddot{\boldsymbol{r}}(t))$, respectively.

2.2. A characterization of world lines

The world lines are special curves in M (for the notion of curves see VI.4.3).

It is evident that if C is a world line then $\mathrm{C} \cap t$ has at most one element for all $t \in \mathrm{I}$ (where I is identified with the affine subspaces in M, directed by **E**, see 1.1.4). We shall use the symbol

$$\mathrm{C} \star t$$

for the unique element of $\mathrm{C} \cap t$ if this latter is not void. Then we have that the world line function r corresponding to C is given by

$$\begin{aligned} \mathrm{Dom}\ r &= \{t \in \mathrm{I} \mid \quad \mathrm{C} \cap t \neq \emptyset\}, \\ r(t) &= \mathrm{C} \star t \qquad\qquad\qquad (t \in \mathrm{Dom}\ r). \end{aligned}$$

It is evident as well that a twice differentiable curve C for which $\mathrm{C} \cap t$ has at most one element for all $t \in \mathrm{I}$ need not be a world line: it can have a spacelike tangent vector.

Every non-zero tangent vector of a world line is timelike. The converse is true as well.

Proposition. Let C be a connected twice differentiable curve in M whose non-zero tangent vectors are timelike; then C is a world line.

Proof. Let $p : \mathbb{R} \rightarrowtail \mathrm{M}$ be a parameterization of C. Then $\boldsymbol{\tau} \cdot (\dot{p}(\alpha)) \neq \mathbf{0}$ for all $\alpha \in \mathrm{Dom}\ \ p$. The function $\tau \circ p : \mathbb{R} \rightarrowtail \mathrm{I}$ is defined in an interval, is twice continuously differentiable, its derivative $\boldsymbol{\tau} \cdot \dot{p}$ is nowhere zero; hence it is strictly monotone, its inverse $z := (\tau \circ p)^{-1}$ is twice continuously differentiable as well and $\dot{z}(t) = 1/\boldsymbol{\tau} \cdot \dot{p}(z(t))$, as it is well known. It is obvious then that $r : = p \circ z$ is a world line function and $\mathrm{Ran}\ r = \mathrm{C}$.

2.3. Classification of world lines

Definition. The twice continuously differentiable world line function r and the corresponding world line are called

(i) *inertial* if $\ddot{r} = \mathbf{0}$,

(ii) *uniformly accelerated* if $\ddot{r}$ is constant,

(iii) *twist-free* if $\ddot{r}(s)$ is parallel to $\ddot{r}(t)$ for all $t, s \in \mathrm{Dom}\ r$.

Proposition. The twice continuously differentiable world line function r is

(i) inertial if and only if there are $x_o \in \mathrm{M}$ and $\boldsymbol{u}_o \in \mathrm{V}(1)$ such that

$$r(t) = x_o + \boldsymbol{u}_o(t - \tau(x_o)) \qquad\qquad (t \in \mathrm{Dom}\ r);$$

(*ii*) uniformly accelerated if and only if there are $x_o \in \mathrm{M}$, $\boldsymbol{u}_o \in \mathrm{V}(1)$ and $\boldsymbol{a}_o \in \frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{I}}$ such that

$$r(t) = x_o + \boldsymbol{u}_o(t - \tau(x_o)) + \frac{1}{2}\boldsymbol{a}_o(t - \tau(x_o))^2 \qquad (t \in \mathrm{Dom}\ r);$$

(*iii*) twist-free if and only if there exist $x_o \in \mathrm{M}$, $\boldsymbol{u}_o \in \mathrm{V}(1)$, $\mathbf{0} \neq \boldsymbol{a}_o \in \frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{I}}$ and a twice continuously differentiable function $\boldsymbol{h} : \mathbf{I} \rightarrowtail \mathbf{I}\otimes\mathbf{I}$ for which $\boldsymbol{h}(\mathbf{0}) = \mathbf{0}$, $\dot{\boldsymbol{h}}(\mathbf{0}) = \mathbf{0}$ and

$$r(t) = x_o + \boldsymbol{u}_o(t - \tau(x_o)) + \boldsymbol{a}_o\boldsymbol{h}(t - \tau(x_o)) \qquad (t \in \mathrm{Dom}\ r).$$

Proof. The validity of the assertions comes from the theory of differential equations; (*i*) and (*ii*) are quite trivial. For (*iii*) observe that r is twist-free if and only if there is a non-zero acceleration value $\boldsymbol{a}_o$ and a continuous function $\alpha : \mathrm{I} \rightarrowtail \mathbb{R}$ (which can be zero) such that $\ddot{r}(t) = \boldsymbol{a}_o\alpha(t)$. If x_o is a point in the range of r, we define $\chi : \mathbf{I} \rightarrowtail \mathbb{R}$ by $\chi(\boldsymbol{t}) := \alpha(\tau(x_o) + \boldsymbol{t})$ which means that $\chi(t - \tau(x_o)) = \alpha(t)$. Then $\boldsymbol{h}$ will be the function whose second derivative is χ and that satisfies the above given initial condition. ■

Observe that a twice continuously differentiable world line function r is twist-free if and only if $\ddot{r}/|\ddot{r}|$ is constant on each interval where the second derivative is not zero.

An inertial world line is uniformly accelerated (with zero acceleration) and a uniformly accelerated world line is twist-free (with constant acceleration).

A world line is inertial if and only if it is a straight line segment.

2.4. Newtonian equation

2.4.1. We shall say some words about the Newtonian equation though it does not belong to the subject of this volume; the Newtonian equation motivates the notion of force fields and potentials which will make us understand the importance of splitting of vectors and covectors (see Section 6).

First of all we have to say something about mass. One usually introduces the unit of mass, $\boldsymbol{kg}$, as a unit independent of the unit of distances, $\boldsymbol{m}$, and of the unit of time periods, $\boldsymbol{s}$. This means in our language that we introduce the measure line $\mathbf{G}$ of mass as a measure line "independent" of $\mathbf{D}$ and $\mathbf{I}$. We shall do so in another book where we wish to treat physical theories in a form suitable for applications, so in a form which applies the SI physical dimensions. However, for the present purposes we choose another possibility.

The results of quantum mechanics showed that Nature establishes a relation among the measure lines $\mathbf{D}$, $\mathbf{I}$ and $\mathbf{G}$. Namely, it is discovered, that the values of angular momentum are integer multiples of a given quantum denoted by $h/4\pi$ where h is known as the Planck constant. Hence we can choose $\mathbb{R}$ for the measure line of angular momentum; a real number (more precisely an integer) n represents the angular momentum value $nh/4\pi$. As it is known, angular momentum is the product of mass, position and velocity; thus its measure line is $\mathbf{G}\otimes\mathbf{D}\otimes\frac{\mathbf{D}}{\mathbf{I}}$ which is identified with $\mathbb{R}$; consequently, $\mathbf{G}\equiv\frac{\mathbf{I}}{\mathbf{D}\otimes\mathbf{D}}$.

In this book, for easier theoretical considerations, we take $\frac{\mathbf{I}}{\mathbf{D}\otimes\mathbf{D}}$ as the measure line of masses. If $\boldsymbol{m}$ is the unit distance and $\boldsymbol{s}$ is the unit time period then $\frac{\boldsymbol{s}}{\boldsymbol{m}^2}$ is the unit mass. One finds the experimental data

$$h/4\pi = (1,05...)10^{-34}\frac{\boldsymbol{m}^2\ \boldsymbol{kg}}{\boldsymbol{s}}$$

hence if we take it equal to the real number one we arrive at the definition

$$\boldsymbol{kg} := (9,4813...)10^{33}\frac{\boldsymbol{s}}{\boldsymbol{m}^2}.$$

2.4.2. Since acceleration values are elements of $\frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{I}}$ and "the product of mass and acceleration equals the force", the force values are elements of $\frac{\mathbf{I}}{\mathbf{D}\otimes\mathbf{D}}\otimes\frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{I}}\equiv\frac{\mathbf{E}}{\mathbf{I}\otimes\mathbf{D}\otimes\mathbf{D}}\equiv\frac{\mathbf{E}^*}{\mathbf{I}}$; moreover, "a force can depend on time, on space and on velocity". Thus we accept that a *force field* is a differentiable mapping

$$\boldsymbol{f}:\mathrm{M}\times\mathrm{V}(1)\rightarrowtail\frac{\mathbf{E}^*}{\mathbf{I}}$$

and the history of the material point with mass m under the action of the force field $\boldsymbol{f}$ is given by the Newtonian equation

$$m\ddot{x}=\boldsymbol{f}(x,\dot{x}),$$

i.e. the world line modelling the history is a solution of this differential equation.

2.4.3. The most important force fields can be derived from potentials; e.g. the gravitational field and the electromagnetic field. Usually the gravitational field is the gradient of a scalar potential and the electromagnetic field is given by the gradient of a scalar potential and the curl of a vector potential. The gravitational force acting on a material point depends only on the spacetime position of the masspoint, the electromagnetic force depends on the velocity of the masspoint as well. To introduce the notion of potential in the spacetime model, we have to rely on these facts. Now we give the convenient definition and we shall show in Section 6 that it is suitable indeed.

A *potential* is a twice differentiable mapping

$$\boldsymbol{K} : \mathrm{M} \rightarrowtail \mathbf{M}^*$$

(in other words, a potential is a twice differentiable covector field).

The *field strength* corresponding to $\boldsymbol{K}$ is $\mathrm{D} \wedge \boldsymbol{K} : \mathrm{M} \rightarrowtail \mathbf{M}^* \wedge \mathbf{M}^*$ (the antisymmetric or exterior derivative of $\boldsymbol{K}$, see VI.3.6).

The force field $\boldsymbol{f}$ has a potential (is derived from a potential) if

— there is an open subset $\mathrm{O} \subset \mathrm{M}$ such that $\mathrm{Dom}\, \boldsymbol{f} = \mathrm{O} \times \mathrm{V}(1)$,

— there is a potential $\boldsymbol{K}$ defined on O such that

$$\boldsymbol{f}(x, \boldsymbol{u}) = \mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \boldsymbol{u} \qquad\qquad (x \in \mathrm{O},\ \boldsymbol{u} \in \mathrm{V}(1)),$$

where $\boldsymbol{F} := \mathrm{D} \wedge \boldsymbol{K}$ and $\mathbf{i} : \mathbf{E} \to \mathbf{M}$ is the embedding. Checking this formula, the reader can seize the opportunity to practise using the dot product.

2.5. Exercises

1. Let r_1 and r_2 be world lines. Characterize the function $r_1 - r_2$.
2. Another formulation of the preceding exercise: give necessary and sufficient conditions for a function $\boldsymbol{z}$ that $r + \boldsymbol{z}$ be a world line for all world lines r.
3. Describe the world lines in the split spacetime model (cf. 2.1.4).

3. Observers

3.1. The notion of an observer

3.1.1. In usual physics the phenomena are always described with respect to a “reference frame” which means a material object and coordinates on it.

Our present aim is to define the corresponding notion in the spacetime model. To do so, we separate the material object and the coordinates on it; in such a way first we arrive at observers and then at reference frames.

The notion of an observer is extremely important because our experimental results are always connected with material objects (experimental devices). It is important as well that we see clearly the connection between absolute and relative notions.

Our experience about a physical phenomenon depends on the experimental devices, i.e. on material objects of observation. This means that our experience and the direct abstractions gained from experience are relative: they reflect not only the properties of the observed phenomenon but some properties of the observers and the relation between the phenomenon and the observer as well.

If we wish to separate the properties of the phenomenon we have to compare the experimental results of different observers concerning the same phenomenon; so we can find out what is the core of these facts and we can get rid of observers. To describe a phenomenon we evidently ought to use absolute notions only, i.e. notions independent of observers. Physical theories must be based on absolute notions.

On the other hand, of course, we must lay down as well how an observer deduces the relative notions from the absolute ones, which means how the observer sees the properties of the phenomenon. This is indispensable from the point of view of experiments.

As a matter of fact, this program was started by the theory of relativity at that time, but owing to the use of inadequate mathematical tools and the complicated setting it has not yet been accomplished. Even nowadays one expresses the absolute notions with the aid of relative notions and not vice versa which would be desirable: to deduce the relative notions from the absolute ones.

3.1.2. An observer as a physical reality is a material object or a set of material objects; recall what is said in the Introduction: the earth, the houses on it form an observer, the car is another observer.

We can imagine that an observer is a collection of material points existing "in close proximity" to each other. The existence of a masspoint in spacetime is described by a world line. Thus an observer would be modelled by a collection of world lines that fill "continuously" a domain of spacetime. How to define a convenient notion of such a continuity? To all points of every world line of the observer we assign the corresponding velocity value; in this way we define a velocity field: a function defined for some world points and having values in V(1). Conversely, given a velocity field (with convenient mathematical properties), we can recover the world lines of the observers: world lines having everywhere the velocity value prescribed by the velocity field. We shall see that the velocity field is extremely suitable for our purposes, hence we prefer it to the collection of world lines.

Definition. An observer is a smooth map $\boldsymbol{U} : \mathrm{M} \rightarrowtail \mathrm{V}(1)$ whose domain is connected.

If $\mathrm{Dom}\,\boldsymbol{U} = \mathrm{M}$, the observer is called *global*. ■

We emphasize that we are dealing with mathematical models; an observer as it is defined is a mathematical model for a physical object. To underline this fact we might use the term "observer model" instead of "observer" but we wish to avoid ponderousness. If necessary, we shall say physical observer for the material objects in question.

3.1.3. Let $\boldsymbol{U}$ be an observer. The integral curves of the differential equation

$$(x : \mathrm{I} \rightarrowtail \mathrm{M})? \qquad \dot{x} = \boldsymbol{U}(x)$$

have exclusively timelike tangent vectors, thus they are world lines (see 2.2).

The maximal integral curves of this differential equation will be called ***U**-lines*; they will play an important role.

If the world line function r is a solution of the above differential equation — i.e. $\operatorname{Ran} r$ is an integral curve of $\boldsymbol{U}$— then $\dot{r}(t) = \boldsymbol{U}(r(t))$ and so $\ddot{r}(t) = \mathrm{D}\boldsymbol{U}(r(t)) \cdot \dot{r}(t) = \mathrm{D}\boldsymbol{U}(r(t)) \cdot \boldsymbol{U}(r(t))$ for all $t \in \operatorname{Dom} r$. This motivates that

$$\boldsymbol{A_U} : \mathrm{M} \rightarrowtail \frac{\mathbf{E}}{\mathbf{I} \otimes \mathbf{I}}, \qquad x \mapsto \mathrm{D}\boldsymbol{U}(x) \cdot \boldsymbol{U}(x)$$

is called the *acceleration field* corresponding to the observer $\boldsymbol{U}$.

3.1.4. Definition. An observer $\boldsymbol{U}$ is called *fit* if all the world line functions giving the $\boldsymbol{U}$-lines have the same domain; this uniquely determined interval of I is the *lifetime* of the observer. ■

It may happen that the maximal integral curves of a global observer are not global world lines (see Exercise 3.4). A global observer $\boldsymbol{U}$ is fit if and only if all $\boldsymbol{U}$-lines are global.

3.1.5. In the arithmetic spacetime model an observer is given by a function $\boldsymbol{V} : \mathbb{R} \times \mathbb{R}^3 \rightarrowtail \mathbb{R}^3$ in the form $(1, \boldsymbol{V}) = (1,\ V^1,\ V^2,\ V^3)$. If we denote the partial derivatives corresponding to $\mathbb{R}$ and $\mathbb{R}^3$ by ∂_0 and $\nabla = (\partial_1,\ \partial_2,\ \partial_3)$, respectively, then the acceleration field of the observer is $(0,\ \partial_0 \boldsymbol{V} + \boldsymbol{V} \cdot \nabla \boldsymbol{V}) = \left(0, (\partial_0 V^i + \sum_{k=1}^{3} V^k \partial_k V^i)_{i=1,2,3}\right)$.

3.2. Splitting of spacetime due to an observer

3.2.1. As it is stated in the Introduction, a physical observer — a material object — establishes space for itself. The points of its space are just the material points that it consists of. In our model these points correspond to the maximal integral curves of the observer. Thus the space of an observer $\boldsymbol{U}$ is just the collection of $\boldsymbol{U}$-lines. Now we are in position to define the space of an observer and to establish how an observer *splits* spacetime into time and space.

Definition. Let $\boldsymbol{U}$ be an observer and let $\mathrm{E}_{\boldsymbol{U}}$ denote the set of maximal integral curves of $\boldsymbol{U}$. $\mathrm{E}_{\boldsymbol{U}}$ is called *the space of the observer* $\boldsymbol{U}$, or the $\boldsymbol{U}$-space. ■

The elements of the $\boldsymbol{U}$-space are world lines. We have to get accustomed to this situation, strange at first sight, but common in mathematics: the elements of a set are sets themselves.

A maximal integral curve of $\boldsymbol{U}$ will be called a $\boldsymbol{U}$*-line* if considered to be a subset of M and will be called a $\boldsymbol{U}$*-space point* if considered to be an element of $\mathrm{E}_{\boldsymbol{U}}$.

By the way, we conceive instants, too, as sets: an instant is identified with the corresponding simultaneous hyperplane.

We measure distances in our physical space, we know what is near, what is far. We define limit procedures regarding our space. These notions must appear in the model.

It can be shown that, in general, the $\boldsymbol{U}$-space can be endowed with a smooth structure in a natural way, thus limits, differentiability etc. will make sense. However, in this book we avoid the general theory of smooth manifolds, that is why, in general, we do not deal with the structure of observer spaces. Later the spaces of some special observers, important from the point of view of applications, will be treated.

3.2.2. Recall from the theory of differential equations that different integral curves of $\boldsymbol{U}$ do not intersect (VI.6.2). Let us introduce the map $C_{\boldsymbol{U}} : \mathrm{Dom}\,\boldsymbol{U} \to \mathrm{E}_{\boldsymbol{U}}$ in such a way that $C_{\boldsymbol{U}}(x)$ is the (unique) $\boldsymbol{U}$-line passing through x.

We shall say as well that $C_{\boldsymbol{U}}(x)$ is the $\boldsymbol{U}$-space point that the world point x is *incident* with.

Then the map

$$H_{\boldsymbol{U}} : \mathrm{Dom}\,\boldsymbol{U} \to \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}, \qquad x \mapsto (\tau(x),\ C_{\boldsymbol{U}}(x))$$

is clearly injective, its inverse is

$$(t, q) \mapsto q \star t \qquad ((t, q) \in \mathrm{Ran}\,H_{\boldsymbol{U}} \subset \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}})$$

where the notation introduced in 2.2 is used.

In this way spacetime points in the domain of $\boldsymbol{U}$ are represented by pairs of time points and $\boldsymbol{U}$-space points. We say that the observer $\boldsymbol{U}$ splits spacetime into time and $\boldsymbol{U}$-space with the aid of $H_{\boldsymbol{U}}$.

Definition. $H_{\boldsymbol{U}}$ is the *splitting* of spacetime *according* to $\boldsymbol{U}$. ■

If $\mathrm{E}_{\boldsymbol{U}}$ is endowed with the smooth structure mentioned previously then $H_{\boldsymbol{U}}$ will be smooth. Its properties will be clarified in special cases.

3.3. Classification of observers

3.3.1. We have considered the room and the car as examples of physical observers. However, much "worse" material objects can be observers as well.

For instance, the stormy sea: the distance of its space points (which are the molecules of the water) and even the direction of their mutual positions vary with time. A ship on the stormy sea is a little better because it does not change its shape, it is rigid. However, it rotates, i.e. the directions of relative positions of its space points vary with time. The slightly waving water is better than the stormy one because it does not whirl. These examples show from what point of view we should classify observers in our spacetime model.

We mention that physical observers, in reality, are never rigid and rotation-free; at least molecular motion contradicts these properties. Besides, a physical observer is never global, it cannot fill all the spacetime. All these notions, as all models, are idealizations, extrapolations for a convenient mathematical description.

Recall the notation introduced in 2.2.

Definition. A fit observer $\boldsymbol{U}$ is called

(i) *rigid* if for all $q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}}$ the distance between $q_1 \star t$ and $q_2 \star t$— in other words $|q_1 \star t - q_2 \star t|$— is the same for all t in the lifetime of $\boldsymbol{U}$;

(ii) *rotation-free* if for all $q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}}$ the direction of the vector $q_1 \star t - q_2 \star t$ is the same for all t in the lifetime of $\boldsymbol{U}$;

(iii) *twist-free* if all $\boldsymbol{U}$-space points are twist-free;

(iv) *inertial* if $\boldsymbol{U}$ is a constant function; in other words, if the $\boldsymbol{U}$-lines are parallel straight line segments in spacetime. ■

Except the inertial observers, it is difficult to give a good illustration of these types of observers. The following figure tries to show a rigid or rotation-free observer.

Suppose q_1 runs in the plane of the sheet. Letting q_2 bend below the plane of the sheet in such a way that its points have the same distances from the corresponding points of q_1, we can draw a picture of a rigid observer which is not rotation-free.

Letting q_2 bend in the plane of the sheet we can draw a picture of a rotation-free observer which is not rigid.

3.3.2. We call attention to the fact that a fit observer whose space points are all inertial (i.e. straight line segments) is not necessarily inertial: it may occur that its integral curves are not parallel (see Exercise 5.4.1).

Evidently, an inertial observer is rigid, rotation-free and twist-free. The converse is not true: see 5.2.

A fit observer $\boldsymbol{U}$ is rigid and rotation-free if and only if for all $q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}}$, $q_1 \star t - q_2 \star t$ is the same for all t in the lifetime of $\boldsymbol{U}$.

3.4. Exercise

The observer

$$(\xi^0, \xi^1, \xi^2, \xi^3) \mapsto (1, -(\xi^1)^2, 0, 0)$$

in the arithmetic spacetime model is global, its maximal integral curve passing through $(\xi^0, \xi^1, \xi^2, \xi^3)$ is

$$\begin{aligned} &\{(t, 0, \xi^2, \xi^3) \mid t \in \mathbb{R}\} && \text{if} \quad \xi^1 = 0, \\ &\{(t, \frac{1}{t - \xi^0 + 1/\xi^1}, \xi^2, \xi^3) \mid t > \xi^0 - 1/\xi^1\} && \text{if} \quad \xi^1 > 0, \\ &\{(t, \frac{1}{t - \xi^0 + 1/\xi^1}, \xi^2, \xi^3) \mid t < \xi^0 - 1/\xi^1\} && \text{if} \quad \xi^1 > 0. \end{aligned}$$

Consequently, most of the maximal integral curves of the observer are not global.

4. Rigid observers

4.1. Inertial observers

4.1.1. Let us consider a global inertial observer $\boldsymbol{U}$ and let $\boldsymbol{u} \in \mathrm{V}(1)$ denote the constant value of $\boldsymbol{U}$.

Recall the linear map $\boldsymbol{\pi_u}$ — the projection onto $\mathbf{E}$ along $\boldsymbol{u} \otimes \mathbf{I}$ — defined in 1.2.8.

The observer space $\mathrm{E}_{\boldsymbol{U}}$ is the set of straight lines directed by $\boldsymbol{u}$; more closely,

$$C_{\boldsymbol{U}}(x) = x + \boldsymbol{u} \otimes \mathbf{I} := \{x + \boldsymbol{ut} \mid \boldsymbol{t} \in \mathbf{I}\}.$$

Note that

$$(x + \boldsymbol{u} \otimes \mathbf{I}) \star t = x + \boldsymbol{u}(t - \tau(x)).$$

As a consequence, $\boldsymbol{U}$ is rigid and rotation-free:

$$\begin{aligned}(x_2 + \boldsymbol{u} \otimes \mathbf{I}) \star t - &(x_1 + \boldsymbol{u} \otimes \mathbf{I}) \star t = \\ &= (x_2 + \boldsymbol{u}(t - \tau(x_2))) - (x_1 + \boldsymbol{u}(t - \tau(x_1))) = \\ &= x_2 - x_1 - \boldsymbol{u}(\boldsymbol{\tau} \cdot (x_2 - x_1)) = \\ &= \boldsymbol{\pi_u} \cdot (x_2 - x_1).\end{aligned}$$

4.1.2. According to the previous formula, if q_2 and q_1 are $\boldsymbol{U}$-space points then $q_2 \star t - q_1 \star t$ is the same vector in $\mathbf{E}$ for all $t \in \mathrm{I}$: more closely, it equals $\boldsymbol{\pi_u} \cdot (x_2 - x_1)$ where x_1 and x_2 are arbitrary elements of q_1 and q_2, respectively. Regarding this vector as the difference of the $\boldsymbol{U}$-space points, we define an affine structure on $\mathrm{E}_{\boldsymbol{U}}$ in a natural way.

Proposition. $\mathrm{E}_{\boldsymbol{U}}$, endowed with the subtraction

$$q_2 - q_1 := \boldsymbol{\pi_u} \cdot (x_2 - x_1) \qquad\qquad (q_1,\ q_2 \in \mathrm{E}_{\boldsymbol{U}},\ x_1 \in q_1,\ x_2 \in q_2)$$

is an affine space over $\mathbf{E}$. ■

Observe that if $x_1 \in q_1$, $x_2 \in q_2$ and $\tau(x_1) = \tau(x_2)$, then $q_2 - q_1 = x_2 - x_1$.

It is worth remarking that

$$(x+\boldsymbol{q})+\boldsymbol{u}\otimes\mathbf{I}=(x+\boldsymbol{u}\otimes\mathrm{I})+\boldsymbol{q} \qquad (x\in\mathrm{M},\ \boldsymbol{q}\in\mathbf{E}),$$

which is not trivial because here the same sign + denotes different operations: the first one refers to the addition between elements of M and $\mathbf{M}$; the second and the third ones denote a set addition between elements of M and $\mathbf{M}$; the fourth one indicates the addition between elements of $\mathrm{E}_{\boldsymbol{U}}$ and $\mathbf{E}$. This formula has the generalization

$$(x+\boldsymbol{x})+\boldsymbol{u}\otimes\mathbf{I}=(x+\boldsymbol{u}\otimes\mathbf{I})+\boldsymbol{\pi}_{\boldsymbol{u}}\cdot\boldsymbol{x} \qquad (x\in\mathrm{M},\ \boldsymbol{x}\in\mathbf{M}).$$

4.1.3. The space of any global inertial observer is a three-dimensional oriented Euclidean affine space (over $\mathbf{E}$). In this way we regain our experience regarding our physical space from the spacetime model (see the Introduction).

Keep in mind that the space of every global inertial observer is an affine space over the *same* vector space $\mathbf{E}$. Now we see why the vectors in $\mathbf{E}$ are called spacelike.

The following assertion is proved without any difficulty.

Proposition. Let $\boldsymbol{U}$ be a global inertial observer whose constant value is $\boldsymbol{u}$. Then the splitting of spacetime according to $\boldsymbol{U}$,

$$\mathrm{M}\to\mathrm{I}\times\mathrm{E}_{\boldsymbol{U}}, \qquad x\mapsto(\tau(x),\,C_{\boldsymbol{U}}(x))=(x+\mathbf{E},\ x+\boldsymbol{u}\otimes\mathbf{I})$$

is an orientation-preserving affine bijection having $\boldsymbol{h}_{\boldsymbol{u}}=(\boldsymbol{\tau},\boldsymbol{\pi}_{\boldsymbol{u}})$ as its underlying linear map. ■

If we consider the elements of I as hyperplanes in M then $\tau(x)=x+\mathbf{E}$; we used this fact in the previous proposition for later purposes.

4.1.4. We have to get accustomed to the fact that a physical notion which seems “structureless”, “as simple as possible” (e.g. a space point of an observer) is modelled by a less simple, structured mathematical object (by a line). In mathematics it is customary that the *elements* of a set are themselves *sets or functions.*

However, we have a tool to reduce some of our mathematical objects to simpler ones. This tool is the vectorization of affine spaces: choosing an arbitrary element (“reference origin”) in an affine space, we can represent every element of the affine space by a vector.

Let $\boldsymbol{U}$ be a global inertial observer with the velocity value $\boldsymbol{u}$. Taking a $t_o\in\mathrm{I}$ and a $q_o\in\mathrm{E}_{\boldsymbol{U}}$ we can establish the vectorization of time and $\boldsymbol{U}$-space:

$$V_o:\mathrm{I}\times\mathrm{E}_{\boldsymbol{U}}\to\mathbf{I}\times\mathbf{E}, \qquad (t,q)\to(t-t_o,\,q-q_o)$$

by which, in particular, we represent $\boldsymbol{U}$-space points by vectors in $\mathbf{E}$ that are simpler objects than straight lines in M.

Observe that choosing t_o and q_o is equivalent to choosing a "spacetime" *reference origin* $o \in \mathrm{M}:\ o := q_o \star t_o,\ t_o = \tau(o),\ q_o = C_{\boldsymbol{U}}(o)$.

Definition. An *inertial observer with origin* is a pair $(\boldsymbol{U}, o)$ where $\boldsymbol{U}$ is a global inertial observer and o is a world point.

The *vectorized splitting* of spacetime corresponding to $(\boldsymbol{U}, o)$ is the map

$$H_{\boldsymbol{U},o} := V_o \circ H_{\boldsymbol{U}} : \mathrm{M} \to \mathbf{I} \times \mathbf{E}, \quad x \to (\tau(x) - \tau(o),\ C_{\boldsymbol{U}}(x) - C_{\boldsymbol{U}}(o)) = \\ = (\boldsymbol{\tau} \cdot (x - o),\ \pi_{\boldsymbol{u}} \cdot (x - o))\,. \quad \blacksquare$$

Note that

$$H_{\boldsymbol{U},o} = \boldsymbol{h}_{\boldsymbol{u}} \circ O_o,$$

where $\boldsymbol{h}_{\boldsymbol{u}} = (\boldsymbol{\tau}, \boldsymbol{\pi}_{\boldsymbol{u}})$ and O_o is the vectorization of M with origin $o : \mathrm{M} \to \mathbf{M}$, $x \mapsto x - o$.

4.1.5. Let us consider the arithmetic spacetime model and the global inertial observer with constant value $(1, \boldsymbol{v})$. The space point of the observer that $(\alpha, \boldsymbol{\xi})$ is incident with is the straight line $(\alpha, \boldsymbol{\xi}) + (1, \boldsymbol{v})\mathbb{R} = \{(\alpha + t, \boldsymbol{\xi} + \boldsymbol{v}t \mid t \in \mathbb{R}\}$.

As concerns the affine structure of the set of such lines we have

$$[(\alpha, \boldsymbol{\xi}) + (1 + \boldsymbol{v})\mathbb{R}] - [(\beta, \boldsymbol{\zeta}) + (1, \boldsymbol{v})\mathbb{R}] = \boldsymbol{\xi} - \boldsymbol{\zeta} - \boldsymbol{v}(\alpha - \beta) \in \mathbb{R}^3.$$

Let the observer in question choose $(0, \mathbf{0})$ as reference origin. Then the observer space will be represented by $\mathbb{R}^3$; the space point $(\alpha, \boldsymbol{\xi}) + (1, \boldsymbol{v})\mathbb{R}$ will correspond to the difference of this straight line and that passing through $(0, \mathbf{0})$— which is $(1, \boldsymbol{v})\mathbb{R}$— ; this difference is exactly $\boldsymbol{\xi} - \boldsymbol{v}\alpha$.

Consequently, the vectorized splitting of spacetime due to this observer is

$$\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3, \qquad (\alpha, \boldsymbol{\xi}) \mapsto (\alpha, \boldsymbol{\xi} - \boldsymbol{v}\alpha).$$

In particular, the splitting of spacetime according to the *basic observer* — the one whose value is the basic velocity value $(1, \mathbf{0})$— with reference origin $(0, \mathbf{0})$is the identity of $\mathbb{R} \times \mathbb{R}^3$: the arithmetic spacetime model is the Cartesian product of vectorized time and vectorized space relative to the basic observer.

In other words, the observer with reference origin makes the correspondence that previously has been accepted as a natural identification. The vectorized splitting of spacetime is described by the formula above.

4.2. Characterization of rigid observers*

4.2.1. Now we derive some mathematical results to characterize some properties of observers. Simple but important relations for deducing our results are the following.

Recall that $C_{\boldsymbol{U}}(x)$ denotes the $\boldsymbol{U}$-line passing through x. Then $t \mapsto C_{\boldsymbol{U}}(x) \star t$ is the corresponding world line function. So we have

$$C_{\boldsymbol{U}}(x) \star \tau(x) = x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(C_{\boldsymbol{U}}(x) \star t\right) = \boldsymbol{U}\left(C_{\boldsymbol{U}}(x) \star t\right).$$

Proposition. Let $\boldsymbol{U}$ be a fit global observer.

(i) $\boldsymbol{U}$ is rigid if and only if

$$(\boldsymbol{U}(x+\boldsymbol{q}) - \boldsymbol{U}(x)) \cdot \boldsymbol{q} = \mathbf{0} \qquad (x \in \mathrm{M},\ \boldsymbol{q} \in \mathbf{E}).$$

(ii) $\boldsymbol{U}$ is rigid and rotation-free if and only if

$$\boldsymbol{U}(x+\boldsymbol{q}) - \boldsymbol{U}(x) = \mathbf{0} \qquad (x \in \mathrm{M},\ \boldsymbol{q} \in \mathbf{E}),$$

which is equivalent to the existence of a smooth map $\boldsymbol{V} : \mathrm{I} \rightarrowtail \boldsymbol{V}(1)$ such that

$$\boldsymbol{U} = \boldsymbol{V} \circ \tau.$$

Proof. Let $q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}}$.

(i) The function

$$t \mapsto |q_1 \star t - q_2 \star t|^2$$

is constant if and only if its derivative

$$t \mapsto 2\left(\boldsymbol{U}(q_1 \star t) - \boldsymbol{U}(q_2 \star t)\right) \cdot (q_1 \star t - q_2 \star t)$$

is zero.

Putting $x := q_2 \star t$, $\boldsymbol{q} := q_1 \star t - q_2 \star t$ in the derivative we infer that the derivative is zero if and only if the equality in the assertion holds (every $x \in \mathrm{M}$ is of the form $q_2 \star t$ for some q_2 and t and every $\boldsymbol{q} \in \mathbf{E}$ is of the form $q_1 \star t - q_2 \star t$ for some q_1).

(ii) The function

$$t \mapsto q_1 \star t - q_2 \star t$$

is constant if and only if its derivative

$$t \mapsto \boldsymbol{U}(q_1 \star t) - \boldsymbol{U}(q_2 \star t)$$

is zero.

Reasoning as previously we get the desired result.

4.2.2. Let $\boldsymbol{U}$ be a global rigid observer. For $t_o, t \in \mathrm{I}$ let us define

$$R_{\boldsymbol{U}}(t,t_o) \;:\; \mathbf{E} \to \mathbf{E}, \qquad \boldsymbol{q} \mapsto C_{\boldsymbol{U}}(x_o + \boldsymbol{q}) \star t \;-\; C_{\boldsymbol{U}}(x_o) \star t,$$

where x_o is an arbitrary element of t_o (i.e. $x \in \mathrm{M}$ and $\tau(x_o) = t_o$).

Proposition. If $\boldsymbol{U}$ is a global rigid observer then $R_{\boldsymbol{U}}(t,t_o)$ is a rotation in $\mathbf{E}$ (a linear orthogonal map with determinant 1) for all $t_o, t \in \mathrm{I}$. Moreover, $R_{\boldsymbol{U}}(t,t_o)$ is independent of x_o appearing in its definition.

The global rigid observer $\boldsymbol{U}$ is rotation-free if and only if $R_{\boldsymbol{U}}(t,t_o) = \mathrm{id}_{\mathbf{E}}$ for all $t_o, t \in \mathbf{I}$.

Proof. Evidently,

$$R_{\boldsymbol{U}}(t,t_o)(\mathbf{0}) = \mathbf{0}.$$

Moreover, since $\boldsymbol{U}$ is rigid, for all $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbf{E}$ we have

$$\begin{aligned}
&\mid R_{\boldsymbol{U}}(t,t_o)(\boldsymbol{q}_1) - R_{\boldsymbol{U}}(t,t_o)(\boldsymbol{q}_2) \mid = \\
=&\mid \left(C_{\boldsymbol{U}}(x_o+\boldsymbol{q}_1)\star t - C_{\boldsymbol{U}}(x_o)\star t\right) - \left(C_{\boldsymbol{U}}(x_o+\boldsymbol{q}_2)\star t - C_{\boldsymbol{U}}(x_o)\star t\right) \mid = \\
=&\mid C_{\boldsymbol{U}}(x_o+\boldsymbol{q}_1)\star t - C_{\boldsymbol{U}}(x_o+\boldsymbol{q}_2)\star t \mid = \mid (x_o+\boldsymbol{q}_1) - (x_o+\boldsymbol{q}_2) \mid = \\
=&\mid \boldsymbol{q}_1 - \boldsymbol{q}_2 \mid .
\end{aligned}$$

As a consequence, $R_{\boldsymbol{U}}(t,t_o)$ is a linear orthogonal map (see V.3.7).

For fixed t_o and fixed $\boldsymbol{q} \in \mathbf{E}$, the function $\mathrm{I} \to \mathbf{E}$, $t \mapsto R_{\boldsymbol{U}}(t,t_o)\cdot\boldsymbol{q}$ is smooth since it is the difference of two solutions of the smooth differential equation $\dot{x} = \boldsymbol{U}(x)$. Consequently, $t \mapsto \det R_{\boldsymbol{U}}(t,t_o)$ is a smooth function. Since the determinants in question can be 1 or -1 only and

$$R_{\boldsymbol{U}}(t_o,t_o) = \mathrm{id}_{\mathbf{E}},$$

all the determinants in question equal 1.

If y_o is another element of t_o, then with the notation $\boldsymbol{q}_o := y_o - x_o \in \mathbf{E}$ we deduce

$$\begin{aligned} &C_{\boldsymbol{U}}(y_o+\boldsymbol{q})\star t - C_{\boldsymbol{U}}(y_o)\star t = C_{\boldsymbol{U}}(x_o+\boldsymbol{q}_o+\boldsymbol{q})\star t - C_{\boldsymbol{U}}(x_o+\boldsymbol{q}_o)\star t = \\ &= (C_{\boldsymbol{U}}(x_o+\boldsymbol{q}_o+\boldsymbol{q})\star t - C_{\boldsymbol{U}}(x_o)\star t) - (C_{\boldsymbol{U}}(x_o+\boldsymbol{q}_o)\star t - C_{\boldsymbol{U}}(x_o\star t) = \\ &= R_{\boldsymbol{U}}(t,t_o)\cdot(\boldsymbol{q}_o+\boldsymbol{q}) - R_{\boldsymbol{U}}(t,t_o)\cdot\boldsymbol{q}_o = R_{\boldsymbol{U}}(t,t_o)\cdot\boldsymbol{q}, \end{aligned}$$

which means that the definition of $R_{\boldsymbol{U}}(t,t_o)$ is independent of x_o.

4.2.3. Proposition. For all $t_o, t, s \in \mathrm{I}$ we have
(i) $R_{\boldsymbol{U}}(t_o,t_o) = \mathrm{id}_{\mathbf{E}}$,
(ii) $R_{\boldsymbol{U}}(t,t_o)^{-1} = R_{\boldsymbol{U}}(t_o,t)$,
(iii) $R_{\boldsymbol{U}}(t,t_o) = R_{\boldsymbol{U}}(t,s)\cdot R_{\boldsymbol{U}}(s,t_o)$.

Proof. (i) is trivial.

The defining formula of $R_{\boldsymbol{U}}(t,t_o)$ can be rewritten in the following form: if q, q_o are $\boldsymbol{U}$-space points then

$$q\star t - q_o\star t = R_{\boldsymbol{U}}(t,t_o)\cdot(q\star t_o - q_o\star t_o) \qquad (t_o, t\in \mathrm{I}). \quad (*)$$

Interchanging t and t_o we get

$$q\star t_o - q_o\star t_o = R_{\boldsymbol{U}}(t_o,t)\cdot(q\star t - q_o\star t)$$

from which we infer (ii).

In a similar way we obtain (iii.) ■

Observe that $(*)$ implies that if $R_{\boldsymbol{U}}(t,t_o)$ is known for a t_o and for all t then every $\boldsymbol{U}$-space point q can be deduced from an arbitrarily chosen q_o.

4.2.4. Let $\boldsymbol{U}$ be a global rigid observer. For fixed $t_o \in \mathrm{I}$, the function $\mathrm{I}\to\mathbf{E}\otimes\mathbf{E}^*$, $t\mapsto R_{\boldsymbol{U}}(t,t_o)$ is smooth (because for all $\boldsymbol{q}\in\mathbf{E}$, $t\mapsto R_{\boldsymbol{U}}(t,t_o)\cdot\boldsymbol{q}$ is smooth); we introduce

$$\dot R_{\boldsymbol{U}}(t,t_o) := \frac{\mathrm{d}R_{\boldsymbol{U}}(t,t_o)}{\mathrm{d}t} \in \frac{\mathbf{E}\otimes\mathbf{E}^*}{\mathbf{I}} \qquad (t, t_o\in\mathrm{I}),$$

which can be regarded as a linear map

$$\dot R_{\boldsymbol{U}}(t,t_o) : \mathbf{E}\to\frac{\mathbf{E}}{\mathbf{I}}, \qquad \boldsymbol{q}\mapsto \frac{\mathrm{d}}{\mathrm{d}t}R_{\boldsymbol{U}}(t,t_o)\cdot\boldsymbol{q}$$

(VI.3.11). We deduce from the defining formula of $R_{\boldsymbol{U}}(t,t_o)$ that

$$\begin{aligned} \dot R_{\boldsymbol{U}}(t,t_o)\cdot\boldsymbol{q} &= \boldsymbol{U}\left(C_{\boldsymbol{U}}(x_o+\boldsymbol{q})\star t\right) - \boldsymbol{U}\left(C_{\boldsymbol{U}}(x_o)\star t\right) = \\ &= \boldsymbol{U}\left(C_{\boldsymbol{U}}(x_o)\star t + R_{\boldsymbol{U}}(t_o,t)\cdot\boldsymbol{q}\right) - \boldsymbol{U}(C_{\boldsymbol{U}}(x_o)\star t) = \\ &= \boldsymbol{U}\left(q\star t + R_{\boldsymbol{U}}(t_o,t)\cdot\boldsymbol{q}\right) - \boldsymbol{U}(q\star t), \end{aligned}$$

where x_o is an arbitrary element of t_o and q is an arbitrary element of $\mathrm{E}_{\boldsymbol{U}}$.

Substituting $R_{\boldsymbol{U}}(t,t_o)^{-1}\cdot\boldsymbol{q}$ for $\boldsymbol{q}$ and introducing the linear map

$$\Omega_{\boldsymbol{U}}(t) := \dot{R}_{\boldsymbol{U}}(t,t_o)\cdot R_{\boldsymbol{U}}(t,t_o)^{-1} : \mathbf{E}\to\frac{\mathbf{E}}{\mathbf{I}}$$

for $t\in\mathrm{I}$, we obtain

$$\Omega_{\boldsymbol{U}}(t)\cdot\boldsymbol{q} = \boldsymbol{U}(q\star t+\boldsymbol{q}) - \boldsymbol{U}(q\star t) \qquad (t\in\mathrm{I}, \boldsymbol{q}\in\mathbf{E}).$$

We know that $\Omega_{\boldsymbol{U}}(t)$ is antisymmetric (see 11.1.10). Since $q\star t$ can be an arbitrary world point, we have proved:

Proposition. If $\boldsymbol{U}$ is a global rigid observer then $\Omega_{\boldsymbol{U}}(t)$ is an antisymmetric linear map for all $t\in\mathrm{I}$; it is independent of t_o appearing in its definition. Moreover,

$$\boldsymbol{U}(x+\boldsymbol{q}) - \boldsymbol{U}(x) = \Omega_{\boldsymbol{U}}(\tau(x))\cdot\boldsymbol{q} \qquad (x\in\mathrm{M},\ \boldsymbol{q}\in\mathbf{E}). \quad (**)$$

The global rigid observer $\boldsymbol{U}$ is rotation-free if and only if $\Omega_{\boldsymbol{U}}(t)=\mathbf{0}$ for all $t\in\mathrm{I}$. ■

Notice that the restriction of $\boldsymbol{U}$ to an arbitrary simultaneous hyperplane t is an affine map whose underlying linear map is $\Omega_{\boldsymbol{U}}(t)$.

$\Omega_{\boldsymbol{U}}(t)$ can be interpreted as the *angular velocity* of the observer at the instant t (see 11.1.10).

4.2.5. For arbitrarily fixed $t_o\in\mathrm{I}$, the function $t\mapsto R_{\boldsymbol{U}}(t,t_o)$ defines the function $t\mapsto\Omega_{\boldsymbol{U}}(t)$ according to the preceding paragraph. Conversely, if the function $t\mapsto\Omega_{\boldsymbol{U}}(t)$ is known, then $t\mapsto R_{\boldsymbol{U}}(t,t_o)$ is determined as the unique solution of the differential equation

$$(X:\mathrm{I}\to\mathbf{E}\otimes\mathbf{E}^*)? \qquad \dot{X}=\Omega_{\boldsymbol{U}}\cdot X$$

with the initial condition

$$X(t_o)=\mathrm{id}_{\mathbf{E}}.$$

4.2.6. We see from the formula $(**)$ of 4.2.4 that the rigid observer $\boldsymbol{U}$ is completely determined by an arbitrarily chosen $\boldsymbol{U}$-space point q_o and by the angular velocity of the observer, i.e. by the function $t\mapsto\Omega_{\boldsymbol{U}}(t)$. Indeed, putting $\boldsymbol{q}:=q_o\star\tau(x)-x$ in that formula we obtain

$$\boldsymbol{U}(x)=\boldsymbol{U}(q_o\star\tau(x))+\Omega_{\boldsymbol{U}}(\tau(x))\cdot(x-q_o\star\tau(x)) \qquad (x\in\mathrm{M})$$

and we know that the values of $\boldsymbol{U}$ on q_o coincide with the derivative of the world line function $t\mapsto q_o\star t$.

4.3. About the spaces of rigid observers*

4.3.1. Proposition. Let $\boldsymbol{U}$ be a fit global observer. $\boldsymbol{U}$ is rigid and rotation-free if and only if $\mathrm{E}_{\boldsymbol{U}}$, equipped with the subtraction

$$q_1 - q_2 := q_1 \star t - q_2 \star t \qquad (q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}},\ t \in \mathrm{I})$$

is an affine space over $\mathbf{E}$.

Proof. If $\boldsymbol{U}$ is rigid and rotation-free then, for all $q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}}$, $q_1 \star t - q_2 \star t$ is the same element of $\mathbf{E}$ for all $t \in \mathrm{I}$. It is not hard to see that the subtraction in the assertion satisfies the requirements listed in the definition of affine spaces.

Conversely, if $\mathrm{E}_{\boldsymbol{U}}$ is an affine space over $\mathbf{E}$ with the given subtraction then, in particular, $q_1 \star t - q_2 \star t$ is independent of t for all $q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}}$, hence $\boldsymbol{U}$ is rigid and rotation-free.

4.3.2. If $\boldsymbol{U}$ is a global rigid and rotation-free observer, then $\mathrm{E}_{\boldsymbol{U}}$ is an affine space, thus the differentiability of the splitting of spacetime according to $\boldsymbol{U}$ makes sense.

Proposition. Let $\boldsymbol{U}$ be a global rigid and rotation-free observer. Then the splitting

$$H_{\boldsymbol{U}} : \mathrm{M} \to \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}, \qquad x \mapsto (\tau(x),\ C_{\boldsymbol{U}}(x))$$

is a smooth bijection,

$$\mathrm{D}H_{\boldsymbol{U}}(x) = \big(\boldsymbol{\tau}, \boldsymbol{\pi}_{\boldsymbol{U}(x)}\big) \qquad (x \in \mathrm{M}),$$

and the inverse of $H_{\boldsymbol{U}}$ is smooth as well.

Proof. For $x \in \mathrm{M}$ and $t \in \mathrm{I}$ we have $C_{\boldsymbol{U}}(x) \star t = x + \boldsymbol{U}(x)(t - \tau(x)) + \mathrm{ordo}(t - \tau(x))$ (VI.3.3). Thus for all $y, x \in \mathrm{M}$ (see Exercise 4.5.1),

$$\begin{aligned} C_{\boldsymbol{U}}(y) - C_{\boldsymbol{U}}(x) &= y - C_{\boldsymbol{U}}(x) \star \tau(y) = \\ &= y - x + \boldsymbol{U}(x)(\tau(y) - \tau(x)) + \mathrm{ordo}(\tau(y) - \tau(x)) \end{aligned}$$

and so

$$\begin{aligned} H_{\boldsymbol{U}}(y) - H_{\boldsymbol{U}}(x) &= (\tau(y) - \tau(x), \quad C_{\boldsymbol{U}}(y) - C_{\boldsymbol{U}}(x)) = \\ &= \big(\boldsymbol{\tau} \cdot (y - x), \quad \boldsymbol{\pi}_{\boldsymbol{U}(x)} \cdot (y - x)\big) + \mathrm{ordo}(\boldsymbol{\tau}(y - x)). \end{aligned}$$

Hence $H_{\boldsymbol{U}}$ is differentiable, its derivative is the one given in the proposition. As a consequence, we see that $H_{\boldsymbol{U}}$ is smooth; its inverse is smooth by the inverse mapping theorem.

4.3.3. The space of a rigid and rotation-free global observer, endowed with a natural subtraction, is an affine space over $\mathbf{E}$. The space of another observer is *not* affine space with *that subtraction* (in fact that subtraction makes no sense for other observers). This does not mean that the space of other observers cannot be endowed with an affine structure in some other way.

Let us consider a fit global observer $\boldsymbol{U}$. For every instant t we can define the *instantaneous* affine structure on $\mathrm{E}_{\boldsymbol{U}}$ by the subtraction $q_1 - q_2 := q_1 \star t - q_2 \star t$. In general, different instants determine different instantaneous affine structures and all instants have the same "right" for establishing an affine structure on the $\boldsymbol{U}$-space. There is no natural way to select an instant and to use the corresponding instantaneous affine structure as the affine structure of $\mathrm{E}_{\boldsymbol{U}}$.

Nevertheless, we can define a natural affine structure on the spaces of *rigid global* observers.

4.3.4. Though the earth rotates, we experience on it an affine structure independent of time. A stick on the earth represents a vector. Evidently, the stick rotates together with the earth. The stick will be represented in the following reasoning by two points (the extremities of the stick) in the observer space. Now we wish to define that two points in the space of a rigid observer determine a vector (rotating together with the observer).

Let $\boldsymbol{U}$ be a rigid global observer. If q_1 and q_2 are points in the observer space $\mathrm{E}_{\boldsymbol{U}}$ then for all $s, s' \in \mathrm{I}$

$$q_1 \star s - q_2 \star s = R_{\boldsymbol{U}}(s, s') \cdot (q_1 \star s' - q_2 \star s').$$

Let us introduce

$$\mathbf{E}_{\boldsymbol{U}} := \{\boldsymbol{\phi} : \mathrm{I} \to \mathbf{E} | \quad \boldsymbol{\phi} \quad \text{is smooth}, \quad \boldsymbol{\phi}(s) = R_{\boldsymbol{U}}(s, s') \cdot \boldsymbol{\phi}(s') \quad \text{for all} \quad s, s' \in \mathrm{I}\}.$$

It is a routine to check that $\mathbf{E}_{\boldsymbol{U}}$, endowed with the usual pointwise addition and pointwise multiplication by real numbers, is a vector space; it is three-dimensional, because $\mathbf{E}_{\boldsymbol{U}} \to \mathbf{E}$, $\boldsymbol{\phi} \mapsto \boldsymbol{\phi}(s)$ is a linear bijection for arbitrary $s \in \mathrm{I}$ (which means in particular, that the function $\boldsymbol{\phi}$ is completely determined by a single one of its values). Moreover, if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are elements of $\mathbf{E}_{\boldsymbol{U}}$, then $\boldsymbol{\phi}(s) \cdot \boldsymbol{\psi}(s)$ is the same for all instants s, thus

$$\mathbf{E}_{\boldsymbol{U}} \times \mathbf{E}_{\boldsymbol{U}} \to \mathbf{D} \otimes \mathbf{D}, \qquad (\boldsymbol{\phi}, \boldsymbol{\psi}) \mapsto \boldsymbol{\phi} \cdot \boldsymbol{\psi} := \boldsymbol{\phi}(s) \cdot \boldsymbol{\psi}(s)$$

is a positive definite symmetric bilinear map which turns $\mathbf{E}_{\boldsymbol{U}}$ into a Euclidean vector space.

Now it is quite evident that $\mathrm{E}_{\boldsymbol{U}}$, endowed with the subtraction

$$q_1 - q_2 := (\mathrm{I} \to \mathbf{E}, \ s \mapsto (q_1 \star s - q_2 \star s))$$

will be an affine space over $\mathbf{E}_{\boldsymbol{U}}$. In other words, the difference of two $\boldsymbol{U}$-space points is exactly the difference of the corresponding world line functions, as the difference of functions is defined.

If $\boldsymbol{U}$ is rotation-free, then $\mathbf{E}_{\boldsymbol{U}}$ consists of the constant functions from I into $\mathbf{E}$ which can be identified with $\mathbf{E}$. So we get back our previous result that the space of a global rigid and rotation-free observer is an affine space over $\mathbf{E}$ in a natural way.

If $\boldsymbol{U}$ is not rotation-free then $\mathbf{E}_{\boldsymbol{U}}$ is a three-dimensional Euclidean affine space in a natural way, but the underlying vector space is not $\mathbf{E}$; in fact the underlying vector space $\mathbf{E}_{\boldsymbol{U}}$ depends on the observer itself.

4.3.5. The space of a global rigid observer is an affine space, thus the differentiability of the splitting of spacetime according to the observer makes sense. This question, reduced to a simpler affine structure, will be studied in the next section.

4.4. Observers with origin*

4.4.1. The vectorization of observer spaces simplifies some formulae for inertial observers and it will be a powerful tool for non-inertial rigid observers.

Let $\boldsymbol{U}$ be a global rigid and rotation-free observer. Choosing an instant t_o and a $\boldsymbol{U}$-space point q_o, we give the corresponding *vectorization* of time and $\boldsymbol{U}$-space:

$$V_o : \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \to \mathbf{I} \times \mathbf{E}, \qquad (t, q) \mapsto (t - t_o,\ \ q - q_o).$$

We see that in this way $\boldsymbol{U}$-space points (curves in M) are represented by spacelike vectors (points in $\mathbf{E}$).

Notice that choosing t_o and q_o is equivalent to choosing a "spacetime reference origin" $o \in \mathrm{M} : o := q_o \star t_o$, $t_o = \tau(o)$, $q_o = C_{\boldsymbol{U}}(o)$. That is why we have used the symbol V_o for the vectorization which can be written in the following form, too:

$$V_o : \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \to \mathbf{I} \times \mathbf{E}, \qquad (t, q,) \mapsto (t - \tau(o), \quad q \star \tau(o) - o)\,,$$

since $q - q_o = q \star t - q_o \star t$ for all $t \in \mathrm{I}$, in particular for $t := \tau(o)$.

If $\boldsymbol{U}$ is not rotation-free, the result of a similar vectorization

$$V_o : \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{U}}, \qquad (t, q) \mapsto (t - t_o, \quad q - q_o)$$

is not simple enough because the elements of $\mathbf{E}_{\boldsymbol{U}}$ are functions. That is why we make a further step by the linear bijection

$$L_o : \mathbf{E}_{\boldsymbol{U}} \to \mathbf{E}, \qquad \boldsymbol{\phi} \mapsto \boldsymbol{\phi}(t_o).$$

Since $L_o \cdot (q - q_o) = (q - q_o)(t_o) = q \star t_o - q_o \star t_o$, we get the *double vectorization* of time and $\boldsymbol{U}$-space:

$$W_o := (\mathrm{id}_{\mathbf{I}} \times L_o) \circ V_o : \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \to \mathbf{I} \times \mathbf{E}, \qquad (t, q) \mapsto (t - \tau(o), \quad q \star \tau(o) - o)\,,$$

which coincides formally with the vectorization of time and space of a rigid and rotation-free observer.

4.4.2. Definition. An *observer with reference origin* is a pair $(\boldsymbol{U}, o)$ where $\boldsymbol{U}$ is a global rigid observer and o is a world point.

If $\boldsymbol{U}$ is rotation-free, the *vectorized splitting* of spacetime corresponding to $(\boldsymbol{U}, o)$ is the map

$$H_{\boldsymbol{U},o} := V_o \circ H_{\boldsymbol{U}} : \mathrm{M} \to \mathbf{I} \times \mathbf{E}, \qquad x \mapsto (\tau(x) - \tau(o), \quad C_{\boldsymbol{U}}(x) - C_{\boldsymbol{U}}(o)) \\ = (\tau(x) - \tau(o), \quad C_{\boldsymbol{U}}(x) \star \tau(o) - o)\,,$$

and if $\boldsymbol{U}$ is not rotation-free then the *double vectorized splitting* of spacetime is the map

$$H_{\boldsymbol{U},o} := W_o \circ H_{\boldsymbol{U}} : \mathrm{M} \to \mathbf{I} \times \mathbf{E}, \qquad x \mapsto (\tau(x) - \tau(o), \quad C_{\boldsymbol{U}}(x) \star \tau(o) - o)\,.$$

4.4.3. Proposition. Let$(\boldsymbol{U}, o)$ be an observer with reference origin.

If $\boldsymbol{U}$ is rotation-free then the vectorized splitting is a smooth bijection whose inverse is smooth as well and

$$\mathrm{D}H_{\boldsymbol{U},o}(x) = (\boldsymbol{\tau}, \boldsymbol{\pi}_{\boldsymbol{U}(x)}) \qquad (x \in \mathrm{M}).$$

If $\boldsymbol{U}$ is not rotation-free, the double vectorized splitting is a smooth bijection whose inverse is smooth as well and

$$\mathrm{D}H_{\boldsymbol{U},o}(x) = \Big(\boldsymbol{\tau}, R_{\boldsymbol{U}}(\tau(x), t_o)^{-1} \cdot \boldsymbol{\pi}_{\boldsymbol{U}(x)}\Big) \qquad (x \in \mathrm{M})$$

where $t_o := \tau(o)$.

Proof. For rotation-free observers the assertion is trivial because of 4.3.2 and because the derivative of V_o is the identity of $\mathbf{I} \times \mathbf{E}$.

For the double vectorization we argue as follows: the map $\mathrm{M} \to \mathbf{E}$, $x \mapsto C_{\boldsymbol{U}}(x) \star t_o - o = R_{\boldsymbol{U}}(\tau(x), t_o)^{-1} \cdot (x - C_{\boldsymbol{U}}(o) \star \tau(x)\ )$ (see formula $(*)$ in 4.2.3.) is clearly differentiable, its derivative is the linear map (see Exercise 11.2.2)

$$\mathbf{M} \to \mathbf{E}, \qquad \boldsymbol{x} \mapsto - R_{\boldsymbol{U}}(\tau(x), t_o)^{-1} \cdot \Omega_{\boldsymbol{U}}(\tau(x)) \cdot (x - C_{\boldsymbol{U}}(o) \star \tau(x))\, \boldsymbol{\tau} \cdot \boldsymbol{x} + \\ + R_{\boldsymbol{U}}(\tau(x), t_o)^{-1} \cdot (\boldsymbol{x} - \boldsymbol{U}(C_{\boldsymbol{U}}(o) \star \tau(x))\boldsymbol{\tau} \cdot \boldsymbol{x}) = \\ = R_{\boldsymbol{U}}(\tau(x), t_o)^{-1} \cdot \boldsymbol{\pi}_{\boldsymbol{U}(x)} \cdot \boldsymbol{x}. \qquad \blacksquare$$

Since W_o is an affine bijection, it follows that the splitting $H_{\boldsymbol{U}} : \mathrm{M} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{U}}$ is smooth and has a smooth inverse as well (cf. 4.3.5.).

4.4.4. Dealing with observers in the arithmetic spacetime model it is extremely convenient to consider observers with reference origin where the reference origin coincides with the origin $(0,\mathbf{0})$ of $\mathbb{R}\times\mathbb{R}^3$. Namely, in this case the (double) vectorized observer spaces are $\mathbb{R}^3$ and the (double) vectorized splitting is a linear map $\mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}\times\mathbb{R}^3$ whose zeroth component is the zeroth projection.

4.5. Exercises

1. If $\mathrm{E}_{\boldsymbol{U}}$ is an affine space over $\mathbf{E}$ with the subtraction given in 4.3.1, then

$$\begin{aligned} C_{\boldsymbol{U}}(x+\boldsymbol{q}) =& C_{\boldsymbol{U}}(x)+\boldsymbol{q}, \\ C_{\boldsymbol{U}}(y)-C_{\boldsymbol{U}}(x) =& C_{\boldsymbol{U}}(y)\star\tau(x)-x= \\ =& y-C_{\boldsymbol{U}}(x)\star\tau(y). \end{aligned}$$

for all $x,y\in\mathrm{M},\boldsymbol{q}\in\mathbf{E}$.

2. Prove that

$$\Omega_{\boldsymbol{U}}(t)=\dot{R}_{\boldsymbol{U}}(t,t) \qquad (t\in\mathrm{I})$$

(see 4.2.4).

3. We know that the derivative at a point of a double vectorization is of the form $(\boldsymbol{\tau},\boldsymbol{R}^{-1}\cdot\boldsymbol{\pi_u}):\mathbf{M}\to\mathbf{I}\times\mathbf{E}$ where $\boldsymbol{u}\in\mathrm{V}(1)$ and $\boldsymbol{R}$ is an orthogonal map $\mathbf{E}\to\mathbf{E}$, i.e. $\boldsymbol{R}^{*\!\!\!*}=\boldsymbol{R}^{-1}$ (see 4.4.3). Recall that the adjoint $\boldsymbol{R}^{*\!\!\!*}$ is identified with the transpose $\boldsymbol{R}^*$ due to the identification $\frac{\mathbf{E}}{\mathbf{D}\otimes\mathbf{D}}\equiv\mathbf{E}^*$. Thus we have $\boldsymbol{R}^*\equiv\boldsymbol{R}^{*\!\!\!*}=\boldsymbol{R}^{-1}$ and so $(\mathbf{i}\cdot\boldsymbol{R})^*=\boldsymbol{R}^{-1}\cdot\mathbf{i}^*$. Prove that

$$(\boldsymbol{\tau},\boldsymbol{R}^{-1}\cdot\boldsymbol{\pi_u})^{*^{-1}}=(\boldsymbol{u},\boldsymbol{R}^{-1}\cdot\mathbf{i}^*).$$

4. Let $\boldsymbol{U}$ be a global rigid observer. Using Proposition 4.2.1. prove that $D\boldsymbol{U}(x)|_{\mathbf{E}}$ is antisymmetric for all $x\in\mathrm{M}$ (which is proved in 4.2.4 in another way).

5. Let $\boldsymbol{U}$ be a fit global observer. Demonstrate that $\boldsymbol{U}$ is rotation-free if and only if there is a smooth map $\alpha:\mathrm{I}\times\mathrm{M}\times\mathbf{E}\to\mathbb{R}$ such that

(*i*) $C_{\boldsymbol{U}}(x+\boldsymbol{q})\star t-C_{\boldsymbol{U}}(x)\star t=\alpha(t,x,\boldsymbol{q})\boldsymbol{q}$ $\qquad (t\in\mathrm{I},x\in\mathrm{M},\boldsymbol{q}\in\mathbf{E})$;

(*ii*) $\alpha(\tau(x),x,\boldsymbol{q})=1$ $\qquad (x\in\mathrm{M},\boldsymbol{q}\in\mathbf{E})$;

(*iii*) $\alpha(t,x,0)=1$ $\qquad (t\in\mathrm{I},x\in\mathrm{M})$.

6. Using the previous result prove that if $\boldsymbol{U}$ is a global rigid and rotation-free observer then there is a smooth map $\beta:\mathrm{M}\to\frac{\mathbb{R}}{\mathbf{I}}$ such that $\mathrm{D}\boldsymbol{U}(x)|_{\mathbf{E}}=\beta(x)\mathrm{id}_{\mathbf{E}}$ for all $x\in\mathrm{M}$.

5. Some special observers

5.1. Why the inertial observers are better than the others

5.1.1. We know that the space of a rigid and rotation-free global observer, even if it is not inertial, is an affine space over $\mathbf{E}$. However, the splitting of spacetime according to non-inertial observers is not affine.

Proposition. Let $\boldsymbol{U}$ be a rigid and rotation-free global observer. The splitting of spacetime according to $\boldsymbol{U}$,

$$H_{\boldsymbol{U}} : \mathrm{M} \to \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}, \qquad x \mapsto (\tau(x), C_{\boldsymbol{U}}(x))$$

is an affine map if and only if $\boldsymbol{U}$ is inertial.

Proof. We have seen that if $\boldsymbol{U}$ is inertial then $H_{\boldsymbol{U}}$ is affine.

We know that $H_{\boldsymbol{U}}$ is differentiable, $\mathrm{D}H_{\boldsymbol{U}}(x) = (\boldsymbol{\tau}, \boldsymbol{\pi}_{\boldsymbol{U}(x)})$ (see 4.3.2). If $H_{\boldsymbol{U}}$ is affine, then $\mathrm{D}H_{\boldsymbol{U}}(x)$ is the same for all $x \in \mathrm{M}$. This means that $\boldsymbol{\pi}_{\boldsymbol{U}(x)}$ does not depend on x which implies that $\boldsymbol{U}$ is a constant map as well.

5.1.2. We can say that if $\mathrm{E}_{\boldsymbol{U}}$ is affine but $\boldsymbol{U}$ is not inertial then the affine structures of M and $\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}$— though they are mathematically isomorphic — are not related from a physical point of view.

If $\mathrm{E}_{\boldsymbol{U}}$ is affine, then $(\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}, \mathrm{I}, \mathrm{pr}_{\mathrm{I}}, \mathbf{D}, \mathbf{b})$ is a non-relativistic spacetime model and so it is isomorphic to the spacetime model $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$; however, the physically meaningful triplet $(H_{\boldsymbol{U}}, \mathrm{id}_{\mathrm{I}}, \mathrm{id}_{\mathbf{D}})$ is an isomorphism between them if and only if $\boldsymbol{U}$ is inertial.

This shows that global inertial observers play an important role in applications. Let $\boldsymbol{U}$ be a global inertial observer and suppose an assertion is formulated for some objects related to $\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}$; then the assertion concerns an absolute fact if it uses only the affine structure of $\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}$. The assertion has not necessarily an absolute content if it uses other properties of $\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}$; for instance, the Cartesian product structure or the affine structure of $E_{\boldsymbol{U}}$ alone.

5.2. Uniformly accelerated observer

5.2.1. The rigid global observer $\boldsymbol{U}$ is called *uniformly accelerated* if its acceleration field is a non-zero constant, i.e. there is a $\mathbf{0} \neq \boldsymbol{a} \in \frac{\mathbf{E}}{\mathbf{I} \otimes \mathbf{I}}$ such that

$$\boldsymbol{A}_{\boldsymbol{U}}(x) := \mathrm{D}\boldsymbol{U}(x) \cdot \boldsymbol{U}(x) = \boldsymbol{a}. \qquad (x \in \mathrm{M}).$$

Equivalently, for all $\boldsymbol{U}$-space points q, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(q \star t) = \boldsymbol{a}$ $(t \in \mathrm{I})$.

We have for all $x \in \mathrm{M}$ and $t \in \mathrm{I}$ that

$$C_{\boldsymbol{U}}(x) \star t = x + \boldsymbol{U}(x)\,(t - \tau(x)) + \frac{1}{2}\boldsymbol{a}(t-\tau(x))^2$$

and

$$\boldsymbol{U}\,(C_{\boldsymbol{U}}(x) \star t) = \frac{\mathrm{d}}{\mathrm{d}t}\,(C_{\boldsymbol{U}}(x) \star t) = \boldsymbol{U}(x) + \boldsymbol{a}\,(t - \tau(x))\,. \qquad (*)$$

Now it follows that for all $x \in \mathrm{M}$, $\boldsymbol{q} \in \mathbf{E}$ and $t \in \mathrm{I}$

$$C_{\boldsymbol{U}}(x+\boldsymbol{q}) \star t - C_{\boldsymbol{U}}(x) \star t = \boldsymbol{q} + (\boldsymbol{U}(x+\boldsymbol{q}) - \boldsymbol{U}(x))\,(t-\tau(x))\,.$$

Since $\boldsymbol{U}$ is rigid, the length of this vector is independent of t, so it equals the length of $\boldsymbol{q}$. Then assertion (i) in proposition 4.2.1 implies that

$$\boldsymbol{U}(x+\boldsymbol{q}) - \boldsymbol{U}(x) = \mathbf{0} \qquad (x \in \mathrm{M}, \boldsymbol{q} \in \mathbf{E})$$

which means, according to the quoted proposition, that $\boldsymbol{U}$ is rotation-free.

5.2.2. $\boldsymbol{U}$ is constant on the simultaneous hyperplanes. Thus $\boldsymbol{U}(C_{\boldsymbol{U}}(x) \star \tau(y)) = \boldsymbol{U}(y)$ for all $x, y \in \mathrm{M}$ and we infer from $(*)$ that

$$\boldsymbol{U}(y) = \boldsymbol{U}(x) + \boldsymbol{a}\,(\boldsymbol{\tau}(y-x)) \qquad (x, y \in \mathrm{M}).$$

As a corollary, the uniformly accelerated observer $\boldsymbol{U}$ is uniquely determined by a single value of $\boldsymbol{U}$ at an arbitrary world point and by the constant value of the acceleration field of $\boldsymbol{U}$.

We see as well that the uniformly accelerated observer is an affine map from M into $\mathrm{V}(1)$ whose underlying linear map is $\boldsymbol{a} \cdot \boldsymbol{\tau}$.

5.2.3. Let the previous observer choose a reference origin o. Then

$$\begin{aligned} C_{\boldsymbol{U}}(x) - C_{\boldsymbol{U}}(o) =& x - C_{\boldsymbol{U}}(o) \star \tau(x) = \\ =& x - o + \boldsymbol{U}(o)\,(\tau(x) - \tau(o)) - \frac{1}{2}\boldsymbol{a}(\tau(x) - \tau(o))^2. \end{aligned}$$

As a consequence, the vectorized splitting of spacetime is

$$\mathrm{M} \to \mathbf{I} \times \mathbf{E}, \quad x \mapsto \left(\boldsymbol{\tau} \cdot (x - o), \quad \boldsymbol{\pi}_{\boldsymbol{U}(o)} \cdot (x - o) - \frac{1}{2}\boldsymbol{a}(\boldsymbol{\tau} \cdot (x - o))^2\right).$$

5.2.4. For $\alpha > 0$, the observer

$$(\xi^0,\, \xi^1,\, \xi^2,\, \xi^3) \mapsto (1,\, \alpha\xi^0,\, 0,\, 0)$$

in the arithmetic spacetime model is uniformly accelerated. Its maximal integral curve passing through $(\xi^0,\, \xi^1,\, \xi^2,\, \xi^3)$ is

$$\begin{aligned} &\left\{\left(t,\, \xi^1 + \alpha\xi^0(t - \xi^0) + \frac{1}{2}\alpha(t - \xi^0)^2,\, \xi^2,\, \xi^3\right) \mid \ t \in \mathbb{R}\right\} = \\ &\qquad = \left\{\left(t,\, \xi^1 + \frac{1}{2}\alpha t^2 - \frac{1}{2}\alpha(\xi^0)^2,\, \xi^2,\, \xi^3\right) \mid \ t \in \mathbb{R}\right\}. \end{aligned}$$

Accordingly, if the observer chooses $(0, \mathbf{0})$ as a reference origin then the vectorized splitting becomes

$$\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3, \qquad (\xi^0,\, \xi^1,\, \xi^2,\, \xi^3) \mapsto \left(\xi^0,\, \xi^1 - \frac{1}{2}\alpha(\xi^0)^2,\, \xi^2,\, \xi^3\right).$$

5.3. Uniformly rotating observer

5.3.1. The global observer $\boldsymbol{U}$ is called *uniformly rotating* if there is a non-zero antisymmetric linear map $\Omega : \mathbf{E} \to \frac{\mathbf{E}}{\mathbf{I}}$ (in other words, $\Omega \in \frac{\mathbf{N} \wedge \mathbf{N}}{\mathbf{I}}$, $\mathbf{N} := \frac{\mathbf{E}}{\mathbf{D}}$), called the *angular velocity,* such that

$$\boldsymbol{U}(x + \boldsymbol{q}) - \boldsymbol{U}(x) = \Omega \cdot \boldsymbol{q} \qquad (x \in \mathrm{M}, \boldsymbol{q} \in \mathbf{E}).$$

Proposition 4.2.1 (i) implies that $\boldsymbol{U}$ is rigid. Moreover, we easily obtain that

$$R_{\boldsymbol{U}}(t, t_o) = e^{(t - t_o)\Omega} \qquad (t_o, t \in \mathrm{I}),$$

because this is the (necessarily unique) solution of the initial value problem given in 4.2.5.

Consequently, 3.4.3 yields that if $o, x \in \mathrm{M}$, $\tau(o) = t_o$ and $t \in \mathrm{I}$ then

$$C_{\boldsymbol{U}}(x) \star t = C_{\boldsymbol{U}}(o) \star t + e^{(t-t_o)\Omega} \cdot (C_{\boldsymbol{U}}(x) \star t_o - o)\,. \qquad (*)$$

Every $\boldsymbol{U}$-line is obtained from a given one and from Ω. This formula becomes simpler if we consider $x \in t_o$:

$$C_{\boldsymbol{U}}(x) \star t = C_{\boldsymbol{U}}(o) \star t + e^{(t-t_o)\Omega} \cdot (x - o).$$

$\boldsymbol{U}$ itself is determined by its values on a given $\boldsymbol{U}$-line q_o and by Ω (4.2.6):

$$\boldsymbol{U}(x) = \boldsymbol{U}\,(q_o \star \tau(x)) + \Omega \cdot (x - q_o \star \tau(x)) \qquad (x \in \mathrm{M}).$$

5.3.2. Reformulating the previous result we can say that a uniformly rotating observer can be given by the history of a point of the observer (by a space point of the observer) and by its angular velocity. If we deal with a uniformly rotating observer then we are to look for its "best" space points to have a simple description of the observer. Even if the observer is given by one of its space points and by its angular velocity, it may happen that we find a "better" space point than the given one.

Now we shall examine a uniformly rotating observer $\boldsymbol{U}$ that has an inertial spacepoint. Then there is an $o \in \mathrm{M}$ and a $\boldsymbol{c} \in \mathrm{V}(1)$ such that $q_o := o + \boldsymbol{c} \otimes \mathrm{I}$ is a $\boldsymbol{U}$-line. $\boldsymbol{U}$ equals $\boldsymbol{c}$ on q_o, thus

$$\boldsymbol{U}(x) = \boldsymbol{c} + \Omega \cdot \boldsymbol{\pi_c} \cdot (x - o) \qquad (x \in \mathrm{M}).$$

We see that $\boldsymbol{U}$ is an affine map, the underlying linear map is $\Omega \cdot \boldsymbol{\pi_c}$ whose range coincides with the range of Ω which is a two- dimensional linear subspace in $\frac{\mathbf{E}}{\mathrm{I}}$.

We know that the kernel of Ω is one-dimensional and orthogonal to Ran Ω (see V.3.9). If $\boldsymbol{e} \in \mathrm{Ker}\ \Omega$, then $\boldsymbol{U}(o + \boldsymbol{e} + \boldsymbol{c}\boldsymbol{t}) = \boldsymbol{c}$ for all $\boldsymbol{t} \in \mathbf{I}$, i.e. $\boldsymbol{U}$ is constant on the inertial world line $o + \boldsymbol{e} + \boldsymbol{c} \otimes \mathbf{I}$ as well. Thus it is a maximal integral curve of $\boldsymbol{U}$, parallel to q_o. It is an easy task to show that

$$\{x \in \mathrm{M} \mid \ \boldsymbol{U}(x) = \boldsymbol{c}\} = o + \mathrm{Ker}\ \Omega + \boldsymbol{c} \otimes \mathbf{I}.$$

The observer has the acceleration field

$$\begin{aligned}\mathbf{A}_{\boldsymbol{U}}(x) =& \mathrm{D}\boldsymbol{U}(x) \cdot \boldsymbol{U}(x) = \Omega \cdot \boldsymbol{\pi_c} \cdot \boldsymbol{U}(x) = \Omega \cdot (\boldsymbol{U}(x) - \boldsymbol{c}) = \\ =& \Omega \cdot \Omega \cdot \boldsymbol{\pi_c} \cdot (x - o) \qquad (x \in \mathrm{M}).\end{aligned}$$

Since Ker $(\Omega^2) = \text{Ker } \Omega$ (Exercise V.3.20.2), the set of acceleration-free world points is $\{x \in \mathrm{M} \mid \boldsymbol{\pi_c} \cdot (x - o) \in \text{Ker } \Omega\}$ which equals $o + \text{Ker } \Omega + \boldsymbol{c} \otimes \mathbf{I}$.

Thus for all $\boldsymbol{e} \in \text{Ker } \Omega$, $o + \boldsymbol{e} + \boldsymbol{c} \otimes \mathbf{I}$ is an inertial $\boldsymbol{U}$-space point and there are no other inertial $\boldsymbol{U}$-space points. The inertial $\boldsymbol{U}$-space points corresponding to different elements of Ker Ω are different. The set

$$\{o + \boldsymbol{e} + \boldsymbol{c} \otimes \mathbf{I} \mid \boldsymbol{e} \in \text{Ker } \Omega\}$$

in $\mathrm{E}_{\boldsymbol{U}}$ is called the *axis of rotation.*

5.3.3. The axis of rotation makes sense for arbitrary uniformly rotating observers (see Exercise 5.4.4).

The earth can be modelled by a uniformly rotating observer. Note that the angle between the axis of rotation and the direction of progression makes no absolute sense. The direction of progression is the direction of the relative velocity with respect to the Sun. The axis of rotation (Ker Ω, an oriented one-dimensional linear subspace in $\mathbf{E}$) and a relative velocity value (an element of $\frac{\mathbf{E}}{\mathbf{I}}$ as we shall see in Section 6.2) make an angle; however, Ker Ω and an absolute velocity value ($\boldsymbol{c}$ in the former treatment) form no angle.

5.3.4. Let the previous observer choose o as a reference origin. Then formula $(*)$ in 5.3.1 yields that

$$C_{\boldsymbol{U}}(x) \star t_o - o = e^{-(\tau(x)-\tau(o))\Omega} \cdot (x - (o + \boldsymbol{c}(\boldsymbol{\tau}(x) - \tau(o)))\,,$$

thus the double vectorized splitting of spacetime becomes

$$\mathrm{M} \to \mathbf{I} \times \mathbf{E}, \qquad x \mapsto \Big(\boldsymbol{\tau} \cdot (x - o), e^{-\boldsymbol{\tau}\cdot(x-o)\Omega} \cdot \boldsymbol{\pi_c} \cdot (x - o)\Big)\,.$$

5.3.5. For $\omega > 0$, the observer

$$(\xi^0,\, \xi^1,\, \xi^2,\, \xi^3) \mapsto (1,\, -\omega\xi^2,\, \omega\xi^1,\, 0)$$

in the arithmetic spacetime model is uniformly rotating. Its maximal integral curve passing through $(\xi^0,\, \xi^1,\, \xi^2,\, \xi^3)$ is

$$\begin{aligned}\big\{\big(t,&\xi^1 \cos\omega(t-\xi^0) - \xi^2 \sin\omega(t-\xi^0),\\ &\xi^1 \sin\omega(t-\xi^0) + \xi^2 \cos\omega(t-\xi^0), \xi^3\big) \mid t \in \mathbb{R})\big\}\,.\end{aligned}$$

If the observer chooses $(0, \mathbf{0})$ as a reference origin, the double vectorized splitting will be

$$\begin{aligned}&\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3,\\ &(\xi^0,\, \xi^1,\, \xi^2,\, \xi^3) \mapsto \big(\xi^0,\, \xi^1 \cos\omega\xi^0 + \xi^2 \sin\omega\xi^0,\, -\xi^1 \sin\omega\xi^0 + \xi^2 \cos\omega\xi^0,\, \xi^3\big)\,.\end{aligned}$$

5.4. Exercises

1. Let $\boldsymbol{U}$ be a global observer. Demonstrate that the following assertions are equivalent:
(i) the acceleration field of $\boldsymbol{U}$ is zero,
(ii) all the integral curves of $\boldsymbol{U}$ are straight lines.
Such an observer need not be inertial. Consider the observer

$$(\xi^0,\ \xi^1,\ \xi^2,\ \xi^3) \mapsto (1,\ 0,\ \xi^1,\ 0)$$

in the arithmetic spacetime model. Give its maximal integral curves. Show that the observer is not rigid.

2. Let $\boldsymbol{U}$ be a global observer. Demonstrate that the following assertions are equivalent:
(i) the acceleration field of $\boldsymbol{U}$ is a non-zero constant,
(ii) all the integral curves of $\boldsymbol{U}$ are uniformly accelerated with the same non-zero acceleration.
Such an observer need not be uniformly accelerated. Consider the observer

$$(\xi^0,\ \xi^1,\ \xi^2,\ \xi^3) \mapsto (1, 0, \xi^0 + \xi^1, 0)$$

in the arithmetic spacetime model.

3. Prove that a global rigid observer whose integral curves are straight lines is inertial.

4. Define the axis of rotation for an arbitrary uniformly rotating observer.

5. Find the axis of rotation of the observer given in 4.3.5.

6. Since M and V(1) are affine spaces, it makes sense that a global observer $\boldsymbol{U} : \mathrm{M} \to \mathrm{V}(1)$ is affine; let $\mathrm{D}\boldsymbol{U} : \mathrm{M} \to \frac{\mathbf{E}}{\mathbf{I}}$ be the underlying linear map (the derivative of $\boldsymbol{U}$ at every point equals the linear map under $\boldsymbol{U}$). The restriction of $\mathrm{D}\boldsymbol{U}$ onto $\mathbf{E}$ will be denoted by $\Omega_{\boldsymbol{U}}$; it is a linear map from $\mathbf{E}$ into $\frac{\mathbf{E}}{\mathbf{I}}$. Prove that for all $x \in \mathrm{M}$ the world line function

$$\mathrm{I} \to \mathrm{M}, \quad t \mapsto x + \boldsymbol{U}(x)(t - \tau(x)) + \frac{1}{2}\mathrm{D}\boldsymbol{U} \cdot \boldsymbol{U}(x)(t - \tau(x))^2 + \\ + \sum_{n=3}^{\infty} \frac{1}{n!}((t - \tau(x))\Omega_{\boldsymbol{U}})^{n-2} \cdot \mathrm{D}\boldsymbol{U} \cdot \boldsymbol{U}(x)(t - \tau(x))^2$$

gives the maximal integral curve passing through x.

7. Let $\boldsymbol{U}$ be an affine observer. Then $\Omega_{\boldsymbol{U}} := \mathrm{D}\,\boldsymbol{U}|_{\mathbf{E}} : \mathbf{E} \to \frac{\mathbf{E}}{\mathbf{I}}$ is a linear map. Prove that

$$\text{(i)} \qquad C_{\boldsymbol{U}}(x + \boldsymbol{q}) = C_{\boldsymbol{U}}(x) + \boldsymbol{q} \qquad (x \in \mathrm{M},\ \boldsymbol{q} \in \mathbf{E})$$

if and only if $\boldsymbol{q} \in \mathrm{Ker}\ \Omega_{\boldsymbol{U}}$;

(ii) $\boldsymbol{U}$ is rigid if and only if $\Omega_{\boldsymbol{U}}$ is antisymmetric.

8. Let $\boldsymbol{U}$ be a rigid affine observer. Then, according to the previous exercise, $\Omega_{\boldsymbol{U}}$ is antisymmetric.

We distinguish four cases:

$$(i)\quad \Omega_{\boldsymbol{U}} \neq \mathbf{0},\ \ \mathrm{D}\boldsymbol{U}\cdot\boldsymbol{u} \neq \mathbf{0} \qquad \text{for all} \quad \boldsymbol{u} \in \mathrm{V}(1);$$

$$(ii)\quad \Omega_{\boldsymbol{U}} \neq \mathbf{0},\ \ \mathrm{D}\boldsymbol{U}\cdot\boldsymbol{c} = \mathbf{0} \qquad \text{for some} \quad \boldsymbol{c} \in \mathrm{V}(1);$$

$$(iii)\quad \Omega_{\boldsymbol{U}} = \mathbf{0},\ \ \mathrm{D}\boldsymbol{U}\cdot\boldsymbol{u} \neq \mathbf{0} \qquad \text{for all} \quad \boldsymbol{u} \in \mathrm{V}(1);$$

$$(iv)\quad \Omega_{\boldsymbol{U}} = \mathbf{0},\ \ \mathrm{D}\boldsymbol{U}\cdot\boldsymbol{c} = \mathbf{0} \qquad \text{for some} \quad \boldsymbol{c} \in \mathrm{V}(1)\ \text{(i.e. } \mathrm{D}\boldsymbol{U} = \mathbf{0}).$$

Demonstrate that

(iv) is an inertial observer,

(iii) is a uniformly accelerated observer,

(ii) is a uniformly rotating observer having an inertial space point,

(i) is a uniformly rotating observer having a uniformly accelerated space point.

(Hint: the kernel of $\Omega_{\boldsymbol{U}} \neq \mathbf{0}$ is one-dimensional, $\boldsymbol{U}$ and $\mathrm{D}\boldsymbol{U}$ are surjections. Hence there is a $\boldsymbol{c} \in \mathrm{V}(1)$ such that $\boldsymbol{a} := \mathrm{D}\boldsymbol{U}\cdot\boldsymbol{c}$ is in the kernel of $\Omega_{\boldsymbol{U}}$. Consequently, there is a world point o such that for all world points x

$$\boldsymbol{U}(x) = \boldsymbol{U}(o) + \mathrm{D}\boldsymbol{U}\cdot(x-o) = \boldsymbol{c} + \Omega_{\boldsymbol{U}}\cdot\pi_{\boldsymbol{c}}\cdot(x-o) + \boldsymbol{a}\boldsymbol{\tau}\cdot(x-o)$$

and so the observer has the acceleration field

$$\boldsymbol{A}_{\boldsymbol{U}}(x) = \boldsymbol{a} + \Omega_{\boldsymbol{U}}\cdot\Omega_{\boldsymbol{U}}\cdot\pi_{\boldsymbol{c}}\cdot(x-o)).$$

9. Take an $o \in \mathrm{M}$ and define the observer

$$\boldsymbol{U}(x) := \frac{x-o}{\boldsymbol{\tau}\cdot(x-o)} \qquad\qquad (x \in o + T^{\to}).$$

Prove that

(i) every $\boldsymbol{U}$-space point is inertial, more closely,

$$C_{\boldsymbol{U}}(x) \star t = o + \frac{x-o}{\boldsymbol{\tau}\cdot(x-o)}(t-\tau(o)) \qquad\qquad (x \in \mathrm{Dom}\,\boldsymbol{U},\ \ t > \tau(o));$$

(ii) the acceleration field corresponding to $\boldsymbol{U}$ is zero which follows from

$$\mathrm{D}\boldsymbol{U}(x) = \frac{\boldsymbol{\pi}_{\boldsymbol{U}(x)}}{\boldsymbol{\tau}\cdot(x-o)};$$

(iii) $\boldsymbol{U}$ is not rigid; the distance between two $\boldsymbol{U}$-space points increases as time passes.

10. Take an $o \in \mathrm{M}$, a $\boldsymbol{u} \in \mathrm{V}(1)$, an $\boldsymbol{s} \in \mathbf{I}^+$ and define the observer

$$\boldsymbol{U}(x) := \boldsymbol{u} + \frac{\boldsymbol{\pi_u} \cdot (x - o)}{\boldsymbol{s}} \qquad (x \in \mathrm{M}).$$

Demonstrate that
(i) $\boldsymbol{U}$ is an affine observer, more closely

$$\mathrm{D}\boldsymbol{U}(x) = \frac{\boldsymbol{\pi_u}}{\boldsymbol{s}} \qquad \text{for all} \qquad x \in \mathrm{M};$$

(ii) the acceleration field corresponding to $\boldsymbol{U}$ is

$$x \mapsto \frac{\boldsymbol{\pi_u} \cdot (x - o)}{\boldsymbol{s}^2} = \frac{\boldsymbol{U}(x) - \boldsymbol{u}}{\boldsymbol{s}};$$

(iii) $C_{\boldsymbol{U}}(o + \boldsymbol{q}) \star t = o + \boldsymbol{u}(t - \tau(o)) + e^{(t-\tau(o))/\boldsymbol{s}}\boldsymbol{q} \qquad (\boldsymbol{q} \in \mathbf{E}).$
(iv) $\boldsymbol{U}$ is rotation-free and is not rigid: the distance between two $\boldsymbol{U}$-space points increases with time.

6. Kinematics

6.1. The history of a masspoint is observed as a motion

6.1.1. The motion of a material point relative to an observer is described by a function assigning to an instant the space point where the material point is at that instant.

Now we are able to give how an observer determines the motion from the history of a material point.

Definition. Let $\boldsymbol{U}$ be a fit observer and let r be a world line function, $\mathrm{Ran}\, r \subset \mathrm{Dom}\, \boldsymbol{U}$. Then

$$r_{\boldsymbol{U}} : \mathrm{I} \rightarrowtail \mathrm{E}_{\boldsymbol{U}}\ , \qquad t \mapsto C_{\boldsymbol{U}}(r(t))$$

is called the *motion relative to* $\boldsymbol{U}$, or the $\boldsymbol{U}$*-motion,* corresponding to the world line function r. ■

6.1.2. If $\boldsymbol{U}$ is a global rigid observer, then $\mathrm{E}_{\boldsymbol{U}}$ is an affine space thus the differentiability of $r_{\boldsymbol{U}}$ makes sense and $r_{\boldsymbol{U}}$ is piecewise twice differentiable.

Given a rigid and rotation-free global observer $\boldsymbol{U}$ and a motion relative to $\boldsymbol{U}$, i.e. a piecewise twice differentiable function $m : \mathrm{I} \rightarrowtail \mathrm{E}_{\boldsymbol{U}}$, we can regain the history, i.e. the world line function r for which $r_{\boldsymbol{U}} = m$ holds. Indeed, for every

t, $m(t)$ is a $\boldsymbol{U}$-space point, i.e. a maximal integral curve of $\boldsymbol{U}$; then $r(t)$ will be the unique element in $t \cap m(t)$. In other words, using the splitting $H_{\boldsymbol{U}}$ we have

$$r(t) = {H_{\boldsymbol{U}}}^{-1}(t, m(t)) = m(t) \star t.$$

Similar considerations can be made for a general global rigid observer.

6.1.3. Let us consider the arithmetic spacetime model. As we know (see 2.1.4), a world line function r in it is given by a function $\boldsymbol{r} : \mathbb{R} \rightarrowtail \mathbb{R}^3$ in the form $r(t) = (t, \boldsymbol{r}(t))$. Paragraph 4.1.5 shows that $\boldsymbol{r}$ is the corresponding motion relative to the basic observer. We see that the history is regained very simply from the motion (in view of the previous considerations it is a consequence of the fact that for the basic observer $(1, \mathbf{0})$, $H_{(1,\mathbf{0})}$ is the identity of $\mathbb{R} \times \mathbb{R}^3$).

Thus if $\boldsymbol{r} : \mathbb{R} \rightarrowtail \mathbb{R}^3$ describes the motion relative to the basic observer then

$$r(t) = (t, \boldsymbol{r}(t)) \qquad (t \in \operatorname{Dom} \boldsymbol{r})$$

is the corresponding world line function.

6.2. Relative velocities

6.2.1. Proposition. Let $\boldsymbol{U}$ be a global rigid and rotation-free observer; if the world line function r is twice differentiable then $r_{\boldsymbol{U}}$ is twice differentiable as well and

$$\dot{r_{\boldsymbol{U}}}(t) = \dot{r}(t) - \boldsymbol{U}(r(t)), \qquad \ddot{r}_{\boldsymbol{U}}(t) = \ddot{r}(t) - \boldsymbol{A}_{\boldsymbol{U}}(r(t)).$$

Proof. Taking into account the relations

$$\begin{aligned} C_{\boldsymbol{U}}(r(s)) - C_{\boldsymbol{U}}(r(t)) =& C_{\boldsymbol{U}}(r(s)) \star s - C_{\boldsymbol{U}}(r(t)) \star s = \\ =& r(s) - r(t) - [C_{\boldsymbol{U}}(r(t)) \star s - C_{\boldsymbol{U}}(r(t)) \star t] \end{aligned}$$

we deduce

$$\dot{r_U}\,(t)=\lim_{s\to t}\frac{C_U\,(r(s))-C_U\,(r(t))}{s-t}=\dot{r}(t)-\boldsymbol{U}(r(t)),$$

from which $\ddot{r}_U(t)=\ddot{r}(t)-\mathrm{D}\boldsymbol{U}(r(t))\cdot\dot{r}(t)$ follows immediately. Since now $\boldsymbol{U}=\boldsymbol{V}\circ\tau$ (see 4.2.1 (*ii*)), we have $\mathrm{D}\boldsymbol{U}=(\mathrm{D}\boldsymbol{V}\circ\tau)\cdot\tau$ and $\boldsymbol{A_U}=\mathrm{D}\boldsymbol{V}\circ\tau$; thus the equality regarding the relative acceleration is verified. ■

The first and the second derivative of $r_{\boldsymbol{U}}$ is accepted as the *relative velocity* and the *relative acceleration* of r with respect to the global rigid and rotation-free observer $\boldsymbol{U}$, respectively.

6.2.2. The preceding result motivates the following definition.

Definition. Let $\boldsymbol{u}$ and $\boldsymbol{u}'$ be elements of $\mathrm{V}(1)$. Then

$$\boldsymbol{v_{u'u}}:=\boldsymbol{u}'-\boldsymbol{u}$$

is called the *relative velocity of* $\boldsymbol{u}'$ *with respect to* $\boldsymbol{u}$.

Proposition. Suppose $\boldsymbol{u}$, $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are elements of $\mathrm{V}(1)$. Then
(*i*) $\boldsymbol{v_{u'u}}$ is in $\frac{\mathbf{E}}{\mathbf{I}}$,
(*ii*) $\boldsymbol{v_{u'u}}=-\boldsymbol{v_{uu'}}$,
(*iii*) $\boldsymbol{v_{u''u}}=\boldsymbol{v_{u''u'}}+\boldsymbol{v_{u'u}}$. ■

These relations are very simple and they are in accordance with our everyday experience:
(*i*) the relative velocity values form a three-dimensional Euclidean vector space, the length of a relative velocity is in $\frac{\mathbf{D}}{\mathbf{I}}$;
(*ii*) if a body moves with a given relative velocity with respect to another body then the second body moves relative to the first one with the opposite velocity.
(*iii*) the sum of relative velocity values in a given order yields the resultant relative velocity value.

6.2.3. Let us imagine that a car is going on a straight road and it is raining. The raindrops hit the road and the car at different angles. What is the relation between the two angles?

Let $\boldsymbol{u}$ and $\boldsymbol{u}'$ be two different elements of $\mathrm{V}(1)$ (the absolute velocity values of the road and of the car, respectively). If $\boldsymbol{w}$ is an element of $\mathrm{V}(1)$, too, $\boldsymbol{w}\neq\boldsymbol{u}$, $\boldsymbol{w}\neq\boldsymbol{u}'$ (the absolute velocity value of the raindrops),

$$\theta(\boldsymbol{w}):=\arccos\frac{\boldsymbol{v_{wu}}\cdot\boldsymbol{v_{u'u}}}{|\boldsymbol{v_{wu}}||\boldsymbol{v_{u'u}}|}\,,\qquad \theta'(\boldsymbol{w}):=\arccos\frac{\boldsymbol{v_{wu'}}\cdot(-\boldsymbol{v_{uu'}})}{|\boldsymbol{v_{wu'}}||\boldsymbol{v_{uu'}}|}$$

are the angle formed by the relative velocity values $\boldsymbol{v_{wu}}$ and $\boldsymbol{v_{u'u}}$ and the angle formed by the relative velocity values $\boldsymbol{v_{wu'}}$ and $-\boldsymbol{v_{u'u}}=\boldsymbol{v_{uu'}}$, respectively (the angles at which the raindrops hit the road and the car, respectively).

A simple calculation yields that

$$\cos\theta(\boldsymbol{w}) = \frac{|\boldsymbol{v_{wu'}}|}{|\boldsymbol{v_{wu}}|}\cos\theta'(\boldsymbol{w}) + \frac{|\boldsymbol{v_{u'u}}|}{|\boldsymbol{v_{wu}}|}.$$

We call attention to an interesting limit case. Suppose $\boldsymbol{u}$ and $\boldsymbol{u}'$ are fixed and $\boldsymbol{w}$ tends to infinity, i.e. it varies in such a way that $|\boldsymbol{v_{wu}}|$ tends to infinity; then $|\boldsymbol{v_{wu'}}|$ tends to infinity as well and the quotient of these quantities tends to the number 1:

$$\lim_{\boldsymbol{w}\to\infty}\cos\theta(\boldsymbol{w}) = \lim_{\boldsymbol{w}\to\infty}\cos\theta'(\boldsymbol{w}),$$

which implies $\lim_{\boldsymbol{w}\to\infty}\theta(\boldsymbol{w}) = \lim_{\boldsymbol{w}\to\infty}\theta'(\boldsymbol{w})$.

Roughly speaking, the raindrops arriving with an "infinitely big" relative velocity hit the road and the car at the same angle. Replacing "raindrops with infinitely big relative velocity" by a "light beam" we get that non-relativistically there is *no aberration of light:* a light beam forms the same angle with the road and the car moving on the road.

We have spoken intuitively; of course the question arises at once: what is the model of a light beam in the non-relativistic spacetime model? What mathematical object in the non-relativistic spacetime model will correspond to a light beam? We shall see that none. A light beam cannot be modelled in the present spacetime model.

6.2.4. We can obtain the results of 6.2.1 by choosing a reference origin o in M, too, for the global rigid and rotation-free observer $\boldsymbol{U}$. Let us put $t_o := \tau(o)$, $q_o := C_{\boldsymbol{U}}(o)$. Evidently, the derivative of the *vectorized motion*

$$\boldsymbol{r_U} : \mathrm{I} \rightarrowtail \mathbf{E}\,, \qquad t \mapsto r_{\boldsymbol{U}}(t) - q_o$$

equals the derivative of $r_{\boldsymbol{U}}$. Since

$$\begin{aligned} r_{\boldsymbol{U}}(t) - q_o =& C_{\boldsymbol{U}}(r(t)) - q_o = C_{\boldsymbol{U}}(r(t)) \star t - q_o \star t = \\ =& r(t) - q_o \star t, \end{aligned}$$

we get immediately

$$\dot{r_{\boldsymbol{U}}}\,(t) = \dot{r}(t) - \boldsymbol{U}(q_o \star t) = \dot{r}(t) - \boldsymbol{U}(r(t)), \qquad (t \in \mathrm{I}),$$

because $\boldsymbol{U}$ is constant on the simultaneous hyperplanes.

We mention, that in practice it is more convenient to use the vectorized motion in such a form that time is vectorized, too:

$$\mathbf{I} \rightarrowtail \mathbf{E}\,, \qquad \boldsymbol{t} \mapsto r_{\boldsymbol{U}}(t_o + \boldsymbol{t}) - q_o = r(t_o + \boldsymbol{t}) - q_o \star (t_o + \boldsymbol{t}).$$

6.3. Motions relative to a rigid observer*

6.3.1. Recall that the space $\mathrm{E}_{\boldsymbol{U}}$ of a rigid global observer $\boldsymbol{U}$ is an affine space over $\mathbf{E}_{\boldsymbol{U}}$ consisting of functions $\mathrm{I} \to \mathbf{E}$ whose values "rotate together with the observer". Thus, in general, it is somewhat complicated to control the affine structure based on these vectors; we can simplify the calculations by performing a double vectorization of the observer space, corresponding to a chosen reference origin o in M. Let $t_o := \tau(o)$, $q_o := C_{\boldsymbol{U}}(o)$ and let $L_o : \mathbf{E}_{\boldsymbol{U}} \to \mathbf{E}$ be the linear bijection introduced in 4.4.1.

Let us take the motion $r_{\boldsymbol{U}}$ corresponding to the world line function r and let us consider the *double vectorized motion*

$$\boldsymbol{r_U} : \mathrm{I} \rightarrowtail \mathbf{E}\,, \qquad t \mapsto L_o \cdot (r_{\boldsymbol{U}}(t) - q_o) = C_{\boldsymbol{U}}(r(t)) \star t_o - o = \\ = R_{\boldsymbol{U}}(t, t_o)^{-1} \cdot (r(t) - q_o \star t)\,.$$

For the sake of simplicity, we shall use the notations $\boldsymbol{r}(t) := \boldsymbol{r_U}(t)$, $R(t) := R_{\boldsymbol{U}}(t, t_o)$, $\Omega(t) := \Omega_{\boldsymbol{U}}(t)$, $\boldsymbol{u}_o(t) := \boldsymbol{U}(q_o \star t)$, $\boldsymbol{a}_o(t) := \boldsymbol{A_U}(q_o \star t)$.

Then the previous formula can be written in the form

$$R(t) \cdot \boldsymbol{r}(t) = r(t) - q_o \star t;$$

differentiating with respect to t and then omitting t from the notation we obtain

$$\dot{R} \cdot \boldsymbol{r} + R \cdot \dot{\boldsymbol{r}} = \dot{r} - \boldsymbol{u}_o$$

yielding

$$R \cdot \dot{\boldsymbol{r}} = -\Omega \cdot R \cdot \boldsymbol{r} + \dot{r} - \boldsymbol{u}_o. \qquad (*)$$

A second differentiation gives

$$\dot{R} \cdot \dot{\boldsymbol{r}} + R \cdot \ddot{\boldsymbol{r}} = -\dot{\Omega} \cdot R \cdot \boldsymbol{r} - \Omega \cdot \dot{R} \cdot \boldsymbol{r} - \Omega \cdot R \cdot \dot{\boldsymbol{r}} + \ddot{r} - \boldsymbol{a}_o$$

from which we infer

$$R \cdot \ddot{\boldsymbol{r}} = -2\Omega \cdot R \cdot \dot{\boldsymbol{r}} - \Omega \cdot \Omega \cdot R \cdot \boldsymbol{r} - \dot{\Omega} \cdot R \cdot \boldsymbol{r} + \ddot{r} - \boldsymbol{a}_o.$$

6.3.2. Let us introduce the notation

$$\omega(t) := R(t)^{-1} \cdot \Omega(t) \cdot R(t) = R(t)^{-1} \cdot \dot{R}(t) \qquad (t \in \mathrm{I}).$$

From $R \cdot \omega = \Omega \cdot R$ we derive that $\dot{R} \cdot \omega + R \cdot \dot{\omega} = \dot{\Omega} \cdot R + \Omega \cdot \dot{R}$, which implies $\Omega \cdot R \cdot \omega + R \cdot \dot{\omega} = \dot{\Omega} \cdot R + \Omega \cdot \Omega \cdot R$; then we can state that

$$\dot{\omega} = R^{-1} \cdot \dot{\Omega} \cdot R.$$

Consequently, the last formula in the preceding paragraph can be written in the form

$$\ddot{\boldsymbol{r}} = -2\omega\cdot\dot{\boldsymbol{r}} - \omega\cdot\omega\cdot\boldsymbol{r} - \dot{\omega}\cdot\boldsymbol{r} + R^{-1}(\ddot{r} - \boldsymbol{a}_{\mathrm{o}}).$$

$-2\omega\cdot\dot{\boldsymbol{r}}$ and $-\omega\cdot\omega\cdot\boldsymbol{r}$ are called the *centripetal acceleration* and the *Coriolis acceleration* with respect to the observer.

6.3.3. Recall that $\boldsymbol{r} : \mathrm{I} \rightarrowtail \mathbf{E}$ denotes the double vectorized motion: $\ddot{\boldsymbol{r}}(t) = L_{\mathrm{o}}\cdot(r_{\boldsymbol{U}}(t) - q_{\mathrm{o}})$; consequently, the relative velocity value at the instant t, $r\dot{_{\boldsymbol{U}}}\ (t) = L_o^{-1}\cdot\dot{\boldsymbol{r}}(t)$ is in $\mathbf{E}_{\boldsymbol{U}}$, i.e. it is a function from I into $\mathbf{E}$ which is uniquely determined by an arbitrary one of its values:

$$r\dot{_{\boldsymbol{U}}}\ (t)(s) = R_{\boldsymbol{U}}(s, t_{\mathrm{o}})\cdot\dot{\boldsymbol{r}}(t) \qquad (s \in \mathrm{I}).$$

Since $\Omega(t)\cdot(r(t) - q_{\mathrm{o}} \star t) = \boldsymbol{U}(r(t)) - \boldsymbol{U}(q_{\mathrm{o}} \star t)$, the formula $(*)$ in 6.3.1 gives

$$r\dot{_{\boldsymbol{U}}}\ (t)(t) = \dot{r}(t) - \boldsymbol{U}(r(t)).$$

The expression on the right-hand side coincides with that for the relative velocity with respect to a rotation-free observer. However, keep in mind that now this expression is only a convenient representative (a value) of the relative velocity and not the relative velocity itself.

6.4. Some motions relative to an inertial observer

6.4.1. Suppose r is an inertial world line function, use the notations of 2.3.1(*iii*) and put $t_{\mathrm{o}} := \tau(x_{\mathrm{o}})$:

$$r(t) = x_{\mathrm{o}} + \boldsymbol{u}_{\mathrm{o}}(t - t_{\mathrm{o}}). \qquad (*)$$

Let $\boldsymbol{U}$ be a global inertial observer with the constant velocity value $\boldsymbol{u}$.

Then applying one of the formulae in 4.1.1 we get

$$\begin{aligned} r_{\boldsymbol{U}}(t) &= (x_{\mathrm{o}} + \boldsymbol{u}_{\mathrm{o}}(t - t_{\mathrm{o}})) + \boldsymbol{u}\otimes\mathbf{I} = (x_{\mathrm{o}} + \boldsymbol{u}\otimes\mathbf{I}) + (\boldsymbol{u} - \boldsymbol{u}_{\mathrm{o}})(t - t_{\mathrm{o}}) = \\ &= q_{x_{\mathrm{o}}} + \boldsymbol{v}_{\boldsymbol{u}_{\mathrm{o}}\boldsymbol{u}}(t - t_{\mathrm{o}}) \end{aligned}$$

where $q_{x_{\mathrm{o}}} := x_{\mathrm{o}} + \boldsymbol{u}\otimes\mathbf{I}$ is the $\boldsymbol{U}$-space point that x_{o} is incident with.

This is a uniform motion along a straight line.

Conversely, suppose that we are given a uniform motion relative to the inertial observer $\boldsymbol{U}$, i.e. there is a $q_{\mathrm{o}} \in \mathrm{E}_{\boldsymbol{U}}$, a $t_{\mathrm{o}} \in \mathrm{I}$ and a $\boldsymbol{v}_{\mathrm{o}} \in \frac{\mathbf{E}}{\mathbf{I}}$ such that

$$r_{\boldsymbol{U}}(t) = q_{\mathrm{o}} + \boldsymbol{v}_{\mathrm{o}}(t - t_{\mathrm{o}}) \qquad (t \in \mathrm{I}).$$

Then the corresponding history is inertial; putting $x_o := q_o \star t_o$, $\boldsymbol{u}_o := \boldsymbol{u} + \boldsymbol{v}_o$, we get the world line function $(*)$ which gives rise to the given motion.

6.4.2. Let r be a twist-free world line function (see 2.3.1(iii)):

$$r(t) = x_o + \boldsymbol{u}_o(t - t_o) + \boldsymbol{a}_o\boldsymbol{h}(t - t_o).$$

If $\boldsymbol{U}$ is a global inertial observer with the velocity value $\boldsymbol{u}$ then

$$r_{\boldsymbol{U}}(t) = q_{x_o} + \boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{u}}(t - t_o) + \boldsymbol{a}_o\boldsymbol{h}(t - t_o),$$

where $q_{x_o} := x_o + \boldsymbol{u} \otimes \mathbf{I}$.

If the world line function is not inertial — $\ddot{\boldsymbol{h}} \neq \mathbf{0}$— then the motion is not uniform. The motion is rectilinear relative to the observer if and only if $\boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{u}}$ is parallel to $\boldsymbol{a}_o$.

6.4.3. Now we see that the property "rectilinear" of a motion is not absolute, in general. The same history can appear as a rectilinear motion to an observer and as a non-rectilinear one to another observer; exceptions are the uniform rectilinear motions, i.e. the inertial histories.

Recall that if $\boldsymbol{U}$ is a global inertial observer then an assertion involving $\boldsymbol{U}$ is absolute if and only if it can be formulated exclusively with the aid of the affine structure of $\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}$.

Let $r_{\boldsymbol{U}} : \mathrm{I} \to \mathrm{E}_{\boldsymbol{U}}$ be a motion. Saying that the motion is rectilinear we state that the range of $r_{\boldsymbol{U}}$ is a straight line in the observer space, i.e. we involve the affine structure of $\mathrm{E}_{\boldsymbol{U}}$ only. This is not an absolute property.

Saying the motion is rectilinear and uniform we state that $\{(t, r_{\boldsymbol{U}}(t) \mid t \in \mathrm{I}\}$ is a straight line in $\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}$; this is an absolute property.

6.4.4. Suppose that the global inertial observer $\boldsymbol{U}$ with constant velocity value $\boldsymbol{u}$ chooses a reference origin o. Then, $q_o := o + \boldsymbol{u} \otimes \mathbf{I}$ is the $\boldsymbol{U}$-space point that o is incident with; hence the vectorized motion corresponding to the world line function r becomes

$$\mathrm{I} \rightarrowtail \mathbf{E}, \qquad t \mapsto r(t) - (o + \boldsymbol{u}(t - t_o)),$$

or

$$\mathbf{I} \rightarrowtail \mathbf{E}, \qquad \boldsymbol{t} \mapsto r(t_o + \boldsymbol{t}) - (o + \boldsymbol{u}\boldsymbol{t}),$$

where $t_o := \tau(o)$.

In particular, if r is the twist-free world line function treated in 6.4.2. and $\tau(x_o) = t_o$ (which can be assumed without loss of generality) then the vectorized motion is

$$\mathbf{I} \to \mathbf{E}, \qquad \boldsymbol{t} \mapsto \boldsymbol{q}_o + \boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{u}}\boldsymbol{t} + \boldsymbol{a}_o\boldsymbol{h}(\boldsymbol{t}),$$

where $\boldsymbol{q}_o := x_o - o$.

Since $\boldsymbol{q}_o = q_{x_o} - q_o$ holds as well, comparing our present result with that of 6.4.2, evidently we have — as it must be by definition — that the vectorized motion equals $\boldsymbol{t} \mapsto r_{\boldsymbol{U}}(t_o + \boldsymbol{t}) - q_o$. The advantage of the vectorized motion is that it is easier to calculate.

6.5. Some motions relative to a uniformly accelerated observer

6.5.1. Let r be the previous twist-free world line function and let us examine the corresponding motion relative to a uniformly accelerated observer $\boldsymbol{U}$ with constant acceleration $\boldsymbol{a}$. We easily obtain by 5.2.1 that

$$C_{\boldsymbol{U}}(r(t)) \star s = x_o + \boldsymbol{u}_o(t - t_o) + \boldsymbol{a}_o(t - t_o) + \boldsymbol{U}(r(t))(s - t) + \frac{1}{2}\boldsymbol{a}(s - t)^2.$$

Then 5.2.2 helps us to transform this expression:

$$\boldsymbol{U}(r(t)) = \boldsymbol{U}(x_o) + \boldsymbol{a}(t - t_o)$$

and so

$$\begin{aligned} C_{\boldsymbol{U}}(r(t)) \star s = x_o + &\boldsymbol{U}(x_o)(s - t_o) + \frac{1}{2}\boldsymbol{a}(s - t_o)^2 + \\ &+ (\boldsymbol{u}_o - \boldsymbol{U}(x_o))(t - t_o) + \left(\boldsymbol{a}_o\boldsymbol{h}(t - t_o) - \frac{1}{2}\boldsymbol{a}(t - t_o)^2\right). \end{aligned}$$

Denoting by q_{x_o} the $\boldsymbol{U}$-space point that x_o is incident with and putting $\boldsymbol{v}_o := \boldsymbol{u}_o - \boldsymbol{U}(x_o)$, we can write:

$$r_{\boldsymbol{U}}(t) = q_{x_o} + \boldsymbol{v}_o(t - t_o) + \left(\boldsymbol{a}_o\boldsymbol{h}(t - t_o) - \frac{1}{2}\boldsymbol{a}(t - t_o)^2\right).$$

In particular, it is a uniformly accelerated motion, if $\ddot{\boldsymbol{h}} = \text{const.}$, i.e. if r is inertial or uniformly accelerated.

6.5.2. Let the previous uniformly accelerated observer $\boldsymbol{U}$ choose a reference origin o. Then the $\boldsymbol{U}$-space point that o is incident with is given by the world line function $t \mapsto q_o \star t := o + \boldsymbol{U}(o)(t - t_o) + \frac{1}{2}\boldsymbol{a}(t - t_o)^2$; hence the vectorized motion corresponding to the world line function r becomes

$$\mathrm{I} \rightarrowtail \mathbf{E}, \qquad t \mapsto r(t) - q_o \star t$$

or

$$\mathbf{I} \rightarrowtail \mathbf{E}, \qquad t \mapsto r(t_o + \boldsymbol{t}) - q_o \star (t_o + \boldsymbol{t}).$$

In particular, the vectorized motion corresponding to the twist-free world line function r treated above is

$$\mathbf{I} \to \mathbf{E}, \qquad \boldsymbol{t} \mapsto \boldsymbol{q}_o + \boldsymbol{v}_o \boldsymbol{t} + \left(\boldsymbol{a}_o \boldsymbol{h}(\boldsymbol{t}) - \frac{1}{2} \boldsymbol{a} \boldsymbol{t}^2 \right),$$

where $\boldsymbol{q}_o := x_o - o$ and $\boldsymbol{v}_o := \boldsymbol{u}_o - \boldsymbol{U}(o) = \boldsymbol{u}_o - \boldsymbol{U}(x_o)$ (recall that $\boldsymbol{U}$ is constant on the simultaneous hyperplanes).

We see in this case, too, that the vectorized motion is $\boldsymbol{t} \mapsto r_{\boldsymbol{U}}(t_o + \boldsymbol{t}) - q_o$, as it must be, but it is more complicated to search out the motion $r_{\boldsymbol{U}}$ and then the vectorized motion than to calculate the vectorized motion directly.

6.6. Some motions relative to a uniformly rotating observer*

6.6.1. Let the uniformly rotating observer $\boldsymbol{U}$ choose a reference origin o. If q_o is the $\boldsymbol{U}$-space point that o is incident with and Ω is the constant angular velocity of the observer, then the double vectorized motion is

$$\mathrm{I} \to \mathbf{E}, \qquad t \mapsto e^{-(t-t_o)\Omega} \cdot (r(t) - q_o \star t).$$

In particular, if q_o is an inertial world line, $q_o = o + \boldsymbol{c} \otimes \mathbf{I}$, and r is an inertial world line function, $r(t) = x_o + \boldsymbol{u}_o(t - t_o)$, where we supposed without loss of generality that $\tau(x_o) = \tau(o) = t_o$, then the double vectorized motion becomes

$$\mathrm{I} \to \mathbf{E}, \qquad t \mapsto e^{-(t-t_o)\Omega} \cdot (\boldsymbol{q}_o + \boldsymbol{v}_{\boldsymbol{u}_o \boldsymbol{c}}(t - t_o))$$

where $\boldsymbol{q}_o := x_o - o$; again it is more convenient to use vectorized time:

$$\mathbf{I} \to \mathbf{E}, \qquad \boldsymbol{t} \mapsto e^{-\boldsymbol{t}\Omega} \cdot (\boldsymbol{q}_o + \boldsymbol{v}_{\boldsymbol{u}_o \boldsymbol{c}} \boldsymbol{t}).$$

If $\boldsymbol{v}_{\boldsymbol{u}_o \boldsymbol{c}} = \mathbf{0}$, i.e. the relative velocity of the material point with respect to the axis of rotation is zero, then the motion relative to the observer is a simple rotation around the axis. If $\boldsymbol{v}_{\boldsymbol{u}_o \boldsymbol{c}} \neq \mathbf{0}$, then the motion is the "rotation of a uniform motion". Anyway, the observed rotation of the inertial masspoint is opposite to the rotation of the observer (take into account the negative sign in the exponent).

6.6.2. In the case of inertial observers and uniformly accelerated observers, the vectorized motion is deduced in a little simpler way than motion. On the other hand, for uniformly rotated observers, it is significantly simpler to get the double vectorized motion than motion itself, as it will be seen from the following deduction.

Let $\boldsymbol{U}$ and r be as in the preceding paragraph. Then $q_o := C_{\boldsymbol{U}}(o) = o + \boldsymbol{c} \otimes \mathbf{I}$ and so

$$\begin{aligned} C_{\boldsymbol{U}}(r(t)) \star s =& q_o \star s + e^{(s-t_o)\Omega} \cdot (C_{\boldsymbol{U}}(r(t)) \star t_o - o) = \\ =& q_o \star s + e^{(s-t_o)\Omega} \cdot e^{-(t-t_o)\Omega} \cdot (C_{\boldsymbol{U}}(r(t)) \star t - q_o \star t) = \\ =& q_o \star s + e^{-(t-t_o)\Omega} \cdot e^{(s-t_o)\Omega} \cdot (x_o - o + \boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{c}}(t - t_o)) \,. \end{aligned}$$

The functions

$$\begin{aligned} \mathrm{I} \to \mathbf{E}, \qquad & s \mapsto \boldsymbol{q}_o(s) := e^{(s-t_o)\Omega} \cdot (x_o - o) \\ \mathrm{I} \to \frac{\mathbf{E}}{\mathbf{I}}, \qquad & s \mapsto \boldsymbol{v}_o(s) := e^{(s-t_o)\Omega} \cdot \boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{c}} \end{aligned}$$

are in $\mathbf{E}_{\boldsymbol{U}}$ and in $\frac{\mathbf{E}_{\boldsymbol{U}}}{\mathbf{I}}$ (they are a vector and a vector of cotype $\mathbf{I}$ in the observer space), respectively. Thus we have got for the motion that

$$r_{\boldsymbol{U}}(t) = q_o + e^{-(t-t_o)\Omega} \cdot (\boldsymbol{q}_o + \boldsymbol{v}_o(t - t_o)) \qquad (t \in I).$$

Originally, the exponent of Ω is a linear map from $\mathbf{E}$ into $\mathbf{E}$. Here it is regarded as a linear map from $\mathbf{E}_{\boldsymbol{U}}$ into $\mathbf{E}_{\boldsymbol{U}}$ defined by

$$\left(e^{-(t-t_o)\Omega} \cdot \boldsymbol{\phi}\right)(s) := e^{-(t-t_o)\Omega} \cdot \boldsymbol{\phi}(s) \qquad (\boldsymbol{\phi} \in \mathbf{E}_{\boldsymbol{U}},\ s \in \mathrm{I}).$$

6.7. Exercise

Let $\boldsymbol{U}$ be a uniformly rotating observer that has an inertial space point. Use the notations of Section 5.3. For $\boldsymbol{q}_o \in \mathbf{E}$ and $\boldsymbol{v}_o \in \frac{\mathbf{E}}{\mathbf{I}}$ define the world line function

$$t \mapsto o + \boldsymbol{c}(t - t_o) + e^{(t-t_o)\Omega} \cdot (\boldsymbol{q}_o + \boldsymbol{v}_o(t - t_o)) \,.$$

Prove that the corresponding motion relative to the observer $\boldsymbol{U}$ is a uniform straight line motion.

7. Some kinds of observation

7.1. Vectors observed by inertial observers

7.1.1. Let C_1 and C_2 be two world lines defined over the same time interval J. The *vector* between C_1 and C_2 at the instant $t \in \mathrm{J}$ is $\mathrm{C}_2 \star t - \mathrm{C}_1 \star t$. The *distance* at t between the two world lines is $|\mathrm{C}_2 \star t - \mathrm{C}_1 \star t|$.

The two world lines represent the history of two material points. A global inertial observer $\boldsymbol{U}$ with constant velocity value $\boldsymbol{u}$ observes the two material points describing their history by the corresponding motions $r_{1,\boldsymbol{U}}$ and $r_{2,\boldsymbol{U}}$. Hence, the *vector observed* by the inertial observer between the material points at the instant t is evidently

$$r_{2,\boldsymbol{U}}(t) - r_{1,\boldsymbol{U}}(t) = (\mathrm{C}_2 \star t + \boldsymbol{u} \otimes \mathbf{I}) - (\mathrm{C}_1 \star t + \boldsymbol{u} \otimes \mathbf{I}) = \mathrm{C}_2 \star t - \mathrm{C}_1 \star t.$$

The observed vector coincides with the (absolute) vector; consequently, the observed distance, too, coincides with the (absolute) distance.

7.1.2. The question arises how a straight line segment in the space of an inertial observer is observed by another observer. The question and the answer are formulated correctly as follows.

Let $\boldsymbol{U}_\mathrm{o}$ and $\boldsymbol{U}$ be global inertial observers with constant velocity values $\boldsymbol{u}_\mathrm{o}$ and $\boldsymbol{u}$, respectively. Let H_o be a subset (a geometrical figure) in the $\boldsymbol{U}_\mathrm{o}$-space. The corresponding figure observed by $\boldsymbol{U}$ at the instant t— called the trace of H_o at t in $\mathrm{E}_{\boldsymbol{U}}$— is the set of $\boldsymbol{U}$-space points that coincide at t with the points of H_o :

$$\{q \star t + \boldsymbol{u} \otimes \mathbf{I} \mid q \in \mathrm{H}_\mathrm{o}\}.$$

Introducing the mapping

$$P_t : \mathrm{E}_{\boldsymbol{U}_\mathrm{o}} \to \mathrm{E}_{\boldsymbol{U}}, \qquad q \mapsto q \star t + \boldsymbol{u} \otimes \mathbf{I},$$

we see that the trace of H_o at t equals $P_t[\mathrm{H}_\mathrm{o}]$. It is quite easy to see (recall the definition of subtraction in observer spaces) that

$$P_t(q_2) - P_t(q_1) = q_2 \star t - q_1 \star t = q_2 - q_1$$

for all $q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}_\mathrm{o}}$. Thus P_t is an affine map whose underlying linear map is the identity of $\mathbf{E}$.

We can say that the observed figure and the original figure are *congruent.* Evidently, every figure in the $\boldsymbol{U}_\mathrm{o}$-space is of the form $q_\mathrm{o} + \mathbf{H}_\mathrm{o}$, where $q_\mathrm{o} \in \mathrm{E}_{\boldsymbol{U}_\mathrm{o}}$ and $\mathbf{H}_\mathrm{o} \subset \mathbf{E}$; then $P_t\,[q_\mathrm{o} + \mathbf{H}_\mathrm{o}] = P_t(q_\mathrm{o}) + \mathbf{H}_\mathrm{o}$.

In particular, a straight line segment in the $\boldsymbol{U}_\mathrm{o}$-space observed at an arbitrary instant by the observer $\boldsymbol{U}$ is a straight line segment parallel to the original one. Moreover, the original and the observed segments have the same length; the original and the observed angle between two segments are equal as well.

7.1.3. It is an important fact that the spaces of different global inertial observers are *different* affine spaces over the *same* vector space $\mathbf{E}$. Thus, though the observer spaces are different, it makes sense that a vector in the space of an inertial observer coincides with a vector in the space of another inertial observer.

Evidently, the coincidence of vectors in different observer spaces is a symmetric and transitive relation (if "your" vector coincides with "my" vector then "mine" coincides with "yours"; if, moreover, "his" vector coincides with "yours" then it coincides with "mine" as well.)

This is a trivial fact here that does not hold in the relativistic spacetime model.

7.2. Measuring rods

7.2.1. A physical observer makes measurements in his space: measures the distance between two points, the length of a line, etc. In practice such measurements are based on measuring rods: one takes a rod, carries it to the figure to be measured, puts it consecutively on convenient places... One supposes that during all this procedure the rod is *absolutely rigid:* it remains a straight line segment and its length does not change.

We are interested in whether the non-relativistic spacetime model allows such measuring rods, i.e. whether we can permit in it the existence of such an absolutely rigid rod.

As we shall see, the answer is positive (in contradistinction to the relativistic case).

7.2.2. The existence of an absolutely rigid rod — if it is meaningful — can be determined uniquely by the history of its extremities. Two world lines C_0 and C_1 correspond to the two extremities of a measuring rod if and only if they are defined on the same interval J and their distance at every instant is the same: $|\mathrm{C}_1 \star t - \mathrm{C}_0 \star t| = \boldsymbol{d}$ for all $t \in \mathrm{J}$.

Then for all $\alpha \in [0,1]$ we can define the world line C_α as follows:

$$\mathrm{C}_\alpha \star t := \mathrm{C}_0 \star t + \alpha\,(\mathrm{C}_1 \star t - \mathrm{C}_0 \star t) \qquad (t \in \mathrm{J}).$$

It is quite evident that the set of world lines, $\{\mathrm{C}_\alpha \mid \alpha \in [0,1]\}$ gives an existence of a rigid rod: at every instant $t \in \mathrm{J}$, $\{\mathrm{C}_\alpha \star t \mid \alpha \in [0,1]\}$ is a straight line segment in $\mathbf{E}$, having the length $\boldsymbol{d}$.

8. Vector splittings

8.1. What is a splitting?

Recall what has been said in 3.1.1: in the experience of a physical observer relative to a phenomenon, and in the notions deduced from experience, the properties of the pheonomenon are mixed with those of the observer. Our aim is to find the *absolute notions* that model some properties or aspects of phenomena independently of observers and then to give how the observers derive *relative notions* from the absolute ones (how the absolute objects are observed).

We know already how spacetime is observed as space and time and how the history of a mass point is observed as a motion. In the following, the splitting of force fields, potentials etc. will be treated: such splittings describe somehow the observed form of force fields, potentials, etc. We begin with the splitting of vectors and covectors according to velocity values and then we define the splitting of vector fields and covector fields according to observers.

8.2. Splitting of vectors

8.2.1. For $\boldsymbol{u} \in \mathrm{V}(1)$ we have already defined

$$\boldsymbol{\pi}_{\boldsymbol{u}} : \mathbf{M} \to \mathbf{E}, \qquad \boldsymbol{x} \mapsto \boldsymbol{x} - (\boldsymbol{\tau} \cdot \boldsymbol{x})\boldsymbol{u}$$

and the linear bijection

$$\boldsymbol{h}_{\boldsymbol{u}} := (\boldsymbol{\tau}, \boldsymbol{\pi}_{\boldsymbol{u}}) : \mathbf{M} \to \mathbf{I} \times \mathbf{E}, \qquad \boldsymbol{x} \mapsto (\boldsymbol{\tau} \cdot \boldsymbol{x}, \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x})$$

having the inverse

$$(\boldsymbol{t}, \boldsymbol{q}) \mapsto \boldsymbol{u}\boldsymbol{t} + \boldsymbol{q}$$

(1.2.8).

Thus

$$\boldsymbol{\pi}_{\boldsymbol{u}} = \mathrm{id}_{\mathbf{M}} - \boldsymbol{u} \otimes \boldsymbol{\tau}, \qquad \boldsymbol{\pi}_{\boldsymbol{u}}^{*} = \mathrm{id}_{\mathbf{M}^*} - \boldsymbol{\tau} \otimes \boldsymbol{u}.$$

Moreover,

$$\boldsymbol{\tau} \cdot \boldsymbol{\pi}_{\boldsymbol{u}} = 0, \qquad \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \mathbf{i} = \mathrm{id}_{\mathbf{E}}, \qquad \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{u} = 0.$$

8.2.2. Definition. $\boldsymbol{\tau} \cdot \boldsymbol{x}$ and $\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x}$ are called the *timelike component* and the $\boldsymbol{u}$*-spacelike component* of the vector $\boldsymbol{x}$. $(\boldsymbol{\tau} \cdot \boldsymbol{x}, \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x})$ is the $\boldsymbol{u}$*-split form* of $\boldsymbol{x}$. $\boldsymbol{h}_{\boldsymbol{u}} := (\boldsymbol{\tau}, \boldsymbol{\pi}_{\boldsymbol{u}})$ is the *splitting* of $\mathbf{M}$ corresponding to $\boldsymbol{u}$, or the $\boldsymbol{u}$*-splitting* of $\mathbf{M}$. ■

Note that $\boldsymbol{h_u} \cdot \boldsymbol{q} = (\mathbf{0}, \boldsymbol{q})$ for all $\boldsymbol{q} \in \mathbf{E}$. In other words, $\mathbf{E}$ is split into $\{\mathbf{0}\} \times \mathbf{E}$ trivially. In applications it is convenient to identify $\{\mathbf{0}\} \times \mathbf{E}$ with $\mathbf{E}$ and to assume that the split form of a spacelike vector $\boldsymbol{q}$ is itself.

8.2.3. If $\mathbf{A}$ is a measure line, $\mathbf{A} \otimes \mathbf{M}$ $\left(\frac{\mathbf{M}}{\mathbf{A}}\right)$ is split into $(\mathbf{A} \otimes \mathbf{I}) \times (\mathbf{A} \otimes \mathbf{E})$ $\left(\frac{\mathbf{I}}{\mathbf{A}} \times \frac{\mathbf{E}}{\mathbf{A}}\right)$ by $\boldsymbol{h_u}$; thus the timelike component and the $\boldsymbol{u}$-spacelike component of a vector of type $\mathbf{A}$ (cotype $\mathbf{A}$) are in $\mathbf{A} \otimes \mathbf{I}$ $\left(\frac{\mathbf{I}}{\mathbf{A}}\right)$ and in $\mathbf{A} \otimes \mathbf{E}$ $\left(\frac{\mathbf{E}}{\mathbf{A}}\right)$, respectively.

In particular, $\boldsymbol{h_u}$ splits $\frac{\mathbf{M}}{\mathbf{I}}$ into $\mathbb{R} \times \frac{\mathbf{E}}{\mathbf{I}}$ and for all $\boldsymbol{u}' \in \mathrm{V}(1)$

$$\boldsymbol{h_u} \cdot \boldsymbol{u}' = (1, \boldsymbol{u}' - \boldsymbol{u}) = (1, \boldsymbol{v_{u'u}});$$

the $\boldsymbol{u}$-spacelike component of the velocity value $\boldsymbol{u}'$ is the relative velocity of $\boldsymbol{u}'$ with respect to $\boldsymbol{u}$.

Thus $\mathrm{V}(1)$ is split into $\{1\} \times \frac{\mathbf{E}}{\mathbf{I}}$; in applications it is often convenient to omit the trivial component $\{1\}$, and to regard only $\boldsymbol{\pi_u}$ instead of $\boldsymbol{h_u}$ as the splitting of $\mathrm{V}(1)$:

$$\mathrm{V}(1) \to \frac{\mathbf{E}}{\mathbf{I}}, \qquad \boldsymbol{u}' \mapsto \boldsymbol{u}' - \boldsymbol{u} = \boldsymbol{v_{u'u}}.$$

8.2.4. The timelike component of a vector is independent of the velocity value $\boldsymbol{u}$ producing the splitting, but the $\boldsymbol{u}$-spacelike components vary with $\boldsymbol{u}$, except when the vector is spacelike (an element of $\mathbf{E}$); then the timelike component is zero and the $\boldsymbol{u}$-spacelike component is the vector itself for all $\boldsymbol{u} \in \mathrm{V}(1)$.

The transformation rule that shows how the $\boldsymbol{u}$-spacelike components of a vector vary with $\boldsymbol{u}$ can be well seen from the following formula giving the $\boldsymbol{u}'$-spacelike component of the vector having the timelike component $\boldsymbol{t}$ and the $\boldsymbol{u}$-spacelike component $\boldsymbol{q}$.

Definition. Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$. Then

$$\boldsymbol{H_{u'u}} := \boldsymbol{h_{u'}} \cdot \boldsymbol{h_u}^{-1} : \mathbf{I} \times \mathbf{E} \to \mathbf{I} \times \mathbf{E}$$

is called the *vector transformation law* from $\boldsymbol{u}$-splitting into $\boldsymbol{u}'$-splitting.

Proposition.

$$\boldsymbol{H_{u'u}} \cdot (\boldsymbol{t}, \boldsymbol{q}) = (\boldsymbol{t}, -\boldsymbol{v_{u'u}}\boldsymbol{t} + \boldsymbol{q}) \qquad (t \in \mathbf{I}, \boldsymbol{q} \in \mathbf{E}).\blacksquare$$

Using the matrix form of the linear maps $\mathbf{I} \times \mathbf{E} \to \mathbf{I} \times \mathbf{E}$ (see IV.3.7), we can write

$$\boldsymbol{H_{u'u}} = \begin{pmatrix} \mathrm{id}_{\mathbf{I}} & \mathbf{0} \\ -\boldsymbol{v_{u'u}} & \mathrm{id}_{\mathbf{E}} \end{pmatrix}.$$

According to the identification $\mathrm{Lin}(\mathbf{I}) \equiv \mathbb{R}$ we have $\mathrm{id}_{\mathbf{I}} \equiv 1$. Moreover, applying the usual convention that the identity of a vector space is denoted by $\mathbf{1}$ (the identity is the operation of multiplication by 1), we obtain

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} = \begin{pmatrix} 1 & \mathbf{0} \\ -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \mathbf{1} \end{pmatrix}.$$

In the lower left position of the matrix a linear map $\mathbf{I} \to \mathbf{E}$ must appear; recall that $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \in \frac{\mathbf{E}}{\mathbf{I}} \equiv \mathrm{Lin}(\mathbf{I}, \mathbf{E})$.

8.2.5. Let us give the transformation rule in a form which is more usual in the literature.

Let $(\boldsymbol{t}, \boldsymbol{q})$ and $(\boldsymbol{t}', \boldsymbol{q}')$ be the $\boldsymbol{u}$-split form and the $\boldsymbol{u}'$-split form of the same vector, respectively. Let $\boldsymbol{v}$ denote the relative velocity of $\boldsymbol{u}'$ with respect to $\boldsymbol{u}$. Then

$$\boldsymbol{t}' = \boldsymbol{t}, \qquad \boldsymbol{q}' = \boldsymbol{q} - \boldsymbol{v}\boldsymbol{t}.$$

Usually one calls this formula — or, rather, a similar formula in the arithmetic spacetime model — the Galilean transformation rule and even one defines Galilean transformations by it.

The transformation rule is a mapping from $\mathbf{I}\times\mathbf{E}$ into $\mathbf{I}\times\mathbf{E}$. A (special) Galilean transformation is to be defined on spacetime vectors, i.e. as a mapping from $\mathbf{M}$ into $\mathbf{M}$. Thus the transformation rule and a Galilean transformation cannot be equal. In the split spacetime model $\mathbf{I} \times \mathbf{E}$ stands for both spacetime vectors and spacetime. Thus, using the split model (or, similarly, the arithmetic spacetime model) one can confuse the transformation rule with a mapping defined on spacetime vectors or on spacetime. This indicates very well that we must not use the split model or the arithmetic model for the composition of general ideas.

Of course, there is some connection between transformation rules and Galilean transformations. We shall see (11.3.7) that there is a special Galilean transformation $\boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}') : \mathbf{M} \to \mathbf{M}$ such that

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} = \boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}') \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1}.$$

8.3. Splitting of covectors

8.3.1. For $\boldsymbol{u} \in \mathrm{V}(1)$, $\mathbf{M}^*$ is split by the transpose of the inverse of $\boldsymbol{h}_{\boldsymbol{u}}$:

$$\boldsymbol{r}_{\boldsymbol{u}} := \left(\boldsymbol{h}_{\boldsymbol{u}}^{-1}\right)^* : \mathbf{M}^* \to (\mathbf{I} \times \mathbf{E})^* \equiv \mathbf{I}^* \times \mathbf{E}^*,$$

where we used the identification described in IV.1.3. Then for all $\boldsymbol{k} \in \mathbf{M}^*$, $(\boldsymbol{t}, \boldsymbol{q}) \in \mathbf{I} \times \mathbf{E}$ we have

$$(\boldsymbol{r_u} \cdot \boldsymbol{k}) \cdot (\boldsymbol{t}, \boldsymbol{q}) = \boldsymbol{k} \cdot \boldsymbol{h_u}^{-1} \cdot (\boldsymbol{t}, \boldsymbol{q}) = \boldsymbol{k} \cdot (\boldsymbol{u}\boldsymbol{t} + \boldsymbol{q}) = (\boldsymbol{k} \cdot \boldsymbol{u})\boldsymbol{t} + \boldsymbol{k} \cdot \boldsymbol{q}.$$

Of course, in the last term $\boldsymbol{k}$ can be replaced by $\boldsymbol{k}|_{\mathbf{E}} = \mathbf{i}^* \cdot \boldsymbol{k}$, where $\mathbf{i} : \mathbf{E} \to \mathbf{M}$ is the canonical embedding. Furthermore, recall that $\boldsymbol{k} \cdot \boldsymbol{u} \in \frac{\mathbb{R}}{\mathbf{I}} \equiv \mathbf{I}^*$ and $(\boldsymbol{k} \cdot \boldsymbol{u})\boldsymbol{t}$ stands for the tensor product of $\boldsymbol{k} \cdot \boldsymbol{u}$ and $\boldsymbol{t}$. Then, in view of our convention regarding the duals of one-dimensional vector spaces (IV.3.8), we can state that $\boldsymbol{r_u} \cdot \boldsymbol{k} = (\boldsymbol{k} \cdot \boldsymbol{u},\ \mathbf{i}^* \cdot \boldsymbol{k})$. Recall that $\mathbf{i}^* \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \mathbf{i}$; moreover, our dot notation convention allows us to interchange the order of $\boldsymbol{k}$ and $\boldsymbol{u}$ to have the more suitable forms

$$\boldsymbol{r_u} \cdot \boldsymbol{k} = (\boldsymbol{k} \cdot \boldsymbol{u}, \boldsymbol{k} \cdot \mathbf{i}) = (\boldsymbol{u} \cdot \boldsymbol{k}, \mathbf{i}^* \cdot \boldsymbol{k}) \qquad (\boldsymbol{k} \in \mathbf{M}^*).$$

Definition. $\boldsymbol{u} \cdot \boldsymbol{k}$ and $\mathbf{i}^* \cdot \boldsymbol{k}$ are called the *$\boldsymbol{u}$-timelike component* and the *spacelike component* of the covector $\boldsymbol{k}$. $(\boldsymbol{u} \cdot \boldsymbol{k},\ \mathbf{i}^* \cdot \boldsymbol{k})$ is the *$\boldsymbol{u}$-split form* of the covector $\boldsymbol{k}$. $\boldsymbol{r_u}$ is the splitting of $\mathbf{M}^*$ corresponding to $\boldsymbol{u}$, or the *$\boldsymbol{u}$-splitting* of $\mathbf{M}^*$. ■

Note that $\boldsymbol{r_u} \cdot (\boldsymbol{e}\boldsymbol{\tau}) = (\boldsymbol{e}, \mathbf{0})$ for all $\boldsymbol{e} \in \mathbf{I}^*$. In other words, $\mathbf{I}^* \cdot \boldsymbol{\tau}$ is split into $\mathbf{I}^* \times \{\mathbf{0}\}$ trivially. In applications it is convenient to identify $\mathbf{I}^* \times \{\mathbf{0}\}$ with $\mathbf{I}^*$ and to consider that the split form of $\boldsymbol{e}\boldsymbol{\tau}$ is simply $\boldsymbol{e}$.

8.3.2. The spacelike component of a covector is independent of the velocity value $\boldsymbol{u}$ establishing the splitting, but the $\boldsymbol{u}$-timelike components vary with $\boldsymbol{u}$, except when the covector is timelike (an element of $\mathbf{I}^* \cdot \boldsymbol{\tau}$; then the spacelike component is zero and the $\boldsymbol{u}$-timelike component coincides with the corresponding element of $\mathbf{I}^*$). The transformation rule that shows how the $\boldsymbol{u}$-timelike components of a covector vary with $\boldsymbol{u}$ can be well seen from the following formula giving the $\boldsymbol{u}'$-timelike component of the covector having the $\boldsymbol{u}$-timelike component $\boldsymbol{e}$ and the spacelike component $\boldsymbol{p}$.

Definition. Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$. Then

$$\boldsymbol{R_{u'u}} := r_{\boldsymbol{u}'} \cdot r_{\boldsymbol{u}}^{-1} : \mathbf{I}^* \times \mathbf{E}^* \to \mathbf{I}^* \times \mathbf{E}^*$$

is called the *covector transformation law* from $\boldsymbol{u}$-splitting into $\boldsymbol{u}'$-splitting.

Proposition.

$$\boldsymbol{R_{u'u}} \cdot (\boldsymbol{e}, \boldsymbol{p}) = (\boldsymbol{e} + \boldsymbol{p} \cdot \boldsymbol{v_{u'u}}, \boldsymbol{p}) \qquad (\boldsymbol{e} \in \mathbf{I}^*,\ \boldsymbol{p} \in \mathbf{E}^*).$$

Proof. It is not hard to see that

$$\boldsymbol{r_u}^{-1}(\boldsymbol{e}, \boldsymbol{p}) = \boldsymbol{e}\boldsymbol{\tau} + \boldsymbol{\pi}_{\boldsymbol{u}}^* \cdot \boldsymbol{p}$$

from which we easily obtain the desired formula. ■

Using the matrix form of the linear maps $\mathbf{I}^* \times \mathbf{E}^* \to \mathbf{I}^* \times \mathbf{E}^*$, we can write

$$\boldsymbol{R}_{\boldsymbol{u}'\boldsymbol{u}} = \begin{pmatrix} \mathrm{id}_{\mathbf{I}^*} & \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \\ \mathbf{0} & \mathrm{id}_{\mathbf{E}^*} \end{pmatrix} \equiv \begin{pmatrix} 1 & \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

In the upper right position a linear map $\mathbf{E}^* \to \mathbf{I}^*$ must appear. The identifications $\mathrm{Lin}(\mathbf{E}^*, \mathbf{I}^*) \equiv \mathbf{I}^* \otimes \mathbf{E} \equiv \frac{\mathbf{E}}{\mathbf{I}}$ justify that $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ stands in that position.

The definitions imply that $\boldsymbol{R}_{\boldsymbol{u}'\boldsymbol{u}} = \left(\boldsymbol{H}^{-1}_{\boldsymbol{u}\boldsymbol{u}'}\right)^*$, which is reflected in the matrix form as well.

8.3.3. In 1.2.8 we have drawn a good picture how vectors are split. Now we give an illustration for splitting of covectors.

Recall that for all $\boldsymbol{u} \in \mathrm{V}(1)$, the surjection $\boldsymbol{\pi}_{\boldsymbol{u}} : \mathbf{M} \to \mathbf{E}$ is the left inverse of the canonical embedding $\mathbf{i} : \mathbf{E} \to \mathbf{M}$, i.e. $\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \mathbf{i} = \mathrm{id}_{\mathbf{E}}$. As a consequence, the injection $\boldsymbol{\pi}^*_{\boldsymbol{u}} : \mathbf{E}^* \to \mathbf{M}^*$ is the right inverse of the surjection $\mathbf{i}^* : \mathbf{M}^* \to \mathbf{E}^*$:

$$\mathbf{i}^* \cdot \boldsymbol{\pi}^*_{\boldsymbol{u}} = \mathrm{id}_{\mathbf{E}^*}.$$

Since $\mathbf{I}^* \cdot \boldsymbol{\tau} = \mathrm{Ker}\ \mathbf{i}^*$ (see 1.2.1),

$$\mathbf{E}^* \cdot \boldsymbol{\pi}_{\boldsymbol{u}} = \mathrm{Ran}\ \boldsymbol{\pi}^*_{\boldsymbol{u}}$$

is a three-dimensional linear subspace in $\mathbf{M}^*$, complementary to $\mathbf{I}^* \cdot \boldsymbol{\tau}$. Evidently, the restriction of $\mathbf{i}^*$ is a linear bijection from $\mathbf{E}^* \cdot \boldsymbol{\pi}_{\boldsymbol{u}}$ onto $\mathbf{E}^*$.

Moreover, we easily find that

$$\mathbf{E}^* \cdot \boldsymbol{\pi}_{\boldsymbol{u}} = \{\boldsymbol{k} \in \mathbf{M}^* \mid \boldsymbol{k} \cdot \boldsymbol{u} = \mathbf{0}\};$$

in other words, $\mathbf{E}^* \cdot \boldsymbol{\pi}_{\boldsymbol{u}}$ is the annullator of $\boldsymbol{u} \otimes \mathbf{I}$.

Then the splitting of covectors according to $\boldsymbol{u}$ is illustrated as follows:

$\boldsymbol{k} - (\boldsymbol{u} \cdot \boldsymbol{k}) \cdot \boldsymbol{\tau}$ is in $\mathbf{E}^* \cdot \boldsymbol{\pi}_{\boldsymbol{u}}$, its image by $\mathbf{i}^*$ is the spacelike component of $\boldsymbol{k}$.

8.4. Vectors and covectors are split in a different way

The splitting of vectors and the splitting of covectors according to $\boldsymbol{u} \in \mathrm{V}(1)$ are essentially different. The timelike component of vectors is independent of $\boldsymbol{u}$, whereas the spacelike component of covectors is independent of $\boldsymbol{u}$. The transformation laws for vectors and covectors are essentially different as well.

The reason of these differences lies in the fact that there is no one-dimensional vector space $\mathbf{A}$ in such a way that $\mathbf{M}^*$ could be canonically identified with $\frac{\mathbf{M}}{\mathbf{A}}$, in contradistinction to the relativistic case.

8.5. Splitting of vector fields and covector fields according to inertial observers

8.5.1. In applications, vectors and covectors appear in two ways: first, as values of functions defined in time; secondly, as values of functions defined in spacetime. The first case can be reduced to the second one: a function defined in time can be considered a function defined in spacetime that is constant on the simultaneous hyperplanes. Thus we shall study *vector fields* and *covector fields,* i.e. functions $\boldsymbol{X} : \mathrm{M} \rightarrowtail \mathbf{M}$ and $\boldsymbol{K} : \mathrm{M} \rightarrowtail \mathbf{M}^*$, respectively.

A global inertial observer $\boldsymbol{U}$ splits vector fields and covector fields in such a way that at every world point x the values of the fields, $\boldsymbol{X}(x)$ and $\boldsymbol{K}(x)$, are split according to the velocity value $\boldsymbol{u}$ of the observer; thus the *half $\boldsymbol{U}$-split form* of the fields will be

$$\begin{aligned} \boldsymbol{h_u} \cdot \boldsymbol{X} : \mathrm{M} \rightarrowtail \mathbf{I} \times \mathbf{E}, &\qquad x \mapsto (\boldsymbol{\tau} \cdot \boldsymbol{X}(x),\ \boldsymbol{\pi_u} \cdot \boldsymbol{X}(x)) , \\ \boldsymbol{r_u} \cdot \boldsymbol{K} : \mathrm{M} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^*, &\qquad x \mapsto (\boldsymbol{u} \cdot \boldsymbol{K}(x),\ \mathbf{i}^* \cdot \boldsymbol{K}(x)) . \end{aligned}$$

However, the observer splits spacetime as well (the observer regards spacetime as time and space); accordingly, instead of world points, instants and $\boldsymbol{U}$-space points will be introduced to get the *completely $\boldsymbol{U}$-split form* of the fields:

$$\begin{aligned} \boldsymbol{h_u} \cdot \boldsymbol{X} \circ H_{\boldsymbol{U}}^{-1} : \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \rightarrowtail \mathbf{I} \times \mathbf{E}, &\qquad (t,q) \mapsto (\boldsymbol{\tau} \cdot \boldsymbol{X}(q \star t),\ \boldsymbol{\pi_u} \cdot \boldsymbol{X}(q \star t)) , \\ \boldsymbol{r_u} \cdot \boldsymbol{K} \circ H_{\boldsymbol{U}}^{-1} : \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^*, &\qquad (t,q) \mapsto (\boldsymbol{u} \cdot \boldsymbol{K}(q \star t),\ \mathbf{i}^* \cdot \boldsymbol{K}(q \star t)) \end{aligned}$$

where $q \star t := H_{\boldsymbol{U}}^{-1}(t,q)$ (see 3.2.2).

8.5.2. Let us examine more closely the split forms of a covector field $\boldsymbol{K}$:

$$(-V_{\boldsymbol{u}}, \boldsymbol{A_u}) := \boldsymbol{r_u} \cdot \boldsymbol{K} : \mathrm{M} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^*, \qquad x \mapsto (\boldsymbol{u} \cdot \boldsymbol{K}(x),\ \mathbf{i}^* \cdot \boldsymbol{K}(x)) ,$$

$$\begin{aligned} (-V_{\boldsymbol{U}}, \boldsymbol{A_U}) := (-V_{\boldsymbol{u}}, \boldsymbol{A_u}) \circ H_{\boldsymbol{U}}^{-1} = \boldsymbol{r_u} \cdot \boldsymbol{K} \circ H_{\boldsymbol{U}}^{-1} : \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^*, \\ (t,q) \mapsto (\boldsymbol{u} \cdot \boldsymbol{K}(q \star t),\ \mathbf{i}^* \cdot \boldsymbol{K}(q \star t)) . \end{aligned}$$

A covector field $\boldsymbol{K}$ is a potential (see 2.4.3). $V_{\boldsymbol{U}}$ and $\boldsymbol{A}_{\boldsymbol{U}}$ are called the corresponding *scalar potential* and *vector potential* according to $\boldsymbol{U}$.

If $\boldsymbol{U}'$ is another global inertial observer with constant velocity value $\boldsymbol{u}'$ then, in view of 8.3.2,

$$V_{\boldsymbol{u}'} = V_{\boldsymbol{u}} - \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{A}_{\boldsymbol{u}}, \qquad \boldsymbol{A}_{\boldsymbol{u}'} = \boldsymbol{A}_{\boldsymbol{u}}.$$

As a consequence,

$$\begin{aligned} V_{\boldsymbol{U}'} \circ H_{\boldsymbol{U}'} &= (V_{\boldsymbol{U}} - \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{A}_{\boldsymbol{U}}) \circ H_{\boldsymbol{U}}, \\ \boldsymbol{A}_{\boldsymbol{U}'} \circ H_{\boldsymbol{U}'} &= \boldsymbol{A}_{\boldsymbol{U}} \circ H_{\boldsymbol{U}}. \end{aligned}$$

8.5.3. Introducing $V := V_{\boldsymbol{u}}$, $V' := V_{\boldsymbol{u}'}$, $\boldsymbol{A} := \boldsymbol{A}_{\boldsymbol{u}}$, $\boldsymbol{A}' := A_{\boldsymbol{u}'}$, $\boldsymbol{v} := \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, we get the formulae

$$V' = V - \boldsymbol{v} \cdot \boldsymbol{A}, \qquad \boldsymbol{A}' = \boldsymbol{A},$$

which are the well-known non-relativistic transformation law for scalar and vector potentials in electromagnetism.

This supports our choice that (absolute) potentials are cotensor fields.

The reader is asked to bear in mind the following remark. One usually says that if an observer perceives scalar potential V and vector potential $\boldsymbol{A}$, then another observer moving with relative velocity $\boldsymbol{v}$ perceives scalar potential $V - \boldsymbol{v} \cdot \boldsymbol{A}$ and vector potential $\boldsymbol{A}$. However, an observer $\boldsymbol{U}$ perceives spacetime as time and $\boldsymbol{U}$-space, perceives the potentials to be functions depending on time and $\boldsymbol{U}$-space; thus, in fact, an observer observes the completely split form of the potentials. In particular, if $\boldsymbol{U}' \neq \boldsymbol{U}$, then $\boldsymbol{A}_{\boldsymbol{U}'} \neq \boldsymbol{A}_{\boldsymbol{U}}$: *the observed vector potentials are different!* Remember, usually one does not distinguish between the half split forms and the completely split forms.

8.5.4. Similarly, one usually says that force is not transformed, a force field is the same for all observers. Of course, this is true for the half split form of force fields; the completely split forms of force fields — which are actually observed — depend on the observers.

A force field

$$\boldsymbol{f} : \mathrm{M} \times \mathrm{V}(1) \rightarrowtail \frac{\mathbf{E}}{\mathbf{I} \otimes \mathbf{D} \otimes \mathbf{D}}$$

has exclusively spacelike values, thus its half split form is $\boldsymbol{f}$ itself for all global inertial observers. On the other hand, $\boldsymbol{f}$ has the completely split form

$$\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \times \frac{\mathbf{E}}{\mathbf{I}} \rightarrowtail \frac{\mathbf{E}}{\mathbf{I} \otimes \mathbf{D} \otimes \mathbf{D}}, \qquad (t, q, \boldsymbol{v}) \mapsto \boldsymbol{f}(q \star t, \boldsymbol{u} + \boldsymbol{v})$$

strongly depending on $\boldsymbol{U}$.

8.5.5. Let the global inertial observer $\boldsymbol{U}$ choose a reference origin o; then $(\boldsymbol{U}, o)$ performs another splitting using $H_{\boldsymbol{U},o}$ instead of $H_{\boldsymbol{U}}$. The half split form of vector fields and covector fields according to $(\boldsymbol{U}, o)$ is the same as the half split form according to $\boldsymbol{U}$; on the other hand, the observer with reference origin obtains functions $\mathbf{I} \times \mathbf{E} \rightarrowtail \mathbf{I} \times \mathbf{E}$ and $\mathbf{I} \times \mathbf{E} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^*$ for the completely split forms of the fields.

8.6. Splitting of vector fields and covector fields according to rigid observers

8.6.1. Recall that the space $\mathrm{E}_{\boldsymbol{U}}$ of a global rigid observer $\boldsymbol{U}$ is an affine space over $\mathbf{E}$ or $\mathbf{E}_{\boldsymbol{U}}$ (Section 4.3), depending on whether $\boldsymbol{U}$ is rotation-free or not. The corresponding splitting of spacetime, $H_{\boldsymbol{U}} : \mathrm{M} \to \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}}$ is a smooth bijection whose inverse is smooth as well.

The splitting of a vector field $\boldsymbol{X}$ according to $\boldsymbol{U}$ is defined by the corresponding formula of coordinatization: at every world point x, the value of the field, $\boldsymbol{X}(x)$, is split — i.e. is mapped from $\mathbf{M}$ into $\mathbf{I} \times \mathbf{E}$ or $\mathbf{I} \times \mathbf{E}_{\boldsymbol{U}}$— by $\mathrm{D}H_{\boldsymbol{U}}(x)$. Similarly, the covector field $\boldsymbol{K}$, is split in such a way that at every world point x the value of the field, $\boldsymbol{K}(x)$ is split — i.e. is mapped from $\mathbf{M}^*$ into $\mathbf{I}^* \times \mathbf{E}^*$ or $\mathbf{I}^* \times \mathbf{E}^*_{\boldsymbol{U}}$— by $\left((\mathrm{D}H_{\boldsymbol{U}}(x))^*\right)^{-1}$. Thus the *half $\boldsymbol{U}$-split forms* of such fields are

$$\begin{aligned}
&\mathrm{M} \rightarrowtail \mathbf{I} \times \mathbf{E} \quad (\text{or } \mathbf{I} \times \mathbf{E}_{\boldsymbol{U}}), && x \mapsto \mathrm{D}H_{\boldsymbol{U}}(x) \cdot \boldsymbol{X}(x),\\
&\mathrm{M} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^* \quad (\text{or } \mathbf{I}^* \times \mathbf{E}^*_{\boldsymbol{U}}), && x \mapsto \left((\mathrm{D}H_{\boldsymbol{U}}(x))^*\right)^{-1} \cdot \boldsymbol{K}(x).
\end{aligned}$$

We get the *completely U-split* forms by substituting $H_{\boldsymbol{U}}^{-1}(t,q) = q \star t$ for x in these formulae.

8.6.2. If $\boldsymbol{U}$ is rotation-free, then, in view of 4.3.2, the half split forms of the fields are

$$\begin{aligned}
&\mathrm{M} \rightarrowtail \mathbf{I} \times \mathbf{E}, && x \mapsto \left(\boldsymbol{\tau} \cdot \boldsymbol{X}(x),\ \boldsymbol{\pi}_{\boldsymbol{U}(x)} \cdot \boldsymbol{X}(x)\right),\\
&\mathrm{M} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^*, && x \mapsto \left(\boldsymbol{U}(x) \cdot \boldsymbol{K}(x),\ \mathbf{i}^* \cdot \boldsymbol{K}(x)\right).
\end{aligned}$$

The values of the fields at x are split by the corresponding value $\boldsymbol{U}(x)$ of the observer.

8.6.3. If $\boldsymbol{U}$ is not rotation-free, $\mathrm{E}_{\boldsymbol{U}}$ is an uneasy object; that is why we let the global rigid observer choose a reference origin o and use the double vectorization $H_{\boldsymbol{U},o}$ of spacetime.

Then the half split forms become

$$\mathrm{M} \rightarrowtail \mathbf{I} \times \mathbf{E}, \quad x \mapsto \mathrm{D}H_{\boldsymbol{U},o}(x) \cdot \boldsymbol{X}(x) = \left(\boldsymbol{\tau} \cdot \boldsymbol{X}(x), R(x)^{-1} \cdot \boldsymbol{\pi}_{\boldsymbol{U}(x)} \cdot \boldsymbol{X}(x)\right),$$

$$\mathrm{M} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^*, \quad x \mapsto \left((\mathrm{D}H_{\boldsymbol{U},o}(x))^*\right)^{-1} \cdot \ \boldsymbol{K}(x) =$$
$$= \Big(\boldsymbol{U}(x)\cdot \boldsymbol{K}(x),\ R(x)^{-1}\cdot \mathbf{i}^* \cdot \boldsymbol{K}(x)\Big),$$

where
$$R(x) := R_{\boldsymbol{U}}(\tau(x), t_o),$$
(see Exercises 4.2.2 and 4.5.3).

8.7. Exercises

1. Give the split form of vector fields and covector fields that depends only on time; more closely, if $\boldsymbol{\chi} : \mathrm{I} \rightarrowtail \mathbf{M}$ and $\boldsymbol{\kappa} : \mathrm{I} \rightarrowtail \mathbf{M}^*$, consider the splitting of the fields $\boldsymbol{X} := \boldsymbol{\chi} \circ \tau$ and $\boldsymbol{K} := \boldsymbol{\kappa} \circ \tau$.

2. We know, it has an absolute meaning that a function $\boldsymbol{\phi}$ defined in spacetime depends only on time: if $\boldsymbol{\phi}$ is constant on the simultaneous hyperplanes.

On the other hand, it does not have an absolute meaning that $\boldsymbol{\phi}$ depends only on space (absolute space does not exist). If $\boldsymbol{U}$ is an observer, it makes sense that $\boldsymbol{\phi}$ depends only on $\boldsymbol{U}$-space, in other words, $\boldsymbol{\phi}$ is $\boldsymbol{U}$-static: if $\boldsymbol{\phi}$ is constant in the $\boldsymbol{U}$-space points (on the $\boldsymbol{U}$-lines), i.e. if the completely split form of $\boldsymbol{\phi}$ depends only on the elements of $\mathrm{E}_{\boldsymbol{U}}$.

Let $o \in \mathrm{M}$, $\boldsymbol{c} \in \mathrm{V}(1)$, $\boldsymbol{C} : \mathbf{E} \to \mathbf{M}$, and let $\boldsymbol{U}$ be the inertial observer with the velocity value $\boldsymbol{u}$. Prove that the vector field $x \mapsto \boldsymbol{C}\left(\boldsymbol{\pi_c} \cdot (x - o)\right)$ is $\boldsymbol{U}$-static if and only if $\boldsymbol{u} = \boldsymbol{c}$.

3. Take the arithmetic spacetime model. Give the completely split form of the vector fields
$$(\xi^0, \boldsymbol{\xi}) \mapsto (|\boldsymbol{\xi}|, \mathbf{0}),$$
$$(\xi^0, \boldsymbol{\xi}) \mapsto (\xi^0 + |\boldsymbol{\xi}|,\ \xi^2 + \xi^3,\ 1 + \xi^1, 0)$$
according to the inertial observer with velocity value $(1, \boldsymbol{v})$.

Consider the previous mappings to be covector fields and give their completely split form.

4. Take the arithmetic spacetime model. Give the completely split form of the vector field
$$(\xi^0, \boldsymbol{\xi}) \mapsto \left(\xi^1 + \xi^2, \cos(\xi^0 - \xi^3), 0, 0\right)$$
according to the uniformly accelerated observer with reference origin treated in 5.2.4 and to the uniformly rotating observer with reference origin treated in 5.3.5. (It is easy to obtain the composition of this vector field and the inverse of the splitting if we use different symbols for the variables; e.g. the splitting due to the uniformly accelerated observer has the inverse $(\zeta^0, \boldsymbol{\zeta}) \mapsto \left(\zeta^0, \zeta^1 + \frac{1}{2}\alpha(\zeta^0)^2, \zeta^2, \zeta^3\right)$.

5. In the split spacetime model the splitting of vectors according to the basic velocity value $(1, \mathbf{0})$ is the identity of $\mathbf{I} \times \mathbf{E}$. The splitting according to $(1, \boldsymbol{v})$ is

$$\mathbf{I} \times \mathbf{E} \to \mathbf{I} \times \mathbf{E}, \qquad (\boldsymbol{t}, \boldsymbol{q}) \mapsto (\boldsymbol{t}, \boldsymbol{q} - \boldsymbol{v}\boldsymbol{t}),$$

which coincides with the transformation rule from $(1, \mathbf{0})$ into $(1, \boldsymbol{v})$. Because of the special structure of the split spacetime model a splitting and a transformation rule — which are in fact different objects — can be equal. To deal with fundamental ideas do not use the split spacetime model or the arithmetic one.

6. In the split spacetime model the splitting of covectors according to the basic velocity value $(1, \mathbf{0})$ is the identity of $\mathbf{I}^* \times \mathbf{E}^*$ and the splitting according to $(1, \boldsymbol{v})$ equals

$$\mathbf{I}^* \times \mathbf{E}^* \to \mathbf{I}^* \times \mathbf{E}^*, \qquad (\boldsymbol{e}, \boldsymbol{p}) \mapsto (\boldsymbol{e} + \boldsymbol{p} \cdot \boldsymbol{v},\ \boldsymbol{p}).$$

Again we see that the splitting coincides with the transformation rule from $(1, \mathbf{0})$ into $(1, \boldsymbol{v})$.

9. Tensor splittings

9.1. Splitting of tensors, cotensors, etc.

9.1.1. The various tensors are split according to $\boldsymbol{u} \in \mathrm{V}(1)$ by the maps

$$\begin{aligned}
\boldsymbol{h_u} \otimes \boldsymbol{h_u} &: \mathbf{M} \otimes \mathbf{M} \to (\mathbf{I} \otimes \mathbf{E}) \otimes (\mathbf{I} \times \mathbf{E}) = \\
&= (\mathbf{I} \otimes \mathbf{I}) \times (\mathbf{I} \otimes \mathbf{E}) \times (\mathbf{E} \otimes \mathbf{I}) \times (\mathbf{E} \otimes \mathbf{E}), \\
\boldsymbol{h_u} \otimes \boldsymbol{r_u} &: \mathbf{M} \otimes \mathbf{M}^* \to (\mathbf{I} \times \mathbf{E}) \otimes (\mathbf{I}^* \times \mathbf{E}^*) = \\
&= (\mathbf{I} \otimes \mathbf{I}^*) \times (\mathbf{I} \otimes \mathbf{E}^*) \times (\mathbf{E} \otimes \mathbf{I}^*) \times (\mathbf{E} \otimes \mathbf{E}^*), \\
\boldsymbol{r_u} \otimes \boldsymbol{h_u} &: \mathbf{M}^* \otimes \mathbf{M} \to (\mathbf{I}^* \times \mathbf{E}^*) \otimes (\mathbf{I} \otimes \mathbf{E}) = \\
&= (\mathbf{I}^* \otimes \mathbf{I}) \times (\mathbf{I}^* \otimes \mathbf{E}) \times (\mathbf{E}^* \otimes \mathbf{I}) \times (\mathbf{E}^* \otimes \mathbf{E}), \\
\boldsymbol{r_u} \otimes \boldsymbol{r_u} &: \mathbf{M}^* \otimes \mathbf{M}^* \to (\mathbf{I}^* \times \mathbf{E}^*) \otimes (\mathbf{I}^* \times \mathbf{E}^*) = \\
&= (\mathbf{I}^* \otimes \mathbf{I}^*) \times (\mathbf{I}^* \otimes \mathbf{E}^*) \times (\mathbf{E}^* \otimes \mathbf{I}^*) \times (\mathbf{E}^* \otimes \mathbf{E}^*).
\end{aligned}$$

Since we know $\boldsymbol{h_u}$ and $\boldsymbol{r_u}$, our task is only to determine the above splittings in a perspicuous way. First recall that the elements of the Cartesian products on the right-hand sides can be well given in a matrix form (see IV.3.7). Second, with the aid of the usual identifications, consider $\boldsymbol{h_u} = (\boldsymbol{\tau}, \boldsymbol{\pi_u}) \in (\mathbf{I} \times \mathbf{E}) \otimes \mathbf{M}^*$, $\boldsymbol{r_u} = (\boldsymbol{u}, \mathbf{i}^*) \in (\mathbf{I}^* \times \mathbf{E}^*) \otimes \mathbf{M}$, take into account the identifications $\mathbf{I} \otimes \mathbf{M}^* \equiv \mathbf{M}^* \otimes \mathbf{I}$, $\boldsymbol{\tau} \equiv \boldsymbol{\tau}^*$ and $\mathbf{I}^* \otimes \mathbf{M} \equiv \mathbf{M} \otimes \mathbf{I}^*$, $\boldsymbol{u} \equiv \boldsymbol{u}^*$ (see IV.3.6), and apply the dot products to have for $\boldsymbol{T} \in \mathbf{M} \otimes \mathbf{M}$:

$$(\boldsymbol{h_u} \otimes \boldsymbol{h_u})(\boldsymbol{T}) = \boldsymbol{h_u} \cdot \boldsymbol{T} \cdot \boldsymbol{h_u^*} = \boldsymbol{h_u} \cdot \boldsymbol{T} \cdot \boldsymbol{r_u^{-1}} = \begin{pmatrix} \boldsymbol{\tau} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau} & \boldsymbol{\tau} \cdot \boldsymbol{T} \cdot \boldsymbol{\pi_u^*} \\ \boldsymbol{\pi_u} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau} & \boldsymbol{\pi_u} \cdot \boldsymbol{T} \cdot \boldsymbol{\pi_u^*} \end{pmatrix} =$$

$$= \begin{pmatrix} \boldsymbol{\tau} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau} & \boldsymbol{\tau} \cdot \boldsymbol{T} - \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau}) \\ \boldsymbol{T} \cdot \boldsymbol{\tau} - \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau}) & \boldsymbol{T} - \boldsymbol{u} \otimes (\boldsymbol{\tau} \cdot \boldsymbol{T}) - (\boldsymbol{T} \cdot \boldsymbol{\tau}) \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau}) \end{pmatrix},$$

for $\boldsymbol{L} \in \mathbf{M} \otimes \mathbf{M}^*$:

$$(\boldsymbol{h_u} \otimes \boldsymbol{r_u})(\boldsymbol{L}) = \boldsymbol{h_u} \cdot \boldsymbol{L} \cdot \boldsymbol{r_u^*} = \boldsymbol{h_u} \cdot \boldsymbol{L} \cdot \boldsymbol{h_u^{-1}} = \begin{pmatrix} \boldsymbol{\tau} \cdot \boldsymbol{L} \cdot \boldsymbol{u} & \boldsymbol{\tau} \cdot \boldsymbol{L} \cdot \mathbf{i} \\ \boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \boldsymbol{u} & \boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i} \end{pmatrix} =$$

$$= \begin{pmatrix} \boldsymbol{\tau} \cdot \boldsymbol{L} \cdot \boldsymbol{u} & \boldsymbol{\tau} \cdot \boldsymbol{L} \cdot \mathbf{i} \\ \boldsymbol{L} \cdot \boldsymbol{u} - \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{L} \cdot \boldsymbol{u}) & \boldsymbol{L} \cdot \mathbf{i} - \boldsymbol{u} \otimes (\boldsymbol{\tau} \cdot \boldsymbol{L} \cdot \mathbf{i}) \end{pmatrix},$$

for $\boldsymbol{P} \in \mathbf{M}^* \otimes \mathbf{M}$:

$$(\boldsymbol{r_u} \otimes \boldsymbol{h_u})(\boldsymbol{P}) = \boldsymbol{r_u} \cdot \boldsymbol{P} \cdot \boldsymbol{h_u^*} = \boldsymbol{r_u} \cdot \boldsymbol{P} \cdot \boldsymbol{r_u^{-1}} = \begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{P} \cdot \boldsymbol{\tau} & \boldsymbol{u} \cdot \boldsymbol{P} \cdot \boldsymbol{\pi_u^*} \\ \mathbf{i}^* \cdot \boldsymbol{P} \cdot \boldsymbol{\tau} & \mathbf{i}^* \cdot \boldsymbol{P} \cdot \boldsymbol{\pi_u^*} \end{pmatrix} =$$

$$= \begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{P} \cdot \boldsymbol{\tau} & \boldsymbol{u} \cdot \boldsymbol{P} - (\boldsymbol{u} \cdot \boldsymbol{P} \cdot \boldsymbol{\tau})\boldsymbol{u} \\ \mathbf{i}^* \cdot \boldsymbol{P} \cdot \boldsymbol{\tau} & \mathbf{i}^* \cdot \boldsymbol{P} - (\mathbf{i}^* \cdot \boldsymbol{P} \cdot \boldsymbol{\tau}) \otimes \boldsymbol{u} \end{pmatrix},$$

for $\boldsymbol{F} \in \mathbf{M}^* \otimes \mathbf{M}^*$:

$$(\boldsymbol{r_u} \otimes \boldsymbol{r_u})(\boldsymbol{F}) = \boldsymbol{r_u} \cdot \boldsymbol{F} \cdot \boldsymbol{r_u^*} = \boldsymbol{r_u} \cdot \boldsymbol{F} \cdot \boldsymbol{h_u^{-1}} = \begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{F} \cdot \boldsymbol{u} & \boldsymbol{u} \cdot \boldsymbol{F} \cdot \mathbf{i} \\ \mathbf{i}^* \cdot \boldsymbol{F} \cdot \boldsymbol{u} & \mathbf{i}^* \cdot \boldsymbol{F} \cdot \mathbf{i} \end{pmatrix}.$$

(To see, e.g. that $\boldsymbol{\pi_u} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau} = \boldsymbol{T} \cdot \boldsymbol{\tau} - \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau})$, take $\boldsymbol{T} = \boldsymbol{x} \otimes \boldsymbol{y}$.)

9.1.2. The splittings corresponding to different velocity values $\boldsymbol{u}$ and $\boldsymbol{u}'$ are different. To compare the different splittings we can deduce transformation rules by giving

$$\boldsymbol{H_{u'u}} \cdot \begin{pmatrix} \boldsymbol{h} & \boldsymbol{b} \\ \boldsymbol{a} & \boldsymbol{A} \end{pmatrix} \cdot \boldsymbol{H_{u'u}}^*,$$

where $\boldsymbol{h} \in \mathbf{I} \otimes \mathbf{I}$, $\boldsymbol{a} \in \mathbf{E} \otimes \mathbf{I}$, $\boldsymbol{b} \in \mathbf{I} \otimes \mathbf{E}$, $\boldsymbol{A} \in \mathbf{E} \otimes \mathbf{E}$, and using similar formulae for the other three cases as well. In general, the transformation rules are rather complicated. We shall study them for antisymmetric tensors and cotensors.

9.2. Splitting of antisymmetric tensors

9.2.1. If the tensor $\boldsymbol{T}$ is antisymmetric — i.e. $\boldsymbol{T} \in \mathbf{M}\wedge\mathbf{M}$— then $\boldsymbol{\tau}\cdot\boldsymbol{T}\cdot\boldsymbol{\tau} = \mathbf{0}$, $\boldsymbol{\tau}\cdot\boldsymbol{T}\cdot\boldsymbol{\pi}_{\boldsymbol{u}}^* = -(\boldsymbol{\pi}_{\boldsymbol{u}}\cdot\boldsymbol{T}\cdot\boldsymbol{\tau})^*$ and $\boldsymbol{\pi}_{\boldsymbol{u}}\cdot\boldsymbol{T}\cdot\boldsymbol{\pi}_{\boldsymbol{u}}^* \in \mathbf{E}\wedge\mathbf{E}$, which (of course) means that the split forms of $\boldsymbol{T}$ are antisymmetric as well. Thus splittings map the elements of $\mathbf{M}\wedge\mathbf{M}$ into elements of the form

$$\begin{pmatrix} \mathbf{0} & -\boldsymbol{a}^* \\ \boldsymbol{a} & \boldsymbol{A} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{0} & -\boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{A} \end{pmatrix},$$

where $\boldsymbol{a} \in \mathbf{E}\otimes\mathbf{I}$, $\boldsymbol{A} \in \mathbf{E}\wedge\mathbf{E}$; $\boldsymbol{a}^* \in \mathbf{I}\otimes\mathbf{E}$ is the transpose of $\boldsymbol{a}$, which is identified with $\boldsymbol{a}$ in the usual identification $\mathbf{I}\otimes\mathbf{E} \equiv \mathbf{E}\otimes\mathbf{I}$. We shall find convenient to write

$$\begin{aligned} (\mathbf{E}\otimes\mathbf{I})\times(\mathbf{E}\wedge\mathbf{E}) &\equiv (\mathbf{I}\times\mathbf{E})\wedge(\mathbf{I}\times\mathbf{E}), \\ (\boldsymbol{a},\boldsymbol{A}) &\equiv \begin{pmatrix} \mathbf{0} & -\boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{A} \end{pmatrix}. \end{aligned}$$

The corresponding formula in 9.1.1 gives us for $\boldsymbol{T} \in \mathbf{M}\wedge\mathbf{M}$

$$\boldsymbol{h}_{\boldsymbol{u}}\cdot\boldsymbol{T}\cdot\boldsymbol{h}_{\boldsymbol{u}}^* = (\boldsymbol{T}\cdot\boldsymbol{\tau},\ \boldsymbol{T} - (\boldsymbol{T}\cdot\boldsymbol{\tau})\wedge\boldsymbol{u}).$$

Definition. $\boldsymbol{T}\cdot\boldsymbol{\tau}$ and $\boldsymbol{T} - (\boldsymbol{T}\cdot\boldsymbol{\tau})\wedge\boldsymbol{u}$ are called the *timelike component* and the $\boldsymbol{u}$*-spacelike component* of the antisymmetric tensor $\boldsymbol{T}$.

9.2.2. Notice the similarity between splittings of vectors and splittings of antisymmetric tensors. The timelike component of $\boldsymbol{T}$ is independent of $\boldsymbol{u}$, the $\boldsymbol{u}$-spacelike component varies with $\boldsymbol{u}$ except when $\boldsymbol{T}$ is spacelike, i.e. is in $\mathbf{E}\wedge\mathbf{E}$; then the timelike component is zero and the $\boldsymbol{u}$-spacelike component is $\boldsymbol{T}$ itself for all $\boldsymbol{u}$.

The following *transformation rule* shows well how the splittings depend on the velocity values.

Proposition. Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$. Then

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}\cdot(\boldsymbol{a},\boldsymbol{A})\cdot{\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}}^* = (\boldsymbol{a}, -\boldsymbol{a}\wedge\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} + \boldsymbol{A}) \qquad (\boldsymbol{a}\in\mathbf{E}\otimes\mathbf{I},\ \boldsymbol{A}\in\mathbf{E}\wedge\mathbf{E}).$$

Proof. Use the matrix forms:

$$\begin{pmatrix} 1 & \mathbf{0} \\ -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & -\boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} 1 & -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{a} \\ \boldsymbol{a} & -\boldsymbol{a}\wedge\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} + \boldsymbol{A} \end{pmatrix}.$$

9.3. Splitting of antisymmetric cotensors

9.3.1. If $\boldsymbol{F} \in \mathbf{M}^* \wedge \mathbf{M}^*$ then $\boldsymbol{u} \cdot \boldsymbol{F} \cdot \boldsymbol{u} = \mathbf{0}$, $\boldsymbol{u} \cdot \boldsymbol{F} \cdot \mathbf{i} = -(\mathbf{i}^* \cdot \boldsymbol{F} \cdot \boldsymbol{u})^*$ and $\mathbf{i}^* \cdot \boldsymbol{F} \cdot \mathbf{i} \in \mathbf{E}^* \wedge \mathbf{E}^*$; the split forms of $\boldsymbol{F}$ are antisymmetric as well. Thus splitting maps the elements of $\mathbf{M}^* \wedge \mathbf{M}^*$ into elements of the form

$$\begin{pmatrix} \mathbf{0} & -\boldsymbol{z}^* \\ \boldsymbol{z} & \boldsymbol{Z} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{0} & -\boldsymbol{z} \\ \boldsymbol{z} & \boldsymbol{Z} \end{pmatrix} \equiv (\boldsymbol{z}, \boldsymbol{Z}) \in (\mathbf{E}^* \otimes \mathbf{I}^*) \times (\mathbf{E}^* \wedge \mathbf{E}^*),$$

where we used notations similar to those in 9.2.1.

The corresponding formula in 9.1.1 gives for $\boldsymbol{F} \in \mathbf{M}^* \wedge \mathbf{M}^*$:

$$\boldsymbol{r_u} \cdot \boldsymbol{F} \cdot \boldsymbol{r_u^*} = (\mathbf{i}^* \cdot \boldsymbol{F} \cdot \boldsymbol{u},\ \mathbf{i}^* \cdot \boldsymbol{F} \cdot \mathbf{i}).$$

Definition. $\mathbf{i}^* \cdot \boldsymbol{F} \cdot \boldsymbol{u}$ and $\mathbf{i}^* \cdot \boldsymbol{F} \cdot \mathbf{i}$ are called the $\boldsymbol{u}$*-timelike component* and the *spacelike component* of the antisymmetric cotensor $\boldsymbol{F}$.

9.3.2. Notice the similarity between splittings of covectors and splittings of antisymmetric cotensors. The spacelike component of $\boldsymbol{F}$ is independent of $\boldsymbol{u}$, the $\boldsymbol{u}$-timelike component varies with $\boldsymbol{u}$ except when $\boldsymbol{F}$ is in $\mathbf{M}^* \wedge (\mathbf{I}^* \cdot \boldsymbol{\tau}) := \{\boldsymbol{k} \wedge (\boldsymbol{e} \cdot \boldsymbol{\tau}) \mid \boldsymbol{k} \in \mathbf{M}^*, \boldsymbol{e} \in \mathbf{I}^*\}$; then the spacelike component is zero and the $\boldsymbol{u}$-timelike component is the same for all $\boldsymbol{u}$.

The following *transformation rule* shows well, how the splittings depend on the velocity values.

Proposition. Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$. Then

$$\boldsymbol{R_{u'u}} \cdot (\boldsymbol{z}, \boldsymbol{Z}) \cdot \boldsymbol{R_{u'u}}^* = (\boldsymbol{z} + \boldsymbol{Z} \cdot \boldsymbol{v_{u'u}}, \boldsymbol{Z}) \qquad (\boldsymbol{z} \in \mathbf{I}^* \otimes \mathbf{E}^*,\ \boldsymbol{Z} \in \mathbf{E}^* \wedge \mathbf{E}^*).$$

Proof. Use the matrix forms:

$$\begin{pmatrix} 1 & \boldsymbol{v_{u'u}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & -\boldsymbol{z} \\ \boldsymbol{z} & \boldsymbol{Z} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{v_{u'u}} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -(\boldsymbol{z} + \boldsymbol{Z} \cdot \boldsymbol{v_{u'u}}) \\ \boldsymbol{z} + \boldsymbol{Z} \cdot \boldsymbol{v_{u'u}} & \boldsymbol{Z} \end{pmatrix}.$$

9.4. Splitting of cotensor fields

9.4.1. A rotation-free rigid observer $\boldsymbol{U}$ splits various tensor fields in such a way that the value of the tensor field at the world point x is split according to $\boldsymbol{U}(x)$; for the sake of definiteness we shall consider cotensor fields. The *half split form* of the cotensor field $\boldsymbol{F} : \mathrm{M} \rightarrowtail \mathbf{M}^* \otimes \mathbf{M}^*$ according to $\boldsymbol{U}$ is

$$\mathrm{M} \rightarrowtail (\mathbf{I}^* \times \mathbf{E}^*) \otimes (\mathbf{I}^* \times \mathbf{E}^*), \qquad x \mapsto \boldsymbol{r}_{\boldsymbol{U}(x)} \cdot \boldsymbol{F}(x) \cdot \boldsymbol{r}_{\boldsymbol{U}(x)}{}^*.$$

The *completely split form* of $\boldsymbol{F}$ according to $\boldsymbol{U}$ is

$$\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \rightarrowtail (\mathbf{I} \times \mathbf{E}^*) \otimes (\mathbf{I}^* \times \mathbf{E}^*), \qquad (t, q) \mapsto \boldsymbol{r}_{\boldsymbol{U}(q\star t)} \cdot \boldsymbol{F}(q \star t) \cdot \boldsymbol{r}_{\boldsymbol{U}(q\star t)},$$

where $q \star t = H_{\boldsymbol{U}}^{-1}(t, q)$.

In particular, if $\boldsymbol{F}$ is antisymmetric, then it has the half split form

$$\mathrm{M} \rightarrowtail (\mathbf{E}^* \otimes \mathbf{I}^*) \times (\mathbf{E}^* \wedge \mathbf{E}^*), \qquad x \mapsto (\mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \boldsymbol{U}(x), \mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \mathbf{i}) .$$

9.4.2. Now let us suppose that $\boldsymbol{U}$ is a global inertial observer with the constant velocity value $\boldsymbol{u}$. Then the antisymmetric cotensor field $\boldsymbol{F}$ has the half split form

$$\begin{aligned}(\boldsymbol{E}_{\boldsymbol{u}}, -\boldsymbol{B}_{\boldsymbol{u}}) := \boldsymbol{r}_{\boldsymbol{u}} \cdot \boldsymbol{F} \cdot \boldsymbol{r}_{\boldsymbol{u}}^* :& \mathrm{M} \rightarrowtail (\mathbf{E}^* \otimes \mathbf{I}^*) \times (\mathbf{E}^* \wedge \mathbf{E}^*), \\ & x \mapsto (\mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \boldsymbol{u},\ \mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \mathbf{i})\end{aligned}$$

and the completely split form

$$\begin{aligned}(\boldsymbol{E}_{\boldsymbol{U}}, -\boldsymbol{B}_{\boldsymbol{U}}) := (\boldsymbol{E}_{\boldsymbol{u}}, -\boldsymbol{B}_{\boldsymbol{u}}) \circ H_{\boldsymbol{U}}^{-1} = & \\ = \boldsymbol{r}_{\boldsymbol{u}} \cdot (\boldsymbol{F} \circ H_{\boldsymbol{U}}^{-1}) \cdot \boldsymbol{r}_{\boldsymbol{u}}^* &: \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \rightarrowtail (\mathbf{E}^* \otimes \mathbf{I}^*) \times (\mathbf{E}^* \wedge \mathbf{E}^*), \\ (t, q) \mapsto (\mathbf{i}^* \cdot \boldsymbol{F}(q \star t) \cdot \boldsymbol{u},\ \mathbf{i}^* \cdot \boldsymbol{F}(q \star t) \cdot \mathbf{i}) .\end{aligned}$$

If $\boldsymbol{U}'$ is another inertial observer with the velocity value $\boldsymbol{u}'$, then 9.3.2 gives ($\boldsymbol{B}_{\boldsymbol{u}}$ is antisymmetric, hence $\boldsymbol{B}_{\boldsymbol{u}} \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{B}_{\boldsymbol{u}}$) that

$$\boldsymbol{E}_{\boldsymbol{u}'} = \boldsymbol{E}_{\boldsymbol{u}} + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{B}_{\boldsymbol{u}}, \qquad \boldsymbol{B}_{\boldsymbol{u}'} = \boldsymbol{B}_{\boldsymbol{u}}.$$

As a consequence,

$$\begin{aligned}\boldsymbol{E}_{\boldsymbol{U}'} \circ H_{\boldsymbol{U}'} =& (\boldsymbol{E}_{\boldsymbol{U}} + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{B}_{\boldsymbol{U}}) \circ H_{\boldsymbol{U}}, \\ \boldsymbol{B}_{\boldsymbol{U}'} \circ H_{\boldsymbol{U}'} =& \boldsymbol{B}_{\boldsymbol{U}} \circ H_{\boldsymbol{U}}.\end{aligned}$$

9.4.3. Introducing $\boldsymbol{E} := \boldsymbol{E}_{\boldsymbol{u}},\ \boldsymbol{E}' := \boldsymbol{E}_{\boldsymbol{u}'},\ \boldsymbol{B} := \boldsymbol{B}_{\boldsymbol{u}},\ \boldsymbol{B}' := \boldsymbol{B}_{\boldsymbol{u}'},\ \ \boldsymbol{v} := \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, we get the formula

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \cdot \boldsymbol{B}, \qquad \boldsymbol{B}' = \boldsymbol{B}$$

which is the well-known non-relativistic transformation law for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. (Here $\boldsymbol{B}$ is an antisymmetric spacelike tensor of cotype $\overset{4}{\otimes} \mathbf{D}$, an element of $\mathbf{E}^* \wedge \mathbf{E}^* = \frac{\mathbf{E} \wedge \mathbf{E}}{\mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D}}$, which can be identified with a vector of cotype $\overset{3}{\otimes} \mathbf{D}$, an element of $\frac{\mathbf{E}}{\mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D}} \equiv \frac{\mathbf{E}^*}{\mathbf{D}}$ (V.3.17.); with the aid of this identification magnetic field is regarded as a vector field and then instead of $\boldsymbol{v} \cdot \boldsymbol{B}$ one has a vectorial product.)

This supports the idea that (absolute) electromagnetic fields exist whose timelike and negative spacelike components according to an observer are the observed electric and magnetic fields, respectively.

One usually says that if an observer perceives electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, then another observer moving with the velocity $\boldsymbol{v}$ perceives electric field $\boldsymbol{E} + \boldsymbol{v} \cdot \boldsymbol{B}$ and magnetic field $\boldsymbol{B}$. However, an observer perceives spacetime as time and $\boldsymbol{U}$-space, perceives the fields as functions depending on time and $\boldsymbol{U}$-space; thus, in fact, an observer observes the completely split form of the fields, and we can repeat the remark at the end of 8.5.3.

9.4.4. Consider the completely split form of a potential $\boldsymbol{K}$ according to the inertial observer $\boldsymbol{U}$ with velocity value $\boldsymbol{u}$:

$$(-V_{\boldsymbol{U}}, \boldsymbol{A}_{\boldsymbol{U}}) := \boldsymbol{r}_{\boldsymbol{u}} \cdot (\boldsymbol{K} \circ H_{\boldsymbol{U}}^{-1}) : \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \rightarrowtail \mathbf{I}^* \times \mathbf{E}^*.$$

Its derivative is

$$\mathrm{D}(-V_{\boldsymbol{U}}, \boldsymbol{A}_{\boldsymbol{U}}) = \boldsymbol{r}_{\boldsymbol{u}} \cdot (\mathrm{D}\boldsymbol{K} \circ H_{\boldsymbol{U}}^{-1}) \cdot \boldsymbol{r}_{\boldsymbol{u}}{}^*$$

having the transpose

$$(\mathrm{D}(-V_{\boldsymbol{U}}, \boldsymbol{A}_{\boldsymbol{U}}))^* = \boldsymbol{r}_{\boldsymbol{u}} \cdot \left((\mathrm{D}\boldsymbol{K} \circ H_{\boldsymbol{U}}^{-1})\right)^* \cdot \boldsymbol{r}_{\boldsymbol{u}}{}^*.$$

Consequently, for the exterior derivatives (see VI.3.6(i)) we have

$$\mathrm{D} \wedge (-V_{\boldsymbol{U}}, \boldsymbol{A}_{\boldsymbol{U}}) = \boldsymbol{r}_{\boldsymbol{u}} \cdot \left((\mathrm{D} \wedge \boldsymbol{K}) \circ H_{\boldsymbol{U}}^{-1}\right) \cdot \boldsymbol{r}_{\boldsymbol{u}}{}^*.$$

Let $\boldsymbol{F} := \mathrm{D} \wedge \boldsymbol{K}$, use the notations of the previous paragraph and let ∂_{o} and ∇ denote the partial derivations with respect to I and $\mathrm{E}_{\boldsymbol{U}}$, respectively. Then (see VI.3.7(ii)) the above equality yields

$$-\partial_{\mathrm{o}} \boldsymbol{A}_{\boldsymbol{U}} - \nabla V_{\boldsymbol{U}} = \boldsymbol{E}_{\boldsymbol{U}}, \qquad -\nabla \wedge \boldsymbol{A}_{\boldsymbol{U}} = \boldsymbol{B}_{\boldsymbol{U}}.$$

9.4.5. Let us consider the force field defined by the potential $\boldsymbol{K}$:

$$\boldsymbol{f}(x, \boldsymbol{u}') = \mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \boldsymbol{u}' \qquad (x \in \mathrm{Dom}\, \boldsymbol{K}, \;\; \boldsymbol{u}' \in \mathrm{V}(1).$$
$$(\boldsymbol{F} := \mathrm{D} \wedge \boldsymbol{K}).$$

According to 9.3.1, the value of the force field at $(x, \boldsymbol{u}')$ is the $\boldsymbol{u}'$-timelike component of the antisymmetric cotensor $\boldsymbol{F}(x)$.

A masspoint at the world point x having the instantaneous velocity value $\boldsymbol{u}'$ "feels" only the $\boldsymbol{u}'$-timelike component of the field; a masspoint always "feels" the time component of the field according to its instantaneous velocity value.

Consider now the inertial observer with velocity value $\boldsymbol{u}$ and use the notations of the previous paragraphs. Then

$$\begin{aligned}\boldsymbol{f}(x,\boldsymbol{u}') =& \mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \boldsymbol{u} + \mathbf{i}^* \cdot \boldsymbol{F}(x)(\boldsymbol{u}' - \boldsymbol{u}) = \\ =& \boldsymbol{E}_{\boldsymbol{u}}(x) + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{B}_{\boldsymbol{u}}(x),\end{aligned}$$

a well-known formula for the Lorentz force in electromagnetism.

9.4.6. If a potential $\boldsymbol{K}$ is timelike, i.e. has values in $\mathbf{I}^* \cdot \boldsymbol{\tau}$, (in fact $\boldsymbol{K}$ is a scalar field: there is a function $V : \mathrm{M} \rightarrowtail \mathbf{I}^*$ such that $\boldsymbol{K} = V \cdot \boldsymbol{\tau}$) then $\mathrm{D} \wedge \boldsymbol{K}$ takes values in $\mathbf{M}^* \wedge (\mathbf{I}^* \cdot \boldsymbol{\tau})$; consequently the corresponding force field does not depend on velocity values; the spacelike component of $\boldsymbol{K}$ is zero and the half split form of $\boldsymbol{K}$ is the same for all observers.

The possibility of (absolute) scalar potentials is a peculiar feature of the non-relativistic spacetime model in contradistinction to relativistic spacetime models. (Newtonian gravitational fields, elastic fields are modelled by such timelike potentials in non-relativistic physics.)

9.4.7. Let us mention the case of a general (rotating) global rigid observer $\boldsymbol{U}$. Then it is convenient to choose a reference origin o for the observer and consider the corresponding double vectorization of spacetime.

We easily infer from the splitting of vector fields and covector fields that the half split forms of various tensor fields according to $(\boldsymbol{U}, o)$ are obtained from the half split forms according to a rotation-free observer in such a way that $R_{\boldsymbol{U}}(\tau(x), t_o)^{-1} \cdot \boldsymbol{\pi}_{\boldsymbol{U}(x)}$ and $R_{\boldsymbol{U}}(\tau(x), t_o)^{-1} \cdot \mathbf{i}^*$ are substituted for $\boldsymbol{\pi}_{\boldsymbol{U}(x)}$ and $\mathbf{i}^*$, respectively (then $\mathbf{i} \cdot R_{\boldsymbol{U}}(\tau(x), t_o)$ is substituted for $\mathbf{i}$).

For instance, the half split form of an antisymmetric cotensor field $\boldsymbol{F}$ becomes

$$\begin{aligned}&\mathrm{M} \rightarrowtail (\mathbf{E}^* \otimes \mathbf{I}^*) \times (\mathbf{E}^* \wedge \mathbf{E}^*), \\ &\qquad x \mapsto \left(R(x)^{-1} \cdot \mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \boldsymbol{U}(x), \quad R(x)^{-1} \cdot \mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \mathbf{i} \cdot R(x)\right),\end{aligned}$$

where $R(x) := R_{\boldsymbol{U}}(\tau(x), t_o)$.

9.5. Exercises

1. Give the $\boldsymbol{u}$-split form of tensors in $\mathbf{E}\otimes\mathbf{E}$, $\mathbf{E}\otimes\mathbf{M}$, $\mathbf{M}\otimes\mathbf{E}$, $\mathbf{E}\otimes\mathbf{M}^*$, $\mathbf{M}^*\otimes\mathbf{E}$, $(\mathbf{I}^*\cdot\boldsymbol{\tau})\otimes\mathbf{M}$, $\mathbf{M}\otimes(\mathbf{I}^*\cdot\boldsymbol{\tau})$, $(\mathbf{I}^*\cdot\boldsymbol{\tau})\otimes\mathbf{M}^*$, $\mathbf{M}^*\otimes(\mathbf{I}^*\cdot\boldsymbol{\tau})$ and derive the transformation rules between their $\boldsymbol{u}'$-splitting and $\boldsymbol{u}$-splitting.

2. Derive the transformation rules for the splitting of arbitrary tensors.

3. A potential in the arithmetic spacetime model is a function $(-V, \mathbf{A}) : \mathbb{R}\times\mathbb{R}^3 \rightarrowtail (\mathbb{R}\times\mathbb{R}^3)^*$ which is the completely split form of the potential according to the basic observer.

The half split form of this potential according to the inertial observer with velocity value $(1, \boldsymbol{v})$ is $(-V + \boldsymbol{v}\cdot\boldsymbol{A}, \boldsymbol{A})$.

Choose $(0, \mathbf{0})$ as a reference origin for the observer and give the completely split form of the potential.

4. An antisymmetric cotensor field in the arithmetic spacetime model is a function $(\boldsymbol{E}, -\boldsymbol{B}) : \mathbb{R}\times\mathbb{R}^3 \rightarrowtail (\mathbb{R}^3)^* \times \left((\mathbb{R}^3)^* \wedge (\mathbb{R}^3)^*\right)$, being the completely split form of the field according to the basic observer:

$$(\boldsymbol{E}, -\boldsymbol{B}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

The half split form of this field according to the inertial observer with velocity value $(1, \boldsymbol{v})$ is $(\boldsymbol{E} + \boldsymbol{v}\cdot\boldsymbol{B},\ \boldsymbol{B})$.

Choose $(0, \mathbf{0})$ as a reference origin for the observer and give the completely split form of the field.

5. Take the uniformly accelerated observer treated in 5.2.4.

The half split forms of the previous potential and field according to this observer are

$$(-V + \alpha t A_1,\ \boldsymbol{A}) \qquad \text{and} \qquad ((E_1,\ E_2 - \alpha t B_3,\ E_3 + \alpha t B_2),\ -\boldsymbol{B}),$$

where t is the time evaluation: $\mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, $(\xi^0, \boldsymbol{\xi}) \mapsto \xi^0$.

Choose $(0, \mathbf{0})$ as a reference origin for the observer and give the completely split forms.

6. Take the uniformly rotating observer treated in 5.3.5.

Let $(-V', \boldsymbol{A}')$ and $(\boldsymbol{E}', -\boldsymbol{B}')$ denote the half split forms of the previous potential and field, respectively, according to this observer. Then

$$\begin{aligned} V' &= V + \omega(x^2 A_1 - x^1 A_2), \\ A_1' &= A_1 \cos\omega t - A_2 \sin\omega t, \\ A_2' &= A_1 \sin\omega t + A_2 \cos\omega t, \\ A_3' &= A_3 \end{aligned}$$

and

$$\begin{aligned}
E_1' &= (E_1 + \omega x^1 B_3)\cos\omega t - (E_2 + \omega x^2 B_3)\sin\omega t,\\
E_2' &= (E_1 + \omega x^1 B_3)\sin\omega t + (E_2 + \omega x^2 B_3)\cos\omega t,\\
E_3' &= E_3 - \omega(x^2 B_2 + x^1 B_1),\\
B_1' &= B_1\cos\omega t - B_2\sin\omega t,\\
B_2' &= B_1\sin\omega t + B_2\cos\omega t,\\
B_3' &= B_3,
\end{aligned}$$

where t is the time evaluation $\mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, $(\xi^0, \boldsymbol{\xi}) \mapsto \xi^0$ and x^i is the evaluation of the i-th space coordinate: $\mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, $(\xi^0, \boldsymbol{\xi}) \to \xi^i$.

10. Reference frames

10.1. The notion of a reference frame

10.1.1. Reference frames are usually fundamental notions in textbooks of physics: the phenomena are always described in reference frames. However, this notion is not exactly defined there.

We have expressed our intention to give an *absolute description* of phenomena, i.e. a description free of reference frames and observers. Observers and reference frames — which must be exactly defined in our framework — have only a practical (not theoretical) importance: it is convenient and suitable to use reference frames for solving actual problems, for achieving numerical characterization of quantities.

It was mentioned in 3.1 that the usual notion of reference frames involves coordinates introduced with the aid of some material objects. The material objects play a more fundamental role; that is why we created the notion of their model: the *observer.* An observer and a *coordinatization* of time and observer space together will form a *reference system* giving rise to a *reference frame.* Coordinatization models the procedure how a physical observer measures time with a clock (having a dial) and introduces numbered reference lines in his space. Then a time point is represented by a number, and a space point is represented by a triplet of numbers.

The reader is supposed to be familiar with the notion of coordinatization which can be found in Section VI.5.

10.1.2. Recall that an observer $\boldsymbol{U}$ makes the splitting $H_{\boldsymbol{U}} = (\tau, C_{\boldsymbol{U}}) : \mathrm{M} \rightarrowtail \mathrm{I}\times\mathrm{E}_{\boldsymbol{U}}$.

Definition. A *reference system* is a triplet $(\boldsymbol{U}, T, S_{\boldsymbol{U}})$ where

(i) $\boldsymbol{U}$ is an observer,
(ii) $T : \mathrm{I} \rightarrowtail \mathbb{R}$ is a strictly monotone increasing mapping,
(iii) $S_{\boldsymbol{U}} : \mathrm{E}_{\boldsymbol{U}} \rightarrowtail \mathbb{R}^3$ is a mapping
such that $(T \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U}} = (T \circ \tau,\ S_{\boldsymbol{U}} \circ C_{\boldsymbol{U}}) : \mathrm{M} \rightarrowtail \mathbb{R} \times \mathbb{R}^3$ is an orientation preserving (local) coordinatization of spacetime. ■

According to the definition, $T \circ \tau$ is smooth which implies by VI.3.5 that T is smooth as well. Because of (ii) the derivative of T— denoted by T'— is everywhere positive,

$$0 < T'(t) \in \mathbf{I}^* \equiv \frac{\mathbb{R}}{\mathrm{I}} \qquad (t \in \mathrm{Dom}\ T),$$

i.e. T is an (orientation preserving) *coordinatization of time.*

If $\boldsymbol{U}$ is a global rigid observer then $\mathrm{E}_{\boldsymbol{U}}$ is an affine space and $C_{\boldsymbol{U}}$ is a smooth map (see 4.4.3); consequently, we can state that $S_{\boldsymbol{U}}$ is a coordinatization of $\mathrm{E}_{\boldsymbol{U}}$. On the contrary, since $\mathrm{E}_{\boldsymbol{U}}$, in general, is not an affine space, and we introduced the notion of coordinatization only for affine spaces, we cannot state that $S_{\boldsymbol{U}}$ is a coordinatization of $\boldsymbol{U}$-space; nevertheless it will be called the *coordinatization of $\boldsymbol{U}$-space.* (We mentioned that in any case $\mathrm{E}_{\boldsymbol{U}}$ can be endowed with a smooth structure; in the framework of smooth structures $S_{\boldsymbol{U}}$ does become a coordinatization.)

10.1.3. Definition. A coordinatization $K : \mathrm{M} \rightarrowtail \mathbb{R}\times\mathbb{R}^3$ is called a *reference frame* if there is a reference system $(\boldsymbol{U}, T, S_{\boldsymbol{U}})$ such that $K = (T \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U}}$.

$\boldsymbol{U}$, T and $S_{\boldsymbol{U}}$ are called *the observer, the time coordinatization and the $\boldsymbol{U}$-space coordinatization* corresponding to the reference frame. ■

As usual, the coordinates of $\mathbb{R} \times \mathbb{R}^3$ are numbered from zero to three. Accordingly, we find convenient to write a coordinatization of spacetime in the form $K = (\kappa^0, \boldsymbol{\kappa}) : \mathrm{M} \rightarrowtail \mathbb{R} \times \mathbb{R}^3$. Using the notations $\mathrm{pr}^0 : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{pr} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ for the canonical projections, we have $\kappa^0 = \mathrm{pr}^0 \circ K$, $\boldsymbol{\kappa} = \mathbf{pr} \circ K$.

The following important relation holds for an arbitrary coordinatization K :

$$\mathrm{D}\boldsymbol{\kappa}(x) \cdot \partial_0 K^{-1}(K(x)) = 0 \qquad\qquad (x \in \mathrm{Dom}\ K).$$

Indeed, according to the definition of partial derivatives (VI.3.8) and the rules of differentiation (VI.3.4), we have

$$\begin{aligned} \partial_0 K^{-1}(K(x)) &= \left(\mathrm{D}K^{-1}\right)(K(x)) \cdot (1, \mathbf{0}) = \mathrm{D}K(x)^{-1} \cdot (1, \mathbf{0}), \qquad (*) \\ \mathrm{D}\boldsymbol{\kappa}(x) &= \mathbf{pr} \cdot \mathrm{D}K(x), \end{aligned}$$

from which we infer the desired equality.

If K is a reference frame then

$$\kappa^0 = T \circ \tau, \qquad \boldsymbol{\kappa} = S_{\boldsymbol{U}} \circ C_{\boldsymbol{U}}$$

and

$$\left(\mathrm{D}\kappa^0\right)(x) = T'(\tau(x))\boldsymbol{\tau}$$

from which we deduce

$$T'(\tau(x)) = \frac{1}{\boldsymbol{\tau}\cdot\partial_0 K^{-1}(K(x))}$$

in the following way:

$$\begin{aligned}
\mathrm{pr}^0\cdot\mathrm{D}K(x) =& T'(\tau(x))\boldsymbol{\tau},\\
\mathrm{pr}^0 =& T'(\tau(x))\boldsymbol{\tau}\cdot\mathrm{D}K(x)^{-1},\\
1 =& T'(\tau(x))\boldsymbol{\tau}\cdot\mathrm{D}K(x)^{-1}\cdot(1,\mathbf{0}),\\
1 =& T'(\tau(x))\boldsymbol{\tau}\cdot\partial_0 K^{-1}(K(x)).
\end{aligned}$$

10.1.4. Proposition. A coordinatization $K = (\kappa^0, \boldsymbol{\kappa}) : \mathrm{M} \rightarrowtail \mathbb{R}\times\mathbb{R}^3$ is a reference frame if and only if

(i) K is orientation-preserving,
(ii) $\partial_0 K^{-1}(K(x))$ is a future-directed timelike vector,
(iii) $\kappa^0(x) < \kappa^0(y)$ is equivalent to $\tau(x) < \tau(y)$ for all $x, y \in \mathrm{Dom}\ K$.

Then

(1)
$$\boldsymbol{U}(x) = \frac{\partial_0 K^{-1}(K(x))}{\boldsymbol{\tau}\cdot\partial_0 K^{-1}(K(x))} = \partial_0 K^{-1}(K(x))\cdot T'(\tau(x)) \qquad (x\in\mathrm{Dom}\ K),$$

(2)
$$T(t) = \kappa^0(x) \qquad (t\cap\mathrm{Dom}\ K\neq\emptyset,\ \ x\in t),$$

(3)
$$S_{\boldsymbol{U}}(q) = \boldsymbol{\kappa}(x) \qquad (q\in\mathrm{E}_{\boldsymbol{U}},\ \ x\in q)$$

for the corresponding reference system $(\boldsymbol{U}, T, S_{\boldsymbol{U}})$.

Proof. If K is a reference frame, $K = (T\times S_{\boldsymbol{U}})\circ H_{\boldsymbol{U}}$, then ($i$) is trivial and ($iii$) follows from $\kappa^0 = T\circ\tau$ and the strictly monotone character of T. As concerns (ii), note that a world line function r satisfies $\dot{r}(t) = \boldsymbol{U}(r(t))$ and takes values in the domain of K if and only if $K(r(t)) = (T(t), \boldsymbol{\xi})$, i.e. $r(t) = K^{-1}(T(t), \boldsymbol{\xi})$ for a $\boldsymbol{\xi}\in\mathbb{R}^3$ and for all $t\in\mathrm{Dom}\ r$. As a consequence, we have

$$\begin{aligned}
\boldsymbol{U}(r(t)) =& \frac{\mathrm{d}}{\mathrm{d}t}K^{-1}(T(t),\boldsymbol{\xi}) = \partial_0 K^{-1}(T(t),\boldsymbol{\xi}))\cdot T'(t) =\\
=& \partial_0 K^{-1}\left(K(r(t))\right)\cdot T'(t)
\end{aligned}$$

implying

$$\boldsymbol{U}(x) = \partial_0 K^{-1}\left(K(x)\right)\cdot T'\left(\tau(x)\right), \qquad (x\in\mathrm{Dom}\ K),$$

which proves (ii), since $T'(\tau(x)) > 0$. It proves equality (1) as well; equalities (2) and (3) are trivial.

Suppose now that $K = (\kappa^0, \boldsymbol{\kappa})$ is a coordinatization that fulfills conditions (i)–(iii).

Then condition (ii) implies that $\boldsymbol{U}$ *defined* by the first equality in (1) is an observer.

According to (iii), K is constant on the simultaneous hyperplanes, thus T is well *defined* by the formula (2). Moreover, T is strictly monotone increasing.

If r is a world line function such that $\dot{r}(t) = \boldsymbol{U}(r(t))$ then according to $(*)$ in the preceding paragraph

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\kappa}(r(t)) = \mathrm{D}\boldsymbol{\kappa}(r(t)) \cdot \boldsymbol{U}(r(t)) = \mathrm{D}\boldsymbol{\kappa}(r(t)) \cdot \frac{\partial_0 K^{-1}\left(K(r(t))\right)}{\boldsymbol{\tau} \cdot \partial_0 K^{-1}\left(K(r(t))\right)} = 0,$$

which means that $\boldsymbol{\kappa} \circ r$ is a constant mapping, in other words, $\boldsymbol{\kappa}$ is constant on the $\boldsymbol{U}$-lines; hence $S_{\boldsymbol{U}}$ is well *defined* by the formula (3).

Finally, it is evident that $K = (T \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U}}$. ■

It is suitable to use $P := K^{-1}$, the parameterization corresponding to K. Then — putting $\phi(P)$ instead of $\phi \circ P$ for any function ϕ — we can rewrite formula (1) in the proposition:

$$\boldsymbol{U}(P) = \frac{\partial_0 P}{\boldsymbol{\tau} \cdot \partial_0 P}.$$

10.1.5. Condition (iii) in the previous proposition can be replaced by
(iii)' for all $x \in \mathrm{Dom}\, K$ there is an $\boldsymbol{e}(x) \in (\mathbf{I}^*)^+$ such that

$$\mathrm{D}\kappa^0(x) = \boldsymbol{e}(x)\boldsymbol{\tau},$$

i.e. the derivative of κ^0 in every point is a positive multiple of $\boldsymbol{\tau}$.

Indeed, if K is a reference frame, then $\boldsymbol{e}(x) = T'(\tau(x))$.

Conversely, if (iii)' holds, then the restriction of κ^0 onto every simultaneous hyperplane t has zero derivative: $\mathrm{D}\left(\kappa^0\big|_t\right)(x) = \mathrm{D}\kappa^0(x)\big|_{\mathbf{E}} = \mathbf{0}$ $(x \in t)$ thus κ^0 is constant on every simultaneous hyperplane which allows us to define T by the formula (1) in the previous proposition.

Moreover, Lagrange's mean value theorem implies that every x in the domain of K has a neighbourhood such that for all y in that neighbourhood there is a z on the straight line segment connecting x and y in such a way that

$$\kappa^0(y) - \kappa^0(x) = \mathrm{D}\kappa^0(z) \cdot (y - x) = \boldsymbol{e}(z)\boldsymbol{\tau} \cdot (y - x),$$

hence $\kappa^0(y) - \kappa^0(x) > 0$ is equivalent to $\boldsymbol{\tau} \cdot (y - x) > 0$ in the neighbourhood in question. Since the domain of K is connected, this relation holds globally as well.

10.2. Galilean reference frames

10.2.1. Now we are interested in what kinds of affine coordinatizations of spacetime can be reference frames.

Let us take an affine coordinatization K of M. Then there are

— an $o \in \mathrm{M}$,

— an ordered basis $(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ of $\mathbf{M}$ such that

$$K(x) = \big(\boldsymbol{k}^i \cdot (x - o) \mid \ i = 0, 1, 2, 3\big) \qquad (x \in \mathrm{M}),$$

where $(\boldsymbol{k}^0,\ \boldsymbol{k}^1,\ \boldsymbol{k}^2,\ \boldsymbol{k}^3)$ is the dual of the basis in question.

Proposition. The affine coordinatization K is a reference frame if and only if

(i) $(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ is a positively oriented basis,

(ii) $\boldsymbol{x}_0$ is a future-directed timelike vector,

(iii) $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are spacelike vectors.

Then the corresponding observer is global and inertial having the constant value

$$\boldsymbol{u} := \frac{\boldsymbol{x}_0}{\boldsymbol{s}},$$

and

$$K(x) = \left(\frac{\boldsymbol{\tau} \cdot (x - o)}{\boldsymbol{s}},\ (\boldsymbol{p}^\alpha \cdot \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (x - o))_{\alpha=1,2,3}\right) \qquad (x \in \mathrm{M}),$$

$$K^{-1}(\xi^0, \boldsymbol{\xi}) = o + \xi^0 \boldsymbol{s}\boldsymbol{u} + \sum_{\alpha=1}^{3} \xi^\alpha \boldsymbol{x}_\alpha \qquad \big((\xi^0, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^3\big)$$

where

$$\boldsymbol{s} := \boldsymbol{\tau} \cdot \boldsymbol{x}_0$$

and $\{\boldsymbol{p}^\alpha := \boldsymbol{k}^\alpha|_{\boldsymbol{E}} = \boldsymbol{k}^\alpha \cdot \mathbf{i} \mid \ (\alpha = 1, 2, 3)\}$ is the dual of the basis $\{\boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{x}_3\}$ of $\mathbf{E}$.

Proof. We show that the present conditions (i)–(iii) correspond to the conditions (i)–(ii) listed in Proposition 10.1.4 and condition (iii)' in 10.1.5.

(i) The coordinatization is orientation-preserving if and only if the corresponding basis is positively oriented;

(ii) $\partial_0 K^{-1}(K(x)) = \boldsymbol{x}_0$;

(iii)' $\mathrm{D}\kappa^0(x) = \boldsymbol{k}^0$ for all $x \in \mathrm{M}$. Since $\boldsymbol{k}^0 \cdot \boldsymbol{x}_\alpha = 0$, $\boldsymbol{k}^0 = \boldsymbol{e}\boldsymbol{\tau}$ for some $\boldsymbol{e} \in \mathbf{I}^*$ if and only if $\boldsymbol{x}_\alpha$-s ($\alpha = 1, 2, 3$) are spacelike; then, because of $\boldsymbol{k}^{\mathbf{0}} \cdot \boldsymbol{x}_{\mathbf{0}} = 1 > 0$, $\boldsymbol{e} = \frac{1}{\boldsymbol{\tau} \cdot \boldsymbol{x}_{\mathbf{0}}} > \mathbf{0}$. ■

We shall use the following names: o is the *origin,* $(\boldsymbol{x}_0,\ \boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{x}_3)$ is the spacetime basis of the affine reference system; moreover, $\boldsymbol{s} := \boldsymbol{\tau} \cdot \boldsymbol{x}_{\mathbf{0}} \in \mathbf{I}^+$ is the *time unit,* $\boldsymbol{u} := \frac{\boldsymbol{x}_0}{\boldsymbol{s}}$ is the *velocity value,* $(\boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{x}_3)$ is the *space basis* of the affine reference frame.

10.2.2. Let us take an affine reference frame K. Then the restriction of $\boldsymbol{K}$ (the linear map under K) onto $\mathbf{E}$ is a linear bijection between $\mathbf{E}$ and $\{0\} \times \mathbb{R}^3$. Let B denote the usual inner product on $\mathbb{R}^3 \equiv \{0\} \times \mathbb{R}^3$; then it makes sense that $\boldsymbol{K}|_{\mathbf{E}} = \boldsymbol{K} \cdot \mathbf{i} : \mathbf{E} \to \{0\} \times \mathbb{R}^3$ is $\mathbf{b}$-B-orthogonal (see V.1.6).

Definition. A reference frame K is called *Galilean* if
— K is affine,
— $\boldsymbol{K} \cdot \mathbf{i} : \mathbf{E} \to \{0\} \times \mathbb{R}^3$ is $\mathbf{b}$-B-orthogonal. ■

Proposition. A reference frame K is Galilean if and only if there are
(i) an $o \in \mathrm{M}$,
(ii) an ordered basis $(\boldsymbol{e}_0,\ \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ of $\mathbf{M}$,
— $(\boldsymbol{e}_0,\ \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ is positively oriented,
— $\boldsymbol{s} := \boldsymbol{\tau} \cdot \boldsymbol{e}_0 > 0$,
— $(\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ is a (necessarily positively oriented) orthogonal basis in $\mathbf{E}$, normed to an $\boldsymbol{m} \in \mathbf{D}^+$, such that

$$K(x) = \left(\frac{\boldsymbol{\tau} \cdot (x - o)}{\boldsymbol{s}},\ \left(\frac{\boldsymbol{e}_\alpha \cdot \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (x - o)}{\boldsymbol{m}^2} \right)_{\alpha = 1,2,3} \right) \qquad (x \in \mathrm{M}),$$

where

$$\boldsymbol{u} := \frac{\boldsymbol{e}_0}{\boldsymbol{s}}$$

is the constant value of the corresponding inertial observer.

Proof. It is quite evident that an affine reference frame is Galilean if and only if the spacelike elements of the corresponding basis in $\mathbf{M}$ are orthogonal to each other and have the same length. We know that the dual of the basis $(\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ becomes $\left(\frac{\boldsymbol{e}_1}{\boldsymbol{m}^2},\ \frac{\boldsymbol{e}_2}{\boldsymbol{m}^2},\ \frac{\boldsymbol{e}_3}{\boldsymbol{m}^2}\right)$ in the identification $\mathbf{E}^* \equiv \frac{\mathbf{E}}{\mathbf{D} \otimes \mathbf{D}}$ which proves the equality regarding K. ■

We shall use the following names for a Galilean reference frame: o is its *origin*, $(\boldsymbol{e}_0,\ \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ is its *spacetime basis*; moreover, $\boldsymbol{s} := \boldsymbol{\tau} \cdot \boldsymbol{e}_0$ is its *time unit,* $(\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ is its *space basis,* $\boldsymbol{m} := |\boldsymbol{e}_\alpha|$ $(\alpha = 1, 2, 3)$ is its *distance unit,* $\boldsymbol{u} := \frac{\boldsymbol{e}_0}{\boldsymbol{s}}$ is *its velocity value.*

10.2.3. Let K be a Galilean reference frame and use the previous notations.

Recalling 1.5.2, we see that the Galilean reference frame establishes an isomorphism between the spacetime model $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$ and the arithmetic spacetime model. More precisely, the coordinatization K and the mappings $B : \mathrm{I} \to \mathbb{R}$, $t \mapsto \frac{t - \tau(o)}{\boldsymbol{s}}$ and $\boldsymbol{Z} : \mathbf{D} \to \mathbb{R}$, $\boldsymbol{d} \mapsto \frac{\boldsymbol{d}}{\boldsymbol{m}}$ constitute an isomorphism.

This isomorphism transforms vectors, covectors and tensors, cotensors, etc. into vectors, covectors, etc. of the arithmetic spacetime model. In particular,

$$\boldsymbol{K} : \mathbf{M} \to \mathbb{R} \times \mathbb{R}^3, \qquad \boldsymbol{x} \mapsto \left(\frac{\boldsymbol{\tau} \cdot \boldsymbol{x}}{\boldsymbol{s}}, \ \left(\frac{\boldsymbol{e}_\alpha \cdot \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x}}{\boldsymbol{m}^2} \right)_{\alpha=1,2,3} \right),$$

is the coordinatization of vectors; note that it maps $\mathbf{E}$ onto $\{0\} \times \mathbb{R}^3$;

$$\left(\boldsymbol{K}^{-1}\right)^* : \mathbf{M}^* \times \mathbb{R} \times \mathbb{R}^3, \qquad \boldsymbol{k} \mapsto (\boldsymbol{k} \cdot \boldsymbol{e}_i \mid \ i = 0,1,2,3),$$

is the coordinatization of covectors; note that it maps $\mathbf{I}^* \cdot \boldsymbol{\tau}$ onto $\mathbb{R} \times \{\mathbf{0}\}$.

We can generalize the coordinatization for vectors (covectors) of type or cotype $\mathbf{A}$, i.e. for elements in $\mathbf{M} \otimes \mathbf{A}$ or $\frac{\mathbf{M}}{\mathbf{A}}$ $\left(\mathbf{M}^* \otimes \mathbf{A}, \ \frac{\mathbf{M}^*}{\mathbf{A}}\right)$, too, where $\mathbf{A}$ is a measure line. For instance, elements of $\frac{\mathbf{M}}{\mathbf{I}}$ or $\frac{\mathbf{M}}{\mathbf{D} \otimes \mathbf{D}}$ are coordinatized by the basis $\left(\frac{\boldsymbol{e}_i}{\boldsymbol{s}} \mid \ i = 0,1,2,3\right)$ and by the basis $\left(\frac{\boldsymbol{e}_i}{\boldsymbol{m}^2} \mid \ i = 0,1,2,3\right)$, respectively:

$$\frac{\mathbf{M}}{\mathbf{I}} \to \mathbb{R} \times \mathbb{R}^3, \qquad \boldsymbol{w} \mapsto \boldsymbol{s} \left(\frac{\boldsymbol{\tau} \cdot \boldsymbol{w}}{\boldsymbol{s}}, \ \left(\frac{\boldsymbol{e}_\alpha \cdot \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{w}}{\boldsymbol{m}^2} \right)_{\alpha=1,2,3} \right),$$

$$\frac{\mathbf{M}}{\mathbf{D} \otimes \mathbf{D}} \to \mathbb{R} \times \mathbb{R}^3, \qquad \boldsymbol{p} \mapsto \boldsymbol{m}^2 \left(\frac{\boldsymbol{\tau} \cdot \boldsymbol{p}}{\boldsymbol{s}}, \ \left(\frac{\boldsymbol{e}_\alpha \cdot \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{p}}{\boldsymbol{m}^2} \right)_{\alpha=1,2,3} \right).$$

10.3. Subscripts and superscripts

10.3.1. In textbooks one generally uses, without a precise definition, Galilean reference frames and the arithmetic spacetime model. Vectors, covectors and tensors, cotensors, etc. are given by coordinates relative to a spacetime basis. Let us survey the usual formalism from our point of view.

Let us take a Galilean reference system and let us use the previous notations.

If $(\boldsymbol{k}^0, \ \boldsymbol{k}^1, \ \boldsymbol{k}^2, \ \boldsymbol{k}^3)$ is the dual of the basis $(\boldsymbol{e}_0, \ \boldsymbol{e}_1, \ \boldsymbol{e}_2, \ \boldsymbol{e}_3)$, then

$$x^i := \boldsymbol{k}^i \cdot \boldsymbol{x} \qquad (i = 0,1,2,3)$$

are the coordinates of the vector $\boldsymbol{x}$; we know that

$$x^0 := \frac{\boldsymbol{\tau} \cdot \boldsymbol{x}}{\boldsymbol{s}}, \qquad x^\alpha := \frac{\boldsymbol{e}_\alpha \cdot \boldsymbol{x}}{\boldsymbol{m}^2} \qquad (\alpha = 1,2,3).$$

The covector $\boldsymbol{k}$ has the coordinates

$$k_i := \boldsymbol{k} \cdot \boldsymbol{e}_i \qquad (i = 0,1,2,3).$$

Let us accept the convention that the coordinates of vectors are denoted by superscripts and the coordinates of covectors are denoted by subscripts, and we shall not indicate that the coordinates run from 0 to 3. Then the symbol $\boldsymbol{x} \sim x^i$ and $\boldsymbol{k} \sim k_i$ will mean that the vector $\boldsymbol{x}$ (covector $\boldsymbol{k}$) has the coordinates x^i (k_i).

We have $\boldsymbol{k} \cdot \boldsymbol{x} = \sum_{i=0}^{3} k_i x^i$. According to the Einstein summation rule we shall omit the symbol of summation as well: $\boldsymbol{k} \cdot \boldsymbol{x} = k_i x^i$.

The various tensors are given by coordinates with respect to the tensor products of the corresponding bases (e.g. $(\boldsymbol{e}_i \otimes \boldsymbol{e}_j \mid i,j = 0,1,2,3)$ or $(\boldsymbol{e}_i \otimes \boldsymbol{k}^j \mid i,j = 0,1,2,3)$), as the following symbols show:

$$
\begin{aligned}
&\boldsymbol{T} \in \mathbf{M} \otimes \mathbf{M}, &\quad& \boldsymbol{T} \sim T^{ij},\\
&\boldsymbol{L} \in \mathbf{M} \otimes \mathbf{M}^*, && \boldsymbol{L} \sim L^i{}_j,\\
&\boldsymbol{P} \in \mathbf{M}^* \otimes \mathbf{M}, && \boldsymbol{P} \sim P_i{}^j,\\
&\boldsymbol{F} \in \mathbf{M}^* \otimes \mathbf{M}^*, && \boldsymbol{F} \sim F_{ij}.
\end{aligned}
$$

Applying the Einstein summation rule we can write, e.g. $\boldsymbol{T} \cdot \boldsymbol{k} \sim T^{ij} k_j$, $\boldsymbol{L} \cdot \boldsymbol{x} \sim L^i{}_j x^j$, $\boldsymbol{L} \cdot \boldsymbol{T} \sim L^i{}_j T^{jk}$, $\mathrm{Tr}\boldsymbol{L} = L^i{}_i$, etc.

We know that $\boldsymbol{x} \cdot \boldsymbol{y}$ makes no sense for $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{M}$; in coordinates this means that $x^i y^i$ makes no sense. Similarly, $\boldsymbol{L} \cdot \boldsymbol{k}$ makes no sense for $\boldsymbol{L} \in \mathbf{M} \otimes \mathbf{M}^*$ and $\boldsymbol{k} \in \mathbf{M}^*$; in coordinates this means that $L^i{}_j k_j$ makes no sense. More precisely, $x^i y^i$, etc. does not *make an absolute sense.* Of course, the value of this expression can be computed, but it depends on the reference frame: taking the coordinates x'^i and y'^i relative to another reference frame and computing $x'^i y'^i$ we get a different value.

We can see that, in general, a summation makes an absolute sense only for equal subscripts and superscripts.

10.3.2. Recall that we have the identification $\mathbf{E}^* \equiv \frac{\mathbf{E}}{\mathbf{D} \otimes \mathbf{D}}$ and under this identification the dual of the orthogonal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ becomes $\left(\frac{\boldsymbol{e}_\alpha}{\boldsymbol{m}^2} \mid \alpha = 1,2,3\right)$. The coordinates of $\boldsymbol{p} \in \mathbf{E}^*$ are $p_\alpha := \boldsymbol{p} \cdot \boldsymbol{e}_\alpha$ ($\alpha = 1,2,3$). If we consider $\boldsymbol{p}$ as an element of $\frac{\mathbf{E}}{\mathbf{D} \otimes \mathbf{D}}$ then it has the coordinates $p^\alpha := \boldsymbol{m}^2 \left(\frac{\boldsymbol{e}_\alpha}{\boldsymbol{m}^2} \cdot \boldsymbol{p}\right) = p_\alpha$ ($\alpha = 1,2,3$).

Similarly, $\boldsymbol{q} \in \mathbf{E}$ has the coordinates $q^\alpha := \frac{\boldsymbol{e}_\alpha}{\boldsymbol{m}^2} \cdot \boldsymbol{q}$ ($\alpha = 1,2,3$). If we consider $\boldsymbol{q}$ as an element of $\mathbf{E}^* \otimes \mathbf{D} \otimes \mathbf{D}$, then its coordinates are $q_\alpha := \frac{1}{\boldsymbol{m}^2}(\boldsymbol{q} \cdot \boldsymbol{e}_\alpha) = q^\alpha$ ($\alpha = 1,2,3$).

Thus dealing exclusively with *spacelike vectors,* we need not distinguish between superscripts and subscripts. We know that $\boldsymbol{q} \cdot \boldsymbol{q}$ makes sense for a spacelike vector $\boldsymbol{q}$, and $\boldsymbol{q} \cdot \boldsymbol{q} \sim q^\alpha q^\alpha = q^\alpha q_\alpha = q_\alpha q_\alpha$.

We emphasize that this is true only if we use an orthogonal and normed basis in $\mathbf{E}$ (see V.3.20).

10.4. Reference systems associated with global rigid observers*

10.4.1. We know that the space $\mathrm{E}_{\boldsymbol{U}}$ of a global rigid observer $\boldsymbol{U}$ is a three-dimensional affine space. Moreover, given $t_o \in \mathrm{I}$ and $q_o \in \mathrm{E}_{\boldsymbol{U}}$, or, equivalently, given $o \in \mathrm{M}$— called the *origin* — such that $o = q_o \star t_o$, $t_o = \tau(o)$, $q_o = C_{\boldsymbol{U}}(o)$— we establish the (double) vectorization

$$\begin{aligned}\mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \to \mathbf{I} \times \mathbf{E}, \qquad (t, q) &\mapsto (t - t_o,\ q \star t_o - q_o \star t_o) \\ &= (t - \tau(o), q \star \tau(o) - o)\,,\end{aligned}$$

which is an orientation-preserving affine bijection.

Then choosing an $\boldsymbol{s} \in \mathbf{I}^+$ (a positively oriented basis in $\mathbf{I}$)— called the *time unit* — and a positively oriented basis $(\boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{x}_3)$ in $\mathbf{E}$— called the space basis — , we can establish coordinatizations of time and $\boldsymbol{U}$-space:

$$T(t) := \frac{t - \tau(o)}{\boldsymbol{s}} \qquad (t \in \mathrm{I}),$$

$$S_{\boldsymbol{U}}(q) := (\boldsymbol{p}^\alpha \cdot (q \star \tau(o) - o) \mid \ \alpha = 1, 2, 3)\,, \qquad (q \in \mathrm{E}_{\boldsymbol{U}}),$$

where $(\boldsymbol{p}^1,\ \boldsymbol{p}^2,\ \boldsymbol{p}^3)$ is the dual of the basis in question.

Evidently, T and $S_{\boldsymbol{U}}$ are orientation-preserving affine bijections; we know that $H_{\boldsymbol{U}}$ is an orientation-preserving smooth bijection whose inverse is smooth as well (see 4.3.2 and 4.4.3), thus $(\boldsymbol{U}, T, S_{\boldsymbol{U}})$ is a reference system. For the corresponding reference frame $K := (T \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U}}$ we have

$$K(x) = \left(\frac{\boldsymbol{\tau} \cdot (x - o)}{\boldsymbol{s}}, (\boldsymbol{p}^\alpha \cdot (C_{\boldsymbol{U}}(x) \star \tau(o) - o))_{\alpha=1,2,3}\right) \qquad (x \in \mathrm{M}),$$

$$K^{-1}(\xi^0, \boldsymbol{\xi}) = C_{\boldsymbol{U}}\left(o + \sum_{\alpha=1}^{3} \xi^\alpha \boldsymbol{x}_\alpha\right) \star \left(\tau(o) + \xi^0 \boldsymbol{s}\right) \qquad \left((\xi^0, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^3\right).$$

T and $S_{\boldsymbol{U}}$ are affine coordinatizations of time and $\boldsymbol{U}$-space. Evidently, $K = (T \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U}}$ is an affine coordinatization of spacetime if and only if $H_{\boldsymbol{U}}$ is an affine map which holds if and only if $\boldsymbol{U}$ is a global inertial observer (see 5.1).

10.4.2. Let us take a uniformly accelerated observer $\boldsymbol{U}$ having the constant acceleration value $\boldsymbol{a}$ (see 5.2).

Then, according to 5.2.3, for the reference frame treated in 10.4.1 we have

$$K(x) = \left(\frac{\boldsymbol{\tau} \cdot (x - o)}{\boldsymbol{s}},\ \boldsymbol{p}^\alpha \cdot \left(\boldsymbol{\pi}_{\boldsymbol{U}(o)} \cdot (x - o) - \frac{1}{2}\boldsymbol{a}(\boldsymbol{\tau} \cdot (x - o))^2\right)_{\alpha=1,2,3}\right)$$

$$(x \in \mathrm{M})$$

and

$$K^{-1}(\xi^0,\boldsymbol{\xi}) = o + \xi^0 \boldsymbol{s}\boldsymbol{U}(o) + \sum_{\alpha=1}^{3} \xi^\alpha \boldsymbol{x}_\alpha + \frac{1}{2}(\xi^0)^2 \boldsymbol{s}^2 \boldsymbol{a} \qquad ((\xi^0,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^3).$$

10.4.3. Let us take a uniformly rotating observer $\boldsymbol{U}$, and let o, $\boldsymbol{c}$ and Ω be the quantities introduced in 5.3.

Then, according to 5.4.3, for the reference frame treated in 7.4.1 we have

$$K(x) = \left(\frac{\boldsymbol{\tau}\cdot(x-o)}{\boldsymbol{s}}, \left(\boldsymbol{p}^\alpha \cdot e^{-(\boldsymbol{\tau}\cdot(x-o))\Omega} \cdot \boldsymbol{\pi}_{\boldsymbol{c}} \cdot (x-o)\right)_{\alpha=1,2,3}\right) \qquad (x \in \mathrm{M}),$$

$$K^{-1}(\xi^0,\boldsymbol{\xi}) = C_{\boldsymbol{U}}(o) \star \left(\tau(o) + \xi^0 \boldsymbol{s}\right) + e^{\xi^0 \boldsymbol{s}\Omega} \cdot \sum_{\alpha=1}^{3} \xi^\alpha \boldsymbol{x}_\alpha \qquad \left((\xi^0,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^3\right).$$

10.4.4. Expressing in words we can say:

Galilean reference system = global inertial observer + measuring time with respect to an initial instant and a time unit + introducing orthogonal (Cartesian) coordinates in the observer space.

Affine reference system = global inertial observer + measuring time with respect to an initial instant and a time unit + introducing (oblique-angled) rectilinear coordinates in the observer space.

Other reference systems treated previously = global rigid observer + measuring time with respect to an initial instant and a time unit + introducing rectilinear coordinates in the observer space.

For the solution of some practical problems we often use reference systems in which curvilinear coordinates (e.g. spherical coordinates or cylindrical coordinates) are introduced in the observer space.

10.5. Equivalent reference frames

10.5.1. In textbooks one usually formulates the principle — without a precise definition — that the Galilean reference frames are equivalent with respect to the description of phenomena. It is very important that then one takes tacitly into consideration Galilean reference frames with the same time unit and the same distance unit.

Reference frames as we defined them are mathematical objects. The physical object modelled by them will be called here a physical reference frame. When could we consider two physical reference frames to be equivalent? The answer is: if the experiments prepared in the same way in the reference frames give the same results. Let us see some illustrative examples.

Take two physical Galilean reference frames in which the time units and distance units are different and perform the following experiment in both systems: let an iron ball of unit diameter moving with unit relative velocity hit perpendicularly a sheet of glass of unit width. It may happen that the ball bounces in one of the reference frames, the glass breaks in the other. The two reference frames are not equivalent.

Take an affine reference frame in which the first space basis element is perpendicular to the other two basis elements; take another affine reference frame in which the first space basis element is not perpendicular to the other two basis elements. Perform the following experiment in both frames: let a ball moving parallelly to the first space axis hit a plane parallel to the other two axes. The ball returns to its initial position in one of the reference frames and does not in the other. The two reference frames are not equivalent.

10.5.2. Recall the notion of automorphisms of the spacetime model (1.5.4). An automorphism is a transformation that leaves invariant (preserves) the structure of the spacetime model. Strict automorphisms do not change time periods and distances.

It is quite natural that two objects transformed into each other by a strict automorphism of the spacetime model are considered equivalent (i.e. identical from a physical point of view).

In the next paragraph we shall study the Noether transformations that involve the strict automorphisms of the spacetime model. Now we recall the basic facts.

Let $\mathcal{SO}(\mathbf{b})$ denote the set of linear maps $\boldsymbol{R} : \mathbf{E} \to \mathbf{E}$ that preserve the Euclidean structure and the orientation of $\mathbf{E}$: $\mathbf{b} \circ (\boldsymbol{R} \times \boldsymbol{R}) = \mathbf{b}$ and $\det\boldsymbol{R} = 1$ (see 11.1.2).

Let us introduce the notation

$$\mathcal{N}^{+\to} := \{L : \mathrm{M} \to \mathrm{M} \mid \ L \quad \text{is affine}, \quad \boldsymbol{\tau} \cdot \boldsymbol{L} = \boldsymbol{\tau}, \quad \boldsymbol{L}|_{\mathbf{E}} \in \mathcal{SO}(\mathbf{b})\}$$

and let us call the elements of $\mathcal{N}^{+\to}$ *proper Noether transformations.* It is quite evident that $(L, \mathrm{id}_{\mathrm{I}},\ \mathrm{id}_{\mathbf{D}})$ is a strict automorphism of the spacetime model if and only if L is a proper Noether transformation (11.6.4).

An affine map $\mathrm{ti}L : \mathrm{I} \to \mathrm{I}$ can be assigned to every proper Noether transformation L in such a way that $\tau \circ L = (\mathrm{ti}L) \circ \tau$ (see 11.6.3).

10.5.3. Definition. The reference frames K and K' are called *equivalent* if there is a proper Noether transformation L such that

$$K' \circ L = K.$$

Two reference systems are *equivalent* if the corresponding reference frames are equivalent.

Proposition. Let $(\boldsymbol{U}, T, S_{\boldsymbol{U}})$ and $(\boldsymbol{U}', T', S_{\boldsymbol{U}'})$ be the reference systems corresponding to the reference frames K and K', respectively. If K and K' are equivalent, $K' \circ L = K$, then
(i) $\boldsymbol{L}^{-1} \cdot \boldsymbol{U}' \circ L = \boldsymbol{U}$, in other words, $\boldsymbol{L} \cdot \boldsymbol{U} = \boldsymbol{U}' \circ L$,
(ii) $T' \circ (\mathrm{ti}L) = T$, in other words, $T'^{-1} \circ T = \mathrm{ti}L$.
(iii) $\left(S_{\boldsymbol{U}'}^{-1} \circ S_{\boldsymbol{U}}\right) \circ C_{\boldsymbol{U}} = C_{\boldsymbol{U}'} \circ L$.

Proof. (i) $K = K' \circ L$, $K^{-1} = L^{-1} \circ K'^{-1}$ and $\boldsymbol{\tau} \cdot \boldsymbol{L} = \boldsymbol{\tau}$ together with 10.1.4 imply

$$\boldsymbol{U}(x) = \frac{\partial_0 K^{-1}(K(x))}{\boldsymbol{\tau} \cdot \partial_0 K^{-1}(K(x))} = \frac{\boldsymbol{L}^{-1} \cdot \partial_0 K'^{-1}\left(K'(L(x))\right)}{\boldsymbol{\tau} \cdot \boldsymbol{L}^{-1} \cdot \partial_0 K'^{-1}\left(K'(L(x))\right)} = \boldsymbol{L}^{-1} \cdot \boldsymbol{U}'(L(x)).$$

(ii) The equalities

$$T \circ \tau = \mathrm{pr}^0 \circ K = \mathrm{pr}^0 \circ K' \circ L = T' \circ \tau \circ L = T' \circ (\mathrm{ti}L) \circ \tau$$

yield the desired relation immediately.
(iii) Consider the equalities

$$S_{\boldsymbol{U}} \circ C_{\boldsymbol{U}} = \mathbf{pr} \circ K = \mathbf{p} \circ K' \circ L = S_{\boldsymbol{U}'} \circ C_{\boldsymbol{U}'} \circ L.$$

10.5.4. Now we shall see that our definition of equivalence of reference frames is in accordance with the intuitive notion expounded in 10.5.1.

Proposition. Two Galilean reference frames are equivalent if and only if they have the same time unit and distance unit, respectively.

Proof. Let the Galilean reference frames K and K' be defined by the origins o and o' and the spacetime bases $(\boldsymbol{e}_0,\ \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ and $(\boldsymbol{e}_0',\ \boldsymbol{e}_1',\ \boldsymbol{e}_2',\ \boldsymbol{e}_3')$, respectively.
Then $L := K'^{-1} \circ K : \mathrm{M} \to \mathrm{M}$ is the affine bijection determined by

$$L(o) = o', \qquad \boldsymbol{L} \cdot \boldsymbol{e}_i = \boldsymbol{e}_i' \qquad\qquad (i = 0, 1, 2, 3).$$

Evidently, L is orientation-preserving. Moreover, $\boldsymbol{\tau} \cdot \boldsymbol{L} = \boldsymbol{\tau}$ if and only if $\boldsymbol{\tau} \cdot \boldsymbol{e}_0 = \boldsymbol{\tau} \cdot \boldsymbol{e}_0'$, and $\boldsymbol{L}|_{\boldsymbol{E}} \in \mathcal{SO}(\mathbf{b})$ if and only if $|\boldsymbol{e}_\alpha| = |\boldsymbol{e}_\alpha'|$ ($\alpha = 1, 2, 3$).

10.6. Exercises

1. Reference frames are coordinatizations, hence we can apply all the notions introduced in VI.5, e.g. the coordinatized form of vector fields.

Let $\boldsymbol{U}$ be the observer corresponding to the reference frame K. Demonstrate that the coordinatized form of $\boldsymbol{U}$ according to K is the constant mapping $(1, \mathbf{0})$. ($\boldsymbol{U}$ is a vector field of cotype $\mathbf{I}$, hence by definition, $(\mathrm{D}K \cdot \boldsymbol{U}) \circ K^{-1}$ is its coordinatized form according to K.)

2. Take a uniformly accelerated observer $\boldsymbol{U}$ having the acceleration value $\boldsymbol{a} \neq \mathbf{0}$. Fix $\boldsymbol{s} \in \mathbf{I}^+$, $\boldsymbol{m} \in \mathbf{D}^+$ and define a Galilean reference frame K with an arbitrary origin and with a spacetime basis such that $\boldsymbol{e}_0 := \boldsymbol{s}\boldsymbol{U}(o)$, $\boldsymbol{e}_1 := \boldsymbol{m}\frac{\boldsymbol{a}}{|\boldsymbol{a}|}$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are arbitrary. Demonstrate that, according to K, $\boldsymbol{U}$ has the coordinatized form

$$(\xi^0,\ \xi^1,\ \xi^2,\ \xi^3) \mapsto (1,\ \alpha\xi^0,\ 0,\ 0),$$

where α is the number for which $|\boldsymbol{a}| = \alpha\frac{\boldsymbol{m}}{\boldsymbol{s}^2}$ holds.

The $\boldsymbol{U}$-line passing through $o + \sum_{i=0}^{3} \xi^i \boldsymbol{e}_i$ becomes

$$\left\{ \left(t, \xi^1 + \alpha\xi^0(t - \xi^0) + \frac{1}{2}\alpha(t - \xi^0)^2, \xi^2, \xi^3 \right) \mid t \in \mathbb{R} \right\}.$$

3. Take a uniformly rotating observer $\boldsymbol{U}$ having the angular velocity Ω and suppose there is an inertial $\boldsymbol{U}$-space point $q_o = o + \boldsymbol{c} \otimes \mathbf{I}$. Fix $\boldsymbol{s} \in \mathbf{I}^+$, $\boldsymbol{m} \in \mathbf{D}^+$ and define a Galilean reference frame with o, $\boldsymbol{e}_0 := \boldsymbol{s}\boldsymbol{U}(o)$, $\boldsymbol{e}_3$ positively oriented in $\operatorname{Ker}\Omega$, $|\boldsymbol{e}_3| = \boldsymbol{m}$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ being arbitrary. Demonstrate that, according to K, $\boldsymbol{U}$ has the coordinatized form

$$(\xi^0,\ \xi^1,\ \xi^2,\ \xi^3) \mapsto (1,\ -\omega\xi^2,\ \omega\xi^1,\ 0),$$

where ω is the number for which $|\Omega| = \omega\frac{1}{\boldsymbol{s}}$ holds.

The $\boldsymbol{U}$-line passing through $o + \sum_{i=0}^{3} \xi^i \boldsymbol{e}_i$ becomes

$$\Big\{ \big(t,\ \xi^1 \cos\omega(t - \xi^0) - \xi^2 \sin\omega(t - \xi^0),$$
$$\xi^1 \sin\omega(t - \xi^0) + \xi^2 \cos\omega(t - \xi^0),\ \xi^3\big) \mid t \in \mathbb{R} \Big\}.$$

4. Prove that two affine reference frames are equivalent if and only if they have the same time unit and the corresponding elements of the space bases have the same length and the same angles between themselves; in other words, the affine reference frames defined by the origins o and o' and the spacetime bases $(\boldsymbol{x}_0,\ \boldsymbol{x}_1,\ \boldsymbol{x}_2,\ x_3)$ and $(\boldsymbol{x}'_0,\ \boldsymbol{x}'_1,\ \boldsymbol{x}'_2,\ \boldsymbol{x}'_3)$, respectively, are equivalent if and only if

$$\boldsymbol{\tau} \cdot \boldsymbol{x}_0 = \boldsymbol{\tau} \cdot \boldsymbol{x}'_0$$

and

$$\boldsymbol{x}_\alpha \cdot \boldsymbol{x}_\beta = \boldsymbol{x}'_\alpha \cdot \boldsymbol{x}'_\beta \qquad (\alpha, \beta = 1, 2, 3).$$

5. Prove that two reference frames defined for uniformly accelerated observers in the form given in 10.4.2 are equivalent if and only if the two acceleration values have the same magnitude, the time units are equal, the corresponding elements of the space bases have the same length and the same angles between themselves, and the acceleration values incline in the same way to the basis elements; in other words, if $\boldsymbol{a}$ and $\boldsymbol{a}'$ are the acceleration values, $\boldsymbol{s}$ and $\boldsymbol{s}'$ are the time units, $(\boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{x}_3)$ and $(\boldsymbol{x}'_1,\ \boldsymbol{x}'_2,\ \boldsymbol{x}'_3)$ are the space bases, then the two reference systems are equivalent if and only if

$$|\boldsymbol{a}| = |\boldsymbol{a}'|, \qquad \boldsymbol{s} = \boldsymbol{s}',$$
$$\boldsymbol{x}_\alpha \cdot \boldsymbol{x}_\beta = \boldsymbol{x}'_\alpha \cdot \boldsymbol{x}'_\beta, \quad \frac{\boldsymbol{x}_\alpha \cdot \boldsymbol{a}}{|\boldsymbol{x}_\alpha||\boldsymbol{a}|} = \frac{\boldsymbol{x}'_\alpha \cdot \boldsymbol{a}'}{|\boldsymbol{x}'_\alpha||\boldsymbol{a}'|} \quad (\alpha, \beta = 1, 2, 3).$$

6. Prove that two reference frames defined for uniformly rotating observers in the form given in 10.4.3 are equivalent if and only if the angular velocities have the same magnitude, the time units are equal, the corresponding elements of the space bases have the same length and the same angles between themselves and the oriented kernels of the angular velocities incline in the same way to the basis elements.

7. Take a global inertial observer and construct a reference system by spherical (cylindrical) coordinatization of the observer space. Find necessary and sufficient conditions that two such reference systems be equivalent.

8. In all the treated reference systems time is coordinatized by an affine map. Construct a reference system based on a global inertial observer in which the time coordinatization is not affine.

11. Spacetime groups*

11.1. The three-dimensional orthogonal groups

11.1.1. $(\mathbf{E}, \mathbf{D}, \mathbf{b})$ is a three-dimensional oriented Euclidean vector space. Recall the notations (see V.2.7)

$$\mathrm{A}(\mathbf{b}) := \{\boldsymbol{A} \in \mathbf{E} \otimes \mathbf{E}^* \mid \boldsymbol{A}^* = -\boldsymbol{A}\} \equiv \frac{\mathbf{E}}{\mathbf{D}} \wedge \frac{\mathbf{E}}{\mathbf{D}},$$
$$\mathcal{O}(\mathbf{b}) := \{\boldsymbol{R} \in \mathbf{E} \otimes \mathbf{E}^* \mid \boldsymbol{R}^* = \boldsymbol{R}^{-1}\}.$$

$\mathrm{A}(\mathbf{b})$ is a three-dimensional subspace in $\mathbf{E} \otimes \mathbf{E}^*$ and $\mathcal{O}(\mathbf{b})$ is a three-dimensional Lie group having $\mathrm{A}(\mathbf{b})$ as its Lie algebra (VII.5).

11.1.2. We know that $|\det \boldsymbol{R}| = 1$ for $\boldsymbol{R} \in \mathcal{O}(\mathbf{b})$ (see V.2.8). We introduce the notations

$$\mathcal{SO}(\mathbf{b}) := \mathcal{O}(\mathbf{b})^+ := \{\boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \mid \det \boldsymbol{R} = 1\},$$
$$\mathcal{O}(\mathbf{b})^- := \{\boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \mid \det \boldsymbol{R} = -1\}.$$

The elements of $\mathcal{SO}(\mathbf{b})$ are called *rotations.*

Since the determinant is a continuous function, $\mathcal{O}(\mathbf{b})^+$ and $\mathcal{O}(\mathbf{b})^-$ are disjoint.

Evidently, $\mathrm{id}_\mathbf{E} \in \mathcal{O}(\mathbf{b})^+$ and $-\mathrm{id}_\mathbf{E} \in \mathcal{O}(\mathbf{b})^-$; moreover, $(-\mathrm{id}_\mathbf{E}) \cdot \mathcal{O}(\mathbf{b})^+ = \mathcal{O}(\mathbf{b})^-$.

The determinant is a continuous function, hence both $\mathcal{O}(\mathbf{b})^+$ and $\mathcal{O}(\mathbf{b})^-$ are closed. Moreover, we know that $\boldsymbol{F} \mapsto \mathrm{Tr}(\boldsymbol{F}^* \cdot \boldsymbol{F})$ is an inner product (real-valued positive definite bilinear form) on $\mathbf{E} \otimes \mathbf{E}^*$ (see V.2.10). Since $\mathrm{Tr}(\boldsymbol{R}^* \cdot \boldsymbol{R}) = \mathrm{Tr}(\mathrm{id}_\mathbf{E}) = 3$ for all $\boldsymbol{R} \in \mathcal{O}(\mathbf{b})$, $\mathcal{O}(\mathbf{b})$ is a bounded set.

Thus we can state, that $\mathcal{O}(\mathbf{b})$, $\mathcal{O}(\mathbf{b})^+$ and $\mathcal{O}(\mathbf{b})^-$ are compact (closed and bounded) sets.

11.1.3. Let $\boldsymbol{R} \in \mathcal{SO}(\mathbf{b})$. For all $\boldsymbol{x} \in \mathbf{E}$ we have $|\boldsymbol{R} \cdot \boldsymbol{x}| = |\boldsymbol{x}|$. As a consequence, $\boldsymbol{R} \cdot \boldsymbol{x} = \alpha \boldsymbol{x}$ implies $\alpha = \pm 1$.

Proposition. For every $\boldsymbol{R} \in \mathcal{SO}(\mathbf{b})$ there is a non-zero $\boldsymbol{x} \in \mathbf{E}$ such that $\boldsymbol{R} \cdot \boldsymbol{x} = \boldsymbol{x}$; moreover,

$$a_{\boldsymbol{R}} := \{\boldsymbol{x} \in \mathbf{E} \mid \boldsymbol{R} \cdot \boldsymbol{x} = \boldsymbol{x}\}$$

is a one-dimensional linear subspace if and only if $\boldsymbol{R} \neq \mathrm{id}_\mathbf{E}$.

Proof. It is trivial that $a_{\boldsymbol{R}} = \mathbf{E}$ for $\boldsymbol{R} = \mathrm{id}_\mathbf{E}$.
IV.3.18 and V.1.5 result in

$$\det(\boldsymbol{R} - \mathrm{id}_\mathbf{E}) = \det(\boldsymbol{R} - \boldsymbol{R}^* \cdot \boldsymbol{R}) = \det(\mathrm{id}_\mathbf{E} - \boldsymbol{R}^*) \det \boldsymbol{R} = -\det(\boldsymbol{R} - \mathrm{id}_\mathbf{E}).$$

Consequently, $\det(\boldsymbol{R} - \mathrm{id}_\mathbf{E}) = 0$, $\boldsymbol{R} - \mathrm{id}_\mathbf{E}$ is not injective, there is a non-zero $\boldsymbol{x}$ such that $(\boldsymbol{R} - \mathrm{id}_\mathbf{E}) \cdot \boldsymbol{x} = \mathbf{0}$.

Let us suppose $a_{\boldsymbol{R}}$ is not one-dimensional, i.e. $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are not parallel vectors such that $\boldsymbol{R} \cdot \boldsymbol{x}_1 = \boldsymbol{x}_1$ and $\boldsymbol{R} \cdot \boldsymbol{x}_2 = \boldsymbol{x}_2$. Then for every element $\boldsymbol{x}$ in the plane spanned by $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ we have $\boldsymbol{R} \cdot \boldsymbol{x} = \boldsymbol{x}$. This means that the plane spanned by $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is invariant for $\boldsymbol{R}$ and the restriction of $\boldsymbol{R}$ onto that plane is the identity. Let $\boldsymbol{y}$ be a non-zero vector orthogonal to the plane spanned by $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Since $\boldsymbol{R}$ preserves orthogonality, $\boldsymbol{R} \cdot \boldsymbol{y}$ must be orthogonal to that plane, i.e. it is parallel to $\boldsymbol{y}$: $\boldsymbol{R} \cdot \boldsymbol{y} = \pm \boldsymbol{y}$. $\boldsymbol{R}$ is orientation-preserving, thus $\boldsymbol{R} \cdot \boldsymbol{y} = \boldsymbol{y}$ must hold. This means that $\boldsymbol{R} = \mathrm{id}_\mathbf{E}$. ■

For $\boldsymbol{R} \neq \mathrm{id}_\mathbf{E}$, $a_{\boldsymbol{R}}$ is called the *axis of rotation* of $\boldsymbol{R}$.

11.1.4. For $\boldsymbol{R} \in \mathcal{SO}(\mathbf{b})$ the symbol $a_{\boldsymbol{R}}^{\perp}$ will stand for the orthogonal complement of $a_{\boldsymbol{R}}$:

$$a_{\boldsymbol{R}}^{\perp} := \{x \in \mathbf{E} \mid x \quad \text{is orthogonal to} \quad a_{\boldsymbol{R}}\}.$$

Evidently, $a_{\boldsymbol{R}}^{\perp} = \{\mathbf{0}\}$ for $\boldsymbol{R} = \mathrm{id}_{\mathbf{E}}$ and $a_{\boldsymbol{R}}^{\perp}$ is a plane for $\boldsymbol{R} \neq \mathrm{id}_{\mathbf{E}}$. Moreover, $a_{\boldsymbol{R}}^{\perp}$ is invariant for $\boldsymbol{R}$.

The restriction of $\boldsymbol{R} \neq \mathrm{id}_{\mathbf{E}}$ onto $a_{\boldsymbol{R}}^{\perp}$ is a rotation in a plane which "evidently" can be characterized by an angle of rotation. This is the content of the following proposition.

Proposition. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are non-zero vectors in $a_{\boldsymbol{R}}^{\perp}$ then

$$\frac{\boldsymbol{x} \cdot \boldsymbol{R} \cdot \boldsymbol{x}}{|\boldsymbol{x}|^2} = \frac{\boldsymbol{y} \cdot \boldsymbol{R} \cdot \boldsymbol{y}}{|\boldsymbol{y}|^2}.$$

Proof. We can exclude the trivial cases $\boldsymbol{R} \cdot \boldsymbol{x} = \boldsymbol{x}$ and $\boldsymbol{R} \cdot \boldsymbol{x} = -\boldsymbol{x}$ for all $\boldsymbol{x} \in a_{\boldsymbol{R}}^{\perp}$ (note that the first case is $\boldsymbol{R} = \mathrm{id}_{\mathbf{E}}$).

It will be convenient to put $\boldsymbol{n} := \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$, $\boldsymbol{k} := \frac{\boldsymbol{y}}{|\boldsymbol{y}|}$ and to consider $\boldsymbol{R}$ to be a linear map on $\frac{\mathbf{E}}{\mathbf{D}}$. Let us introduce the notation

$$S_{\boldsymbol{R}} := \left\{ \boldsymbol{n} \in \frac{\mathbf{E}}{\mathbf{D}} \mid \boldsymbol{n} \quad \text{is orthogonal to} \quad a_{\boldsymbol{R}}, \quad |\boldsymbol{n}| = 1 \right\}.$$

The proof consists of several simple steps whose details are left to the reader.

(*i*) Let $\boldsymbol{n}$ and $\boldsymbol{k}$ be elements of $S_{\boldsymbol{R}}$ orthogonal to each other. Then, excluding the trivial case,

$$\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{k} \neq 0.$$

Indeed,

$$1 = \det\boldsymbol{R} = (\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{k}) - (\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{k})(\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{n})$$

and because of the Cauchy inequality (apart from the trivial case), $(\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{k}) < 1$.

(*ii*) $\boldsymbol{R} \cdot \boldsymbol{n} \neq \boldsymbol{R}^{-1} \cdot \boldsymbol{n}$. Indeed, suppose $\boldsymbol{R} \cdot \boldsymbol{n} = \boldsymbol{R}^{-1} \cdot \boldsymbol{n}$. Then we get from the previous formula that

$$\begin{aligned} 1 &= (\boldsymbol{n} \cdot \boldsymbol{R}^{-1} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{k}) - (\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{k})(\boldsymbol{k} \cdot \boldsymbol{R}^{-1} \cdot \boldsymbol{n}) = \\ &= (\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{k}) - (\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{k})(\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{k}), \end{aligned}$$

which implies $(\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{k}) > 1$ contradicting the Cauchy inequality.

(*iii*) $\mathbf{0} \neq \boldsymbol{R} \cdot \boldsymbol{n} - \boldsymbol{R}^{-1} \cdot \boldsymbol{n}$ is orthogonal to $\boldsymbol{n}$, hence it is parallel to $\boldsymbol{k}$.

(*iv*) $\boldsymbol{R} \cdot \boldsymbol{n} + \boldsymbol{R}^{-1} \cdot \boldsymbol{n}$ is orthogonal to $\boldsymbol{R} \cdot \boldsymbol{n} - \boldsymbol{R}^{-1} \cdot \boldsymbol{n}$, hence it is orthogonal to $\boldsymbol{k}$ as well. Consequently,

$$\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{k} + \boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{n} = \mathbf{0}.$$

(*v*) $\boldsymbol{R} \cdot \boldsymbol{n} = (\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{n})\boldsymbol{n} + (\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{n})\boldsymbol{k}$, and from a similar relation for $\boldsymbol{R} \cdot \boldsymbol{k}$ we have

$$0 = (\boldsymbol{R} \cdot \boldsymbol{n}) \cdot (\boldsymbol{R} \cdot \boldsymbol{k}) = (\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{n})(\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{k}) + (\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{R} \cdot \boldsymbol{k});$$

then we infer from (i) and (iii) that $\boldsymbol{n}\cdot\boldsymbol{R}\cdot\boldsymbol{n}=\boldsymbol{k}\cdot\boldsymbol{R}\cdot\boldsymbol{k}$.

(vi) If $\boldsymbol{m}\in S_{\boldsymbol{R}}$ then $\boldsymbol{m}=\alpha\boldsymbol{n}+\beta\boldsymbol{k}$, $\alpha^2+\beta^2=1$ and $\boldsymbol{m}\cdot\boldsymbol{R}\cdot\boldsymbol{m}=\boldsymbol{n}\cdot\boldsymbol{R}\cdot\boldsymbol{n}$. ■

Now let us return to $\mathbf{0}\neq\boldsymbol{x}\in\mathbf{E}$, orthogonal to $a_{\boldsymbol{R}}$. The Cauchy inequality gives $|\boldsymbol{x}\cdot\boldsymbol{R}\cdot\boldsymbol{x}|\leq|\boldsymbol{x}|^2$; thus

$$\alpha_{\boldsymbol{R}}:=\arccos\frac{\boldsymbol{x}\cdot\boldsymbol{R}\cdot\boldsymbol{x}}{|\boldsymbol{x}|^2}\in[0,\pi]$$

is meaningful, which is called the *angle of rotation of* $\boldsymbol{R}$.

Observe that

— $\alpha_{\boldsymbol{R}}=0$ if and only if $\boldsymbol{R}\cdot\boldsymbol{x}=\boldsymbol{x}$ for all $\boldsymbol{x}$ orthogonal to $a_{\boldsymbol{R}}$, i.e. $\boldsymbol{R}=\mathrm{id}_{\mathbf{E}}$,

— $\alpha_{\boldsymbol{R}}=\pi$ if and only if $\boldsymbol{R}\cdot\boldsymbol{x}=-\boldsymbol{x}$ for all $\boldsymbol{x}$ orthogonal to $a_{\boldsymbol{R}}$.

11.1.5. Proposition. Let $\boldsymbol{R}\neq\mathrm{id}_{\mathbf{E}}$ and $\alpha_{\boldsymbol{R}}\neq\pi$. Take an arbitrary non-zero $\boldsymbol{x}\in a_{\boldsymbol{R}}^{\perp}$. Let $\boldsymbol{y}\in a_{\boldsymbol{R}}^{\perp}$ be orthogonal to $\boldsymbol{x}$, $|\boldsymbol{y}|=|\boldsymbol{x}|$, and suppose $(\boldsymbol{x},\boldsymbol{y})$ and $(\boldsymbol{x},\boldsymbol{R}\cdot\boldsymbol{x})$ are equally oriented bases in $a_{\boldsymbol{R}}^{\perp}$. Then

$$\boldsymbol{R}\cdot\boldsymbol{x}=(\cos\alpha_{\boldsymbol{R}})\boldsymbol{x}+(\sin\alpha_{\boldsymbol{R}})\boldsymbol{y}.$$

Proof. Since $\boldsymbol{R}\cdot\boldsymbol{x}=\frac{\boldsymbol{x}\cdot\boldsymbol{R}\cdot\boldsymbol{x}}{|\boldsymbol{x}|^2}\boldsymbol{x}+\frac{\boldsymbol{y}\cdot\boldsymbol{R}\cdot\boldsymbol{x}}{|\boldsymbol{y}|^2}\boldsymbol{y}$, we easily find that $\cos^2\alpha_{\boldsymbol{R}}+\left(\frac{\boldsymbol{y}\cdot\boldsymbol{R}\cdot\boldsymbol{x}}{|\boldsymbol{y}|^2}\right)^2=1$. As a consequence of the equal orientation of $(\boldsymbol{x},\boldsymbol{y})$ and $(\boldsymbol{x},\boldsymbol{R}\cdot\boldsymbol{x})$, we have $\frac{\boldsymbol{y}\cdot\boldsymbol{R}\cdot\boldsymbol{x}}{|\boldsymbol{x}||\boldsymbol{y}|}>0$ which implies that this expression equals $\sin\alpha_{\boldsymbol{R}}$ (because $\alpha_{\boldsymbol{R}}$ is between 0 and π).

11.1.6. Let $\boldsymbol{R}\neq\mathrm{id}_{\mathbf{E}}$ and $\alpha_{\boldsymbol{R}}\neq\pi$. Then $\boldsymbol{x}$ and $\boldsymbol{R}\cdot\boldsymbol{x}$ are linearly independent if $\boldsymbol{x}$ is a non-zero vector orthogonal to $a_{\boldsymbol{R}}$. It is not hard to see that if $\boldsymbol{y}$ is another non-zero vector orthogonal to $a_{\boldsymbol{R}}$, then the pairs $(\boldsymbol{x},\boldsymbol{R}\cdot\boldsymbol{x})$ and $(\boldsymbol{y},\boldsymbol{R}\cdot\boldsymbol{y})$ are equally oriented bases in $a_{\boldsymbol{R}}^{\perp}$. As a consequence, $(\boldsymbol{R}\cdot\boldsymbol{x})\wedge\boldsymbol{x}$ and $(\boldsymbol{R}\cdot\boldsymbol{y})\wedge\boldsymbol{y}$ are positive multiples of each other.

Since $|(\boldsymbol{R}\cdot\boldsymbol{x})\wedge\boldsymbol{x}|^2=|\boldsymbol{x}|^4-(\boldsymbol{x}\cdot\boldsymbol{R}\cdot\boldsymbol{x})^2=|\boldsymbol{x}|^4\sin^2\alpha_{\boldsymbol{R}}$, we have that for $\boldsymbol{R}\neq\mathrm{id}_{\mathbf{E}}$, $\alpha_{\boldsymbol{R}}\neq\pi$

$$\log\boldsymbol{R}:=\frac{(\boldsymbol{R}\cdot\boldsymbol{x})\wedge\boldsymbol{x}}{|\boldsymbol{x}|^2\sin\alpha_{\boldsymbol{R}}}\alpha_{\boldsymbol{R}}\in\mathrm{A}(\mathbf{b})\qquad(\mathbf{0}\neq\boldsymbol{x}\in a_{\boldsymbol{R}}^{\perp})$$

is independent of $\boldsymbol{x}$. Moreover, put

$$\log(\mathrm{id}_{\mathbf{E}}):=\mathbf{0}\in\mathrm{A}(\mathbf{b}).$$

It is easy to see that

(i) Ker $(\log\boldsymbol{R})=a_{\boldsymbol{R}}$,

(ii) $|\log\boldsymbol{R}|=\alpha_{\boldsymbol{R}}$,

(iii) if $\boldsymbol{R} \neq \mathrm{id}_{\mathbf{E}}$ then for an arbitrary non-zero $\boldsymbol{x} \in a_{\boldsymbol{R}}^{\perp}$, $(\boldsymbol{x},\ \boldsymbol{R}\cdot\boldsymbol{x})$ and $(\boldsymbol{x},\ (\log\boldsymbol{R})\cdot\boldsymbol{x})$ form equally oriented bases in $a_{\boldsymbol{R}}^{\perp}$.

In this way, assuming the notations

$$\mathrm{N} := \{\boldsymbol{R} \in \mathcal{SO}(\mathbf{b}) \mid \alpha_{\boldsymbol{R}} \neq \pi\}, \qquad \mathrm{P} := \{\boldsymbol{A} \in \mathrm{A}(\mathbf{b}) \mid |\boldsymbol{A}| < \pi\}$$

we defined a mapping $\log : \mathrm{N} \to \mathrm{P}$; we shall show that log is a bijection whose inverse is the restriction of the exponential mapping (see VII.3.7).

11.1.7. Proposition. For $\mathbf{0} \neq \boldsymbol{A} \in \mathrm{A}(\mathbf{b})$ putting $\alpha := |\boldsymbol{A}|$, $\boldsymbol{A}_{\mathrm{o}} := \frac{\boldsymbol{A}}{|\boldsymbol{A}|}$, we have

$$\mathrm{e}^{\boldsymbol{A}} = -{\boldsymbol{A}_{\mathrm{o}}}^2\cos\alpha + \boldsymbol{A}_{\mathrm{o}}\sin\alpha + \left(\mathrm{id}_{\mathbf{E}} + {\boldsymbol{A}_{\mathrm{o}}}^2\right).$$

Proof. Recall that $\boldsymbol{A}^3 = -\alpha^2\boldsymbol{A}$ (see V.3.10); thus

$$\begin{aligned}
\mathrm{e}^{\boldsymbol{A}} = \sum_{n=0}^{\infty}\frac{\boldsymbol{A}^n}{n!} &= \mathrm{id}_{\mathbf{E}} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \frac{\boldsymbol{A}^4}{4!} + \frac{\boldsymbol{A}^5}{5!} + \frac{\boldsymbol{A}^6}{6!} + \frac{\boldsymbol{A}^7}{7!} + \ldots. = \\
&= \left(\mathrm{id}_{\mathbf{E}} + \frac{\boldsymbol{A}^2}{\alpha^2}\right) - \frac{\boldsymbol{A}^2}{\alpha^2} + \frac{\boldsymbol{A}^2}{2!} - \frac{\alpha^2\boldsymbol{A}^2}{4!} + \frac{\alpha^4\boldsymbol{A}^2}{6!} - \ldots. \\
&\quad + \boldsymbol{A} - \frac{\alpha^2\boldsymbol{A}}{3!} + \frac{\alpha^4\boldsymbol{A}}{5!} - \frac{\alpha^6\boldsymbol{A}}{7!} + \ldots.
\end{aligned}$$

which yields the desired result by $\boldsymbol{A} = \alpha\boldsymbol{A}_{\mathrm{o}}$. ■

Note that for $\boldsymbol{A} \neq \mathbf{0}$, $\mathrm{id}_{\mathbf{E}} + \boldsymbol{A}_{\mathrm{o}}^2$ is the orthogonal projection onto the plane orthogonal to the kernel of $\boldsymbol{A}$.

As a consequence, if $\boldsymbol{A} \neq \mathbf{0}$ then

(i) $\mathrm{e}^{\boldsymbol{A}}\cdot\boldsymbol{x} = \boldsymbol{x}$ for $\boldsymbol{x} \in \mathrm{Ker}\ \boldsymbol{A}$ (the axis of rotation of $\mathrm{e}^{\boldsymbol{A}}$ is the kernel of $\boldsymbol{A}$);

(ii) $\mathrm{e}^{\boldsymbol{A}}\cdot\boldsymbol{x} = (\cos\alpha)\boldsymbol{x} + (\sin\alpha)\boldsymbol{A}_{\mathrm{o}}\cdot\boldsymbol{x}$ for $\boldsymbol{x}$ orthogonal to Ker $\boldsymbol{A}$ (the angle of rotation of $\mathrm{e}^{\boldsymbol{A}}$ is $\alpha := |\boldsymbol{A}|$);

11.1.8. Proposition. For $\boldsymbol{R} \in \mathrm{N}$

$$\mathrm{e}^{\log\boldsymbol{R}} = \boldsymbol{R}$$

and for $\boldsymbol{A} \in \mathrm{P}$

$$\log(\mathrm{e}^{\boldsymbol{A}}) = \boldsymbol{A}.$$

Proof. Evidently, for $\boldsymbol{R} = \mathrm{id}_{\mathbf{E}}$ and for $\boldsymbol{A} = \mathbf{0}$ the equalities hold.

If $\boldsymbol{R} \neq \mathrm{id}_{\mathbf{E}}$ and $\boldsymbol{x}$ is in $a_{\boldsymbol{R}}$ then, obviously, $\mathrm{e}^{\ln\boldsymbol{R}}\cdot\boldsymbol{x} = \boldsymbol{x} = \boldsymbol{R}\cdot\boldsymbol{x}$. If $\boldsymbol{x}$ is orthogonal to the axis of rotation of $\boldsymbol{R}$, then

$$\mathrm{e}^{\log\boldsymbol{R}}\cdot\boldsymbol{x} = (\cos\alpha_{\boldsymbol{R}})\boldsymbol{x} + \sin\alpha_{\boldsymbol{R}}\frac{\log\boldsymbol{R}}{\alpha_{\boldsymbol{R}}}\cdot\boldsymbol{x} = \boldsymbol{R}\cdot\boldsymbol{x},$$

in view of 11.1.6 and 11.1.7.

According to the previous proposition, for $\boldsymbol{A} \neq \mathbf{0}$, the axis of rotation of $\mathrm{e}^{\boldsymbol{A}}$ is the kernel of $\boldsymbol{A}$; the angle of rotation of $\mathrm{e}^{\boldsymbol{A}}$ is $|\boldsymbol{A}|$. Thus if $\boldsymbol{x} \in \mathrm{Ker}\ \boldsymbol{A}$ then $\log(\mathrm{e}^{\boldsymbol{A}}) \cdot \boldsymbol{x} = \mathbf{0} = \boldsymbol{A} \cdot \boldsymbol{x}$. If $\boldsymbol{x}$ is orthogonal to the kernel of $\boldsymbol{A}$, then $\log(\mathrm{e}^{\boldsymbol{A}}) = \frac{(\mathrm{e}^{\boldsymbol{A}} \cdot \boldsymbol{x}) \wedge \boldsymbol{x}}{|\boldsymbol{x}|^2 \sin|\boldsymbol{A}|}$ and an easy calculation based on the formula in 11.1.6 yields that $\log(\mathrm{e}^{\boldsymbol{A}}) \cdot \boldsymbol{x} = \boldsymbol{A} \cdot \boldsymbol{x}$.

11.1.9. It is trivial that the closure of N is $\mathcal{SO}(\mathbf{b})$. It is not hard to see that exponential mapping $\mathrm{A}(\mathbf{b}) \to \mathcal{SO}(\mathbf{b})$ maps the closure of P onto $\mathcal{SO}(\mathbf{b})$. However, the exponential mapping on the closure of P is not injective: if $|\boldsymbol{A}| = \pi$ then $\mathrm{e}^{\boldsymbol{A}} = \mathrm{e}^{-\boldsymbol{A}}$.

Since the closure of P is connected and the exponential mapping is continuous, $\mathcal{SO}(\mathbf{b})$ is connected as well.

(However, $\mathcal{SO}(\mathbf{b})$ is not simply connected: it is homeomorphic to a set which is obtained from the closure of P by "sticking" together the diametrical points of the boundary of P.)

The one-parameter subgroup of $\mathcal{SO}(\mathbf{b})$ corresponding to $\boldsymbol{A} \in \mathrm{A}(\mathbf{b})$ is $\mathbb{R} \to \mathcal{SO}(\mathbf{b})$, $t \mapsto \mathrm{e}^{t\boldsymbol{A}}$. If $\boldsymbol{A} \neq \mathbf{0}$, then all the elements of the one-parameter subgroup are rotations around the same axis Ker $\boldsymbol{A}$.

Since the exponential mapping is surjective, every element of $\mathcal{SO}(\mathbf{b})$ is in a one-parameter subgroup.

11.1.10. In physical applications we meet $\frac{\mathrm{A}(\mathbf{b})}{\mathbf{I}}$ instead of $\mathrm{A}(\mathbf{b})$. If $\Omega \in \frac{\mathrm{A}(\mathbf{b})}{\mathbf{I}}$, then we can give a function $R : \mathrm{I} \to \mathcal{SO}(\mathbf{b})$, $t \mapsto e^{(t-t_o)\Omega}$, where t_o is a fixed element of I. Then every value of such a function is a rotation around the same axis; the angle of rotation of $R(t)$ is $(t - t_o)|\Omega|$. Thus $|\Omega|$ is interpreted as the magnitude of the angular velocity and Ω itself as the *angular velocity* of the rotation.

We know that R is differentiable, $\dot{R} = \Omega \cdot R$, from which we infer that

$$\Omega = \dot{R} \cdot R^{-1}.$$

In general, consider a differentiable function $R : \mathrm{I} \rightarrowtail \mathcal{SO}(\mathbf{b}) \subset \mathbf{E} \otimes \mathbf{E}^*$. Its derivative at t, $\dot{R}(t)$, is a linear map from I into $\mathbf{E} \otimes \mathbf{E}^*$ that takes values in the tangent space of $\mathcal{SO}(\mathbf{b})$ at $R(t)$ which is $R(t) \cdot \mathrm{A}(\mathbf{b}) = \{R(t) \cdot \boldsymbol{A} \mid \boldsymbol{A} \in \mathrm{A}(\mathbf{b})\}$ (see VII.3.3). In other words, $\dot{R}(t) \in \frac{R(t) \cdot \mathrm{A}(\mathbf{b})}{\mathbf{I}}$, i.e. $R(t)^{-1} \cdot \dot{R}(t) \in \mathrm{A}(\mathbf{b})$. Then V.2.11(*ii*) implies that $R(t) \cdot \left(R(t)^{-1} \cdot \dot{R}(t)\right) \cdot R(t)^{-1}$ is in $\mathrm{A}(\mathbf{b})$ as well;

$$\Omega(t) := \dot{R}(t) \cdot R(t)^{-1} \in \mathrm{A}(\mathbf{b})$$

is called the *angular velocity value* at t, and the function $\Omega : \mathrm{I} \rightarrowtail \mathrm{A}(\mathbf{b})$ is the *angular velocity.*

Evidently, R is the solution of the differential equation

$$(X : \mathrm{I} \rightarrowtail \mathcal{SO}(\mathbf{b}))? \qquad \dot{X} = \Omega \cdot X.$$

11.2. Exercises

1. Let us coordinatize $\mathcal{SO}(\mathbf{b})$ by the *Euler angles* as follows.

Let $(\boldsymbol{n}_1,\ \boldsymbol{n}_2,\ \boldsymbol{n}_3)$ be a positively oriented orthonormal basis in $\frac{\mathbf{E}}{\mathbf{D}}$. If $\boldsymbol{R} \cdot \boldsymbol{n}_3$ is not parallel to $\boldsymbol{n}_3$, put $\boldsymbol{n} := \frac{\boldsymbol{n}_3 \times (\boldsymbol{R} \cdot \boldsymbol{n}_3)}{|\boldsymbol{n}_3 \times (\boldsymbol{R} \cdot \boldsymbol{n}_3)|}$ and

$$\begin{aligned}
\vartheta_{\boldsymbol{R}} &:= \arccos(\boldsymbol{n}_3 \cdot \boldsymbol{R} \cdot \boldsymbol{n}_3),\\
\boldsymbol{\psi}_{\boldsymbol{R}} &:= \mathrm{sign}(\boldsymbol{n} \cdot \boldsymbol{n}_2) \arccos(\boldsymbol{n} \cdot \boldsymbol{n}_1),\\
\varphi_{\boldsymbol{R}} &:= \mathrm{sign}(\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{n}_2) \arccos(\boldsymbol{n} \cdot \boldsymbol{R} \cdot \boldsymbol{n}_1)
\end{aligned}$$

where $\mathrm{sign}x := \frac{x}{|x|}$ if $0 \neq x \in \mathbb{R}$ and $\mathrm{sign}0 := 1$.

Prove that if R_i denotes the one-parameter subgroup of rotations around $\boldsymbol{n}_i$ $(i = 1, 2, 3)$ then

$$\boldsymbol{R} = R_3(\varphi_{\boldsymbol{R}}) \cdot R_1(\vartheta_{\boldsymbol{R}}) \cdot R_3(\boldsymbol{\psi}_{\boldsymbol{R}}).$$

2. Let $R : \mathrm{I} \rightarrowtail \mathcal{SO}(\mathbf{b})$ be a differentiable function and put $R^{-1} : \mathrm{I} \rightarrowtail \mathcal{SO}(\mathbf{b})$, $t \mapsto R(t)^{-1}$. Using $R \cdot R^{-1} = \mathrm{id}_{\mathbf{E}}$ prove that R^{-1} is also differentiable and

$$\left(R^{-1}\right)^{\cdot} = -R^{-1} \cdot \dot{R} \cdot R^{-1}.$$

3. Prove that for $\mathbf{0} \leq \boldsymbol{r} \in \mathbf{D}$, $\{\boldsymbol{x} \in \mathbf{E} \mid\ |\boldsymbol{x}| = \boldsymbol{r}\}$ is an orbit of $\mathcal{SO}(\mathbf{b})$ and all its orbits are of this kind.

11.3. The Galilean group

11.3.1. We shall deal with linear maps from $\mathbf{M}$ into $\mathbf{M}$, permanently using the identification $\mathrm{Lin}(\mathbf{M}) \equiv \mathbf{M} \otimes \mathbf{M}^*$. The restriction of a linear map $\boldsymbol{L} : \mathbf{M} \to \mathbf{M}$ ($\boldsymbol{L} \in \mathbf{M} \otimes \mathbf{M}^*$) onto $\mathbf{E}$ equals $\boldsymbol{L} \cdot \mathbf{i}$ where $\mathbf{i} : \mathbf{E} \to \mathbf{M}$ ($\mathbf{i} \in \mathbf{M} \otimes \mathbf{E}^*$) is the canonical embedding. The symbol $\boldsymbol{L} \cdot \mathbf{i} \in \mathbf{i} \cdot \mathcal{O}(\mathbf{b})$ means that the restriction of $\boldsymbol{L}$ onto $\mathbf{E}$ is in $\mathcal{O}(\mathbf{b})$, i.e. there is an $\boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \subset \mathbf{E} \otimes \mathbf{E}^*$ such that $\boldsymbol{L} \cdot \mathbf{i} = \mathbf{i} \cdot \boldsymbol{R}$.

First we define the Galilean group and then studying it we find its physical meaning.

Definition.

$$\mathcal{G} := \{\boldsymbol{L} \in \mathbf{M} \otimes \mathbf{M}^* \mid\ \boldsymbol{\tau} \cdot \boldsymbol{L} = \pm \boldsymbol{\tau},\ \boldsymbol{L} \cdot \mathbf{i} \in \mathbf{i} \cdot \mathcal{O}(\mathbf{b})\}$$

is called the *Galilean group*; its elements are the *Galilean transformations.*

If $\boldsymbol{L}$ is a Galilean transformation then

$$\mathrm{ar}\boldsymbol{L} := \begin{cases} +1 & \text{if} \boldsymbol{\tau} \cdot \boldsymbol{L} = \boldsymbol{\tau} \\ -1 & \text{if} \boldsymbol{\tau} \cdot \boldsymbol{L} = -\boldsymbol{\tau} \end{cases}$$

is the *arrow* of $\boldsymbol{L}$ and

$$\mathrm{sign}\boldsymbol{L} := \begin{cases} +1 & \text{if} \boldsymbol{L} \cdot \mathbf{i} \in \mathbf{i} \cdot \mathcal{O}(\mathbf{b})^{+} \\ -1 & \text{if} \boldsymbol{L} \cdot \mathbf{i} \in \mathbf{i} \cdot \mathcal{O}(\mathbf{b})^{-} \end{cases}$$

is the *sign* of $\boldsymbol{L}$.

Let us put

$$\begin{aligned} \mathcal{G}^{+\rightarrow} :=& \{\boldsymbol{L} \in \mathcal{G} \mid \ \mathrm{sign}\boldsymbol{L} = \mathrm{ar}\boldsymbol{L} = 1\}, \\ \mathcal{G}^{+\leftarrow} :=& \{\boldsymbol{L} \in \mathcal{G} \mid \ \mathrm{sign}\boldsymbol{L} = -\mathrm{ar}\boldsymbol{L} = 1\}, \\ \mathcal{G}^{-\rightarrow} :=& \{\boldsymbol{L} \in \mathcal{G} \mid \ \mathrm{sign}\boldsymbol{L} = -\mathrm{ar}\boldsymbol{L} = -1\}, \\ \mathcal{G}^{-\leftarrow} :=& \{\boldsymbol{L} \in \mathcal{G} \mid \ \mathrm{sign}\boldsymbol{L} = \mathrm{ar}\boldsymbol{L} = -1\}. \end{aligned}$$

$\mathcal{G}^{+\rightarrow}$ is called the *proper Galilean group.* ■

(i) The condition $\boldsymbol{\tau} \cdot \boldsymbol{L} = \pm\boldsymbol{\tau}$ implies that $\mathbf{E}$ is invariant for the linear map $\boldsymbol{L} : \mathbf{M} \to \mathbf{M}$.

(ii) The condition $\boldsymbol{L} \cdot \mathbf{i} \in \mathbf{i} \cdot \mathcal{O}(\mathbf{b})$ means that there is a (necessarily unique) $\boldsymbol{R_L}$ in $\mathcal{O}(\mathbf{b})$ such that

$$\boldsymbol{L} \cdot \mathbf{i} = \mathbf{i} \cdot \boldsymbol{R_L}.$$

(iii) The Galilean transformations are linear bijections: if $\boldsymbol{L} \cdot \boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{\tau} \cdot \boldsymbol{x} = \mathbf{0}$, i.e. $\boldsymbol{x}$ is in $\mathbf{E}$; the restriction of $\boldsymbol{L}$ onto $\mathbf{E}$ is injective, thus $\boldsymbol{x} = \mathbf{0}$.

(iv) It is quite trivial that $\mathcal{G}$ is indeed a group: the product of its elements as well as the inverse of its elements are Galilean transformations.

11.3.2. Proposition. The Galilean group is a six-dimensional Lie group having the Lie algebra

$$\mathbf{La}(\mathcal{G}) = \{\boldsymbol{H} \in \mathbf{M} \otimes \mathbf{M}^* \mid \ \boldsymbol{\tau} \cdot \boldsymbol{H} = \mathbf{0}, \ \boldsymbol{H} \cdot \mathbf{i} \in \mathrm{A}(\mathbf{b})\}.$$

Proof. According to the previous remark, $\mathcal{G}$ is a subgroup of $\mathcal{G}\ell(\mathbf{M})$ which is sixteen-dimensional.

We have to show that the Galilean group is a six-dimensional smooth submanifold of $\mathcal{G}\ell(\mathbf{M})$.

Observe that if $\boldsymbol{L} \in \mathcal{G}$, then

$$\boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i} = \boldsymbol{R_L}$$

for all $\boldsymbol{u} \in \mathrm{V}(1)$, where $\boldsymbol{R_L}$ is given in the previous remark.

Thus we can give

$$\phi_{\boldsymbol{u}} : \mathcal{G}\ell(\mathbf{M}) \to (\mathbf{I} \otimes \mathbf{M}^*) \times \mathrm{S}(\mathbf{b}), \qquad \boldsymbol{L} \mapsto \big(\boldsymbol{\tau} \cdot \boldsymbol{L},\ (\boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i})^{\circledast} \cdot (\boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i})\big)$$

which is evidently a smooth map and $\mathcal{G}$ is the preimage of $\{(\pm\boldsymbol{\tau}, \mathrm{id}_{\mathbf{E}})\}$ by $\phi_{\boldsymbol{u}}$ ($\mathrm{S}(\mathbf{b}) := \{\boldsymbol{S} \in \mathbf{E} \in \otimes\mathbf{E}^* \mid \boldsymbol{S}^{\circledast} = \boldsymbol{S}\}$ is a six-dimensional linear subspace).

The derivative of $\phi_{\boldsymbol{u}}$ at $\boldsymbol{L}$ is the linear map

$$\begin{aligned} \mathrm{D}\phi_{\boldsymbol{u}}(\boldsymbol{L}) :\ & \mathbf{M} \otimes \mathbf{M}^* \to (\mathbf{I} \otimes \mathbf{M}^*) \times \mathrm{S}(\mathbf{b}), \\ & \boldsymbol{H} \mapsto \big(\boldsymbol{\tau} \cdot \boldsymbol{H},\ (\boldsymbol{\pi_u} \cdot \boldsymbol{H} \cdot \mathbf{i})^{\circledast} \cdot (\boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i}) + (\boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i})^{\circledast} \cdot (\boldsymbol{\pi_u} \cdot \boldsymbol{H} \cdot \mathbf{i})\big) \end{aligned}$$

which is surjective: $(\boldsymbol{h}, \boldsymbol{T}) \in (\mathbf{I} \otimes \mathbf{M}^*) \times \mathrm{S}(\mathbf{b})$ is the image by $\mathrm{D}\phi_{\boldsymbol{u}}(\boldsymbol{L})$ of $\boldsymbol{u} \otimes \boldsymbol{h} + (1/2)(\boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i})^{\circledast^{-1}} \cdot \boldsymbol{T}$.

Thus, being a six-dimensional submanifold in $\mathcal{G}\ell(\mathbf{M})$, the Galilean group is a Lie group; its Lie algebra is $\mathrm{Ker}\ \mathrm{D}\phi_{\boldsymbol{u}}(\mathrm{id}_{\mathbf{M}})$.

If $\mathrm{D}\phi_{\boldsymbol{u}}(\mathrm{id}_{\mathbf{M}})(\boldsymbol{H}) = \mathbf{0}$, then $\boldsymbol{\tau} \cdot \boldsymbol{H} = \mathbf{0}$, and $\boldsymbol{\pi_u} \cdot \boldsymbol{H} \cdot \mathbf{i}$ is in $\mathrm{A}(\mathbf{b})$. Since the first condition means that $\boldsymbol{H} \in \mathbf{E} \otimes \mathbf{M}^*$, we have $\boldsymbol{\pi_u} \cdot \boldsymbol{H} \cdot \mathbf{i} = \boldsymbol{H} \cdot \mathbf{i}$. Hence the kernel of $\mathrm{D}\phi_{\boldsymbol{u}}(\mathrm{id}_{\mathbf{M}})$ is the linear subspace given in our proposition.

11.3.3. The mappings $\mathcal{G} \to \{-1, 1\}$, $\boldsymbol{L} \mapsto \mathrm{ar}\boldsymbol{L}$ and $\mathcal{G} \to \{-1, 1\}$, $\boldsymbol{L} \mapsto \mathrm{sign}\boldsymbol{L}$ are continuous group homomorphisms. As a consequence, the Galilean group is disconnected. We shall see in 11.4.3 that the proper Galilean group $\mathcal{G}^{+\to}$ is connected. It is quite trivial that if $\boldsymbol{L} \in \mathcal{G}^{+\leftarrow}$ then $\boldsymbol{L} \cdot \mathcal{G}^{+\to} = \mathcal{G}^{+\leftarrow}$ and similar assertions hold for $\mathcal{G}^{-\to}$ and $\mathcal{G}^{-\leftarrow}$ as well. Consequently, the Galilean group has four connected components, the four subsets given in Definition 11.2.1.

From these four components only $\mathcal{G}^{+\to}$— the proper Galilean group — is a subgroup; nevertheless, the union of an arbitrary component and of the proper Galilean group is a subgroup as well.

$\mathcal{G}^{\to} := \mathcal{G}^{+\to} \cup \mathcal{G}^{-\to}$ is called the *orthochronous Galilean group.*

If $\boldsymbol{L} \in \mathcal{G}$, then $\boldsymbol{L}$ preserves or reverses the "orientation" of timelike vectors according to whether $\mathrm{ar}\boldsymbol{L} = 1$ or $\mathrm{ar}\boldsymbol{L} = -1$:

$$\begin{aligned} \text{if} \quad \mathrm{ar}\boldsymbol{L} = 1 \qquad & \text{then} \quad \boldsymbol{L}(\mathrm{T}^{\to}) = \mathrm{T}^{\to}, \quad \boldsymbol{L}(\mathrm{T}^{\leftarrow}) = \mathrm{T}^{\leftarrow}, \\ \text{if} \quad \mathrm{ar}\boldsymbol{L} = -1 \qquad & \text{then} \quad \boldsymbol{L}(\mathrm{T}^{\to}) = \mathrm{T}^{\leftarrow}, \quad \boldsymbol{L}(\mathrm{T}^{\leftarrow}) = \mathrm{T}^{\to}. \end{aligned}$$

Moreover, $\boldsymbol{L}$ preserves or reverses the orientation of $\mathbf{E}$ according to whether $\mathrm{sign}\boldsymbol{L} = 1$ or $\mathrm{sign}\boldsymbol{L} = -1$.

The orientation of $\mathbf{E}$ given in 1.2.4 shows that the elements of $\mathcal{G}^{+\to}$ and $\mathcal{G}^{-\leftarrow}$ preserve the orientation of $\mathbf{M}$, whereas the elements of $\mathcal{G}^{+\leftarrow}$ and $\mathcal{G}^{-\to}$ reverse the orientation.

11.3.4. $\mathbf{M}$ is of even dimension, thus $-\mathrm{id}_{\mathbf{M}}$ is orientation-preserving. Evidently, $-\mathrm{id}_{\mathbf{M}}$ is in $\mathcal{G}^{-\leftarrow}$; it is called the *inversion of spacetime vectors.* We have that $\mathcal{G}^{-\leftarrow} = (-\mathrm{id}_{\mathbf{M}}) \cdot \mathcal{G}^{+\to}$.

We have seen previously that the elements of $\mathcal{G}^{+\leftarrow}$ invert in some sense the timelike vectors and do not invert the spacelike vectors; the elements of $\mathcal{G}^{-\rightarrow}$ invert in some sense the spacelike vectors and do not invert the timelike vectors. However, we cannot select an element of $\mathcal{G}^{+\leftarrow}$ and an element of $\mathcal{G}^{-\rightarrow}$ that we could consider the time inversion and the space inversion.

For each $\boldsymbol{u} \in \mathrm{V}(1)$ we can give a $\boldsymbol{u}$-timelike inversion and a $\boldsymbol{u}$-spacelike inversion as follows.

The $\boldsymbol{u}$-timelike inversion $\boldsymbol{T_u} \in \mathcal{G}^{+\leftarrow}$ inverts the vectors parallel to $\boldsymbol{u}$ and leaves invariant the spacelike vectors:

$$\boldsymbol{T_u} \cdot \boldsymbol{u} := -\boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{T_u} \cdot \boldsymbol{q} := \boldsymbol{q} \quad \text{for} \quad \boldsymbol{q} \in \mathbf{E}.$$

In general,

$$\boldsymbol{T_u} \cdot \boldsymbol{x} = -\boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{x}) + \boldsymbol{\pi_u} \cdot \boldsymbol{x} = -2\boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{x}) + \boldsymbol{x} \qquad (\boldsymbol{x} \in \mathbf{M})$$

i.e.

$$\boldsymbol{T_u} = \mathrm{id}_{\mathbf{M}} - 2\boldsymbol{u} \otimes \boldsymbol{\tau}.$$

The $\boldsymbol{u}$-spacelike inversion $\boldsymbol{P_u} \in \mathcal{G}^{-\rightarrow}$ inverts the spacelike vectors and leaves invariant the vectors parallel to $\boldsymbol{u}$:

$$\boldsymbol{P_u} \cdot \boldsymbol{u} := \boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{P_u} \cdot \boldsymbol{q} := -\boldsymbol{q} \qquad \text{for} \qquad \boldsymbol{q} \in \mathbf{E}.$$

In general,

$$\boldsymbol{P_u} \cdot \boldsymbol{x} = \boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{x}) - \boldsymbol{\pi_u} \cdot \boldsymbol{x} = 2\boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{x}) - \boldsymbol{x} \qquad (\boldsymbol{x} \in \mathbf{M}),$$

i.e.

$$\boldsymbol{P_u} = 2\boldsymbol{u} \otimes \boldsymbol{\tau} - \mathrm{id}_{\mathbf{M}}.$$

We easily deduce the following equalities:

$$\begin{gathered} \boldsymbol{T_u}^{-1} = \boldsymbol{T_u}, \qquad \boldsymbol{P_u}^{-1} = \boldsymbol{P_u}, \\ -\boldsymbol{T_u} = \boldsymbol{P_u}, \\ \boldsymbol{T_u} \cdot \boldsymbol{P_u} = \boldsymbol{P_u} \cdot \boldsymbol{T_u} = -\mathrm{id}_{\mathbf{M}}. \end{gathered}$$

11.3.5. The three-dimensional orthogonal group *is not a subgroup* of the Galilean group: $\mathcal{O}(\mathbf{b})$ cannot be a subgroup of $\mathcal{G}$ because the elements of $\mathcal{G}$ are linear maps defined on $\mathbf{M}$ whereas the elements of $\mathcal{O}(\mathbf{b})$ are linear maps defined on $\mathbf{E}$ ($\mathbf{E} \otimes \mathbf{E}^*$ is not a subset of $\mathbf{M} \otimes \mathbf{M}^*$).

It is quite obvious that

$$\mathcal{G}^{\rightarrow} \to \mathcal{O}(\mathbf{b}), \qquad \boldsymbol{L} \mapsto \boldsymbol{R_L}$$

(where $\boldsymbol{L} \cdot \mathbf{i} = \mathbf{i} \cdot \boldsymbol{R_L}$) is a surjective Lie group homomorphism.

For every $\boldsymbol{u} \in \mathrm{V}(1)$,

$$\mathcal{O}(\mathbf{b})_{\boldsymbol{u}} := \{\boldsymbol{L} \in \mathcal{G}^{\rightarrow} \mid \ \boldsymbol{L} \cdot \boldsymbol{u} = \boldsymbol{u}\},$$

called the group of *$\boldsymbol{u}$-spacelike orthogonal transformations,* is a subgroup of $\mathcal{G}^{\rightarrow}$; the restriction of the above Lie group homomorphism to $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ is a bijection between $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ and $\mathcal{O}(\mathbf{b})$.

Indeed, if $\boldsymbol{L}\cdot\boldsymbol{u} = \boldsymbol{u}$ and $\boldsymbol{R_L} = \mathrm{id}_{\mathbf{E}}$ then $\boldsymbol{L}$ is the identity on the complementary subspaces $\boldsymbol{u} \otimes \mathbf{I}$ and $\mathbf{E}$, thus $\boldsymbol{L} = \mathrm{id}_{\mathbf{M}}$: the group homomorphism from $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ into $\mathcal{O}(\mathbf{b})$ is injective.

If $\boldsymbol{R} \in \mathcal{O}(\mathbf{b})$ then

$$\boldsymbol{R_u} := \boldsymbol{u} \otimes \boldsymbol{\tau} + \boldsymbol{R} \cdot \boldsymbol{\pi_u}$$

is a Galilean transformation in $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ and $\boldsymbol{\pi_u} \cdot \boldsymbol{R_u} \cdot \mathbf{i} = \boldsymbol{R}$ (recall that $\boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i} = \boldsymbol{R_L}$ for all Galilean transformations $\boldsymbol{L}$) : the group homomorphism from $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ onto $\mathcal{O}(\mathbf{b})$ is surjective.

11.3.6. The kernel of the surjection $\mathcal{G}^{\rightarrow} \to \mathcal{O}(\mathbf{b})$, i.e.

$$\mathcal{V} := \{L \in \mathcal{G}^{\rightarrow} \mid \ \boldsymbol{R_L} = \mathrm{id}_{\mathbf{E}}\} = \{\boldsymbol{L} \in \mathcal{G}^{\rightarrow} \mid \ \boldsymbol{L} \cdot \mathbf{i} = \mathbf{i}\}$$

is called the *special Galilean group.* Observe that $\mathcal{V}$ is in $\mathcal{G}^{+\rightarrow}$.

The special Galilean group is a three-dimensional Lie group having the Lie algebra

$$\mathbf{La}(\mathcal{V}) = \{\boldsymbol{H} \in \mathbf{M} \otimes \mathbf{M}^* \mid \ \boldsymbol{\tau} \cdot \boldsymbol{H} = \mathbf{0}, \qquad \boldsymbol{H} \cdot \mathbf{i} = \mathbf{0}\}.$$

Proposition. If $\boldsymbol{L} \in \mathcal{V}$, then there is a unique $\boldsymbol{v_L} \in \frac{\mathbf{E}}{\mathbf{I}}$ such that

$$\boldsymbol{L} \cdot \boldsymbol{x} = \boldsymbol{v_L}(\boldsymbol{\tau} \cdot \boldsymbol{x}) + \boldsymbol{x} \qquad\qquad (\boldsymbol{x} \in \mathbf{M}),$$

i.e.

$$\boldsymbol{L} = \mathrm{id}_{\mathbf{M}} + \boldsymbol{v_L} \otimes \boldsymbol{\tau}.$$

The correspondence $\mathcal{V} \to \frac{\mathbf{E}}{\mathbf{I}}$, $\boldsymbol{L} \mapsto \boldsymbol{v_L}$ is a bijective group homomorphism regarding the additive structure of $\frac{\mathbf{E}}{\mathbf{I}}$ (i.e. $\boldsymbol{v_{L\cdot K}} = \boldsymbol{v_L} + \boldsymbol{v_K}$ for all $\boldsymbol{L}, \boldsymbol{K} \in \mathcal{V}$).

Proof. Let $\boldsymbol{L}$ be an element of $\mathcal{V}$. Let us take an arbitrary $\boldsymbol{u} \in \mathrm{V}(1)$ and put $\boldsymbol{v_L} := L \cdot \boldsymbol{u} - \boldsymbol{u}$. We claim that $\boldsymbol{v_L}$ does not depend on $\boldsymbol{u}$. Indeed, if $\boldsymbol{u}' \in \mathrm{V}(1)$ then

$$(\boldsymbol{L} \cdot \boldsymbol{u} - \boldsymbol{u}) - (\boldsymbol{L} \cdot \boldsymbol{u}' - \boldsymbol{u}') = \boldsymbol{L} \cdot (\boldsymbol{u} - \boldsymbol{u}') - (\boldsymbol{u} - \boldsymbol{u}') = \mathbf{0},$$

because $\boldsymbol{L} \cdot (\boldsymbol{u} - \boldsymbol{u}') = \boldsymbol{u} - \boldsymbol{u}'$. Moreover, $\boldsymbol{\tau} \cdot (\boldsymbol{L} \cdot \boldsymbol{u} - \boldsymbol{u}) = \mathbf{0}$, thus $\boldsymbol{v_L}$ is in $\frac{\mathbf{E}}{\mathbf{I}}$.

This means that $\boldsymbol{L} \cdot \boldsymbol{u} = \boldsymbol{u} + \boldsymbol{v_L}$ for all $\boldsymbol{u} \in \mathrm{V}(1)$.

Then we find that for $\boldsymbol{x} \in \mathbf{M}$

$$\boldsymbol{L} \cdot \boldsymbol{x} = \boldsymbol{L} \cdot (\boldsymbol{u}(\boldsymbol{\tau} \cdot \boldsymbol{x}) + \boldsymbol{\pi_u} \cdot \boldsymbol{x}) = (\boldsymbol{u} + \boldsymbol{v_L})\boldsymbol{\tau} \cdot \boldsymbol{x} + \boldsymbol{\pi_u} \cdot \boldsymbol{x} = \boldsymbol{v_L}(\boldsymbol{\tau} \cdot \boldsymbol{x}) + \boldsymbol{x}.$$

This formula assures, too, that $\boldsymbol{L} \mapsto \boldsymbol{v_L}$ is a group homomorphism.

If $\boldsymbol{v_L} = \mathbf{0}$ then $\boldsymbol{L} \cdot \boldsymbol{u} = \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathrm{V}(1)$ implying $\boldsymbol{L} = \mathrm{id}_{\mathrm{M}}$; thus the correspondence from $\mathcal{V}$ into $\frac{\mathbf{E}}{\mathbf{I}}$ is injective. Evidently, if $\boldsymbol{v}$ is in $\frac{\mathbf{E}}{\mathbf{I}}$ then $\mathrm{id}_{\mathrm{M}} + \boldsymbol{v} \otimes \boldsymbol{\tau}$ is a special Galilean transformation: the correspondence is surjective. ■

In view of our result, the special Galilean group is a three-dimensional commutative group.

11.3.7. (*i*) If $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$, then the special Galilean transformation

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) := \mathrm{id}_{\mathrm{M}} + (\boldsymbol{u}' - \boldsymbol{u}) \otimes \boldsymbol{\tau},$$

i.e. the one corresponding to $\boldsymbol{v_{u'u}} = \boldsymbol{u}' - \boldsymbol{u}$ is the unique one with the property

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{u} = \boldsymbol{u}'.$$

Let us recall the splitting of $\mathbf{M}$ according to $\boldsymbol{u}$ and $\boldsymbol{u}'$; then we easily find that

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) = \boldsymbol{h_{u'}}^{-1} \cdot \boldsymbol{h_u}.$$

(*ii*) The product of the $\boldsymbol{u}'$-timelike inversion and the $\boldsymbol{u}$-timelike inversion is a special Galilean transformation:

$$\boldsymbol{T_{u'}} \cdot \boldsymbol{T_u} = (\mathrm{id}_{\mathrm{M}} - 2\boldsymbol{u}' \otimes \boldsymbol{\tau}) \cdot (\mathrm{id}_{\mathrm{M}} - 2\boldsymbol{u} \otimes \boldsymbol{\tau}) = \mathrm{id}_{\mathrm{M}} + 2\boldsymbol{v_{u'u}} \otimes \boldsymbol{\tau}.$$

We know that $\boldsymbol{T_u}^{-1} = \boldsymbol{T_u} = -\boldsymbol{P_u}$; then we can assert that

$$\begin{aligned}\boldsymbol{T_{u'}} \cdot \boldsymbol{T_u}^{-1} = \boldsymbol{P_{u'}} \cdot \boldsymbol{P_u}^{-1} =& \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})^2 = \\ =& \boldsymbol{L}(\boldsymbol{u} + 2\boldsymbol{v_{u'u}}, \boldsymbol{u}) = \boldsymbol{L}(\boldsymbol{u} - 2\boldsymbol{v_{uu'}}, \boldsymbol{u}).\end{aligned}$$

11.3.8. Originally the Galilean tranformations are defined to be linear maps from $\mathbf{M}$ into $\mathbf{M}$. In the usual way, we can consider them to be linear maps from $\frac{\mathbf{M}}{\mathbf{I}}$ into $\frac{\mathbf{M}}{\mathbf{I}}$ as we already did in the preceding paragraphs as well.

$\mathrm{V}(1)$ is invariant under orthochronous Galilean transformations. Moreover, the restriction of an orthochronous Galilean transformation $\boldsymbol{L}$ onto $\mathrm{V}(1)$ is an affine bijection whose underlying linear map — which is the restriction of $\boldsymbol{L}$ onto $\frac{\mathbf{E}}{\mathbf{I}}$— preserves the Euclidean structure.

Conversely, if $\boldsymbol{F}$ is a Euclidean transformation of $\mathrm{V}(1)$— an affine bijection whose underlying linear map preserves the Euclidean structure — then $\mathbf{M} \to \mathbf{M}$,

$\boldsymbol{x} \mapsto \boldsymbol{F}\cdot(\boldsymbol{x}/\boldsymbol{\tau}\cdot\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{x}$ is an orthochronous Galilean tranformation whose restriction onto V(1) coincides with $\boldsymbol{F}$.

Thus we can state that the orthochronous Galilean group is canonically isomorphic to the group of Euclidean transformations of V(1).

11.4. The split Galilean group

11.4.1. The Galilean transformations, being elements of $\mathbf{M} \otimes \mathbf{M}^*$, are split by velocity values according to 8.1.1. Since $\boldsymbol{\tau} \cdot \boldsymbol{L} = (\mathrm{ar}\boldsymbol{L})\boldsymbol{\tau}$ and $\boldsymbol{\pi_u} \cdot \boldsymbol{L} \cdot \mathbf{i} = \boldsymbol{R_L}$ for a Galilean transformation $\boldsymbol{L}$ and for $\boldsymbol{u} \in \mathrm{V}(1)$, we have

$$\boldsymbol{h_u} \cdot \boldsymbol{L} \cdot \boldsymbol{h_u}^{-1} = \begin{pmatrix} \mathrm{ar}\boldsymbol{L} & \mathbf{0} \\ \boldsymbol{L}\cdot\boldsymbol{u} - (\mathrm{ar}\boldsymbol{L})\boldsymbol{u} & \boldsymbol{R_L} \end{pmatrix}.$$

Writing $\boldsymbol{L} \cdot \boldsymbol{u} - (\mathrm{ar}\boldsymbol{L})\boldsymbol{u} = (\mathrm{ar}\boldsymbol{L})\left((\mathrm{ar}\boldsymbol{L})\boldsymbol{u} - \boldsymbol{u}\right)$, we see that the following definition describes the split form of Galilean transformations.

Definition. The *split Galilean group* is

$$\left\{ \begin{pmatrix} \alpha & \mathbf{0} \\ \alpha\boldsymbol{v} & \boldsymbol{R} \end{pmatrix} \mid \quad \alpha \in \{-1, 1\}, \quad \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}}, \quad \boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \right\}.$$

Its elements are called *split Galilean transformations.* ■

The split Galilean transformations can be regarded as linear maps $\mathbf{I} \times \mathbf{E} \to \mathbf{I} \times \mathbf{E}$; the one in the definition makes the correspondence

$$(\boldsymbol{t}, \boldsymbol{q}) \mapsto (\alpha\boldsymbol{t},\ \alpha\boldsymbol{v}\boldsymbol{t} + \boldsymbol{R}\cdot\boldsymbol{q}).$$

The split Galilean group is a six-dimensional Lie group having the Lie algebra

$$\left\{ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{v} & \boldsymbol{A} \end{pmatrix} \mid \quad \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}}, \quad \boldsymbol{A} \in \mathrm{A}(\mathbf{b}) \right\}.$$

11.4.2. The splitting $\boldsymbol{h_u}$ according to $\boldsymbol{u}$ establishes a Lie-group isomorphism between the Galilean group and the split Galilean group. The isomorphisms corresponding to different $\boldsymbol{u}'$ and $\boldsymbol{u}$ are different.

The $\frac{\mathbf{E}}{\mathbf{I}}$ component in the split form of Galilean transformations, in general, varies according to the velocity value establishing the splitting.

The following *transformation rule* shows well how the splitting depends on the velocity values.

Let $\boldsymbol{u}', \boldsymbol{u} \in \mathrm{V}(1)$. Recall the notation

$$\boldsymbol{H_{u'u}} := \boldsymbol{h_{u'}} \cdot \boldsymbol{h_u}^{-1} = \begin{pmatrix} 1 & \mathbf{0} \\ -\boldsymbol{v_{u'u}} & \mathrm{id}_{\mathbf{E}} \end{pmatrix}.$$

Then

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \begin{pmatrix} \alpha & \mathbf{0} \\ \alpha \boldsymbol{v} & \boldsymbol{R} \end{pmatrix} \cdot \boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}^{-1} = \begin{pmatrix} \alpha & \mathbf{0} \\ \alpha(\boldsymbol{v} - \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) + \boldsymbol{R} \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \boldsymbol{R} \end{pmatrix}. \qquad \blacksquare$$

11.4.3. The splittings send the proper Galilean group into

$$\left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{v} & \boldsymbol{R} \end{pmatrix} \mid \quad \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}}, \quad \boldsymbol{R} \in \mathcal{SO}(\mathbf{b}) \right\}$$

which is evidently a connected set. Since the splittings are Lie group isomorphisms, $\mathcal{G}^{+\to}$ is connected as well.

11.4.4. If $\boldsymbol{L}$ is a special Galilean transformation and $\boldsymbol{v}_{\boldsymbol{L}}$ is the corresponding element of $\frac{\mathbf{E}}{\mathbf{I}}$, then $\boldsymbol{L}$ has the split form

$$\begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{v}_{\boldsymbol{L}} & \mathrm{id}_{\mathbf{E}} \end{pmatrix}$$

for all $\boldsymbol{u} \in \mathrm{V}(1)$: the splitting is independent of the velocity value. In other words, every $\boldsymbol{u} \in \mathrm{V}(1)$ makes the same bijection between the special Galilean group $\mathcal{V}$ and the group

$$\left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{v} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \mid \quad \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}} \right\}.$$

Observe that for all $\boldsymbol{u}', \boldsymbol{u} \in \mathrm{V}(1)$, the vector transformation law is the split form of a special Galilean transformation:

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} = \boldsymbol{h}_{\boldsymbol{u}'} \cdot {\boldsymbol{h}_{\boldsymbol{u}}}^{-1} = \boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}') \cdot {\boldsymbol{h}_{\boldsymbol{u}}}^{-1}.$$

11.4.5. The Lie algebra of the Galilean group, too, consists of elements of $\mathbf{M} \otimes \mathbf{M}^*$, thus they are split by velocity values in the same way as the Galilean transformations; evidently, their split forms will be different.

If $\boldsymbol{H}$ is in the Lie algebra of the Galilean group and $\boldsymbol{u} \in \mathrm{V}(1)$, then

$$\boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{H} \cdot {\boldsymbol{h}_{\boldsymbol{u}}}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{H} \cdot \boldsymbol{u} & \boldsymbol{H} \cdot \mathbf{i} \end{pmatrix}.$$

The splitting according to $\boldsymbol{u}$ establishes a Lie algebra isomorphism between the Lie algebra of the Galilean group and the Lie algebra of the split Galilean group. The isomorphisms corresponding to different $\boldsymbol{u}'$ and $\boldsymbol{u}$ are different:

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{v} & \boldsymbol{H} \end{pmatrix} \cdot {\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{v} + \boldsymbol{H} \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \boldsymbol{H} \end{pmatrix}.$$

11.5. Exercises

1. Prove that for all $\boldsymbol{t} \in \mathbf{I}$, $\{\boldsymbol{x} \in \mathbf{M} \mid \boldsymbol{\tau} \cdot \boldsymbol{x} = \boldsymbol{t}\}$ is an orbit of the special Galilean group and all orbits are of this form. The orbits of the special Galilean group and the orbits of the proper Galilean group coincide. What are the orbits of the (orthochronous) Galilean group?

2. Beside the trivial linear subspaces $\{\mathbf{0}\}$ and $\mathbf{M}$ there is no subspace invariant for all the special Galilean transformations.

3. The transpose of a Galilean transformation is a linear bijection $\mathbf{M}^* \to \mathbf{M}^*$. Demonstrate that the *transposed Galilean group* $\{\boldsymbol{L}^* \mid \boldsymbol{L} \in \mathcal{G}\}$ leaves $\mathbf{I}^* \cdot \boldsymbol{\tau}$ invariant; more closely, if $\boldsymbol{L} \in \mathcal{G}$ and $\boldsymbol{e} \in \mathbf{I}^* \cdot \boldsymbol{\tau}$, then $\boldsymbol{L}^* \cdot \boldsymbol{e} = (\mathrm{ar}\boldsymbol{L})\boldsymbol{e}$.

Furthermore, if $\boldsymbol{k} \in \mathbf{M}^*$, and $\boldsymbol{L}^* \cdot \boldsymbol{k}$ is parallel to $\boldsymbol{k}$ for all Galilean transformations $\boldsymbol{L}$, then $\boldsymbol{k}$ is in $\mathbf{I}^* \cdot \boldsymbol{\tau}$.

4. The subgroup generated by $\{\boldsymbol{T_u} \mid \boldsymbol{u} \in \mathrm{V}(1)\}$ is the special Galilean group.

5. Prove that

$$\boldsymbol{h_u} \cdot \boldsymbol{T_u} \cdot \boldsymbol{h_u}^{-1} = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \mathrm{id}_{\mathbf{E}} \end{pmatrix}, \qquad \boldsymbol{h_u} \cdot \boldsymbol{P_u} \cdot \boldsymbol{h_u}^{-1} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\mathrm{id}_{\mathbf{E}} \end{pmatrix}.$$

Find $\boldsymbol{h_{u'}} \cdot \boldsymbol{T_u} \cdot \boldsymbol{h_{u'}}^{-1}$ and $\boldsymbol{h_{u'}} \cdot \boldsymbol{P_u} \cdot \boldsymbol{h_{u'}}^{-1}$.

6. The $\boldsymbol{u}$-splitting of the Galilean group sends the special Galilean group into the group

$$\left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{v} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \mid \quad \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}} \right\}$$

whose Lie algebra is

$$\left\{ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{v} & \mathbf{0} \end{pmatrix} \mid \quad \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}} \right\}.$$

The $\boldsymbol{u}$-splitting of special Galilean transformations does not depend on $\boldsymbol{u}$.

7. The Lie algebra of $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ equals

$$\{\boldsymbol{H} \in \mathbf{La}(\mathcal{G}) \mid \boldsymbol{H} \cdot \boldsymbol{u} = \mathbf{0}\}.$$

8. The $\boldsymbol{u}$-splitting sends the subgroup $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ into the group

$$\left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R} \end{pmatrix} \mid \quad \boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \right\}$$

having the Lie algebra

$$\left\{ \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A} \end{pmatrix} \mid \quad \boldsymbol{A} \in \mathrm{A}(\mathbf{b}) \right\}.$$

Find the $\boldsymbol{u}'$-splitting of $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ for $\boldsymbol{u}' \neq \boldsymbol{u}$.

9. Recall the notation introduced in 11.3.7 and prove that
— $\boldsymbol{L}(\boldsymbol{u}',\boldsymbol{u})^{-1} = \boldsymbol{L}(\boldsymbol{u},\boldsymbol{u}')$,
— $\boldsymbol{L}(\boldsymbol{u}''\boldsymbol{u}') \cdot \boldsymbol{L}(\boldsymbol{u}',\boldsymbol{u}) = \boldsymbol{L}(\boldsymbol{u}'',\boldsymbol{u})$.
10. For all $\boldsymbol{u} \in \mathrm{V}(1)$ and for all Galilean transformations $\boldsymbol{L}$ we have that

$$\boldsymbol{R}(\boldsymbol{L},\boldsymbol{u}) := (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}(\boldsymbol{u},(\mathrm{ar}\boldsymbol{L})\cdot\boldsymbol{u})\cdot\boldsymbol{L} = (\mathrm{ar}\boldsymbol{L})\boldsymbol{L} + (\boldsymbol{u} - (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u})\otimes\boldsymbol{\tau}$$

is in $\mathcal{O}(\mathbf{b})_{\boldsymbol{u}}$ and $\boldsymbol{R}\left(\boldsymbol{L},\boldsymbol{u}\right)|_{\mathbf{E}_{\boldsymbol{u}}} = \boldsymbol{R}_{\boldsymbol{L}}$. In other words, given an arbitrary $\boldsymbol{u} \in \mathrm{V}(1)$, every Galilean transformation $\boldsymbol{L}$ is the product of a special Galilean transformation and a $\boldsymbol{u}$-spacelike orthogonal transformation, multiplied by the arrow of $\boldsymbol{L}$:

$$\boldsymbol{L} = (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}((\mathrm{ar}\boldsymbol{L}(\boldsymbol{u}),\boldsymbol{u})\cdot\boldsymbol{R}(\boldsymbol{L},\boldsymbol{u}).$$

11.6. The Noether group

11.6.1. Now we shall deal with affine maps $L : \mathrm{M} \to \mathrm{M}$; as usual, the linear map under L is denoted by $\boldsymbol{L}$.

Definition.

$$\mathcal{N} := \{L : \mathrm{M} \to \mathrm{M} \mid \ L \quad \text{is affine}, \quad \boldsymbol{L} \in \mathcal{G}\}$$

is called the *Noether group*; its elements are the *Noether transformations.*

If L is a Noether transformation, then

$$\mathrm{ar}L := \mathrm{ar}\boldsymbol{L}, \qquad \mathrm{sign}L := \mathrm{sign}\boldsymbol{L}.$$

$\mathcal{N}^{+\to}$, $\mathcal{N}^{+\leftarrow}$, $\mathcal{N}^{-\to}$ and $\mathcal{N}^{-\leftarrow}$ are the subsets of $\mathcal{N}$ consisting of elements whose underlying linear maps belong to $\mathcal{G}^{+\to}$, $\mathcal{G}^{+\leftarrow}$, $\mathcal{G}^{-\to}$ and $\mathcal{G}^{-\leftarrow}$, respectively.

$\mathcal{N}^{+\to}$ is called the *proper Noether group*. ■

The Noether group is the affine group over the Galilean group; according to VII.3.2(*ii*), we can state the following.

Proposition. The Noether group is a ten-dimensional Lie group; its Lie algebra consists of the affine maps $H : \mathrm{M} \to \mathbf{M}$ whose underlying linear map is in the Lie algebra of the Galilean group:

$$\mathbf{La}(\mathcal{N}) = \{H \in \mathrm{Aff}(\mathrm{M},\mathbf{M}) \mid \ \boldsymbol{\tau}\cdot\boldsymbol{H} = \mathbf{0}, \quad \boldsymbol{H}\cdot\mathbf{i} \in \mathrm{A}(\mathbf{b})\}\,. \quad ■$$

The proper Galilean group is a connected subgroup of the Galilean group. As regards $\mathcal{N}^{+\leftarrow}$, etc. we can repeat what was said about the components of the Galilean group.

$\mathcal{N}^{\to} := \mathcal{N}^{+\to} \cup \mathcal{N}^{-\to}$ is called the *orthochronous Noether group.*

11.6.2. We can say that the elements of $\mathcal{N}^{-\leftarrow}$ invert spacetime in some sense but there is no element that we could call the spacetime inversion.

For every $o \in \mathrm{M}$ we can give the *o-centered* spacetime inversion in such a way that first M is vectorized by O_o, then the vectors are inverted ($-\mathrm{id}_{\mathbf{M}}$ is applied), finally the vectorization is removed :

$$I_o := O_o{}^{-1} \circ (\mathrm{id}_{\mathbf{M}}) \circ O_o,$$

i.e.

$$I_o(x) := o - (x - o) \qquad (x \in \mathrm{M}).$$

Similarly, we can say that in some sense the elements of $\mathcal{N}^{-\rightarrow}$ contain spacelike inversion and do not contain timelike inversion; the elements of $\mathcal{N}^{+\leftarrow}$ contain timelike inversion and do not contain spacelike inversion. However, the space inversion and the time inversion do not exist.

For every $o \in \mathrm{M}$ and $\boldsymbol{u} \in \mathrm{V}(1)$ we can give the *o-centered $\boldsymbol{u}$-timelike inversion* and the *o-centered $\boldsymbol{u}$-spacelike inversion* as follows:

$$T_{\boldsymbol{u},o}(x) := o + \boldsymbol{T_u} \cdot (x - o), \qquad P_{\boldsymbol{u},o}(x) := o + \boldsymbol{P_u} \cdot (x - o)$$

$$(x \in \mathrm{M}).$$

11.6.3. Let L be a Noether transformation. If x and y are simultaneous then $L(x)$ and $L(y)$ are simultaneous as well:

$$\tau(L(x)) - \tau L(y)) = \boldsymbol{\tau} \cdot \boldsymbol{L} \cdot (x - y) = (\mathrm{ar}\boldsymbol{L})\boldsymbol{\tau} \cdot (x - y) = \mathbf{0}.$$

Recall that I is identified with the set of hyperplanes of M, directed by $\mathbf{E}$. Thus for a Noether transformation L we can define the mapping

$$\mathrm{ti}L : \mathrm{I} \to \mathrm{I}, \qquad t \mapsto L[t].$$

Observe that

$$(\mathrm{ti}L) \circ \tau = \tau \circ L$$

or, in other words,

$$(\mathrm{ti}L)(t) = \tau(L(x)) \qquad (x \in t),$$

from which we get immediately that

$$(\mathrm{ti}L)(t) - (\mathrm{ti}L)(s) = (\mathrm{ar}L)(t - s) \qquad (t, s \in \mathrm{I}).$$

Thus $\mathrm{ti}L$ is an affine map over $(\mathrm{ar}L)\mathrm{id}_{\mathbf{I}}$.

According to Exercises VI.2.5.6–7, if ar$L = 1$, then ti$L$ is a translation, i.e. there is a unique $\boldsymbol{t} \in \mathbf{I}$ such that $(\mathrm{ti}L)(t) = t + \boldsymbol{t}$; if ar$L = -1$, then ti$L$ is an inversion, i.e. there is a unique $t_o \in \mathrm{I}$ such that $(\mathrm{ti}L)(t) = t_o - (t - t_o)$.

11.6.4. The Noether transformations are mappings of spacetime. They play a fundamental role because the proper Noether transformations can be considered to be the strict automorphisms of the spacetime model.

Proposition. $(F, B, \mathrm{id}_{\mathbf{D}})$ is a strict automorphism of the non-relativistic space time model $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$ if and only if F is a proper Noether transformation and $B = \mathrm{ti}F$.

Proof. Let $(F, B, \mathrm{id}_{\mathbf{D}})$ be a strict automorphism. Then $\tau \circ F = B \circ \tau$ and $\boldsymbol{B} = \mathrm{id}_{\mathbf{I}}$ imply $\boldsymbol{\tau} \circ \boldsymbol{F} = \boldsymbol{\tau}$. Moreover, $\mathbf{b} \circ (\boldsymbol{F} \times \boldsymbol{F}) = \mathbf{b}$ means that the restriction of $\boldsymbol{F}$ onto $\mathbf{E}$ is orthogonal. Thus $\boldsymbol{F}$ is an orthochronous Galilean transformation and F is an orthochronous Noether transformation. Since F must be orientation-preserving, F is a proper Noether transformation. $\tau \circ F = B \circ \tau$ implies that $B = \mathrm{ti}F$.

Conversely, it is evident that if F is a proper Noether transformation, then $(F, \mathrm{ti}F, \mathrm{id}_{\mathbf{D}})$ is a strict automorphism.

11.6.5. Let us denote the translation group of $\mathbf{I}$ by $\mathcal{T}n(\mathbf{I})$ and consider it as an affine transformation group of I : $\boldsymbol{t} \in \mathbf{I}$ acts as $\mathrm{I} \to \mathrm{I}$, $t \mapsto t + \boldsymbol{t}$. In this respect $\mathbf{0} \in \mathbf{I}$ equals the identity map of I. It is quite obvious now that

$$\mathcal{N}^{\to} \to \mathcal{T}n(\mathbf{I}), \qquad L \mapsto \mathrm{ti}L$$

is a surjective Lie group homomorphism. Its kernel,

$$\mathcal{N}_i := \{L \in \mathcal{N}^{\to} \mid \mathrm{ti}L = \mathbf{0}\ (= \mathrm{id}_{\mathrm{I}})\} = \{L \in \mathcal{N}^{\to} \mid \tau \circ L = \tau\}$$

is called the *instantaneous Noether group.* It is a nine-dimensional Lie group having the Lie algebra

$$\mathbf{La}(\mathcal{N}_i) = \{H \in \mathrm{Aff}(\mathrm{M}, \mathbf{M}) \mid \boldsymbol{\tau} \circ H = \mathbf{0}, \quad \boldsymbol{H} \cdot \mathbf{i} \in \mathrm{A}(\mathbf{b})\}.$$

The instantaneous Noether transformations leave every instant invariant.

$\mathcal{T}n(\mathbf{I})$ is not a subgroup of $\mathcal{N}$. For every $\boldsymbol{u} \in \mathrm{V}(1)$,

$$\mathcal{T}n(\mathbf{I})_{\boldsymbol{u}} := \{\mathrm{id}_{\mathrm{M}} + \boldsymbol{u}\boldsymbol{t} \mid \ \boldsymbol{t} \in \mathbf{I}\}$$

is a subgroup of the orthochronous Noether group, called the group of *$\boldsymbol{u}$-timelike translations.* The restriction of the homomorphism $L \mapsto \mathrm{ti}L$ onto $\mathcal{T}n(\mathbf{I})_{\boldsymbol{u}}$ is a bijection between $\mathcal{T}n(\mathbf{I})_{\boldsymbol{u}}$ and $\mathcal{T}n(\mathbf{I})$.

In other words, given $\boldsymbol{u} \in \mathrm{V}(1)$, we can assign to every $\boldsymbol{t} \in \mathbf{I}$ the Noether transformation

$$x \mapsto x + \boldsymbol{u}\boldsymbol{t}$$

called the $\boldsymbol{u}$-timelike translation by $\boldsymbol{t}$.

11.6.6. The Galilean group is not a subgroup of the Noether group. The mapping $\mathcal{N} \to \mathcal{G}$, $L \mapsto \boldsymbol{L}$ is a surjective Lie group homomorphism whose kernel is $\mathcal{T}n(\mathbf{M})$, the translation group of M,

$$\mathcal{T}n(\mathbf{M}) = \{T_{\boldsymbol{x}} | \boldsymbol{x} \in \mathbf{M}\} = \{L \in \mathcal{N} | \boldsymbol{L} = id_{\mathbf{M}}\}\,.$$

As we know, its Lie algebra is $\mathbf{M}$ regarded as the set of constant maps from M into $\mathbf{M}$ (VII.3.3).

For every $o \in \mathrm{M}$,

$$\mathcal{G}_o := \{L \in \mathcal{N} \mid \ L(o) = o\}\,,$$

called the group of *o-centered Galilean transformations,* is a subgroup of the Noether group and even of the instantaneous Noether group; the restriction of the homomorphism $L \mapsto \boldsymbol{L}$ onto $\mathcal{G}_o$ is a bijection between $\mathcal{G}_o$ and $\mathcal{G}$.

In other words, given $o \in \mathrm{M}$, we can assign to every Galilean transformation $\boldsymbol{L}$ the Noether transformation

$$x \mapsto o + \boldsymbol{L} \cdot (x - o),$$

called the *o-centered Galilean transformation by* $\boldsymbol{L}$.

The subgroup of *o-centered special Galilean transformations*

$$\mathcal{V}_o := \{L \in \mathcal{N}_o \mid \ \boldsymbol{L} \in \mathcal{V}\}$$

has a special importance.

11.6.7. The three-dimensional orthogonal group is not a subgroup of the Noether group. The mapping $\mathcal{N}^{\to} \to \mathcal{O}(\mathbf{b})$, $L \mapsto \boldsymbol{R_L}$ (where $\boldsymbol{L} \cdot \mathbf{i} = \mathbf{i} \cdot \boldsymbol{R_L}$) is a surjective Lie group homomorphism having the kernel

$$\mathcal{H} := \{L \in \mathcal{N}^{\to} \mid \ \boldsymbol{L} \cdot \mathbf{i} = \mathbf{i}\} = \{L \in \mathcal{N}^{\to} \mid \ L \in \mathcal{V}\}$$

is called the *special Noether group.* Observe that $\mathcal{H}$ is in $\mathcal{N}^{+\to}$.

The special Noether group is a seven-dimensional Lie group having the Lie algebra

$$\{H \in \mathbf{La}(\mathcal{N}) \mid \ \boldsymbol{H} \in \mathbf{La}(\mathcal{V})\} = \{H \in \mathrm{Aff}(\mathrm{M}, \mathbf{M}) \mid \ \boldsymbol{\tau} \cdot \boldsymbol{H} = \mathbf{0}, \quad \boldsymbol{H} \cdot \mathbf{i} = \mathbf{0}\}\,.$$

For every $\boldsymbol{u} \in \mathrm{V}(1)$ and $o \in \mathrm{M}$,

$$\mathcal{O}(\mathbf{b})_{\boldsymbol{u},o} := \{L \in \mathcal{N}^{\rightarrow} \mid L(o) = o, \quad \boldsymbol{L} \cdot \boldsymbol{u} = \boldsymbol{u}\},$$

called the group of *o-centered* $\boldsymbol{u}$*-spacelike orthogonal transformations,* is a subgroup of $\mathcal{N}^{\rightarrow}$ and even of the instantaneous Noether group $\mathcal{N}_i$. The restriction of the homomorphism $\mathcal{N}^{\rightarrow} \to \mathcal{O}(\mathbf{b})$ onto $\mathcal{O}(\mathbf{b})_{\boldsymbol{u},o}$ is a bijection between $\mathcal{O}(\mathbf{b})_{\boldsymbol{u},o}$ and $\mathcal{O}(\mathbf{b})$.

In other words, given $(\boldsymbol{u}, o) \in \mathrm{V}(1) \times \mathrm{M}$, we can assign to every $\boldsymbol{R} \in \mathcal{O}(\mathbf{b})$ the Noether transformation

$$x \mapsto o + \boldsymbol{u}\boldsymbol{\tau} \cdot (x - o) + \boldsymbol{R} \cdot \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (x - o),$$

called the *o-centered* $\boldsymbol{u}$*-spacelike orthogonal transformation by* $\boldsymbol{R}$.

11.6.8. The Neumann group

$$\mathcal{C} := \{L \in \mathcal{N}_i \mid \boldsymbol{L} \cdot \mathbf{i} = \mathbf{i}\} = \mathcal{H} \cap \mathcal{N}_i$$

is an important subgroup of the special Noether group. It is a six-dimensional Lie group having the Lie algebra

$$\{H \in \mathbf{La}(\mathcal{N}_i) \mid \boldsymbol{H} \in \mathbf{La}(\mathcal{V})\} = \{H \in \mathrm{Aff}(\mathrm{M}, \mathbf{M}) \mid \boldsymbol{\tau} \circ H = 0, \quad \boldsymbol{H} \cdot \mathbf{i} = \mathbf{0}\}.$$

Proposition. The Neumann group is a commutative normal subgroup of the Noether group.

Proof. Let K and L be arbitrary Neumann transformations. Since they are instantaneous Noether transformations, for all world points x we have that $L(x) - x$ and $K(x) - x$ are in $\mathbf{E}$. As a consequence, $L(x) - x = \boldsymbol{K} \cdot (L(x) - x) = KL(x) - K(x)$ and similarly, $K(x) - x = LK(x) - L(x)$ from which we conclude that $KL(x) - LK(x) = \mathbf{0}$, i.e. $KL = LK$, the Neumann group is commutative.

Now we have to show that if L is an arbitrary Neumann transformation and G is an arbitrary Noether transformation then $G^{-1}LG$ is a Neumann transformation, too. The range of $\boldsymbol{G} \cdot \mathbf{i}$ is in $\mathbf{E}$, hence $\boldsymbol{L} \cdot \boldsymbol{G} \cdot \mathbf{i} = \boldsymbol{G} \cdot \mathbf{i}$ and so $\boldsymbol{G}^{-1} \cdot \boldsymbol{L} \cdot \boldsymbol{G} \cdot \mathbf{i} = \mathbf{i}$ which ends the proof.

11.7 The vectorial Noether group

11.7.1. Recall that for an arbitrary world point o, the vectorization of M with origin o, $O_o : \mathrm{M} \to \mathbf{M}$, $x \mapsto x - o$, is an affine bijection.

With the aid of such a vectorization we can "vectorize" the Noether group as well: if L is a Noether transformation then $O_o \circ L \circ {O_o}^{-1}$ is an affine transformation of $\mathbf{M}$, represented by the matrix (see VI.2.4(*ii*) and Exercise VI.2.5)

$$\begin{pmatrix} 1 & \mathbf{0} \\ L(o) - o & \boldsymbol{L} \end{pmatrix}.$$

The Lie algebra of the Noether group consists of affine maps $H : \mathrm{M} \to \mathbf{M}$ where $\mathbf{M}$ is considered to be a *vector space* (the *sum* of such maps is a part of the Lie algebra structure). Thus the vectorization $H \circ {O_o}^{-1}$ is an affine map $\mathbf{M} \to \mathbf{M}$ where the range is considered to be a vector space. Then it is represented by the matrix (see VI.2.4(*iii*))

$$\begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{H}(o) & \boldsymbol{H} \end{pmatrix}.$$

11.7.2. Definition. The *vectorial Noether group* is

$$\left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{L} \end{pmatrix} \middle| \quad \boldsymbol{a} \in \mathbf{M}, \quad \boldsymbol{L} \in (\mathcal{G}) \right\}. \quad \blacksquare$$

The vectorial Noether group is a ten-dimensional Lie group, its Lie algebra is the vectorization of the Lie algebra of the Noether group:

$$\left\{ \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{H} \end{pmatrix} \middle| \quad \boldsymbol{a} \in \mathbf{M}, \quad \boldsymbol{H} \in \mathbf{La}(\mathcal{G}) \right\}.$$

An advantage of this matrix representation is that the commutator of two Lie algebra elements can be computed by the difference of their products in different orders.

11.7.3. A vectorization of the Noether group is a Lie group isomorphism between the Noether group and the vectorial Noether group. The following transformation rule shows how the vectorizations depend on the world points serving as origins of the vectorization. Let o and o' be two world points; then

$$T_{o-o'} := O_{o'} \circ {O_o}^{-1} = \begin{pmatrix} 1 & 0 \\ o - o' & \mathrm{id}_{\mathbf{M}} \end{pmatrix}$$

and

$$T_{o-o'} \begin{pmatrix} 1 & 0 \\ \boldsymbol{a} & \boldsymbol{L} \end{pmatrix} {T_{o-o'}}^{-1} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{a} + (\boldsymbol{L} - \mathrm{id}_{\mathbf{M}})(o' - o) & \boldsymbol{L} \end{pmatrix} \quad (\boldsymbol{a} \in \mathbf{M},\ \boldsymbol{L} \in \mathcal{G}).$$

As concerns the corresponding Lie algebra isomorphisms, we have

$$\begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{H} \end{pmatrix} T_{o-o'}{}^{-1} = \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{a} + \boldsymbol{H}(o'-o) & \boldsymbol{H} \end{pmatrix} \qquad (\boldsymbol{a} \in \mathbf{M},\ \boldsymbol{H} \in \mathbf{La}(\mathcal{G})).$$

11.8. The split Noether group

11.8.1. With the aid of the splitting corresponding to $\boldsymbol{u} \in \mathrm{V}(1)$, we send the transformations of $\mathbf{M}$ into the transformations of $\mathbf{I} \times \mathbf{E}$. Composing a vectorization and a splitting, we convert Noether transformations into affine transformations of $\mathbf{I} \times \mathbf{E}$.

For $o \in \mathrm{M}$ and $\boldsymbol{u} \in \mathrm{V}(1)$ put

$$h_{\boldsymbol{u},o} := \boldsymbol{h}_{\boldsymbol{u}} \circ O_o : \mathrm{M} \to \mathbf{I} \times \mathbf{E}, \quad x \mapsto (\boldsymbol{\tau} \cdot (x-o),\ \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (x-o)).$$

Embedding the affine transformations of $\mathbf{I} \times \mathbf{E}$ into the linear transformations of $\mathbb{R} \times (\mathbf{I} \times \mathbf{E})$ (see VI.2.4(ii)) and using the customary matrix representation of such linear maps, we get

$$h_{\boldsymbol{u},o} \circ L \circ h_{\boldsymbol{u},o}{}^{-1} = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{\tau} \cdot (L(o)-o) & \mathrm{ar}L & \mathbf{0} \\ \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (L(o)-o) & \boldsymbol{L} \cdot \boldsymbol{u} - (\mathrm{ar}L)\boldsymbol{u} & \boldsymbol{R}_{\boldsymbol{L}} \end{pmatrix}.$$

The Lie algebra elements of the Noether group are converted into affine maps $\mathbf{I} \times \mathbf{E} \to \mathbf{I} \times \mathbf{E}$ where the range is regarded as a vector space. Then we can represent such maps in a matrix form as well:

$$h_{\boldsymbol{u}} \circ H \circ h_{\boldsymbol{u},o}{}^{-1} = \begin{pmatrix} 0 & 0 & \mathbf{0} \\ \boldsymbol{\tau} \cdot H(o) & 0 & \mathbf{0} \\ \boldsymbol{\pi}_{\boldsymbol{u}} \cdot H(o) & \boldsymbol{H} \cdot \boldsymbol{u} & \boldsymbol{H} \cdot \mathrm{i} \end{pmatrix}.$$

11.8.2. Definition. The *split Noether group* is

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t} & \alpha & \mathbf{0} \\ \boldsymbol{q} & \alpha\boldsymbol{v} & \boldsymbol{R} \end{pmatrix} \middle|\ \alpha \in \{-1, 1\},\ \boldsymbol{t} \in \mathbf{I},\ \boldsymbol{q} \in \mathbf{E},\ \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}},\ \boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \right\}. \quad \blacksquare$$

The split Noether group is a ten-dimensional Lie group having the Lie algebra

$$\left\{ \begin{pmatrix} 0 & 0 & \mathbf{0} \\ \boldsymbol{t} & 0 & \mathbf{0} \\ \boldsymbol{q} & \boldsymbol{v} & \boldsymbol{A} \end{pmatrix} \middle|\ \boldsymbol{t} \in \mathbf{I},\quad \boldsymbol{q} \in \mathbf{E},\quad \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}},\quad \boldsymbol{A} \in \mathrm{A}(\mathbf{b}) \right\}.$$

Keep in mind that the group multiplication of split Noether transformations coincides with the usual matrix multiplication and the commutator of Lie algebra elements is the difference of their two products.

11.8.3. Every $\boldsymbol{u} \in \mathrm{V}(1)$ and $o \in \mathrm{M}$ establishes a Lie group isomorphism between the Noether group and the split Noether group. Evidently, for different elements of $\mathrm{V}(1) \times \mathrm{M}$, the isomorphisms are different. The transformation rule that shows how the isomorphism depends on $(\boldsymbol{u}, o)$ can be obtained by a combination of the transformation rules 11.7.3 and 11.4.2.

Though the Noether group and the split Noether group are isomorphic (they have the same Lie group structure), they are not "identical": there is no "canonical" isomorphism between them that we could use to identify them.

The split Noether group is the Noether group of the split non-relativistic spacetime model $(\mathbf{I} \times \mathbf{E}, \mathbf{I}, \mathrm{pr}_{\mathbf{I}}, \mathbf{D}, \mathbf{b})$. The spacetime model $(\mathrm{M}, \mathrm{I}, \tau, \mathbf{D}, \mathbf{b})$ and the corresponding split spacetime model are isomorphic, but they cannot be identified, as we pointed out in 1.5.3.

11.8.4. It is a routine to check that the isomorphism established by an arbitrary $(\boldsymbol{u}, o) \in \mathrm{V}(1) \times \mathrm{M}$ sends the subgroups of the Noether group listed below on the left-hand side into the subgroups of the split Noether group listed below on the right-hand side:

$\mathcal{T}n(\mathbf{E})$
$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \boldsymbol{q} & \mathbf{0} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{q} \in \mathbf{E} \right\},$$

$\mathcal{T}n(\mathbf{M})$
$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t} & 1 & \mathbf{0} \\ \boldsymbol{q} & \mathbf{0} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{t} \in \mathbf{I},\ \boldsymbol{q} \in \mathbf{E} \right\},$$

$\mathcal{C}$ (Neumann group)
$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \boldsymbol{q} & \boldsymbol{v} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{q} \in \mathbf{E},\ \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}} \right\},$$

$\mathcal{H}$ (special Noether group)
$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t} & 1 & \mathbf{0} \\ \boldsymbol{q} & \boldsymbol{v} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{t} \in \mathbf{I},\ \boldsymbol{q} \in \mathbf{E},\ \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}} \right\},$$

$\mathcal{N}_i$ (instantaneous Noether group)
$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \boldsymbol{q} & \boldsymbol{v} & \boldsymbol{R} \end{pmatrix} \,\middle|\, \boldsymbol{q} \in \mathbf{E},\ \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}},\ \boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \right\},$$

It is emphasized that the isomorphism established by an arbitrary $(\boldsymbol{u}, o)$ makes a correspondence between the listed subgroups; of course, the correspondences due to different $(\boldsymbol{u}, o)$ and $(\boldsymbol{u}', o')$ are different.

Moreover, the isomorphism established by $(\boldsymbol{u}, o)$ makes correspondences between the following subgroups, too:

$$\mathcal{T}n(\mathbf{I})_{\boldsymbol{u}} \quad \begin{array}{c}(\boldsymbol{u}\text{-timelike}\\ \text{translations})\end{array} \quad \left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t} & 1 & \mathbf{0} \\ 0 & \mathbf{0} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{t} \in \mathbf{I} \right\},$$

$$\begin{array}{c}\mathcal{O}(\mathbf{b})_{\boldsymbol{u},o} \ (o\text{-centered}\\ \boldsymbol{u}\text{-spacelike orthogonal}\\ \text{transformations})\end{array} \quad \left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & \boldsymbol{R} \end{pmatrix} \,\middle|\, \boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \right\},$$

$$\mathcal{G}_o \quad \begin{array}{c}(o\text{-centered}\\ \text{Galilean}\\ \text{transformations})\end{array} \quad \left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & \alpha & \mathbf{0} \\ 0 & \alpha\boldsymbol{v} & \boldsymbol{R} \end{pmatrix} \,\middle|\, \alpha \in \{-1, 1\},\ \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}},\ \boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \right\}$$

$$\begin{array}{c}\mathcal{V}_o \quad (o\text{-centered special}\\ \text{Galilean transformations})\end{array} \left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ 0 & \boldsymbol{v} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}} \right\},$$

and now it is emphasized that the isomorphism established by $(\boldsymbol{u}', o')$, in general, does not make a correspondence between the listed subgroups.

11.8.5. Corresponding to the structure of the split Noether group, the following four subgroups are called its *fundamental subgroups:*

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t} & 1 & \mathbf{0} \\ 0 & \mathbf{0} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{t} \in \mathbf{I} \right\}, \qquad \left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \boldsymbol{q} & \mathbf{0} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{q} \in \mathbf{E} \right\},$$

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{v} & \mathrm{id}_{\mathbf{E}} \end{pmatrix} \,\middle|\, \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}} \right\}, \qquad \left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{R} \end{pmatrix} \,\middle|\, \boldsymbol{R} \in \mathcal{O}(\mathbf{b}) \right\}.$$

The isomorphism established by $(\boldsymbol{u}, o) \in \mathrm{V}(1) \times \mathrm{M}$ assigns these subgroups to the subgroups $\mathcal{T}n(\mathrm{I})_{\boldsymbol{u}}$, $\mathcal{T}n(\mathbf{E})$, $\mathcal{V}_o$ and $\mathcal{O}(\mathbf{b})_{\boldsymbol{u},o}$, respectively.

It is worth repeating the actual form of the corresponding Noether transformations:

$$\begin{aligned} \mathcal{T}n(\mathbf{I})_{\boldsymbol{u}} &: x \mapsto x + \boldsymbol{u}\boldsymbol{t} && (\boldsymbol{t} \in \mathbf{I}), \\ \mathcal{T}n(\mathbf{E}) &: x \mapsto x + \boldsymbol{q} && (\boldsymbol{q} \in \mathbf{E}), \\ \mathcal{V}_o &: x \mapsto x + \boldsymbol{v}\boldsymbol{\tau} \cdot (x - o) && (\boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}}), \\ \mathcal{O}(\mathbf{b})_{\boldsymbol{u},o} &: x \mapsto o + \boldsymbol{u}\boldsymbol{\tau} \cdot (x - o) + \boldsymbol{R} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}(x - o) && (\boldsymbol{R} \in \mathcal{O}(\mathbf{b})). \end{aligned}$$

11.8.6. Taking a linear bijection $\mathbf{I} \to \mathbb{R}$ and an orthogonal linear bijection $\mathbf{E} \to \mathbb{R}^3$, we can transfer the split Noether group into the following affine tranformation group of $\mathbb{R} \times \mathbb{R}^3$,

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \eta & \alpha & \mathbf{0} \\ \boldsymbol{\xi} & \alpha\boldsymbol{\nu} & \boldsymbol{\rho} \end{pmatrix} \middle| \ \alpha \in \{-1, 1\}, \ \eta \in \mathbb{R}, \ \boldsymbol{\xi} \in \mathbb{R}^3, \ \boldsymbol{\nu} \in \mathbb{R}^3, \ \boldsymbol{\rho} \in \mathcal{O}(3) \right\},$$

which we call the *arithmetic Noether group.* This is the Noether group of the arithmetic spacetime model ($\mathcal{O}(3)$ denotes the orthogonal group of $\mathbb{R}^3$ endowed with the usual inner product).

In conventional treatments one considers the arithmetic spacetime model (without an explicit definition) and the arithmetic Noether group which is called there Galilean group. The special form of such transformations yields that one speaks about *the* time inversion ($\alpha = -1$, $\eta = 0$, $\boldsymbol{\xi} = \mathbf{0}$, $\boldsymbol{\nu} = \mathbf{0}$, $\boldsymbol{\rho} = \mathbf{0}$), *the* time translations ($\alpha = 1$, $\eta \in \mathbb{R}$, $\boldsymbol{\xi} = \mathbf{0}$, $\boldsymbol{\nu} = \mathbf{0}$, $\boldsymbol{\rho} = 0$), *the* space rotations ($\alpha = 1$, $\eta = 0$, $\boldsymbol{\xi} = \mathbf{0}$, $\boldsymbol{\nu} = 0$, $\boldsymbol{\rho} \in \mathcal{SO}(3)$) etc., whereas we know well that such Noether transformations do not exist: there are o-centered $\boldsymbol{u}$-timelike inversions, $\boldsymbol{u}$-timelike translations and o-centered $\boldsymbol{u}$-spacelike rotations, etc.

11.9. Exercises

1. Let L be a Noether transformation for which $\boldsymbol{L} = -\mathrm{id}_{\mathbf{M}}$. Then there is a unique $o \in \mathrm{M}$ such that L is the o-centered spacetime inversion.
2. A Noether transformation L is instantaneous, i.e. is in $\mathcal{N}_i$ if and only if all the hyperplanes $t \in \mathrm{I}$ are invariant for L.
3. Prove that for all $o \in \mathrm{M}$,

$$O_o \circ \mathcal{N}_o \circ O_o{}^{-1} = \left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{L} \end{pmatrix} \middle| \ \boldsymbol{L} \in \mathcal{N} \right\}.$$

4. Find $h_{\boldsymbol{u},o} \cdot T_{\boldsymbol{u},o} \cdot h_{\boldsymbol{u},o}{}^{-1}$ and $h_{\boldsymbol{u},o} \cdot P_{\boldsymbol{u},o} \cdot h_{\boldsymbol{u},o}{}^{-1}$.
5. Prove that the subgroup generated by $\{T_{\boldsymbol{u},o} |\ \boldsymbol{u} \in \mathrm{V}(1),\ o \in \mathrm{M}\}$ equals $\{L \in \mathcal{N} |\ \boldsymbol{L} \cdot \mathbf{i} = \mathbf{i}\}$.
6. For all $\boldsymbol{u} \in \mathrm{V}(1)$, $o \in \mathrm{M}$ we have

$$(\mathrm{ti}T_{\boldsymbol{u},o})(t) = \tau(o) - (t - \tau(o)) = t - 2(t - \tau(o)) \qquad (t \in \mathrm{I}).$$

7. Prove that the derived Lie algebra of the Noether group, i.e. $[\mathbf{La}(\mathcal{N}), \mathbf{La}(\mathcal{N})]$ equals the Lie algebra of the instantaneous Noether group.
8. Let $L \in \mathcal{T}n(\mathbf{M})$. Then $h_{\boldsymbol{u},o} \cdot L \cdot h_{\boldsymbol{u},o}{}^{-1}$ is the same for all $\boldsymbol{u}$ and o if and only if $L \in \mathcal{T}n(\mathbf{E})$.

9. Take a $\boldsymbol{u} \in \mathrm{V}(1)$ and an $o \in \mathrm{M}$. If $\boldsymbol{t} \in \mathbf{I}$, $\boldsymbol{q} \in \mathbf{E}$, $\boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}}$, $\boldsymbol{A} \in \mathrm{A}(\mathbf{b})$, then the maps $\mathrm{M} \to \mathbf{M}$

$$H(x) := \begin{cases} (i) & \boldsymbol{ut} \\ (ii) & \boldsymbol{q} \\ (iii) & \boldsymbol{v\tau} \cdot (x - o) \\ (iv) & \boldsymbol{A} \cdot \boldsymbol{\pi_u} \cdot (x - o) \end{cases} \qquad (x \in \mathrm{M})$$

are elements of the Lie algebra of the Noether group. Prove that

$$\mathrm{e}^{H}(x) = \begin{cases} (i) & x + \boldsymbol{ut} \\ (ii) & x + \boldsymbol{q} \\ (iii) & x + \boldsymbol{v\tau} \cdot (x - o) \\ (iv) & o + \boldsymbol{u\tau} \cdot (x - o) + \mathrm{e}^{\boldsymbol{A}} \cdot \boldsymbol{\pi_u} \cdot (x - o) \end{cases} \qquad (x \in \mathrm{M}).$$

10. Compute the product of two split Noether transformations:

$$\begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t} & \alpha & \mathbf{0} \\ \boldsymbol{q} & \alpha\boldsymbol{v} & \boldsymbol{R} \end{pmatrix} \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t}' & \alpha' & \mathbf{0} \\ \boldsymbol{q}' & \alpha'\boldsymbol{v}' & \boldsymbol{R}' \end{pmatrix}.$$

11. Let L be a Noether transformation.

If r is a world line function, then $L \circ r \circ (\mathrm{ti}L)^{-1}$ is a world line function, too.

If C is a world line, then $L[\mathrm{C}]$ is a world line, too; moreover, if $C = \mathrm{Ran}\, r$, then $L[\mathrm{C}] = \mathrm{Ran}\, \left(L \circ r \circ (\mathrm{ti}L)^{-1}\right)$.

II. SPECIAL RELATIVISTIC SPACETIME MODELS

1. Fundamentals

1.1. Heuristic considerations

1.1.1. According to the non-relativistic spacetime model, the relative velocity of masspoints can be arbitrarily large: the relative velocities form a Euclidean vector space. However, experience shows that relative velocities cannot exceed the light speed in vacuum. Experience indicates as well that only "sluggish" mechanical phenomena are suitably described by the formulae of a non-relativistic spacetime model, i.e. when the relative velocities of masspoints are low compared to the light speed.

A simple example convinces us that the non-relativistic spacetime model is not right for the correct treatment of electromagnetic phenomena. Let us consider a light signal, a well-known electromagnetic phenomenon. According to our experience, an inertial observer sees a light signal propagating along a straight line with a uniform relative velocity. Let us try to model a light signal in the non-relativistic spacetime model. Evidently, the model would have to be a straight line. There are two possibilities: the straight line is a world line or is contained in a simultaneous hyperplane. The first possibility is excluded because then there would be an inertial observer relative to which the light signal is at rest, which is in contrast with our experience. The second possibility is excluded as well, because then the light signal would propagate with an "infinite" relative velocity (there is no time elapse during the propagation).

Thus wishing to describe correctly "brisk" mechanical phenomena and electromagnetic phenomena, we have to leave the non-relativistic spacetime model and to construct a new spacetime model.

1.1.2. The rectilinear and uniform propagation of light suggests that the affine structure of spacetime can be retained, i.e. spacetime will be modelled again by a four-dimensional oriented affine space M (over the vector space $\mathbf{M}$).

It follows then that we have to reject absolute time. Of course, something must be introduced instead of absolute time; *we accept absolute propagation of light.* Next we explain what the absolute propagation of light means.

1.1.3. According to our experience, light — independently of its source — propagates isotropically (with the same speed in every direction) relative to all inertial observers.

Let us say so: the propagation of a light signal starting "from a given place at a given instant" i.e. in a given spacetime point is independent of the source. This means that a subset $\mathrm{Z}(x)$ of M can be assigned to every $x \in \mathrm{M}$: the set of world points that can be reached from x by a light signal. Experience attests that the propagation of light signals starting from the same place relative to an observer does not depend on time (yesterday and today the propagation is the same) and light signals starting from different places propagate "congruently": a simple translation in space sends different propagations into each other. Accordingly we accept that $\mathrm{Z}(x)$-s are parallel translations of each other which implies that there is a subset $\mathrm{L}^{\rightarrow}$ of $\mathbf{M}$ such that $\mathrm{Z}(x) = x + \mathrm{L}^{\rightarrow}$ for all $x \in \mathbf{M}$.

The light signals starting from an arbitrary world point are half lines in such a way that if y is accessible by a light signal starting from x, then x is not accessible from y.

This means that

(i) if $\boldsymbol{x} \in \mathrm{L}^{\rightarrow}$ and $\alpha \in \mathbb{R}^{+}$ then $\alpha\boldsymbol{x} \in \mathrm{L}^{\rightarrow}$,

(ii) if $\boldsymbol{x} \in \mathrm{L}^{\rightarrow}$ then $-\boldsymbol{x} \notin \mathrm{L}^{\rightarrow}$:

$\mathrm{L}^{\rightarrow}$ is a cone and does not contain a line.

1.1.4. Absolute time in the non-relativistic spacetime model is equivalent to assigning to every world point x the set of world points simultaneous with x,

$$\tau(x) = x + \mathbf{E},$$

where $\mathbf{E}$ is a three-dimensional linear subspace of $\mathbf{M}$.

We introduce absolute light propagation in the special relativistic spacetime model by assigning to every world point x the set of world points accessible by light signals starting from x,

$$\mathrm{Z}(x) = x + \mathrm{L}^{\rightarrow},$$

where $\mathrm{L}^{\rightarrow}$ is a cone without being a linear subspace. We shall see later that $\mathrm{L}^{\rightarrow}$ — called the future light cone — is a three-dimensional submanifold, the boundary of an open convex cone.

1.1.5. We want to include in the model that observers experience a Euclidean structure on their space. In the non-relativistic case the Euclidean structure was related to simultaneity (with respect to absolute time). Here the Euclidean structure will be related to the isotropic propagation of light, a property that is not reflected in $\mathrm{L}^{\rightarrow}$ yet.

To get an inspiration, how to proceed, let us take the following heuristic consideration. Let us accept that the time and the space of an inertial observer can be represented by $\mathbb{R}$ and $\mathbb{R}^3$, respectively; let the units be chosen in such a way that the light speed is 1 (i.e. if $\boldsymbol{s}$ is the time unit, then the distance unit is the distance covered by a light signal in $1\boldsymbol{s}$). Then

$$\{(\xi^0, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^3 \mid |\boldsymbol{\xi}| = \xi^0,\ \xi^0 > 0\}$$

represents the set of spacetime points accessible by a light signal from $(0, \mathbf{0})$, where $|\ |$ denotes the usual Euclidean norm on $\mathbb{R}^3$.

The Euclidean norm derives from an inner product; that is why it is suitable to take the bilinear form

$$\mathrm{G} : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (\xi, \eta) \mapsto -\xi^0\eta^0 + \sum_{i=1}^{3} \xi^i\eta^i,$$

and to write the the above set in the form

$$\{\xi \in \mathbb{R}^4 \mid \quad \mathrm{G}(\xi, \xi) = 0, \quad \xi^0 > 0\}.$$

G is a Lorentz form. The reader, having studied Section V.4 and being familiar with Lorentz forms will notice that the condition $\xi^0 > 0$ selects one of the arrow classes of $\{\xi \neq 0 \mid \mathrm{G}(\xi, \xi) = 0\}$.

Now it seems natural to accept that in our spacetime model the accessibility by light signals is described by an arrow-oriented Lorentz form. More closely, we introduce the measure line $\mathbf{I}$ of spacetime distances and we suppose that there is an arrow-oriented Lorentz form $\mathbf{g} : \mathbf{M} \times \mathbf{M} \to \mathbf{I} \otimes \mathbf{I}$ such that $\mathrm{L}^{\to}$ is one of the arrow classes treated in V.4.13.

1.2. Definition of the spacetime model

1.2.1. Definition. A *special relativistic spacetime model* is a triplet $(\mathrm{M}, \mathbf{I}, \mathbf{g})$, where

— M is an oriented four-dimensional real affine space (over the vector space $\mathbf{M}$),

— $\mathbf{I}$ is an oriented one-dimensional real vector space,

— $\mathbf{g} : \mathbf{M} \times \mathbf{M} \to \mathbf{I} \otimes \mathbf{I}$ is an arrow oriented Lorentz form. ■

We shall use the following names:

M is *spacetime* or *world,*

$\mathbf{I}$ is the *measure line of spacetime distances,*

$\mathbf{g}$ is the *Lorentz form.*

Elements of M are called *world points.* Elements of $\mathbf{M}$ are called *world vectors.*

1.2.2. If $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ is a special relativistic spacetime model, then $(\mathbf{M}, \mathbf{I}, \mathbf{g})$ is an oriented and arrow-oriented Minkowskian vector space. The results and formulae of Section V.4 will be used all over this part. Remember to distinguish between $\boldsymbol{x}^2 := \boldsymbol{x}\cdot\boldsymbol{x}$ and $|\boldsymbol{x}|^2 := |\boldsymbol{x}\cdot\boldsymbol{x}|$; since $\mathbf{I}$ is oriented, the pseudo-length $|\boldsymbol{x}| := \sqrt{|\boldsymbol{x}|^2}$ is meaningful. Moreover, recall

$$\begin{aligned}\mathrm{S} :=&\{\boldsymbol{x}\in\mathbf{M} \mid \boldsymbol{x}^2 > 0\},\\ \mathrm{T} :=&\{\boldsymbol{x}\in\mathbf{M} \mid \boldsymbol{x}^2 < 0\},\\ \mathrm{L} :=&\{\boldsymbol{x}\in\mathbf{M} \mid \boldsymbol{x}^2 = 0, \boldsymbol{x}\neq 0\};\end{aligned}$$

the elements of $\mathrm{S}_0 := \mathrm{S}\cup\{\mathbf{0}\}$, T and L are called *spacelike, timelike* and *lightlike,* respectively.

Furthermore, the arrow orientation indicates the arrow classes $\mathrm{T}^{\rightarrow}$ and $\mathrm{L}^{\rightarrow}$; for every $\boldsymbol{x}\in\mathrm{T}^{\rightarrow}$ and $\boldsymbol{y}\in\mathrm{T}^{\rightarrow}\cup\mathrm{L}^{\rightarrow}$ we have $\boldsymbol{x}\cdot\boldsymbol{y} < 0$. Then $\mathrm{T}^{\leftarrow} := -\mathrm{T}^{\rightarrow}$ and $L^{\leftarrow} := -\mathrm{L}^{\rightarrow}$ are the other arrow classes and

$$\mathrm{T} = \mathrm{T}^{\rightarrow}\cup\mathrm{T}^{\leftarrow}, \qquad \mathrm{L} = \mathrm{L}^{\rightarrow}\cup\mathrm{L}^{\leftarrow}.$$

$\mathrm{T}^{\rightarrow}$ and $\mathrm{L}^{\rightarrow}$ are the *future time cone* and the *future light cone,* respectively; their elements are called *future-directed.* $\mathrm{T}^{\leftarrow}$ and $L^{\leftarrow}$ are the corresponding *past cones* with *past-directed* elements.

We often illustrate the world vectors in the plane of the page:

This illustration is based on the following: represent $\mathbb{R}\times\mathbb{R}$ in the plane in the usual way by horizontal and vertical axes, called zeroth and first; draw the sets S, $\mathrm{T}^{\rightarrow}$, $\mathrm{L}^{\rightarrow}$, etc. corresponding to the Lorentz form

$$\big((\xi^0,\xi^1),(\eta^0,\eta^1)\big) \mapsto -\xi^0\eta^0 + \xi^1\eta^1$$

and to the arrow orientation determined by the condition $\xi^0 > 0$; cancel the coordinate axes.

We know that T consists of two disjoint open subsets, the two arrow classes which can be well seen in the illustration. On the other hand, S is connected, in spite of the illustration. Keep in mind this slight inaccuracy of the illustration.

1.2.3. Spacetime, too, will be illustrated in the plane of the page. If x is a world point, $x + (T^{\rightarrow} \cup \mathrm{L}^{\rightarrow})$ and $x + (T^{\leftarrow} \cup L^{\leftarrow})$ are called the *future-like* and the *past-like* part of M, with respect to x.

If $y \in x + (T^{\rightarrow} \cup \mathrm{L}^{\rightarrow})$ — or, equivalently, $y - x \in (T^{\rightarrow} \cup L^{\rightarrow})$ — then we say y is *future-like with respect to* x (x is *past-like with respect to* y), or y is *later* than x (x is earlier than y).

We say that the world points x and y are *spacelike separated, timelike separated, lightlike separated,* if $y - x$ is in S, T, L, respectively.

1.3. Structure of world vectors and covectors

1.3.1. The Euclidean structure of our space is deeply fixed in our mind, therefore we must be careful when dealing with **M** which has not a Euclidean structure; especially when illustrating it in the Euclidean plane of the page. For instance, keep in mind that the centre line of the cone $\mathrm{L}^{\rightarrow}$ makes no sense (the centre line would be the set of points that have the same distance from every generatrix of the cone but distance is not meaningful here). The following considerations help us to take in the situation.

Put

$$\mathrm{V}(1) := \left\{ \boldsymbol{u} \in \frac{\mathbf{M}}{\mathbf{I}} \;\middle|\; \boldsymbol{u}^2 = -1,\ \boldsymbol{u} \otimes \mathbf{I}^+ \subset \mathrm{T}^{\rightarrow} \right\}.$$

We shall see in 2.3.4 that the elements of V(1) can be interpreted as *velocity values.*

According to our convention, V(1) is illustrated as follows:

Three elements of $\mathrm{V}(1)$ appear in the Figure. Observe that it makes no sense that

— $\boldsymbol{u}_1$ is in the centre line of $\mathrm{T}^{\to}$ (there is no centre line of $\mathrm{T}^{\to}$),

— the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is less than the angle between $\boldsymbol{u}_1$ and $\boldsymbol{u}_3$ (there is no angle between the elements of $\mathrm{V}(1)$),

— $\boldsymbol{u}_2$ is longer than $\boldsymbol{u}_1$ (the elements of $\mathrm{V}(1)$ have no length).

The reversed Cauchy inequality (see V.4.7) involves the following important and frequently used relation:

$$-\boldsymbol{u}\cdot\boldsymbol{u}' \geq 1$$

for $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$ and equality holds if and only if $\boldsymbol{u} = \boldsymbol{u}'$.

1.3.2. For $\boldsymbol{u} \in \mathrm{V}(1)$ put

$$\begin{aligned}
\boldsymbol{\tau_u} &: \mathbf{M} \to \mathbf{I}, \qquad \boldsymbol{x} \mapsto -\boldsymbol{u}\cdot\boldsymbol{x},\\
\mathbf{E}_{\boldsymbol{u}} &:= \mathrm{Ker}\ \boldsymbol{\tau_u} = \{\boldsymbol{x} \in \mathbf{M} \mid \boldsymbol{u}\cdot\boldsymbol{x} = 0\},\\
\mathbf{i}_{\boldsymbol{u}} &:= \mathbf{E}_{\boldsymbol{u}} \to \mathbf{M}, \quad \boldsymbol{x} \mapsto \boldsymbol{x}.
\end{aligned}$$

Since $\boldsymbol{u}$ is timelike, $\mathbf{E}_{\boldsymbol{u}}$ is a three-dimensional linear subspace consisting of spacelike vectors. According to our convention, $\mathbf{E}_{\boldsymbol{u}}$ is represented by a line that inclines to $\mathrm{L}^{\to}$ with the same angle as $\boldsymbol{u}$:

We emphasize that "inclination to $\mathrm{L}^{\to}$" makes no sense in the structure of the spacetime model; it makes sense only in the rules of the illustration we have chosen.

$\mathbf{E}_{\boldsymbol{u}}$ and $\boldsymbol{u}\otimes\mathbf{I}$ are complementary subspaces in $\mathbf{M}$, thus every vector $\boldsymbol{x}$ can be uniquely decomposed into the sum of components in $\boldsymbol{u}\otimes\mathbf{I}$ and in $\mathbf{E}_{\boldsymbol{u}}$, respectively:

$$\boldsymbol{x} = \boldsymbol{u}(\boldsymbol{\tau}_{\boldsymbol{u}}\cdot\boldsymbol{x}) + \big(\boldsymbol{x} - \boldsymbol{u}(\boldsymbol{\tau}_{\boldsymbol{u}}\cdot\boldsymbol{x})\big) = \boldsymbol{u}(-\boldsymbol{u}\cdot\boldsymbol{x}) + \big(\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{x})\big).$$

The linear map

$$\boldsymbol{\pi}_{\boldsymbol{u}}:\ \mathbf{M}\to\mathbf{E}_{\boldsymbol{u}},\qquad \boldsymbol{x}\mapsto\boldsymbol{x}+\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{x})$$

is the projection onto $\mathbf{E}_{\boldsymbol{u}}$ along $\boldsymbol{u}$. It is illustrated as follows:

The dashed line is to express that $\mathrm{V}(1)$ is in fact a subset of $\frac{\mathbf{M}}{\mathbf{I}}$ and not of $\mathbf{M}$.

1.3.3. For all $\boldsymbol{u} \in \mathrm{V}(1)$, the restriction $\mathbf{b}_{\boldsymbol{u}}$ of the Lorentz form $\mathbf{g}$ onto $\mathbf{E}_{\boldsymbol{u}} \times \mathbf{E}_{\boldsymbol{u}}$ is positive definite. Thus $(\mathbf{E}_{\boldsymbol{u}}, \mathbf{I}, \mathbf{b}_{\boldsymbol{u}})$ is a three-dimensional Euclidean vector space.

Accordingly, the pseudo-length of vectors in $\mathbf{E}_{\boldsymbol{u}}$ is in fact a *length* and the *angle* between non-zero vectors in $\mathbf{E}_{\boldsymbol{u}}$ make sense; of course, similar notions for vectors in $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{A}}$ can be introduced where $\mathbf{A}$ is a measure line. Moreover, all the results obtained in I.1.2.5 can be applied.

It is trivial that every spacelike vector is contained in some $\mathbf{E}_{\boldsymbol{u}}$:

$$\mathrm{S}_0 = \bigcup_{\boldsymbol{u} \in \mathrm{V}(1)} \mathbf{E}_{\boldsymbol{u}}.$$

Consequently, the pseudo-length of a spacelike vector will be said *length* or *magnitude.* However, we call attention to the fact that this length satisfies the triangle inequality only for two spacelike vectors spanning a spacelike linear subspace (see Exercise V.4.20.2)

1.3.4. The orientation of $\mathbf{M}$ and the arrow orientation of $\mathbf{g}$ determine a unique orientation of $\mathbf{E}_{\boldsymbol{u}}$.

Definition. Let $\boldsymbol{u} \in \mathrm{V}(1)$. An ordered basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ of $\mathbf{E}_{\boldsymbol{u}}$ is called *positively oriented* if $(\boldsymbol{u}\boldsymbol{t}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a positively oriented basis of $\mathbf{M}$ for some (hence for all) $\boldsymbol{t} \in \mathbf{I}^+$.

1.3.5. Proposition. Let $\boldsymbol{u} \in \mathrm{V}(1)$. Then

$$\boldsymbol{h}_{\boldsymbol{u}} := (\boldsymbol{\tau}_{\boldsymbol{u}}, \boldsymbol{\pi}_{\boldsymbol{u}}) : \ \mathbf{M} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}, \quad \boldsymbol{x} \mapsto \big(-\boldsymbol{u} \cdot \boldsymbol{x},\, \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x})\big)$$

is an orientation-preserving linear bijection and

$$\boldsymbol{h}_{\boldsymbol{u}}^{-1}(\boldsymbol{t}, \boldsymbol{q}) = \boldsymbol{u}\boldsymbol{t} + \boldsymbol{q} \qquad (\boldsymbol{t} \in \mathbf{I},\, \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}}). \qquad \blacksquare$$

Keep in mind that $\boldsymbol{x} = \boldsymbol{u}(-\boldsymbol{u} \cdot \boldsymbol{x}) + \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x}$ results in the following important formula:

$$\boldsymbol{x}^2 = -(\boldsymbol{u} \cdot \boldsymbol{x})^2 + |\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x}|^2 \qquad (\boldsymbol{x} \in \mathbf{M}).$$

1.3.6. Note the striking similarity between the previous formulae and the formulae of the non-relativistic spacetime model treated in I.1.2. However, behind the resemblance to it there is an important difference: in the non-relativistic case a single three-dimensional subspace $\mathbf{E}$ appears whereas in the special relativistic case every $\boldsymbol{u} \in \mathrm{V}(1)$ indicates its own three-dimensional subspace. Correspondingly, instead of a single $\boldsymbol{\tau}$, now there is a $\boldsymbol{\tau}_{\boldsymbol{u}}$ for all $\boldsymbol{u}$. The range of $\boldsymbol{h}_{\boldsymbol{u}}$ is the same set in the non-relativistic case, whereas it depends on $\boldsymbol{u}$ in the relativistic case.

A further very important difference is that $\mathbf{M}$ and $\mathbf{M}^*$ are different vector spaces in the non-relativistic case, whereas they are "nearly the same" in the relativistic case. More precisely, we have the identification (see V.1.3).

$$\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}} \equiv \mathbf{M}^*,$$

which is established by the Lorentz form $\mathbf{g}$. According to our dot product notation, $\mathbf{g}$ does not appear in the formulae. That is why we accept the notation

$$\mathbf{g} := \mathrm{id}_{\mathbf{M}} \in \mathbf{M}\otimes\mathbf{M}^*$$

as well, which will facilitate the comparison of our formulae with those of usually employed in textbooks. Then, for instance, we can write

$$\boldsymbol{\pi}_{\boldsymbol{u}} = \mathbf{g} + \boldsymbol{u}\otimes\boldsymbol{u}.$$

Of course, we make the identification

$$\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}\otimes\mathbf{I}} \equiv \mathbf{E}_{\boldsymbol{u}}^*,$$

too.

According to these identifications we have

$$\boldsymbol{\tau}_{\boldsymbol{u}} \in \mathbf{I}\otimes\mathbf{M}^* \equiv \frac{\mathbf{M}}{\mathbf{I}}, \qquad \mathbf{i}_{\boldsymbol{u}} \in \mathbf{M}\otimes\mathbf{E}_{\boldsymbol{u}}^* \equiv \frac{\mathbf{M}\otimes\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}\otimes\mathbf{I}},$$
$$\boldsymbol{\tau}_{\boldsymbol{u}}^* \in \mathbf{M}^*\otimes\mathbf{I} \equiv \frac{\mathbf{M}}{\mathbf{I}}, \mathbf{i}_{\boldsymbol{u}}^* \in \mathbf{E}_{\boldsymbol{u}}^*\otimes\mathbf{M} \equiv \frac{\mathbf{E}_{\boldsymbol{u}}\otimes\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}.$$

$$\boldsymbol{\pi}_{\boldsymbol{u}} \in \mathbf{E}_{\boldsymbol{u}}\otimes\mathbf{M}^* \equiv \frac{\mathbf{E}_{\boldsymbol{u}}\otimes\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}.$$

Moreover,

$$\boldsymbol{\tau}_{\boldsymbol{u}}\cdot\mathbf{i}_{\boldsymbol{u}} = 0, \qquad \boldsymbol{\pi}_{\boldsymbol{u}}\cdot\mathbf{i}_{\boldsymbol{u}} = \mathrm{id}_{\mathbf{E}_{\boldsymbol{u}}}$$

and the identifications yield the relations

$$\boldsymbol{\tau}_{\boldsymbol{u}} \equiv \boldsymbol{\tau}_{\boldsymbol{u}}^* \equiv -\boldsymbol{u}, \qquad \mathbf{i}_{\boldsymbol{u}}^* \equiv \boldsymbol{\pi}_{\boldsymbol{u}}.$$

The reader is asked to prove the first formula; as concerns the second one, see the following equalities for $\boldsymbol{x}\in\mathbf{M}$, $\boldsymbol{q}\in\mathbf{E}_{\boldsymbol{u}}$:

$$(\mathbf{i}_{\boldsymbol{u}}^*\cdot\boldsymbol{x})\cdot\boldsymbol{q} = \boldsymbol{x}\cdot\mathbf{i}_{\boldsymbol{u}}\cdot\boldsymbol{q} = \boldsymbol{x}\cdot\boldsymbol{q} = (\boldsymbol{\pi}_{\boldsymbol{u}}\cdot\boldsymbol{x})\cdot\boldsymbol{q}.$$

1.3.7. For $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$ put

$$\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} := \frac{\boldsymbol{u}'}{-\boldsymbol{u}' \cdot \boldsymbol{u}} - \boldsymbol{u}.$$

We shall see later that this is the relative velocity of $\boldsymbol{u}'$ with respect to $\boldsymbol{u}$.

It is an easy task to show that $|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2 = |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2 = 1 - \frac{1}{(\boldsymbol{u}' \cdot \boldsymbol{u})^2}$; as a consequence of the reversed Cauchy inequality, $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = \boldsymbol{0}$ if and only if $\boldsymbol{u} = \boldsymbol{u}'$. Moreover, if $\boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$ then $\boldsymbol{q} \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = 0$ which proves the following.

Proposition. $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$ is a two-dimensional linear subspace if and only if $\boldsymbol{u} \neq \boldsymbol{u}'$ and in this case $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \otimes \mathbf{I}$ ($\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'} \otimes \mathbf{I}$) is a one-dimensional linear subspace of $\mathbf{E}_{\boldsymbol{u}}$ ($\mathbf{E}_{\boldsymbol{u}'}$), orthogonal to $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$.

(In other words, $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$ and $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \otimes \mathbf{I}$ ($\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'} \otimes \mathbf{I}$) are orthogonal complementary subspaces in $\mathbf{E}_{\boldsymbol{u}}$ ($\mathbf{E}_{\boldsymbol{u}'}$)).

1.3.8. For different $\boldsymbol{u}$ and $\boldsymbol{u}'$, $\mathbf{E}_{\boldsymbol{u}}$ and $\mathbf{E}_{\boldsymbol{u}'}$ are different linear subspaces; however, we can give a distinguished bijection between them which will play a fundamental role concerning observer spaces.

Let $\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})$ be the linear map from $\mathbf{E}_{\boldsymbol{u}}$ onto $\mathbf{E}_{\boldsymbol{u}'}$ defined in such a way that it leaves invariant the elements of $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$ and maps the orthogonal complements of this subspace into each other. More precisely,

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{q} := \begin{cases} \boldsymbol{q} & \text{if } \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'} \\ -\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}\boldsymbol{t} & \text{if } \boldsymbol{q} = \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\boldsymbol{t} \quad (\boldsymbol{t} \in \mathbf{I}) \end{cases}.$$

It is not difficult to see that $\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})$ is an orientation-preserving $\mathbf{b}_{\boldsymbol{u}} - \mathbf{b}_{\boldsymbol{u}'}$-orthogonal linear bijection between $\mathbf{E}_{\boldsymbol{u}}$ and $\mathbf{E}_{\boldsymbol{u}'}$. We can extend it to a linear bijection $\mathbf{M} \to \mathbf{M}$ by the requirement

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{u} := \boldsymbol{u}'$$

(recall that the dot product notation allows us to apply linear maps $\mathbf{M} \to \mathbf{M}$ to elements of $\frac{\mathbf{M}}{\mathbf{I}}$).

This linear bijection can be given by a simple formula. Now we give this formula and then characterize its properties. Recall $\mathbf{g} := \mathrm{id}_{\mathbf{M}}$ and for $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$, $\boldsymbol{u}' \otimes \boldsymbol{u} \in \frac{\mathbf{M}}{\mathbf{I}} \otimes \frac{\mathbf{M}}{\mathbf{I}} \equiv \frac{\mathbf{M} \otimes \mathbf{M}}{\mathbf{I} \otimes \mathbf{I}} \equiv \mathbf{M} \otimes \mathbf{M}^* \equiv \mathrm{Lin}(\mathbf{M}, \mathbf{M})$.

Definition. Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$. Then

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) := \mathbf{g} + \frac{(\boldsymbol{u}' + \boldsymbol{u}) \otimes (\boldsymbol{u}' + \boldsymbol{u})}{1 - \boldsymbol{u}' \cdot \boldsymbol{u}} - 2\boldsymbol{u}' \otimes \boldsymbol{u}$$

is called the *Lorentz boost* from $\boldsymbol{u}$ to $\boldsymbol{u}'$.

Proposition. (*i*) $\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})$ is an orientation- and arrow-preserving $\mathbf{g}$-orthogonal linear map from $\mathbf{M}$ into $\mathbf{M}$;
(*ii*) $\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{u} = \boldsymbol{u}'$;
(*iii*) $\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})$ maps $\mathbf{E}_{\boldsymbol{u}}$ onto $\mathbf{E}_{\boldsymbol{u}'}$, more closely,
— $\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{q} = \boldsymbol{q}$ if $\boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$,
— $\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = -\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$;
(*iv*) $\boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}) = \mathbf{g}, \qquad \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})^{-1} = \boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}')$
and $\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})$ is the unique linear map for which (*i*)–(*iii*) hold.

1.3.9. Since the Lorentz boosts map the corresponding spacelike subspaces onto each other in a "handsome" manner, we might expect that executing the Lorentz boost from $\boldsymbol{u}$ to $\boldsymbol{u}'$ and then the Lorentz boost from $\boldsymbol{u}'$ to $\boldsymbol{u}''$ we should get the Lorentz boost from $\boldsymbol{u}$ to $\boldsymbol{u}''$; however, this occurs only in some special cases.

Proposition. Let $\boldsymbol{u}$, $\boldsymbol{u}'$, $\boldsymbol{u}''$ be elements of $\mathrm{V}(1)$. Then

$$\boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}') \cdot \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) = \boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u})$$

if and only if the three elements of $\mathrm{V}(1)$ are coplanar.

Proof. Suppose the equality holds. Then for all $\boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}''}$

$$\begin{aligned}
\boldsymbol{q} &= \boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}') \cdot \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{q} = \\
&= \left(\mathbf{g} + \frac{(\boldsymbol{u}'' + \boldsymbol{u}') \otimes (\boldsymbol{u}'' + \boldsymbol{u}')}{1 - \boldsymbol{u}'' \cdot \boldsymbol{u}'} - 2\boldsymbol{u}'' \otimes \boldsymbol{u}'\right) \cdot \left(\boldsymbol{q} + \frac{(\boldsymbol{u}' + \boldsymbol{u})\boldsymbol{u}' \cdot \boldsymbol{q}}{1 - \boldsymbol{u}' \cdot \boldsymbol{u}}\right) = \\
&= \boldsymbol{q} + (\boldsymbol{u}' \cdot \boldsymbol{q}) \left(\frac{\boldsymbol{u}'' + \boldsymbol{u}'}{1 - \boldsymbol{u}'' \cdot \boldsymbol{u}'} + \frac{\boldsymbol{u}' + \boldsymbol{u}}{1 - \boldsymbol{u}' \cdot \boldsymbol{u}} + \right. \\
&\qquad\qquad \left. + \frac{(\boldsymbol{u}'' + \boldsymbol{u}')(\boldsymbol{u}'' \cdot \boldsymbol{u}' + \boldsymbol{u}'' \cdot \boldsymbol{u} + \boldsymbol{u}' \cdot \boldsymbol{u} - 1)}{(1 - \boldsymbol{u}'' \cdot \boldsymbol{u}')(1 - \boldsymbol{u}' \cdot \boldsymbol{u})}\right),
\end{aligned}$$

from which we deduce that

— either $\boldsymbol{u}' \cdot \boldsymbol{q} = \mathbf{0}$ for all $\boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}''}$, implying that $\boldsymbol{u}'$ is in the two-dimensional subspace spanned by $\boldsymbol{u}$ and $\boldsymbol{u}''$, i.e. the three elements of $\mathrm{V}(1)$ are coplanar,

— or the last expression in parentheses is zero which implies again that the three elements of $\mathrm{V}(1)$ are coplanar. ■

Observe that $\boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}') \cdot \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})$ maps $\mathbf{E}_{\boldsymbol{u}}$ onto $\mathbf{E}_{\boldsymbol{u}''}$; as a consequence, if it is a Lorentz boost, it must equal $\boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u})$. Thus our result implies that, in general, the product of Lorentz boosts is not a Lorentz boost.

1.4. The arithmetic spacetime model

1.4.1. Let us take the Minkowskian vector space $(\mathbb{R}^{1+3}, \mathbb{R}, \mathrm{G})$ treated in V.4.19 and endowed with the standard orientation and arrow orientation. Considering $\mathbb{R}^{1+3}$ to be an affine space, we easily find that $(\mathbb{R}^{1+3}, \mathbb{R}, \mathrm{G})$ is a special relativistic spacetime model which we call the *arithmetic special relativistic spacetime model.*

As in the arithmetic non-relativistic spacetime model, the same object, $\mathbb{R}^{1+3}$, represents the affine space of world points and the vector space of world vectors (and even the vector space of covectors). We follow our non-relativistic convention that the world points will be denoted by Greek letters whereas world vectors (and covectors) will be denoted by Latin letters.

We find convenient to write the elements of the *affine space* $\mathbb{R}^{1+3}$ in the form (ξ^i); the elements of the vector space $\mathbb{R}^{1+3}$ in the form $(x^i) = (x^0, \boldsymbol{x})$, and the elements of $(\mathbb{R}^{1+3})^*$ in the form $(k_i) = (k_0, \boldsymbol{k})$.

Recall that the identification $(\mathbb{R}^{1+3})^* \equiv \mathbb{R}^{1+3}$ established by G gives

$$x_0 = -x^0, \qquad x_\alpha = x^\alpha \qquad (\alpha = 1, 2, 3).$$

Correspondingly, the dot product of (x^i) and (y^i) equals

$$x^i y_i,$$

where the Einstein summation rule is applied: a summation is carried out from 0 to 3 for identical subscripts and superscripts.

1.4.2. In the arithmetic spacetime model

$$\mathrm{V}(1) = \left\{ (u^i) \in \mathbb{R}^{1+3} \,|\, u^i u_i = -1, \quad u^0 > 0 \right\}.$$

The simplest element $(1, \mathbf{0})$ of $\mathrm{V}(1)$ is called the *basic velocity value.* $\boldsymbol{\pi}_{(1,\mathbf{0})}$ is the canonical projection $\mathbb{R}^{1+3} \to \{0\} \times \mathbb{R}^3$.

For an arbitrary element (u^i) of $\mathrm{V}(1)$ we can define

$$v^\alpha := \frac{u^\alpha}{u^0} \qquad (\alpha = 1, 2, 3), \qquad \boldsymbol{v} := (v^1, v^2, v^3) \in \mathbb{R}^3;$$

then with the usual norm $|\ |$ on $\mathbb{R}^3$ we have $|\boldsymbol{v}| < 1$ and

$$u^0 = \frac{1}{\sqrt{1 - |\boldsymbol{v}|^2}}$$

and

$$(u^i) = \frac{1}{\sqrt{1 - |\boldsymbol{v}|^2}} (1, \boldsymbol{v}). \qquad (*)$$

We easily find that $\boldsymbol{v}$ is exactly the relative velocity of (u^i) with respect to the basic velocity value (see 1.3.7)

1.4.3. It is then obvious that

$$\mathbf{E}_{(u^i)} = \left\{(x^i) \in \mathbb{R}^{1+3} \,|\, x^0 = \boldsymbol{x} \cdot \boldsymbol{v}\right\}.$$

Unlike the non-relativistic case, $\boldsymbol{\pi}_{(u^i)}$ for a general (u^i) in V(1) is an uneasy object because it maps onto a three-dimensional linear subspace in $\mathbb{R}^{1+3}$ which is different from $\{0\} \times \mathbb{R}^3$. Thus the values of $\boldsymbol{\pi}_{(u^i)}$ cannot be given directly by triplets of real numbers. However, as it is known, in textbooks one usually deals with triplets (and quartets) of real numbers. We can achieve this by always referring to the space of the basic observer with the aid of the corresponding Lorentz boost, i.e. instead of $\boldsymbol{\pi}_{(u^i)}$ taking $\boldsymbol{L}\big((1,\mathbf{0}),(u^i)\big) \cdot \boldsymbol{\pi}_{(u^i)}$, whose range is $\{0\} \times \mathbb{R}^3$.

1.4.4. The Lorentz boost from $u'^i)$ to (u^i) is given by the matrix

$$L^i_k := g^i_k + \frac{(u^i + u'^i)(u_k + u'_k)}{1 - u^j u'_j} - 2u^i u'_k.$$

If (u'^i) is the basic velocity value then it becomes

$$\begin{pmatrix} u^0 & u^1 & u^2 & u^3 \\ u^1 & 1 + \dfrac{(u^1)^2}{1+u^0} & \dfrac{u^1u^2}{1+u^0} & \dfrac{u^1u^3}{1+u^0} \\ u^2 & \dfrac{u^1u^2}{1+u^0} & 1 + \dfrac{(u^2)^2}{1+u^0} & \dfrac{u^2u^3}{1+u^0} \\ u^3 & \dfrac{u^1u^3}{1+u^0} & \dfrac{u^2u^3}{1+u^0} & 1 + \dfrac{(u^3)^2}{1+u^0} \end{pmatrix}.$$

Using the formula $(*)$ in 1.4.2 and the notation

$$\kappa := \frac{1}{\sqrt{1 - |v|^2}},$$

we find that

$$\boldsymbol{L}\big((1,\mathbf{0}),(u^i)\big) =$$

$$= \kappa \begin{pmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & \frac{1}{\kappa} + \frac{\kappa}{1+\kappa}(v^1)^2 & \frac{\kappa}{1+\kappa}v^1v^2 & \frac{\kappa}{1+\kappa}v^1v^3 \\ v^2 & \frac{\kappa}{1+\kappa}v^1v^2 & \frac{1}{\kappa} + \frac{\kappa}{1+\kappa}(v^2)^2 & \frac{\kappa}{1+\kappa}v^2v^3 \\ v^3 & \frac{\kappa}{1+\kappa}v^1v^3 & \frac{\kappa}{1+\kappa}v^2v^3 & \frac{1}{\kappa} + \frac{\kappa}{1+\kappa}(v^3)^2 \end{pmatrix}.$$

This shows what a complicated form $\boldsymbol{L}\big((1,\mathbf{0}),(u^i)\big)\cdot\boldsymbol{\pi}_{(u^i)}$ has; later (see 7.1.4) we give it in detail.

1.4.5. The previous matrix is the usual "Lorentz transformation". Most frequently one considers the special case $v^2 = v^3 = 0$, $v := v^1$; then $\kappa = \frac{1}{\sqrt{1-v^2}}$ and $\frac{\kappa^2 v^2}{1+\kappa} = \kappa - 1$, thus the previous matrix reduces to

$$\kappa \begin{pmatrix} 1 & v & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1/\kappa & 0 \\ 0 & 0 & 0 & 1/\kappa \end{pmatrix}.$$

1.5. Classification of physical quantities

1.5.1. We introduce notions similar to those in the non-relativistic spacetime model. Let $\mathbf{A}$ be a measure line. Then the elements of

$$\begin{array}{rll}
\mathbf{A} & \text{are called} & \textit{scalars of type} \quad \mathbf{A}, \\
\mathbf{A}\otimes\mathbf{M} & \text{are called} & \textit{vectors of type} \quad \mathbf{A}, \\
\dfrac{\mathbf{M}}{\mathbf{A}} & \text{are called} & \textit{vectors of cotype} \quad \mathbf{A}, \\
\mathbf{A}\otimes(\mathbf{M}\otimes\mathbf{M}) & \text{are called} & \textit{tensors of type} \quad \mathbf{A}, \\
\dfrac{\mathbf{M}\otimes\mathbf{M}}{\mathbf{A}} & \text{are called} & \textit{tensors of cotype} \quad \mathbf{A}.
\end{array}$$

Covectors of type $\mathbf{A}$, etc. are defined similarly with $\mathbf{M}^*$ instead of $\mathbf{M}$.

In particular, the elements of $\mathbf{M}\otimes\mathbf{M}$ and $\mathbf{M}^*\otimes\mathbf{M}^*$ are called *tensors* and *cotensors,* respectively; the elements of $\mathbf{M}\otimes\mathbf{M}^*$ and $\mathbf{M}^*\otimes\mathbf{M}$ are *mixed tensors.*

A very important feature of the special relativistic spacetime model is that covectors can be identified with vectors of cotype $\mathbf{I}\otimes\mathbf{I}$. As a consequence, e.g. a covector of type $\mathbf{A}$ is identified with a vector of type $\frac{\mathbf{A}}{\mathbf{I}\otimes\mathbf{I}}$.

1.5.2. According to our convention, the dot product between vectors (covectors) of different types makes sense. For instance, for $\boldsymbol{u} \in \mathrm{V}(1) \subset \frac{\mathbf{M}}{\mathbf{I}}$ and for

$$\begin{array}{rlll}
\boldsymbol{z} \in \mathbf{A}\otimes\mathbf{M} & \text{we have} & \boldsymbol{u}\cdot\boldsymbol{z} \in \mathbf{I}\otimes\mathbf{A}, & \boldsymbol{z}^2 \in (\mathbf{A}\otimes\mathbf{A})\otimes(\mathbf{I}\otimes\mathbf{I}), \\
\boldsymbol{w} \in \dfrac{\mathbf{M}}{\mathbf{A}} & \text{we have} & \boldsymbol{u}\cdot\boldsymbol{w} \in \dfrac{\mathbf{I}}{\mathbf{A}}, & \boldsymbol{w}^2 \in \dfrac{\mathbf{I}\otimes\mathbf{I}}{\mathbf{A}\otimes\mathbf{A}}.
\end{array}$$

In particular, $\boldsymbol{z}^2 \in \mathbb{R}$ for $\boldsymbol{z} \in \frac{\mathbf{M}}{\mathbf{I}}$.

Since $(\mathbf{A}\otimes\mathbf{A})\otimes(\mathbf{I}\otimes\mathbf{I}) \equiv (\mathbf{A}\otimes\mathbf{I})\otimes(\mathbf{A}\otimes\mathbf{I})$ has a natural orientation, we can speak of its positive and negative elements. Thus a vector $\boldsymbol{z}$ of type $\mathbf{A}$ is called

$$\begin{aligned}
&\textit{spacelike} \text{ if } \boldsymbol{z}^2 > \mathbf{0} \text{ or } \boldsymbol{z} = \mathbf{0},\\
&\textit{timelike} \text{ if } \boldsymbol{z}^2 < \mathbf{0},\\
&\textit{lightlike} \text{ if } \boldsymbol{z}^2 = \mathbf{0}, \boldsymbol{z} \neq \mathbf{0}.
\end{aligned}$$

It can be easily shown that $\boldsymbol{z}$ is spacelike if and only if $\boldsymbol{z} \in \mathbf{A}\otimes \mathrm{S}_0$, etc. Moreover, a measure line $\mathbf{A}$ is oriented, hence $\mathbf{A}^+$ makes sense. Consequently, we define that a timelike (lightlike) vector $\boldsymbol{z}$ of type $\mathbf{A}$ is *future-directed* if $\boldsymbol{z} \in \mathbf{A}^+ \otimes T^{\rightarrow}$ $(\boldsymbol{z} \in \mathbf{A}^+ \otimes \mathrm{L}^{\rightarrow})$.

1.6. Comparison of spacetime models

1.6.1. Definition. The special relativistic spacetime model $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ is *isomorphic* to the special relativistic spacetime model $(\mathrm{M}', \mathbf{I}', \mathbf{g}')$if there are

(i) an orientation- and arrow-preserving affine bijection $F : \mathrm{M} \to \mathrm{M}'$,

(ii) an orientation-preserving linear bijection $\boldsymbol{Z} : \mathbf{I} \to \mathbf{I}'$

such that

$$\mathbf{g}' \circ (\boldsymbol{F}\times\boldsymbol{F}) = (\boldsymbol{Z}\otimes\boldsymbol{Z})\circ\mathbf{g},$$

where $\boldsymbol{F}$ is the linear map under F. The pair $(F, \boldsymbol{Z})$ is an *isomorphism* between the two spacetime models.

If the two models coincide, an isomorphism is called an *automorphism.* An automorphism of the form $(F, \mathrm{id}_{\mathbf{I}})$ is *strict.* ■

Three diagrams illustrate the isomorphism:

$$\begin{array}{ccc}
\mathrm{M} & \qquad \mathbf{I} & \qquad \begin{array}{ccc} \mathbf{M}\times\mathbf{M} & \xrightarrow{\mathbf{g}} & \mathbf{I}\otimes\mathbf{I} \end{array}\\
F\downarrow & \qquad \downarrow\boldsymbol{Z} & \qquad \begin{array}{ccc} \boldsymbol{F}\times\boldsymbol{F}\downarrow & & \downarrow\boldsymbol{Z}\otimes\boldsymbol{Z} \end{array}\\
\mathrm{M}' & \qquad \mathbf{I}' & \qquad \begin{array}{ccc} \mathbf{M}'\times\mathbf{M}' & \xrightarrow[\mathbf{g}']{} & \mathbf{I}'\otimes\mathbf{I}'. \end{array}
\end{array}$$

The definition is quite natural and simple, needs no comment.

1.6.2. Proposition. The special relativistic spacetime model $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ is isomorphic to the arithmetic spacetime model.

Proof. Take

(i) a positive element $\boldsymbol{s}$ of $\mathbf{I}$,

(ii) a positively oriented $\mathbf{g}$-orthogonal basis $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, normed to $\boldsymbol{s}$, of $\mathbf{M}$, for which $\boldsymbol{e}_0$ is future-directed,

(iii) an element o of M.

Then

$$F : \mathrm{M} \to \mathbb{R}^4, \qquad x \mapsto \left(\frac{\boldsymbol{e}_k \cdot (x - o)}{\boldsymbol{e}_k^2} \,\middle|\, k = 0, 1, 2, 3 \right),$$
$$\boldsymbol{Z} : \mathbf{I} \to \mathbb{R}, \qquad \boldsymbol{t} \mapsto \frac{\boldsymbol{t}}{\boldsymbol{s}}$$

is an isomorphism. ■

This isomorphism has the inverse

$$\mathbb{R}^4 \to \mathrm{M}, \qquad \xi \mapsto o + \sum_{k=0}^{3} \xi^k \boldsymbol{e}_k,$$
$$\mathbb{R} \to \mathbf{I}, \qquad \alpha \mapsto \alpha \boldsymbol{s}.$$

1.6.3. An important consequence of the previous result is that *any two special relativistic spacetime models are isomorphic,* i.e. are of the same kind. The special relativistic spacetime model as a mathematical structure is unique. This means that there is a unique "special relativistic physics".

Note: the special relativistic spacetime models are of the same kind but, in general, are not identical. They are isomorphic, but, in general, there is no "canonical" isomorphism between them, we cannot identify them by a distinguished isomorphism. The situation is the same as that we encountered for non-relativistic spacetime models.

Since all special relativistic spacetime models are isomorphic, we can use an arbitrary one for investigation and application. However, an actual model can have additional structures. For instance, in the arithmetic spacetime model, spacetime is a vector space, V(1) has a distinguished element. This model tempts us to multiply world points by real numbers (though this has no physical meaning and that is why it is not meaningful in the abstract spacetime model), to speak about time and space, consider spacetime as the Cartesian product of time and space (whereas neither time nor space exists), etc.

To avoid such confusions, we should keep away from similar specially constructed models for investigation and general application of the special relativistic spacetime model. However, for solving special problems, for executing some particular calculations, we can choose a convenient actual model, like in the non-relativistic case.

1.6.4. Present day physics uses tacitly the arithmetic spacetime model. One represents time points by real numbers, space points by triplets of real numbers. To obtain such representations, one chooses a unit for time periods, an initial time point, a distance unit, an initial space point and an orthogonal spatial basis whose elements have unit length.

However, all the previous notions in usual circumstances have merely a heuristic sense. The isomorphism established in 1.6.2 will give these notions a mathematically precise meaning. We shall see later that $\boldsymbol{s}$ is the time unit (and the distance unit), $\boldsymbol{e}_0$ characterizes an observer which produces its own time and space, the spacelike vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ correspond to the spatial basis, o includes the initial time point and space point in some way.

1.7. The $\boldsymbol{u}$-split spacetime model

1.7.1. The arithmetic spacetime model is useful for solving particular problems, for executing practical calculations. Moreover, at present, one usually expounds theories, too, in the frame of the arithmetic spacetime model, so we have to translate every notion in the arithmetic language. As in the non-relativistic case, it is convenient to introduce an "intermediate" spacetime model between the abstract and the arithmetic ones.

1.7.2. Let $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ be a special relativistic spacetime model and use the notations introduced in this chapter. Take a $\boldsymbol{u} \in \mathrm{V}(1)$ and define the Lorentz form

$$\mathbf{g}_{\boldsymbol{u}} : (\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}) \times (\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}) \to \mathbf{I} \otimes \mathbf{I}, \qquad \big((\boldsymbol{t}', \boldsymbol{q}'), (\boldsymbol{t}, \boldsymbol{q})\big) \mapsto -\boldsymbol{t}'\boldsymbol{t} + \boldsymbol{q}' \cdot \boldsymbol{q}.$$

Put

$$\begin{aligned} \mathrm{S} &:= \big\{(\boldsymbol{t}, \boldsymbol{q}) \,\big|\, |\boldsymbol{q}| > |\boldsymbol{t}|\big\}, \\ \mathrm{T} &:= \big\{(\boldsymbol{t}, \boldsymbol{q}) \,\big|\, |\boldsymbol{q}| < |\boldsymbol{t}|\big\}, \\ \mathrm{L} &:= \big\{(\boldsymbol{t}, \boldsymbol{q}) \,\big|\, |\boldsymbol{q}| = |\boldsymbol{t}| \neq \mathbf{0}\big\}. \end{aligned}$$

Endow $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$ with the product orientation and $\mathbf{g}_{\boldsymbol{u}}$ with the arrow orientation determined by

$$\mathrm{T}^{\rightarrow} := \big\{(\boldsymbol{t}, \boldsymbol{q}) \in T \,\big|\, \boldsymbol{t} > \mathbf{0}\big\}.$$

Then $(\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}, \mathbf{I}, \mathbf{g}_{\boldsymbol{u}})$ is a special relativistic spacetime model, called the *$\boldsymbol{u}$-split special relativistic spacetime model.*

It is quite obvious that for all $o \in \mathrm{M}$,

$$\begin{aligned} &\mathrm{M} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}, \qquad && x \mapsto \boldsymbol{h}_{\boldsymbol{u}} \cdot (x - o), \\ &\mathbf{I} \to \mathbf{I}, && \boldsymbol{t} \mapsto \boldsymbol{t} \end{aligned}$$

is an isomorphism between the two special relativistic spacetime models.

1.7.3. In the $\boldsymbol{u}$-split model

$$\mathrm{V}(1) = \left\{ (\alpha, \boldsymbol{h}) \in \mathbb{R} \times \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}} \;\middle|\; -\alpha^2 + |\boldsymbol{h}|^2 = -1,\ \alpha > 0 \right\} =$$
$$= \left\{ \frac{1}{\sqrt{1 - |\boldsymbol{v}|^2}}(1, \boldsymbol{v}) \;\middle|\; \boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}},\ |\boldsymbol{v}| < 1 \right\}.$$

There is a simplest element (the *basic velocity value*) in it: $(1, \mathbf{0})$.

1.7.4. The split non-relativistic spacetime model is simple because the actual form of the fundamental notions — $\boldsymbol{\tau}$, $\mathbf{i}$, $\boldsymbol{\pi_u}$ and $\boldsymbol{h_u}$ — is simple for all $\boldsymbol{u} \in \mathrm{V}(1)$. This follows from the fact that there is a single three-dimensional subspace of spacelike vectors which appears in the split model as a Cartesian factor.

On the other hand, taken a $\boldsymbol{u}_{\mathrm{o}} \in \mathrm{V}(1)$, the $\boldsymbol{u}_{\mathrm{o}}$-split relativistic spacetime model is not so simple because the actual form of the fundamental notions — $\mathbf{E}_{\boldsymbol{u}}$, $\boldsymbol{\pi_u}$ and $\boldsymbol{h_u}$ for $\boldsymbol{u} \neq \boldsymbol{u}_{\mathrm{o}}$ — is rather complicated. This follows from the fact that there is not a distinguished three-dimensional subspace of spacelike vectors; the spacelike subspace corresponding to $\boldsymbol{u}$ differs from the one corresponding to $\boldsymbol{u}_{\mathrm{o}}$ and only the subspace $\mathbf{E}_{\boldsymbol{u}_{\mathrm{o}}}$ appears as a Cartesian factor in the $\boldsymbol{u}_{\mathrm{o}}$-split model.

We can exploit the Cartesian product structure of the $\boldsymbol{u}_{\mathrm{o}}$-split model by always referring to $\mathbf{E}_{\boldsymbol{u}_{\mathrm{o}}}$ with the aid of the Lorentz boost $\boldsymbol{L}(\boldsymbol{u}_{\mathrm{o}}, \boldsymbol{u})$ (cf. the arithmetic spacetime model, 1.4.2).

1.8. About the two types of spacetime models

Let us summarize the essential features of the non-relativistic spacetime model and the special relativistic spacetime model.

The affine structure of spacetime is the same in both models.

In the non-relativistic model there is a τ which gives absolute simultaneity implying a single three-dimensional subspace of spacelike vectors, and then there is a Euclidean structure $\mathbf{b}$ on the subspace of spacelike vectors.

In the special relativistic model there is a Lorentz structure $\mathbf{g}$ which gives the absolute propagation of light and induces the Euclidean structure on the three-dimensional spacelike subspaces.

1.9. Exercises

1. To be later (future-like) is a transitive relation on M : if y is future-like with respect to x and z is future-like with respect to y then z is future-like with respect to x.

2. V(1) is a three-dimensional submanifold of M; its tangent space at $\boldsymbol{u}$ is $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ (see Exc.VI.4.14.3). For every $\boldsymbol{u} \in \mathrm{V}(1)$,

$$\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}} \to \mathrm{V}(1), \qquad \boldsymbol{h} \mapsto \boldsymbol{u}\sqrt{1+|\boldsymbol{h}|^2} + \boldsymbol{h},$$

and

$$\left\{ \boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}} \,\middle|\, |\boldsymbol{v}| < 1 \right\} \to \mathrm{V}(1), \qquad \boldsymbol{v} \mapsto \frac{\boldsymbol{u}+\boldsymbol{v}}{\sqrt{1-|\boldsymbol{v}|^2}}$$

are global parametrizations of V(1) having the inverses

$$\boldsymbol{u}' \mapsto \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{u}' = \boldsymbol{u}' + (\boldsymbol{u} \cdot \boldsymbol{u}')\boldsymbol{u} = \frac{\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}$$

and

$$\boldsymbol{u}' \mapsto \frac{\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{u}'}{-\boldsymbol{u} \cdot \boldsymbol{u}'} = \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}},$$

respectively.

3. Prove that for all $\boldsymbol{u} \in \mathrm{V}(1)$

$$\mathbb{R} \times \left\{ \boldsymbol{n} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}} \,\middle|\, |\boldsymbol{n}| = 1 \right\} \to \mathrm{V}(1), \qquad (\alpha, \boldsymbol{n}) \mapsto \frac{\boldsymbol{u} + \boldsymbol{n}\mathrm{th}\alpha}{\sqrt{1-\mathrm{th}^2\alpha}} = \boldsymbol{u}\mathrm{ch}\alpha + \boldsymbol{n}\mathrm{sh}\alpha$$

is a smooth map which is a bijection between $\mathbb{R}_0^+ \times \left\{ \boldsymbol{n} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}} \,\middle|\, |\boldsymbol{n}| = 1 \right\}$ and V(1), having the inverse

$$\boldsymbol{u}' \mapsto \left(\mathrm{arch}(-\boldsymbol{u} \cdot \boldsymbol{u}'), \frac{\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{u}'}{\sqrt{(\boldsymbol{u} \cdot \boldsymbol{u}')^2 - 1}} \right).$$

4. Let $\boldsymbol{u} \in \mathrm{V}(1)$, $\boldsymbol{n} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $|\boldsymbol{n}| = 1$. Take $\alpha, \beta \in \mathbb{R}$ and put

$$\begin{aligned}
\boldsymbol{u}' &:= \boldsymbol{u}\mathrm{ch}\alpha + \boldsymbol{n}\mathrm{sh}\alpha, \quad \boldsymbol{n}' := \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{u}\mathrm{sh}\alpha + \boldsymbol{n}\mathrm{ch}\alpha \\
\boldsymbol{u}'' &:= \boldsymbol{u}'\mathrm{ch}\beta + \boldsymbol{n}'\mathrm{sh}\beta = \boldsymbol{u}\mathrm{ch}(\alpha+\beta) + \boldsymbol{n}\mathrm{sh}(\alpha+\beta), \\
\boldsymbol{u}''' &:= \boldsymbol{u}\mathrm{ch}\beta + \boldsymbol{n}\mathrm{sh}\beta.
\end{aligned}$$

Prove that

$$\boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}') = \boldsymbol{L}(\boldsymbol{u}''', \boldsymbol{u}).$$

(Hint: $\boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}') \cdot \boldsymbol{u} = \boldsymbol{u}'''$ and $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'''} = \mathbf{E}_{\boldsymbol{u}'} \cap \mathbf{E}_{\boldsymbol{u}''}$.)

5. Use the notations of the preceding exercise and prove that

$$\boldsymbol{L}(\boldsymbol{u}''', \boldsymbol{u}) \cdot \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) = \boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u})$$

i.e.

$$\boldsymbol{L}(\boldsymbol{u}\mathrm{ch}\beta + \boldsymbol{n}\mathrm{sh}\beta, \boldsymbol{u}) \cdot \boldsymbol{L}(\boldsymbol{u}\mathrm{ch}\alpha + \boldsymbol{n}\mathrm{sh}\alpha, \boldsymbol{u}) = \boldsymbol{L}\big(\boldsymbol{u}\mathrm{ch}(\alpha+\beta) + \boldsymbol{n}\mathrm{sh}(\alpha+\beta),\, \boldsymbol{u}\big).$$

6. Let $\boldsymbol{u} \in \mathrm{V}(1)$, $\boldsymbol{m}, \boldsymbol{n} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $|\boldsymbol{m}| = |\boldsymbol{n}| = 1$, $\boldsymbol{m} \cdot \boldsymbol{n} = 0$. Take an $0 \neq \alpha \in \mathbb{R}$ and put

$$\boldsymbol{u}' := \boldsymbol{u}\mathrm{ch}\alpha + \boldsymbol{n}\mathrm{sh}\alpha, \qquad \boldsymbol{u}'' := \boldsymbol{u}\mathrm{ch}\alpha + \boldsymbol{m}\mathrm{sh}\alpha.$$

Then $\boldsymbol{n}' := \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{u}\mathrm{sh}\alpha + \boldsymbol{n}\mathrm{ch}\alpha$ and $\boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{n}$. Prove that $\boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}') \cdot \boldsymbol{n}'$ is not parallel to $\boldsymbol{n}$.

7. Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$. Then $\boldsymbol{u}' \otimes \mathbf{I}$ and $\mathbf{E}_{\boldsymbol{u}}$ are complementary subspaces. The *projection onto* $\mathbf{E}_{\boldsymbol{u}}$ *along* $\boldsymbol{u}' \otimes \mathbf{I}$ is the linear map

$$\boldsymbol{P}_{\boldsymbol{u}\boldsymbol{u}'} := \mathbf{g} + \frac{\boldsymbol{u}' \otimes \boldsymbol{u}}{-\boldsymbol{u}' \cdot \boldsymbol{u}} : \mathbf{M} \to \mathbf{M}, \qquad \boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{u}' \frac{\boldsymbol{u} \cdot \boldsymbol{x}}{-\boldsymbol{u}' \cdot \boldsymbol{u}}.$$

Prove that

(i) the restriction of $\boldsymbol{P}_{\boldsymbol{u}\boldsymbol{u}'}$ onto $\mathbf{E}_{\boldsymbol{u}'}$ is a bijection between $\mathbf{E}_{\boldsymbol{u}'}$ and $\mathbf{E}_{\boldsymbol{u}}$;

(ii) the restriction of $\boldsymbol{P}_{\boldsymbol{u}\boldsymbol{u}'}$ onto $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$ is the identity;

(iii) $\boldsymbol{P}_{\boldsymbol{u}\boldsymbol{u}'} \cdot \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'} = \sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2}\, \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$.

2. World lines

2.1. History of a masspoint: world line

2.1.1. As in the non-relativistic spacetime model, the history of a masspoint will be described by a curve in the special relativistic spacetime model as well. However, it is not obvious here, what kind of curves can be allowed.

Our heuristic considerations regarding the affine structure of spacetime imply that the history of a free masspoint has to be described by a straight line. We can discover simply that such a straight line must be directed by a timelike vector. Indeed, it cannot be lightlike because this would mean that there is a light signal resting with respect to the masspoint. Suppose that the straight line is directed by a spacelike vector, choose two different points on the line and draw the corresponding future light cones: the cones intersect each other. As a consequence, two light signals emitted successively by the masspoint would meet which contradicts our experience.

A simple generalization — in accordance with I.2.2 — yields that the existence of a masspoint must be described by a curve whose tangent vectors are timelike.

We call attention to the fact that up to now we have spoken about light signals and masspoint histories in a heuristic sense. The following definition gives these notions a precise meaning in the spacetime model.

2.1.2. Definition. 1. A straight line segment in M, directed by a lightlike vector, is called a *light signal.*

2. A *world line* is a connected twice differentiable curve in M whose tangent vectors are timelike.

Proposition. Let C be a world line. Then $y-x$ is timelike for every $x, y \in \mathrm{C}$, $x \neq y$. In other words,

$$\mathrm{C} \setminus \{x\} \subset x + \mathrm{T} \qquad (x \in \mathrm{C}).$$

Proof. Suppose the statement is not true: there is an $x \in \mathrm{C}$ such that $\mathrm{C} \setminus \{x\}$ is not contained in $x + \mathrm{T}$. Let $p : \mathbb{R} \rightarrowtail \mathrm{M}$ be a parametrization of C, $p(0) = x$. Then there is a $0 \neq \alpha \in \mathrm{Dom}\, p$ such that $p(\alpha) - p(0)$ is not timelike. For the sake of definiteness we can assume $\alpha > 0$. Then

$$a := \inf \{\alpha \in \mathrm{Dom}\, p \mid \alpha > 0,\ p(\alpha) - p(0) \notin \mathrm{T}\} > 0.$$

Indeed, if this infimum were zero then there would be a sequence $\alpha_n > 0$ ($n \in \mathbb{N}$) such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\frac{p(\alpha_n)-p(0)}{\alpha_n} \notin \mathrm{T}$ for all n implying $\dot{p}(0) = \lim\limits_{n\to\infty} \frac{p(\alpha_n)-p(0)}{\alpha_n} \notin \mathrm{T}$ because the set of timelike vectors is open (the complement of T is closed). Because of the same reason, $p(a)-p(0) = \lim\limits_{n\to\infty} \big(p(\alpha_n)-p(0)\big) \notin \mathrm{T}$.

Thus $p(\alpha) - p(0)$ is timelike for $0 < \alpha < a$ and $p(a) - p(0)$ is not timelike. Since p is continuous, $p(a) - p(0)$ must be in the closure of T, i.e. it is lightlike: $\big(p(a) - p(0)\big)^2 = 0$.

Lagrange's mean value theorem, applied to the function $[0,a] \to \mathbf{I}\otimes\mathbf{I}$, $\alpha \mapsto \big(p(\alpha)-p(0)\big)^2$ assures a $c \in]0,a[$ such that $2\big(p(c)-p(0)\big)\cdot \dot{p}(c) = 0$. Since $\dot{p}(c)$ is timelike, this means that $p(c)-p(0)$ is spacelike, a contradiction.

2.1.3. The previous result and the arrow orientation (which gives rise to the relation to be earlier, see 1.2.3) allow us to define an order — an orientation — on a world line as follows.

Proposition. Let $p : \mathbb{R} \rightarrowtail \mathrm{M}$ be a parametrization of the world line C. Then one of the following two possibilities occurs:

(i) $\alpha < \beta$ if and only if $p(\alpha)$ is earlier than $p(\beta)$,

(ii) $\alpha < \beta$ if and only if $p(\alpha)$ is later than $p(\beta)$

for all $\alpha, \beta \in \mathrm{Dom}\, p$.

Proof. $\dot{p}$ is a continuous function having values in T and defined on an interval, thus its range is connected which means that the range of $\dot{p}$ is contained either in $\mathrm{T}^{\rightarrow}$ or in $\mathrm{T}^{\leftarrow}$.

(i) Suppose $\mathrm{Ran}\,\dot{p} \subset \mathrm{T}^{\rightarrow}$ and select an arbitrary α from the domain of p. Then $\{\beta \in \mathrm{Dom}\, p \mid \alpha < \beta\} \to \mathrm{T}$, $\beta \mapsto \frac{p(\beta)-p(\alpha)}{\beta-\alpha}$ is a continuous function defined on an interval, hence its range is contained in $\mathrm{T}^{\rightarrow}$ or in $\mathrm{T}^{\leftarrow}$. Since $\lim\limits_{\beta\to\alpha} \frac{p(\beta)-p(\alpha)}{\beta-\alpha} = \dot{p}(\alpha) \in \mathrm{T}^{\rightarrow}$ and $\mathrm{T}^{\rightarrow}$ is open, we conclude that $\frac{p(\beta)-p(\alpha)}{\beta-\alpha} \in \mathrm{T}^{\rightarrow}$, which implies that $p(\beta)-p(\alpha)$ is in $\mathrm{T}^{\rightarrow}$, i.e. $p(\alpha)$ is earlier than $p(\beta)$ for all $\alpha < \beta$.

(ii) Similar considerations yield the desired result if $\mathrm{Ran}\,\dot{p} \subset \mathrm{T}^{\leftarrow}$.

Definition. A parametrization p of a world line is called *progressive* (*regressive*) if $\alpha < \beta$ implies that $p(\alpha)$ is earlier (later) than $p(\beta)$ for all $\alpha, \beta \in \mathrm{Dom}\, p$.

A world line is considered oriented by progressive parametrizations. ■

The reader easily verifies that the orientation is correctly defined: if p and q are progressive parametrizations of a world line, then $p^{-1}\circ q : \mathbb{R} \rightarrowtail \mathbb{R}$ is strictly monotone increasing.

Note that the proposition holds and the definition can be applied also for parametrizations that are defined on an oriented one-dimensional affine space.

2.1.4. If x and y are different points of a world line then they are timelike separated.

Conversely, if x and y are timelike separated world points then there is a world line C such that $x, y \in \mathrm{C}$. Indeed, the straight line passing through x and y is such a world line. Note the important fact that there are many world lines containing x and y.

2.2. Proper time of world lines

2.2.1. Masspoint is an abstraction of a "small" material object. Imagine a piece of quartz oscillator as a masspoint; it "feels" that time passes during its history: the progress of time is measured by the number of oscillations. Since absolute time does not exist, it is evident that each history has its own proper time that passes. This means physically that the oscillations depend on the history. Take two "small" quartz crystals resting on the table. Let one of them continue to rest and seize the other, shake it for a while, then put it back on the table. Count the number of oscillations in each crystal during their separation: the two numbers can be different.

2.2.2. We already know what is later and earlier on a world line. Now we should like to measure how much later (or earlier) a point of a world line is than another, i.e. we want to measure the time passed between two points of a world line.

Our experience indicates that time passes "uniformly" for an inertial masspoint. According to the affine structure of spacetime, the history of an inertial masspoint will be described by a straight line segment.

Let x, y and z be points of a straight world line such that z is later than x and y is later than z. Then $y-x$ is parallel to $z-x$, thus there is an $\alpha \in \mathbb{R}^+$ such that $y-x = \alpha(z-x)$. The uniform flow of time suggests that α times more time passed between x and y than between x and z. But we do not know yet *how much* time passed between the world points. To measure the time passed between x and y (along the straight world line) we ought to measure somehow the "length" of the vector $y-x$. The Lorentz form $\mathbf{g}$ offers a possibility: we accept that $|y-x|$ is the time passed between x and y along the straight world line. Note that $\mathbf{g}$ is not positive definite (is not a Euclidean form), thus the pseudo-length defined by $\mathbf{g}$ has strange properties (see V.4.10) which will be important in the sequel.

Take now a "world line" consisting of two consecutive non-parallel straight line segments (according to our present definition, such a line is not a world line because it is not differentiable in one point, that is why we put the quotation mark; we use such broken world lines for our heuristic consideration and later we permit them by a precise definition, too). Let z be the breaking point, let x be earlier than z, z earlier than y. Then we measure the time passed between x and y along the broken world line by the sum of the time passed along the straight line segments: $|z-x|+|y-z|$.

The generalization to a broken world line consisting of several straight line segments is trivial.

Let now C be an arbitrary world line, $x, y \in \mathrm{C}$, x is earlier then y. We can approximate the time passed between x and y along C by the time passed along broken lines approximating C.

Take a progressive parametrization p of the world line C. Then an approximation of the time passed between x and y along C has the form

$$\sum_{k=1}^{n} |p(\alpha_{k+1}) - p(\alpha_k)|$$

which nearly equals

$$\sum_{k=1}^{n} |\dot{p}(\alpha_k)|(\alpha_{k+1} - \alpha_k).$$

We recognize an integral approximating sum. This suggests us the following definition (the reader is asked to study Section VI.7).

Definition. Let x and y be timelike separated world points or $x = y$. If C is a world line passing through x and y (i.e. $x, y \in \mathrm{C}$) then

$$\mathbf{t}_{\mathrm{C}}(x,y) := \int_x^y |d\mathrm{C}|$$

is called the *time passed between x and y along C.*

The time passed between x and y along a straight line is called the *inertial time between x and y* and is denoted by $\mathbf{t}(x,y)$. ■

Evidently,

$$\mathbf{t}(x,y) = \begin{cases} \ \ |y-x| & \text{if } x \text{ is earlier than } y \\ -|y-x| & \text{if } y \text{ is earlier than } x. \end{cases}$$

2.2.3. The time passed between two world points along different world lines can be different. The longest time passes along the inertial world line:

Proposition. Let x be a world point earlier than the world point y. If C is a world line containing x and y then

$$\mathbf{t}_{\mathrm{C}}(x,y) \le \mathbf{t}(x,y)$$

and equality holds if and only if C is a straight line segment between x and y.

Proof. Let $z \in \mathrm{C}$, z is earlier than y and later than x. Then the reversed triangle inequality (V.4.10) results in $\mathbf{t}(x,z) + \mathbf{t}(z,y) \leq \mathbf{t}(x,y)$, where equality holds if and only if z is on the straight line passing through x and y. As a consequence, the time passed between x and y along a broken line (defined to be the sum of times passed along the corresponding straight line segments) is smaller than the inertial time between x and y. The definition of $\mathbf{t}_{\mathrm{C}}(x,y)$ as an integral involves that $\mathbf{t}_{\mathrm{C}}(x,y)$ can be obtained as the infimum of times passed between x and y along broken lines.

2.2.4. Note that the Lorentz form $\mathbf{g}$ — besides the determination of the light cone and the Euclidean structure on spacelike subspaces — has got a new and important role: the determination of time passing along world lines.

We emphasize that the integral formula for the time passing along a world line is a *definition* and not a statement.

2.2.5. We call attention to the fact that in our customary illustration the same time period passed along different inertial world lines is represented, in general, by segments of different lengths.

The same length corresponds to the same time period on two inertial world lines if and only if the two illustrating straight lines have the same inclination to the two lines of $\mathrm{L}^{\to}$:

2.3. World line functions

2.3.1. Definition. Let C be a world line, $x_o \in \mathrm{C}$. Then the mapping

$$\mathrm{C} \to \mathbf{I}, \qquad x \mapsto \mathbf{t}_\mathrm{C}(x_o, x)$$

is called the *proper time* of C starting from x_o. ■

Since every tangent vector $\boldsymbol{x} \neq \mathbf{0}$ of the world line C is timelike i.e. $|\boldsymbol{x}| \neq \mathbf{0}$, according to Proposition VI.7.5, the inverse of the proper time,

$$r : \mathbf{I} \rightarrowtail \mathrm{M}$$

defined by

$$r(\mathbf{t}_\mathrm{C}(x_o, x)) = x \qquad (x \in \mathrm{C})$$

and having the property

$$\mathbf{t}_\mathrm{C}(x_o, r(\boldsymbol{t})) = \boldsymbol{t} \qquad (\boldsymbol{t} \in \mathrm{Dom}\ r)$$

is a progressive parametrization of C, called the *proper time parametrization* of C starting from x_o. We know that for all $\boldsymbol{t} \in \mathrm{Dom}\ r$

$$\dot{r}(\boldsymbol{t}) \in \frac{\mathbf{M}}{\mathbf{I}}, \qquad \dot{r}(\boldsymbol{t}) \ \text{ is future-directed timelike,}$$

moreover, Proposition VI.7.5 implies

$$|\dot{r}(\boldsymbol{t})| = 1;$$

all these mean that

$$\dot{r}(\boldsymbol{t}) \in \mathrm{V}(1) \qquad\qquad (\boldsymbol{t} \in \mathrm{Dom}\ r).$$

2.3.2. According to the previous considerations, if C is a world line then there is a parametrization $r : \mathbf{I} \rightarrowtail \mathrm{M}$ of C (i.e. r is defined on an interval, is twice differentiable, its range is C) such that $\dot{r}(\boldsymbol{t}) \in \mathrm{V}(1)$ for all $\boldsymbol{t} \in \mathrm{Dom}\, r$.

From the properties of integration on curves we derive that

$$\mathbf{t}_{\mathrm{C}}(x, y) = \int_{r^{-1}(x)}^{r^{-1}(y)} |\dot{r}(\boldsymbol{t})| d\boldsymbol{t} = r^{-1}(y) - r^{-1}(x).$$

As a consequence, if r_1 and r_2 are parametrizations with the above property then there is a $\boldsymbol{t}_o \in \mathbf{I}$ such that $\mathrm{Dom}\, r_2 = \boldsymbol{t}_o + \mathrm{Dom}\, r_1$ and $r_2(\mathbf{t}) = r_1(\boldsymbol{t} - \boldsymbol{t}_o)$ $(\boldsymbol{t} \in \mathrm{Dom}\, r_2)$.

Indeed, choosing an element x_o of C and putting $\boldsymbol{t}_o := r_2^{-1}(x_o) - r_1^{-1}(x_o)$ we get $r_1^{-1}(x) = r_2^{-1}(x) - \boldsymbol{t}_o$ which gives the desired result with the notation $\boldsymbol{t} := r_2^{-1}(x)$.

2.3.3. Our results suggest how to introduce the notion of world line functions which allows us to admit piecewise differentiability as in the non-relativistic case.

Definition. A function $r : \mathbf{I} \rightarrowtail \mathrm{M}$ is called a *world line function* if

(i) Dom r is an interval,

(ii) r is piecewise twice continuously differentiable,

(iii) $\dot{r}(\boldsymbol{t})$ is in V(1) for all $\boldsymbol{t} \in \mathrm{Dom}\, r$ where r is differentiable.

A subset C of M is a *world line* if it is the range of a world line function.

The world line function r and the world line Ran r is *global* if $\mathrm{Dom}\, r = \mathbf{I}$. ■

2.3.4. If r is a world line function then differentiating the constant mapping $\boldsymbol{t} \mapsto \dot{r}(\boldsymbol{t}) \cdot \dot{r}(\boldsymbol{t}) = -1$ defined on the differentiable pieces of Dom r we get that

$$\dot{r}(\boldsymbol{t}) \cdot \ddot{r}(\boldsymbol{t}) = 0, \qquad \text{i.e.} \qquad \ddot{r}(\boldsymbol{t}) \in \frac{\mathbf{E}_{\dot{r}(\boldsymbol{t})}}{\mathbf{I} \otimes \mathbf{I}},$$

and the same is true for right and left derivatives where r is not differentiable.

The functions $\dot{r} : \mathbf{I} \rightarrowtail \mathrm{V}(1)$ and $\ddot{r} : \mathbf{I} \rightarrowtail \frac{\mathbf{M}}{\mathbf{I} \otimes \mathbf{I}}$ can be interpreted as the (absolute) *velocity* and the (absolute) *acceleration* of the material point whose history is described by r.

That is why we call the elements of V(1) *velocity values* and the spacelike elements in $\frac{\mathbf{M}}{\mathbf{I} \otimes \mathbf{I}}$ *acceleration values.*

2.3.5. Recall that V(1) is a three-dimensional smooth submanifold of $\frac{\mathbf{M}}{\mathbf{I}}$. The elements of $\left\{ \boldsymbol{v} \in \frac{\mathbf{M}}{\mathbf{I}} \,\middle|\, 0 < \boldsymbol{v}^2 < 1 \right\}$ will be called *relative velocity values*; later we shall see the motivation of this name.

Note the following important facts.

(i) The velocity values are timelike vectors of cotype $\mathbf{I}$, in particular they are future-directed. The velocity values do not form either a vector space or an affine space. The pseudo-length of every velocity value is 1. There is no zero velocity value. Velocity values have no angles between themselves.

(ii) Relative velocity values are spacelike vectors of cotype $\mathbf{I}$. They do not form a vector space. The *magnitude* of a relative velocity value (see 1.3.3) is a real number less than 1. A relative velocity can be smaller than another; there is a *zero* relative velocity value. If $\boldsymbol{u} \in \mathrm{V}(1)$ then $\left\{ \boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}} \,\middle|\, |\boldsymbol{v}| < 1 \right\}$ is an open ball in a three-dimensional Euclidean vector space and consists of relative velocity values. The *angle* between such relative velocities makes sense.

(iii) Acceleration values are the spacelike vectors of cotype $\mathbf{I} \otimes \mathbf{I}$. They do not form a vector space. The *magnitude* of an acceleration value is meaningful, it is an element of $\frac{\mathbb{R}}{\mathbf{I}}$. An acceleration value can be smaller than another; there is a *zero* acceleration value. If $\boldsymbol{u} \in \mathrm{V}(1)$, then $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I} \otimes \mathbf{I}}$ is a three-dimensional Euclidean vector space consisting of acceleration values. The *angle* between such acceleration values makes sense.

"Quickness" makes no absolute sense; it is not meaningful that a material object exists more quickly than another. A velocity value characterizes somehow an actual *tendency* of the history of a material point. Material objects can move slowly or quickly *relative to each other.*

2.4. Classification of world lines

2.4.1. We would like to classify the world lines as we did it in the non-relativistic case. The notion of inertial world line is straightforward. However, the uniformly accelerated world line and the twist-free world line give us some trouble.

If we copied the non-relativistic definition, i.e. we required that the acceleration of a world line function r be constant, $\ddot{r} = \boldsymbol{a}$, where $\boldsymbol{a}$ is a spacelike element of $\frac{\mathbf{M}}{\mathbf{I} \otimes \mathbf{I}}$, then there is a $\boldsymbol{c} \in \mathrm{V}(1)$ such that $\dot{r}(\boldsymbol{t}) = \boldsymbol{c} + \boldsymbol{a}\boldsymbol{t}$ ($\boldsymbol{t} \in \mathrm{Dom}\ r$). Since $\dot{r}$ and $\ddot{r}$ are $\mathbf{g}$-orthogonal, $\boldsymbol{c} + \boldsymbol{a}\boldsymbol{t}$ and $\boldsymbol{a}$ are $\mathbf{g}$-orthogonal: $\boldsymbol{c} \cdot \boldsymbol{a} + |\boldsymbol{a}|^2 \boldsymbol{t} = \mathbf{0}$ for all $\boldsymbol{t} \in \mathrm{Dom}\ r$ which implies $\boldsymbol{a} = \mathbf{0}$. There would be no uniformly accelerated world lines except the inertial ones.

The problem lies in the fact that the actual acceleration values of a world line belong to a subspace $\mathbf{g}$-orthogonal to the corresponding velocity values; if the velocity value changes then the corresponding subspace changes as well: changing velocity involves changing acceleration.

Nevertheless, we have not to give up the notion of uniform acceleration. We established a natural mapping between two subspaces $\mathbf{g}$-orthogonal to two velocity values: the corresponding Lorentz boost (see 1.3.8). Then we may require that the world line function r is uniformly accelerated if $\ddot{r}(\boldsymbol{s})$ is mapped into $\ddot{r}(\boldsymbol{t})$ by the Lorentz boost from $\dot{r}(\boldsymbol{s})$ to $\dot{r}(\boldsymbol{t})$.

A similar requirement for $\frac{\ddot{r}}{|\ddot{r}|}$ leads us to twist-free world line functions.

2.4.2. Definition. A twice continuously differentiable world line function r and the corresponding world line is called

(i) *inertial* if $\ddot{r} = 0$,

(ii) *uniformly accelerated* if $\boldsymbol{L}\big(\dot{r}(\boldsymbol{t}), \dot{r}(\boldsymbol{s})\big) \cdot \ddot{r}(\boldsymbol{s}) = \ddot{r}(\boldsymbol{t})$ for all $\boldsymbol{t}, \boldsymbol{s} \in \mathrm{Dom}\ r$,

(iii) *twist-free* if $|\ddot{r}(\boldsymbol{t})| \boldsymbol{L}\big(\dot{r}(\boldsymbol{s}), \dot{r}(\boldsymbol{t})\big) \cdot \ddot{r}(\boldsymbol{s}) = |\ddot{r}(\boldsymbol{s})| \ddot{r}(\boldsymbol{t})$ for all $\boldsymbol{t}, \boldsymbol{s} \in \mathrm{Dom}\ r$. ■

It is quite evident that a twice continuously differentiable world line function r is inertial if and only if there are an $x_o \in \mathrm{M}$ and a $\boldsymbol{u}_o \in \mathrm{V}(1)$ such that

$$r(\boldsymbol{t}) = x_o + \boldsymbol{u}_o \boldsymbol{t} \qquad\qquad (\boldsymbol{t} \in \mathrm{Dom}\ r).$$

2.4.3. Let r be a twice continuously differentiable world line function and put

$$\boldsymbol{u} := \dot{r} : \mathbf{I} \rightarrowtail \mathrm{V}(1).$$

If r is uniformly accelerated, then, by definition,

$$\dot{\boldsymbol{u}}(\boldsymbol{s}) - \frac{\big(\boldsymbol{u}(\boldsymbol{t}) + \boldsymbol{u}(\boldsymbol{s})\big)\big(\boldsymbol{u}(\boldsymbol{t}) \cdot \dot{\boldsymbol{u}}(\boldsymbol{s})\big)}{1 - \boldsymbol{u}(\boldsymbol{t}) \cdot \boldsymbol{u}(\boldsymbol{s})} = \dot{\boldsymbol{u}}(\boldsymbol{t}) \qquad (\boldsymbol{t}, \boldsymbol{s} \in \mathrm{Dom}\ r). \qquad (*)$$

Fix an $\boldsymbol{s} \in \mathrm{Dom}\ r$, put $\boldsymbol{u}_o := \boldsymbol{u}(\boldsymbol{s}) \in \mathrm{V}(1)$, $\boldsymbol{a}_o := \dot{\boldsymbol{u}}(\boldsymbol{s}) \in \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I} \otimes \mathbf{I}}$ to have the following first-order differential equation for $\boldsymbol{u}$:

$$\dot{\boldsymbol{u}} = \boldsymbol{a}_o + \frac{(\boldsymbol{u} + \boldsymbol{u}_o)(\boldsymbol{u} \cdot \boldsymbol{a}_o)}{1 - \boldsymbol{u} \cdot \boldsymbol{u}_o}.$$

Unfortunately, it is rather complicated.

Another differential equation can be derived, too, by using $\boldsymbol{u}(\boldsymbol{s}) \cdot \dot{\boldsymbol{u}}(\boldsymbol{s}) = 0$ and observing that $|\ddot{r}| = |\dot{\boldsymbol{u}}| =: \alpha$ is constant (the Lorentz boosts are $\mathbf{g}$-orthogonal maps). We obtain the equality

$$\dot{\boldsymbol{u}}(\boldsymbol{s}) - \dot{\boldsymbol{u}}(\boldsymbol{t}) = \frac{\big(\boldsymbol{u}(\boldsymbol{t}) + \boldsymbol{u}(\boldsymbol{s})\big)\big(\boldsymbol{u}(\boldsymbol{t}) - \boldsymbol{u}(\boldsymbol{s})\big) \cdot \dot{\boldsymbol{u}}(\boldsymbol{s})}{1 - \boldsymbol{u}(\boldsymbol{t}) \cdot \boldsymbol{u}(\boldsymbol{s})}$$

from $(*)$; dividing it by $\boldsymbol{s} - \boldsymbol{t}$ and letting $\boldsymbol{s}$ tend to $\boldsymbol{t}$ we get the extremely simple second-order differential equation

$$\ddot{\boldsymbol{u}} = \alpha^2 \boldsymbol{u}$$

whose general solution has the form

$$\boldsymbol{u}(\boldsymbol{t}) = \boldsymbol{u}_o \mathrm{ch}\alpha\boldsymbol{t} + \frac{\boldsymbol{a}_o}{\alpha}\mathrm{sh}\alpha\boldsymbol{t} \qquad (\boldsymbol{t} \in \mathbf{I}), \qquad (**)$$

where $\boldsymbol{u}_o \in \mathrm{V}(1)$, $\boldsymbol{a}_o \in \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I}\otimes\mathbf{I}}$, $|\boldsymbol{a}_o| = \alpha$.

Equality $(**)$ has been derived from $(*)$. It is not hard to see that $\boldsymbol{t} \mapsto \boldsymbol{u}(\boldsymbol{t})$ defined by $(**)$ satisfies $(*)$, i.e. $(*)$ and $(**)$ are equivalent.

Finally, a simple integration results in the following.

Proposition. The twice continuously differentiable world line function r is uniformly accelerated if and only if there are an $x_o \in \mathrm{M}$, a $\boldsymbol{u}_o \in \mathrm{V}(1)$ and an $\boldsymbol{a}_o \in \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I}\otimes\mathbf{I}}$ such that

$$r(\boldsymbol{t}) = x_o + \boldsymbol{u}_o \frac{\mathrm{sh}|\boldsymbol{a}_o|\boldsymbol{t}}{|\boldsymbol{a}_o|} + \boldsymbol{a}_o \frac{\mathrm{ch}|\boldsymbol{a}_o|\boldsymbol{t} - 1}{|\boldsymbol{a}_o|^2} \qquad (\boldsymbol{t} \in \mathrm{Dom}\, r).$$

2.4.4. If the twice differentiable world line function r is twist-free, then there are $\boldsymbol{u}_o \in \mathrm{V}(1)$, $\boldsymbol{n}_o \in \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I}}$, $|\boldsymbol{n}_o| = 1$ such that for $\boldsymbol{u} := \dot{r}$ the following differential equation holds:

$$\dot{\boldsymbol{u}} = |\dot{\boldsymbol{u}}| \left(\boldsymbol{n}_o + \frac{(\boldsymbol{u} + \boldsymbol{u}_o)(u \cdot \boldsymbol{n}_o)}{1 - \boldsymbol{u} \cdot \boldsymbol{u}_o} \right).$$

The method applied to uniformly accelerated world line functions to derive another differential equation works here as well. The reader is asked to perform the calculations to have

$$\boldsymbol{u}|\dot{\boldsymbol{u}}|^4 = \ddot{\boldsymbol{u}}|\dot{\boldsymbol{u}}|^2 - \dot{\boldsymbol{u}}(\dot{\boldsymbol{u}} \cdot \ddot{\boldsymbol{u}})$$

or

$$\boldsymbol{u}|\dot{\boldsymbol{u}}|^2 = \left(\mathbf{g} - \frac{\dot{\boldsymbol{u}} \otimes \dot{\boldsymbol{u}}}{|\dot{\boldsymbol{u}}|^2} \right) \cdot \ddot{\boldsymbol{u}}$$

provided that $\dot{\boldsymbol{u}}$ is nowhere zero ($\mathbf{g}$ is the identity map of $\mathbf{M}$).

2.5. World horizons

2.5.1. The light signals starting from a world point x are in $x + \mathrm{L}^{\to}$, mass-points existing in x continue their existence in $x + \mathrm{T}^{\to}$: every phenomenon occurring in x can influence only the occurrences in $x + (\mathrm{T}^{\rightarrow} \cup \mathrm{L}^{\rightarrow})$, the future-like part of spacetime with respect to x.

Conversely, only the occurrences in $x+(\mathrm{T}^{\leftarrow} \cup \mathrm{L}^{\leftarrow})$ can influence an occurrence in x.

Consider a world line C. If $x+(\mathrm{T}^{\rightarrow} \cup \mathrm{L}^{\rightarrow})$ does not meet C, then an occurrence in x cannot influence the masspoint whose history is described by C; in other words, the masspoint cannot have information about the occurrence in x. That is why we call

$$\big\{x \in \mathrm{M} \mid \mathrm{C} \cap \big(x + (\mathrm{T}^{\rightarrow} \cup \mathrm{L}^{\rightarrow})\big) = \emptyset\big\} = \{x \in \mathrm{M} \mid (\mathrm{C} - x) \cap (\mathrm{T}^{\rightarrow} \cup \mathrm{L}^{\rightarrow}) = \emptyset\}$$

the *indifferent region* of spacetime with respect to C.

It can be shown that it is a closed set (Exercise 2.7.3) whose boundary is called the *world horizon* of the world line C.

Obviously the indifferent region is void if and only if the world horizon is void.

2.5.2. Consider a world line function r. Then a world point x is not indifferent to the corresponding world line if and only if there is a $\boldsymbol{t} \in \operatorname{Dom} r$ such that $r(\boldsymbol{t}) - x \in \mathrm{T}^{\rightarrow} \cup \mathrm{L}^{\rightarrow}$ i.e.

$$\big(r(\boldsymbol{t}) - x\big)^2 \leq 0$$

and

$$\boldsymbol{u} \cdot \big(r(\boldsymbol{t}) - x\big) < 0$$

for an arbitrary $\boldsymbol{u} \in \mathrm{V}(1)$.

2.5.3. The world horizon of an inertial world line is empty.

Indeed, take the inertial world line $x_o + \boldsymbol{u}_o \otimes \mathbf{I}$, an arbitrary world point x and look for $\boldsymbol{t} \in \mathbf{I}$ satisfying

$$\begin{aligned} (x_o + \boldsymbol{u}_o\boldsymbol{t} - x)^2 &\leq \mathbf{0}, \\ \boldsymbol{u}_o \cdot (x_o + \boldsymbol{u}_o\boldsymbol{t} - x) &< 0. \end{aligned}$$

Since $(x_o - x)^2 = |\boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x_o - x)|^2 - |\boldsymbol{u}_o \cdot (x_o - x)|^2$, the inequalities can be written in the form

$$\begin{aligned} |\boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x_o - x)|^2 - |\boldsymbol{t} - \boldsymbol{u}_o \cdot (x_o - x)|^2 &\leq \mathbf{0}, \\ \boldsymbol{t} - \boldsymbol{u}_o \cdot (x_o - x) &> \mathbf{0}; \end{aligned}$$

they are satisfied for every

$$\boldsymbol{t} > \boldsymbol{u}_o \cdot (x_o - x) + |\boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x_o - x)|.$$

2.5.4. The indifferent region of spacetime with respect to the uniformly accelerated global world line described by

$$\boldsymbol{t} \mapsto x_o + \boldsymbol{u}_o \frac{\operatorname{sh}|\boldsymbol{a}_o|\boldsymbol{t}}{|\boldsymbol{a}_o|} + \boldsymbol{a}_o \frac{\operatorname{ch}|\boldsymbol{a}_o|\boldsymbol{t} - 1}{|\boldsymbol{a}_o|^2} \qquad (\boldsymbol{t} \in \mathbf{I})$$

is

$$\left\{x \in \mathrm{M} \middle| \big(|\boldsymbol{a}_o|\boldsymbol{u}_o + \boldsymbol{a}_o\big) \cdot (x_o - x) \geq 1\right\}. \qquad (*)$$

Indeed, according to 2.5.2, the world point x is not indifferent if and only if there is a $\boldsymbol{t}$ for which

$$\boldsymbol{x}^2 - \frac{\operatorname{sh}^2 \alpha\boldsymbol{t}}{\alpha^2} + \frac{(\operatorname{ch}\alpha\boldsymbol{t} - 1)^2}{\alpha^2} + 2\boldsymbol{u}_o \cdot \boldsymbol{x}\frac{\operatorname{sh}\alpha\boldsymbol{t}}{\alpha} + 2\boldsymbol{a}_o \cdot \boldsymbol{x}\frac{\operatorname{ch}\alpha\boldsymbol{t} - 1}{\alpha^2} \leq \mathbf{0},$$
$$\boldsymbol{u}_o \cdot \boldsymbol{x} - \frac{\operatorname{sh}\alpha\boldsymbol{t}}{\alpha} < \mathbf{0}$$

where

$$\boldsymbol{x} := x_o - x, \qquad \alpha := |\boldsymbol{a}_o|.$$

The second inequality holds if $\boldsymbol{t}$ is large enough.

The first inequality can be written in the form

$$\boldsymbol{x}^2 + 2\boldsymbol{u}_o \cdot \boldsymbol{x}\frac{\operatorname{sh}\alpha\boldsymbol{t} - \operatorname{ch}\alpha\boldsymbol{t} + 1}{\alpha} + 2(\boldsymbol{a}_o \cdot \boldsymbol{x} + \alpha\boldsymbol{u}_o \cdot \boldsymbol{x} - 1)\frac{\operatorname{ch}\alpha\boldsymbol{t} - 1}{\alpha^2} \leq \mathbf{0}.$$

Since $\operatorname{sh}\alpha\boldsymbol{t} - \operatorname{ch}\alpha\boldsymbol{t}$ tends to zero and $\operatorname{ch}\alpha\boldsymbol{t}$ tends to plus infinity as $\boldsymbol{t}$ tends to plus infinity, we see that if $(\boldsymbol{a}_o \cdot \boldsymbol{x} + \alpha\boldsymbol{u} \cdot \boldsymbol{x} - 1) < \mathbf{0}$ then both inequalities hold if $\boldsymbol{t}$ is large enough, i.e. an x out of the set $(*)$ is not indifferent with respect to the world line.

Take now an x in the set $(*)$ such that $\boldsymbol{u}_o \cdot \boldsymbol{x} = \mathbf{0}$. Then $\boldsymbol{x}$ is a non-zero spacelike vector, thus $\boldsymbol{x}^2 > \mathbf{0}$ and we see that the previous inequality does not hold for any $\boldsymbol{t}$ because $\operatorname{ch}\alpha\boldsymbol{t} - 1 \geq 0$: x is indifferent with respect to the world line.

To end the proof, note that $\alpha\boldsymbol{u}_o + \boldsymbol{t}_o$ is a lightlike vector, hence x is indifferent if and only if $x + \lambda(\alpha\boldsymbol{u}_o + \boldsymbol{a}_o)$ is indifferent for some $\lambda \in \mathbf{I}$. Choose λ in such a way that $\boldsymbol{u}_o \cdot \big(x_o - x - \lambda(\alpha\boldsymbol{u}_o + \boldsymbol{a}_o)\big) = 0$.

2.6. Newtonian equation

2.6.1. In the special relativistic spacetime model there is a single measure line, $\mathbf{I}$. Time periods and distances are measured by the elements of $\mathbf{I}$. There is no natural way to introduce different measure lines for time periods and distances. This reflects the experimental fact that light speed is a universal constant; thus a time unit indicates a distance unit as well: the distance covered by a light signal during the unit time period. The SI physical dimensions are extraneous to special relativity.

The light speed in the SI units is

$$c = (2{,}9979\ldots)10^8 \frac{\boldsymbol{m}}{\boldsymbol{s}}.$$

Measuring distances by light signals we arrive at the definition

$$\boldsymbol{m} := (3{,}3356\ldots)10^{-9}\boldsymbol{s}.$$

Now it is totally senseless to introduce a measure line for masses; using the Planck constant and the formulae of I.2.4.1 and the definition above we get that $\mathbf{I}^* \equiv \frac{\mathbb{R}}{\mathbf{I}}$ is the measure line of masses and

$$\boldsymbol{kg} := (8{,}5214\ldots)10^{50}\frac{1}{\boldsymbol{s}}.$$

2.6.2. Since acceleration values are elements of $\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$ and "the product of mass and acceleration equals the force", the force values are elements of $\mathbf{I}^* \otimes \frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}} \equiv \frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I}} \equiv \frac{\mathbf{M}^*}{\mathbf{I}}$; moreover, we take into account that the momentary acceleration value of a masspoint is $\mathbf{g}$-orthogonal to the corresponding velocity value.

Thus we accept that a *force field* is a differentiable mapping

$$\boldsymbol{f} : \ \mathrm{M} \times \mathrm{V}(1) \rightarrowtail \frac{\mathbf{M}^*}{\mathbf{I}}$$

such that

$$\boldsymbol{u} \cdot \boldsymbol{f}(x, \boldsymbol{u}) = \mathbf{0} \qquad \big((x, \boldsymbol{u}) \in \mathrm{Dom}\ \boldsymbol{f}\big).$$

The history of the material point with mass m under the action of the force field $\boldsymbol{f}$ is described by the Newtonian equation

$$m\ddot{x} = \boldsymbol{f}(x, \dot{x})$$

i.e. the world line function modelling the history is a solution of this differential equation.

2.6.3. Some of the most important force fields in special relativity, too, can be derived from potentials; e.g. the electromagnetic field. However, the gravitational field cannot be described by a potential; this problem will be discussed later (Chapter III).

A *potential* is a twice differentiable mapping

$$\boldsymbol{K} : \mathrm{M} \rightarrowtail \mathbf{M}^*$$

(in other words, a potential is a twice differentiable covector field).

The *field strength* corresponding to $\boldsymbol{K}$ is $\mathrm{D} \wedge \boldsymbol{K} : \mathrm{M} \rightarrowtail \mathbf{M}^* \wedge \mathbf{M}^*$ (the antisymmetric or exterior derivative of $\boldsymbol{K}$, see VI.3.6).

The force field $\boldsymbol{f}$ has a potential (is derived from a potential) if

— there is an open subset $\mathrm{O} \subset \mathrm{M}$ such that $\mathrm{Dom}\, \boldsymbol{f} = \mathrm{O} \times \mathrm{V}(1)$,

— there is a potential $\boldsymbol{K}$ defined on O such that

$$\boldsymbol{f}(x, \boldsymbol{u}) = \boldsymbol{F}(x) \cdot \boldsymbol{u} \qquad (x \in \mathrm{O}, \boldsymbol{u} \in \mathrm{V}(1))$$

where $\boldsymbol{F} := \mathrm{D} \wedge \boldsymbol{K}$.

It is worth mentioning: $\boldsymbol{F}(x)$ is antisymmetric, hence $\boldsymbol{u} \cdot \boldsymbol{F}(x) \cdot \boldsymbol{u} = \mathbf{0}$, as it must be for a force field.

2.6.4. In the non-relativistic spacetime model a force field can be independent of either of its variables, in particular, it can be a constant map. In the present case, on the contrary, a non-zero force field cannot be independent of velocity, in particular, it cannot be a constant map.

We could try to define a constant force field in such a way that the corresponding Lorentz boosts map its values into each other, i.e. $\boldsymbol{f}$ would be constant if

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{f}(x, \boldsymbol{u}) = \boldsymbol{f}(x, \boldsymbol{u}')$$

for all possible x, $\boldsymbol{u}$ and $\boldsymbol{u}'$. However, such a non-zero field cannot exist (Exercise 2.7.5): *there is no non-zero special relativistic constant force field*!

2.7. Exercises

1. Prove that the uniformly accelerated world line function given in 2.4.3 satisfies

$$r(\boldsymbol{t}) = x_o + \boldsymbol{u}_o\boldsymbol{t} + \frac{\boldsymbol{a}_o}{2}\boldsymbol{t}^2 + \mathrm{Ordo}(\boldsymbol{t}^3).$$

2. Let $\boldsymbol{u}_o \in \mathrm{V}(1)$, $\boldsymbol{a}_o \in \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I}\otimes\mathbf{I}}$ and $\beta : \mathbf{I} \rightarrowtail \mathbf{I}$ a continuously differentiable function defined on an interval. Demonstrate that the world line function r for which

$$\dot{r} = \boldsymbol{u}_o\sqrt{\beta^2 + 1} + \boldsymbol{a}_o\beta$$

holds is twist-free.

3. The indifferent region of spacetime with respect to the world line C has the complement

$$\bigcup_{z\in C}\{z + (\mathrm{T}^{\leftarrow} \cup \mathrm{L}^{\leftarrow})\}.$$

Using $\mathrm{L}^{\rightarrow} + \mathrm{T}^{\rightarrow} = \mathrm{T}^{\rightarrow}$ show that it equals

$$\bigcup_{z\in C}\{z + \mathrm{T}^{\leftarrow}\}$$

which, being a union of open sets, is open. Consequently, the indifferent part of spacetime with respect to C is closed.

4. Let r be a global world line function and put $\boldsymbol{u} := \dot{r}$. Prove that the world horizon of the corresponding world line is empty if one of the following conditions holds:

(i) there exist $\lim\limits_{\boldsymbol{t}\to\infty} \boldsymbol{u}(\boldsymbol{t})$,

(ii) $\boldsymbol{u}$ is periodic, i.e. there is a $\boldsymbol{t}_o > \mathbf{0}$ such that $\boldsymbol{u}(\boldsymbol{t}+\boldsymbol{t}_o) = \boldsymbol{u}(\boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbf{I}$.

(Hint: (i) V(1) is closed, hence the limit belongs to it. (ii) Put $\boldsymbol{z}_o := \int_0^{\boldsymbol{t}_o} \boldsymbol{u}(\boldsymbol{t})d\boldsymbol{t}$, $\boldsymbol{u}_o := \frac{\boldsymbol{z}_o}{|\boldsymbol{z}_o|}$ and consider the inertial world line $r(\boldsymbol{t}_o) + \boldsymbol{u}_o \otimes \mathbf{I}$.)

5. Let $\boldsymbol{\phi} : \mathrm{V}(1) \to \mathbf{M}$ be a function such that

$$\boldsymbol{u}\cdot\boldsymbol{\phi}(\boldsymbol{u}) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\phi}(\boldsymbol{u}') := \boldsymbol{L}(\boldsymbol{u}',\boldsymbol{u})\cdot\boldsymbol{\phi}(\boldsymbol{u}) \qquad \big(\boldsymbol{u}',\boldsymbol{u} \in \mathrm{V}(1)\big).$$

Prove that $\boldsymbol{\phi} = \mathbf{0}$. (Hint: $\boldsymbol{L}(\boldsymbol{u}'',\boldsymbol{u}')\cdot\boldsymbol{L}(\boldsymbol{u}',\boldsymbol{u})\cdot\boldsymbol{\phi}(\boldsymbol{u}) = \boldsymbol{L}(\boldsymbol{u}'',\boldsymbol{u})\cdot\boldsymbol{\phi}(\boldsymbol{u})$ must hold; applying Proposition 1.3.9 find appropriate $\boldsymbol{u}''$ and $\boldsymbol{u}'$ for a fixed $\boldsymbol{u}$ in such a way that the equality fails.)

3. Inertial observers

3.1. Observers

3.1.1. We can repeat word by word what we said in I.3.11 to motivate the following definition.

Definition. An *observer* is a smooth map $\boldsymbol{U} : \mathrm{M} \rightarrowtail \mathrm{V}(1)$ whose domain is connected.

If $\mathrm{Dom}\,\boldsymbol{U} = \mathrm{M}$, the observer is *global.*

The observer is called *inertial* if it is a constant map. ∎

$\mathrm{V}(1)$ is a subset of $\frac{\mathbf{M}}{\mathbf{I}}$; the differentiability (smoothness) of a map from M into $\mathrm{V}(1)$ means the differentiability (smoothness) of the map from M into $\frac{\mathbf{M}}{\mathbf{I}}$.

3.1.2. Let $\boldsymbol{U}$ be an observer. The integral curves of the differential equation

$$(x : \mathbf{I} \rightarrowtail \mathrm{M})? \qquad \dot{x} = \boldsymbol{U}(x)$$

are evidently world lines.

As in the non-relativistic case,

$$\boldsymbol{A}_{\boldsymbol{U}} := \mathrm{D}\boldsymbol{U} \cdot \boldsymbol{U} : \mathrm{M} \rightarrowtail \frac{\mathbf{M}}{\mathbf{I} \otimes \mathbf{I}}$$

is the *acceleration field* corresponding to $\boldsymbol{U}$.

3.1.3. Again we can repeat the arguments confirming that the space of an observer is the set of its maximal integral curves.

Definition. Let $\boldsymbol{U}$ be an observer. Then $\mathrm{E}_{\boldsymbol{U}}$, the set of maximal integral curves of $\boldsymbol{U}$, is the *space* of the observer $\boldsymbol{U}$ or the $\boldsymbol{U}$*-space.* ∎

Again a maximal integral curve of $\boldsymbol{U}$ is called a $\boldsymbol{U}$-line if considered to be a subset of M and is called a $\boldsymbol{U}$-space point if considered to be an element of $\mathrm{E}_{\boldsymbol{U}}$.

$C_{\boldsymbol{U}}(x)$ will stand for the (unique) $\boldsymbol{U}$-line passing through x; we say that $C_{\boldsymbol{U}}(x)$ is the $\boldsymbol{U}$-space point that x is *incident* with.

3.1.4. There is no absolute time in the special relativistic spacetime model. We could think that — on the analogy of the observer space — the time of an observer can be defined in a natural way.

How do we try to introduce the observer time? We ought to determine somehow which world points have the same instant from the point of view of the observer, i.e. which world points are considered to be simultaneous.

We know that every $\boldsymbol{U}$-line has its proper time: in every $\boldsymbol{U}$-space point the own time of the point passes. Evidently, we expect that simultaneity is related to the proper time of space points. However, in general, "time passes differently in different space points" and that is why simultaneity cannot be defined in a natural way.

The exact meaning of the above phrase in parentheses will be clarified later.

Inertial observers are good exceptions: the lines of an inertial observer are evidently parallel straight line segments: "time passes in the same uniform way in each space point".

3.2. The time of an inertial observer

3.2.1. Our experience that light propagates isotropically with respect to an arbitrary inertial observer (with the same speed in all directions) suggests the following method for determining simultaneity.

A clock at a space point says the time, another clock at another space point says the time, too. We should like to synchronize them: "when one of them says 12 then (at the same moment) let the other say 12 as well". We make such a synchronization by the everyday method: a radio signal (i.e. in fact a light signal) is emitted by the clock that says 12 at the studio and, hearing the signal, we set our clock. Of course hearing the signal we do not set the clock to 12 because we know that some time passed between emission and reception. Knowing the distance between the studio and our place we know the time passed owing to the constancy of light speed. How do we measure the distance between the studio and our place? With the aid of a radar, i.e. by means of light signals. Let the radar be at the studio. It emits a light signal toward us, the light signal is reflected by us, the radar receives the reflected signal and measures the time passed between emission and reception. Knowing this time passed we know the distance owing to the constancy of light speed.

We can simplify the procedure in such a way that the time signal and the radar signal be the same. Let us see this simplified version.

Take two different space points of the observer. Put a source of light in one of them and locate a mirror in the other. Emit a light signal toward the mirror and receive the reflected signal. Since light travels the same time there and back, the reflection at the mirror is simultaneous with that time point at the source which halves the interval between emission and reception.

3.2.2. Let $\boldsymbol{U}$ be a global inertial observer having the constant velocity value $\boldsymbol{u}$.

We want to determine the condition that the world point y is simultaneous with the world point x, according to $\boldsymbol{U}$; x and y symbolize the middle point between emission and reception, and the reflection at the mirror, respectively.

The world point y is to be simultaneous with x according to $\boldsymbol{U}$ if there is a $\boldsymbol{t} \in \mathbf{I}$ such that $y - (x - \boldsymbol{ut})$ and $y - (x + \boldsymbol{ut})$ are lightlike vectors:

$$\big(y - (x - \boldsymbol{ut})\big)^2 = \mathbf{0}, \qquad \big(y - (x + \boldsymbol{ut})\big)^2 = \mathbf{0},$$
$$(y - x)^2 + 2(y - x) \cdot \boldsymbol{ut} - \boldsymbol{t}^2 = \mathbf{0}, \qquad (y - x)^2 - 2(y - x) \cdot \boldsymbol{ut} - \boldsymbol{t}^2 = \mathbf{0}$$

which give

$$y - x \in \mathbf{E}_{\boldsymbol{u}}; \quad \text{in other words,} \quad y \in x + \mathbf{E}_{\boldsymbol{u}}.$$

All these have been heuristic considerations to support the following definition.

Definition. Let $\boldsymbol{U}$ be a global inertial observer having the constant velocity value $\boldsymbol{u}$.

The set of world points *simultaneous* with x, according to $\boldsymbol{U}$, is $x + \mathbf{E}_{\boldsymbol{u}}$, the hyperplane passing through x and directed by $\mathbf{E}_{\boldsymbol{u}}$.

The set of hyperplanes directed by $\mathbf{E}_{\boldsymbol{u}}$, denoted by $\mathrm{I}_{\boldsymbol{U}}$, is called the *time* of the observer or the $\boldsymbol{U}$*-time.* Its elements are the $\boldsymbol{U}$*-instants.* ∎

It is important that simultaneity with respect to $\boldsymbol{U}$ is a symmetric relation on M: if y is $\boldsymbol{U}$-simultaneous with x, then x is $\boldsymbol{U}$-simultaneous with y.

We emphasize that $\boldsymbol{U}$-time is defined only for inertial $\boldsymbol{U}$.

In the non-relativistic case there is an absolute time giving absolute simultaneity and it is convenient to identify instants with the corresponding simultaneous hyperplanes.

Here we define simultaneity (with respect to an inertial observer) by hyperplanes, and then we define an instant (with respect to the observer in question) to be a simultaneous hyperplane.

Evidently, different inertial observers determine different simultaneities.

3.2.3. Simultaneity with respect to an inertial observer is in accordance with the time passing in the observer space points: the same time passes in different space points between simultaneous occurrences.

A hyperplane $t \in \mathrm{I}_{\boldsymbol{U}}$ (a $\boldsymbol{U}$-instant) and a line $q \in \mathbf{E}_{\boldsymbol{U}}$ (a $\boldsymbol{U}$-space point) meet in a single world point which will be denoted by

$$q \star t.$$

Proposition. Let $\boldsymbol{U}$ be an inertial observer having the constant velocity value $\boldsymbol{u}$. Take two $\boldsymbol{U}$-space points q and q' and two $\boldsymbol{U}$-instants t and s. Then the time passed along q between $x := q \star t$ and $y := q \star s$ (the inertial time $\mathbf{t}(x, y)$ between x and y) equals the time passed along q' between $x' := q \star t$ and $y' := q' \star s$ (the inertial time $\mathbf{t}(x', y')$ between x' and y').

Proof. We have that $x' - x$ and $y' - y$ are $\mathbf{g}$-orthogonal to $\boldsymbol{u}$ and

$$y - x = \boldsymbol{u}\mathbf{t}(x, y), \qquad y' - x' = \boldsymbol{u}\mathbf{t}(x', y').$$

Multiplying the equality

$$\big((y' - x) =\big) \quad (y' - x') + (x' - x) = (y' - y) + (y - x)$$

by $-\boldsymbol{u}$ we obtain

$$\mathbf{t}(x', y') = \mathbf{t}(x, y). \qquad \blacksquare$$

3.2.4. The previous result offers the possibility to define the *time passed between* two $\boldsymbol{U}$-time instants t and s as the time passed between t and s in an arbitrary $\boldsymbol{U}$-space point.

More closely, take an arbitrary world point x in the hyperplane t, find the unique world point y in s such that the straight line passing through x and y is a $\boldsymbol{U}$-line, and then let $s - t := \mathbf{t}(x, y) = -\boldsymbol{u} \cdot (y - x)$ be the time passed between t and s.

Note that $-\boldsymbol{u} \cdot (y - x)$ is the same for all $y \in s$ and $x \in t$, thus avoiding the notation $\mathbf{t}(x, y)$ we can omit the requirement that x and y be on the same $\boldsymbol{U}$-line.

Proposition. $\mathrm{I}_{\boldsymbol{U}}$, the $\boldsymbol{U}$-time, endowed with the subtraction

$$s - t := \boldsymbol{\tau_u} \cdot (y - x) = -\boldsymbol{u} \cdot (y - x) \qquad (x \in t, y \in s)$$

is an affine space over $\mathbf{I}$. ■

The proof is immediate.

Thus the time of a global inertial observer is a one-dimensional oriented affine space.

3.3. The space of an inertial observer

3.3.1. Our heuristic notion about the affine structure of the space of a physical observer means that we assign a vector to two space points; the assignment supposes simultaneity.

The simultaneity introduced previously offers us indeed a natural way to define an affine structure on the space of an inertial observer.

First we prove an analogue of Proposition 3.2.3.

Proposition. Let $\boldsymbol{U}$ be a global inertial observer with the constant velocity value $\boldsymbol{u}$. Take two $\boldsymbol{U}$-space points q and q' and two $\boldsymbol{U}$-instants t and s. Then the vector between $x' := q' \star t$ and $x := q \star t$ equals the vector between $y' := q' \star s$ and $y := q \star s$.

Proof. By Proposition 3.2.3 we have $\mathbf{t}(x, y) = \mathbf{t}(x', y') =: \boldsymbol{t}$. Consequently, $y - x = y' - x' = \boldsymbol{ut}$ and then

$$(y' - y) + (y - x) = (y' - x') + (x' - x)$$

gives the desired result:

$$y' - y = x' - x.$$

3.3.2. The previous result offers the possibility to define the *vector between* the $\boldsymbol{U}$-space points q' and q to be the vector between the world points x' and x that are simultaneous and incident with q' and q, respectively.

Note that $x' - x = \boldsymbol{\pi_u} \cdot (x' - x)$, if x' and x are simultaneous with respect to $\boldsymbol{U}$, and $\boldsymbol{\pi_u} \cdot (x' - x)$ is the same for all x' in q' and x in q. As a consequence, using $\boldsymbol{\pi_u} \cdot (x' - x)$, we can omit the requirement of simultaneity.

Proposition. $\mathrm{E}_{\boldsymbol{U}}$, the space of the global inertial observer $\boldsymbol{U}$, endowed with the subtraction

$$q' - q := \boldsymbol{\pi_u} \cdot (x' - x) \qquad (x' \in q', x \in q)$$

is an affine space over $\mathbf{E}_{\boldsymbol{u}}$. ■

The proof is immediate.

Recall that $(\mathbf{E}_{\boldsymbol{u}}, \mathbf{I}, \mathbf{b}_{\boldsymbol{u}})$ is a Euclidean vector space. Thus we can say that the space of a global inertial observer is a three-dimensional oriented Euclidean affine space.

3.4. Splitting of spacetime

3.4.1. Let us take a global inertial obvserver $\boldsymbol{U}$ with the constant velocity value $\boldsymbol{u}$.

The observer assigns to every world point x the $\boldsymbol{U}$-time point $\tau_{\boldsymbol{U}}(x)$, the set of world points simultaneous with x according to $\boldsymbol{U}$: $\tau_{\boldsymbol{U}}(x) = x + \mathbf{E}_{\boldsymbol{u}}$, as well as the $\boldsymbol{U}$-space point $C_{\boldsymbol{U}}(x)$ that x is incident with: $C_{\boldsymbol{U}}(x) = x + \boldsymbol{u} \otimes \mathbf{I}$.

It is worth listing the following relations regarding the affine structures of $\mathrm{I}_{\boldsymbol{U}}$ and of $\mathrm{E}_{\boldsymbol{U}}$ as well as the mappings $\tau_{\boldsymbol{U}} : \mathrm{M} \to \mathrm{I}_{\boldsymbol{U}}$ and $C_{\boldsymbol{U}} : \mathrm{M} \to \mathbf{E}_{\boldsymbol{U}}$:

(*i*) $(y + \mathbf{E}_{\boldsymbol{u}}) - (x + \mathbf{E}_{\boldsymbol{u}}) = -\boldsymbol{u} \cdot (y - x)$ $\qquad (x, y \in \mathrm{M})$,

(*ii*) $(x + \boldsymbol{x} + \mathbf{E}_{\boldsymbol{u}}) = (x + \mathbf{E}_{\boldsymbol{u}}) - \boldsymbol{u} \cdot \boldsymbol{x}$ $\qquad (x \in \mathrm{M},\ \boldsymbol{x} \in \mathbf{M})$,

(*iii*) $x + \mathbf{E}_{\boldsymbol{u}} = y + \mathbf{E}_{\boldsymbol{u}}$ if and only if $y - x$ is $\mathbf{g}$-orthogonal to $\boldsymbol{u}$,

and

(*iv*) $(x' + \boldsymbol{u} \otimes \mathbf{I}) - (x + \boldsymbol{u} \otimes \mathbf{I}) = \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (x' - x)$ $\qquad (x', x \in \mathrm{M})$,

(*v*) $(x + \boldsymbol{x}) + \boldsymbol{u} \otimes \mathbf{I} = (x + \boldsymbol{u} \otimes \mathbf{I}) + \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x}$ $\qquad (x \in \mathrm{M},\ \boldsymbol{x} \in \mathbf{M})$,

(*vi*) $x + \boldsymbol{u} \otimes \mathbf{I} = x' + \boldsymbol{u} \otimes \mathbf{I}$ if and only if $x' - x$ is parallel to $\boldsymbol{u}$;

moreover,

(*vii*) $(y + \mathbf{E}_{\boldsymbol{u}}) \cap (x + \boldsymbol{u} \otimes \mathbf{I}) = \{x + \boldsymbol{u}(-\boldsymbol{u} \cdot (y - x))\}$ $\qquad (x, y \in \mathrm{M})$

or, in another form,

$$(y + \mathbf{E}_{\boldsymbol{u}}) \star (x + \boldsymbol{u} \otimes \mathbf{I}) = x + \boldsymbol{u}\big(-\boldsymbol{u} \cdot (y - x)\big)$$

3.4.2. It is trivial by the previous formulae (*i*) and (*iv*) that

$$\tau_{\boldsymbol{U}} :\ \mathrm{M} \to \mathrm{I}_{\boldsymbol{U}}, \qquad x \mapsto x + \mathbf{E}_{\boldsymbol{u}}$$

is an affine map over $\boldsymbol{\tau}_{\boldsymbol{u}} = -\boldsymbol{u}$ and

$$C_{\boldsymbol{U}} :\ \mathrm{M} \to \mathrm{E}_{\boldsymbol{U}}, \qquad x \mapsto x + \boldsymbol{u} \otimes \mathbf{I}$$

is an affine map over $\boldsymbol{\pi}_{\boldsymbol{u}}$.

Definition.

$$H_{\boldsymbol{U}} := (\boldsymbol{\tau}_{\boldsymbol{U}}, C_{\boldsymbol{U}}) :\ \mathrm{M} \to \mathrm{I}_{\boldsymbol{U}} \times \mathrm{E}_{\boldsymbol{U}}, \qquad x \mapsto (x + \mathbf{E}_{\boldsymbol{u}},\ x + \boldsymbol{u} \otimes \mathbf{I})$$

is the *splitting of spacetime according to* the global inertial observer $\boldsymbol{U}$.

Proposition. The splitting $H_{\boldsymbol{U}}$ is an orientation-preserving affine bijection over the linear map $\boldsymbol{h}_{\boldsymbol{u}} = (\boldsymbol{\tau}_{\boldsymbol{u}}, \boldsymbol{\pi}_{\boldsymbol{u}})$ (cf. 1.3.5) and

$$H_{\boldsymbol{U}}^{-1}(t, q) = q \star t \qquad (t \in \mathrm{I}_{\boldsymbol{U}},\ q \in \mathrm{E}_{\boldsymbol{U}}).$$

3.4.3. We can simplify a number of formulae and calculations by choosing a $\boldsymbol{U}$-time point t_o and a $\boldsymbol{U}$-space point q_o and vectorizing $\boldsymbol{U}$-time and $\boldsymbol{U}$-space:

$$\mathrm{I}_{\boldsymbol{U}} \to \mathbf{I}, \qquad t \mapsto t - t_o,$$

$$\mathrm{E}_{\boldsymbol{U}} \to \mathbf{E}_{\boldsymbol{u}}, \qquad q \mapsto q - q_o.$$

Choosing t_o and q_o is equivalent to choosing a "spacetime reference origin" $o \in \mathrm{M} : \{o\} \in q_o \cap t_o,\ \tau_{\boldsymbol{U}}(o) = t_o,\ C_{\boldsymbol{U}}(o) = q_o$.

The pair $(\boldsymbol{U}, o)$ is called a global inertial *observer with reference origin.* We can establish the *vectorized splitting* of spacetime due to $(\boldsymbol{U}, o)$:

$$H_{\boldsymbol{U},o} :\ \mathrm{M} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}, \quad x \mapsto \big(\tau_{\boldsymbol{U}}(x) - \tau_{\boldsymbol{U}}(o),\ C_{\boldsymbol{U}}(x) - C_{\boldsymbol{U}}(o)\big) =$$
$$= \big(-\boldsymbol{u} \cdot (x - o),\ \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (x - o)\big).$$

Thus, if O_o denotes the vectorization of M with origin o then

$$H_{\boldsymbol{U},o} = \boldsymbol{h}_{\boldsymbol{u}} \circ O_o.$$

3.5. Exercise

Define the *basic observer* in the arithmetic spacetime model.

Choose the zero in $\mathbb{R}^{1+3}$ to be a reference origin for the basic observer. Then vectorized splitting of spacetime is the identity map of $\mathbb{R}^{1+3}$.

4. Kinematics

4.1. Motions relative to an inertial observer

4.1.1. Let $\boldsymbol{U}$ be a global inertial observer with constant velocity value $\boldsymbol{u}$.

Take a world line function r.

Then the function $\tau_{\boldsymbol{U}} \circ r : \mathbf{I} \rightarrowtail \mathrm{I}_{\boldsymbol{U}}$ assigns $\boldsymbol{U}$-time points to proper time points of r. This function is piecewise twice differentiable and its derivative

$$(\tau_{\boldsymbol{U}} \circ r)^{\cdot} = \boldsymbol{\tau}_{\boldsymbol{u}} \cdot \dot{r} = -\boldsymbol{u} \cdot \dot{r}$$

is everywhere positive (see 1.3.1). Consequently, $\tau_{\boldsymbol{U}} \circ r$ is strictly monotone increasing, has a monotone increasing inverse

$$z_{\boldsymbol{U}} := (\tau_{\boldsymbol{U}} \circ r)^{-1} : \mathrm{I}_{\boldsymbol{U}} \rightarrowtail \mathbf{I}$$

which gives the proper time points of r corresponding to $\boldsymbol{U}$-time points; moreover, its derivative comes from the inverse of the derivative of $\tau_{\boldsymbol{U}} \circ r$:

$$\dot{z}_{\boldsymbol{U}}(t) = \frac{1}{-\boldsymbol{u} \cdot \dot{r}(z_{\boldsymbol{U}}(t))} \qquad (t \in \mathrm{Dom}\ z_{\boldsymbol{U}}).$$

4.1.2. The history of a material point is described by a world line function r. A global inertial observer $\boldsymbol{U}$ observes this history as a motion described by a function $r_{\boldsymbol{U}}$ assigning to $\boldsymbol{U}$-time points the $\boldsymbol{U}$-space points where the material point is at that $\boldsymbol{U}$-time point.

To establish that function, select a $\boldsymbol{U}$-time point t; find the corresponding proper time point $z_{\boldsymbol{U}}(t)$ and the spacetime position $r(z_{\boldsymbol{U}}(t))$ of the material point; look for the $\boldsymbol{U}$-space point $C_{\boldsymbol{U}}(r(z_{\boldsymbol{U}}(t)))$ that the world point in question is incident with.

Definition.

$$r_{\boldsymbol{U}} : \mathrm{I}_{\boldsymbol{U}} \rightarrowtail \mathrm{E}_{\boldsymbol{U}}, \qquad t \mapsto C_{\boldsymbol{U}}(r(z_{\boldsymbol{U}}(t))) = r(z_{\boldsymbol{U}}(t)) + \boldsymbol{u} \otimes \mathbf{I}$$

is called the *motion relative to* $\boldsymbol{U}$, or the $\boldsymbol{U}$*-motion,* corresponding to the world line function r.

4.1.3. The question arises, whether the history, i.e. the world line function, can be regained from the motion. Later a positive answer will be given (Section 4.4).

4.1.4. Some formulae and calculations become simpler if we use a vectorization of $\boldsymbol{U}$-time and $\boldsymbol{U}$-space, i.e. we introduce a reference origin o (see 3.4.3.).

Then $\tau_{\boldsymbol{U}} \circ r - \tau_{\boldsymbol{U}}(o) = -\boldsymbol{u} \cdot (r - o) : \mathbf{I} \rightarrowtail \mathbf{I}$ is differentiable, its derivative equals the derivative of $\tau_{\boldsymbol{U}} \circ r$, hence it is strictly monotone increasing, its inverse

$$z_{\boldsymbol{U},o} := \big(-\boldsymbol{u} \cdot (r - o)\big)^{-1} : \mathbf{I} \rightarrowtail \mathbf{I}$$

is monotone increasing as well and

$$\dot{z}_{\boldsymbol{U},o}(\boldsymbol{t}) = \frac{1}{-\boldsymbol{u} \cdot \dot{r}(z_{\boldsymbol{U},o}(\boldsymbol{t}))} \qquad (\boldsymbol{t} \in \mathrm{Dom}\ z_{\boldsymbol{U},o}).$$

The *motion relative to* $(\boldsymbol{U}, o)$ is

$$r_{\boldsymbol{U},o} : \mathbf{I} \rightarrowtail \mathbf{E}_{\boldsymbol{u}}, \qquad \boldsymbol{t} \mapsto r_{\boldsymbol{U}}(\boldsymbol{t}) - C_{\boldsymbol{U}}(o) = \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \big(r(z_{\boldsymbol{U},o}(t)) - o\big).$$

4.2. Relative velocities

4.2.1. Proposition. Let $\boldsymbol{U}$ be a global inertial observer and let r be a differentiable world line function; then $r_{\boldsymbol{U}}$ is differentiable and

$$\dot{r}_{\boldsymbol{U}} = \left(\frac{\dot{r}}{-\boldsymbol{u} \cdot \dot{r}} - \boldsymbol{u}\right) \circ z_{\boldsymbol{U}}.$$

Proof. Recalling that $C_{\boldsymbol{U}} : \mathrm{M} \to \mathrm{E}_{\boldsymbol{U}}$ is an affine map over $\boldsymbol{\pi}_{\boldsymbol{u}}$, we obtain

$$\dot{r}_{\boldsymbol{U}}(t) = \frac{\mathrm{d}}{\mathrm{d}t} C_{\boldsymbol{U}}(r(z_{\boldsymbol{U}}(t))) = \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \dot{r}_{\boldsymbol{U}}(z_{\boldsymbol{U}}(t)) \dot{z}_{\boldsymbol{U}}(t);$$

then taking into account the formula in 4.1.1 for the derivative of $z_{\boldsymbol{U}}$, we easily find the desired equality. ■

It is evident that, choosing a reference origin o, we have

$$\dot{r}_{\boldsymbol{U},o} = \left(\frac{\dot{r}}{-\boldsymbol{u} \cdot \dot{r}} - \boldsymbol{u}\right) \circ z_{\boldsymbol{U},o}.$$

4.2.2. Since $r_{\boldsymbol{U}}$ describes the motion, relative to the observer $\boldsymbol{U}$, of a material point, $\dot{r}_{\boldsymbol{U}}$ is the relative velocity function of the material point. This suggests the following definition.

Definition. Let $\boldsymbol{u}$ and $\boldsymbol{u}'$ be elements of $\mathrm{V}(1)$. Then

$$\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} := \frac{\boldsymbol{u}'}{-\boldsymbol{u} \cdot \boldsymbol{u}'} - \boldsymbol{u}$$

is called the *relative velocity of* $\boldsymbol{u}'$ *with respect to* $\boldsymbol{u}$.

Proposition. For all $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$

(*i*) $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ is in $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$,

(*ii*) $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = -\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ if and only if $\boldsymbol{u} = \boldsymbol{u}'$,

(*iii*) $|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2 = |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2 = 1 - \frac{1}{(\boldsymbol{u}\cdot\boldsymbol{u}')^2} < 1$.

Proof. (*i*) is trivial, (*iii*) is demonstrated by a simple calculation.
Suppose

$$\frac{\boldsymbol{u}'}{-\boldsymbol{u}\cdot\boldsymbol{u}'} - \boldsymbol{u} = \frac{\boldsymbol{u}}{-\boldsymbol{u}'\cdot\boldsymbol{u}} - \boldsymbol{u}';$$

multiply the equality by $\boldsymbol{u}$ to have

$$0 = \frac{-1}{-\boldsymbol{u}'\cdot\boldsymbol{u}} - \boldsymbol{u}'\cdot\boldsymbol{u}, \qquad (\boldsymbol{u}'\cdot\boldsymbol{u})^2 = 1.$$

According to the reversed Cauchy inequality (see 1.3.1) this is equivalent to $\boldsymbol{u} = \boldsymbol{u}'$. ∎

Earlier we obtained that $\mathbf{E}_{\boldsymbol{u}}$ and $\mathbf{E}_{\boldsymbol{u}'}$ are different if and only if $\boldsymbol{u} \neq \boldsymbol{u}'$ and in this case $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \otimes \mathbf{I}$ ($\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'} \otimes \mathbf{I}$) is a one-dimensional linear subspace in $\mathbf{E}_{\boldsymbol{u}}$ ($\mathbf{E}_{\boldsymbol{u}'}$), orthogonal to $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$ (see 1.3.7) which offers an alternative proof of (*ii*).

4.2.3. Let us take now two global inertial observers with constant velocity values $\boldsymbol{u}$ and $\boldsymbol{u}'$. Then $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ and $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ are the relative velocities of the observers with respect to each other. Then (*iii*) of the previous proposition implies that $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = \mathbf{0}$ if and only if $\boldsymbol{u} = \boldsymbol{u}'$. Moreover, (*ii*) says that *in contradistinction to the non-relativistic case and to our habitual "evidence", the relative velocity of* $\boldsymbol{u}'$ *with respect to* $\boldsymbol{u}$ *is not the opposite of the relative velocity of* $\boldsymbol{u}$ *with respect to* $\boldsymbol{u}'$, *except the trivial case* $\boldsymbol{u} = \boldsymbol{u}'$.

It is worth emphasizing this fact because in most of the textbooks one takes it for granted that $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ and $-\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ are equal: "if an observer moves with velocity $\boldsymbol{v}$ relative to another then the second observer moves with velocity $-\boldsymbol{v}$ relative to the first one".

Nevertheless, no harm comes because vectors are given there by components with respect to convenient bases and then the components of $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ and $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ become opposite to each other.

The reason of non-equality of $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ and $-\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ is that the spaces of *different* inertial observers are affine spaces over *different* vector spaces.

However, we have a nice relation between the two vector spaces in question: the Lorentz boost from $\boldsymbol{u}$ to $\boldsymbol{u}'$ maps $\mathbf{E}_{\boldsymbol{u}}$ onto $\mathbf{E}_{\boldsymbol{u}'}$ in a natural way and maps $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ into $-\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$.

Having the equality

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = -\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$$

(see 1.3.8), we already know how to choose bases in $\mathbf{E}_{\boldsymbol{u}}$ and in $\mathbf{E}_{\boldsymbol{u}'}$ to get the mentioned usual relation between the components of relative velocities: take an arbitrary ordered basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ in $\mathbf{E}_{\boldsymbol{u}}$ and the basis $\big(\boldsymbol{e}'_i := \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{e}_i \,|\, i = 1, 2, 3\big)$ in $\mathbf{E}_{\boldsymbol{u}'}$. Now,

$$\text{if} \qquad \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = \sum_{i=1}^{3} v^i \boldsymbol{e}_i \qquad \text{then} \qquad \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'} = \sum_{i=1}^{3} (-v^i) \boldsymbol{e}'_i.$$

4.2.4. We often shall use the equalities

$$-\boldsymbol{u} \cdot \boldsymbol{u}' = \frac{1}{\sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}$$

and

$$\boldsymbol{u}' = \frac{\boldsymbol{u} + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}}{\sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}$$

deriving from 4.2.2 (*iii*) and (*i*) and from the definition of $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$.

4.2.5. The relative velocities in the non-relativistic spacetime model form a Euclidean vector space. Here the relative velocities with respect to a fixed $\boldsymbol{u} \in \mathrm{V}(1)$ form the unit open ball in the Euclidean vector space $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$:

$$\mathrm{B}_{\boldsymbol{u}} := \left\{ \boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}} \,\middle|\, |\boldsymbol{v}|^2 < 1 \right\}.$$

The set of all relative velocities is $\bigcup_{\boldsymbol{u} \in \mathrm{V}(1)} \mathrm{B}_{\boldsymbol{u}}$, a complicated subset of $\frac{\mathbf{M}}{\mathbf{I}}$.

4.3. Addition of relative velocities

4.3.1. As a consequence of the structure of relative velocites, the "addition of relative velocities" is not a vector addition, i.e. if $\boldsymbol{u}$, $\boldsymbol{u}'$, $\boldsymbol{u}''$ are different elements of $\mathrm{V}(1)$ then — in contradistinction to the non-relativistic case — we have

$$\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}} \neq \boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'} + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}.$$

The left-hand side is an element of $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$; the right-hand side is the sum of elements in $\frac{\mathbf{E}_{\boldsymbol{u}'}}{\mathbf{I}}$ and in $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ which indicates that they cannot be, in general, equal.

We might think that the convenient Lorentz boost helps us; however,

$$\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}} \neq \boldsymbol{L}(\boldsymbol{u},\boldsymbol{u}') \cdot \boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'} + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$$

because the length of the vector on the right-hand side can be greater than 1.

4.3.2. To find the formula for the addition of relative velocities — i.e. to express $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}}$ by means of $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}$ and $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ — we need some auxiliary formulae:

$$\boldsymbol{u}' = \frac{\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2} + \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}}{-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2},$$

$$\boldsymbol{u} = \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2} + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}}{-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2},$$

$$\frac{1}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}}|^2}} = \frac{1-\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}\cdot\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}|^2}\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2}}.$$

Then starting from the equality

$$(\boldsymbol{u}''=) \quad \frac{\boldsymbol{u}'+\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}|^2}} = \frac{\boldsymbol{u}+\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}}}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}}|^2}}$$

we arrive at the following result.

Proposition. Let $\boldsymbol{u}$, $\boldsymbol{u}'$ and $\boldsymbol{u}''$ be elements of V(1). Then

$$\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}} = \\ = \frac{\left(\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'} - \frac{\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}\cdot\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}}{|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2}\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}\right)\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2} + \left(1-\frac{\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}\cdot\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}}{|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2}\right)\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}}{1-\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}\cdot\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}}. \quad \blacksquare$$

4.3.3. It is important that $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}}$ cannot be expressed as a function of $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}$ and $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ or as a function of $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}$ and $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$; we need $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ or $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ as well.

We can derive other formulae, too; e.g. we can involve $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}$, $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}''}$ and $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ instead of $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}$, $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ and $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$.

It is worth mentioning the special case when $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}$ is parallel to $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$: $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'} = -\alpha\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ for some positive real number α. Then

$$\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}} = \frac{\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} + (-\alpha\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}})}{1+\alpha|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}.$$

4.3.4. Putting $\boldsymbol{v}'' := \boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}}$, $\boldsymbol{v}' := \boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}$, $\boldsymbol{v} := \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, $-\boldsymbol{v} := \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ (but this is impossible!) in the expression for $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}}$ in Proposition 4.3.2, we recognize the formula of usual treatments for the addition of relative velocities. In particular, if $\boldsymbol{v}_{\boldsymbol{u}''\boldsymbol{u}'}$ is parallel to $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$, we get the most frequently cited Einstein formula

$$\boldsymbol{v}'' = \frac{\boldsymbol{v} + \boldsymbol{v}'}{1 + |\boldsymbol{v}|\,|\boldsymbol{v}'|}.$$

4.4. History regained from motion

4.4.1. Given a motion relative an inertial observer $\boldsymbol{U}$, i.e. a piecewise twice differentiable function $m : \mathrm{I}_{\boldsymbol{U}} \rightarrowtail \mathrm{E}_{\boldsymbol{U}}$, can we determine the corresponding world line function r such that $m = r_{\boldsymbol{U}}$?

Since $r_{\boldsymbol{U}} = C_{\boldsymbol{U}} \circ r \circ z_{\boldsymbol{U}}$ and $\tau_{\boldsymbol{U}} \circ r$ is the inverse of $z_{\boldsymbol{U}}$, we have

$$(\mathrm{id}_{\mathrm{I}_{\boldsymbol{U}}}, r_{\boldsymbol{U}}) = (\tau_{\boldsymbol{U}} \circ r \circ z_{\boldsymbol{U}},\ C_{\boldsymbol{U}} \circ r \circ z_{\boldsymbol{U}}) = H_{\boldsymbol{U}} \circ r \circ z_{\boldsymbol{U}}.$$

Consequently, given the motion m,

$$r := H_{\boldsymbol{U}}^{-1} \circ (\mathrm{id}_{\mathrm{I}_{\boldsymbol{U}}}, m) \circ z_{\boldsymbol{U}}^{-1}$$

will be the corresponding world line function.

Similarly, if the vectorized motion $\boldsymbol{m} : \mathbf{I} \rightarrowtail \mathbf{E}_{\boldsymbol{u}}$ is known, then

$$r := H_{\boldsymbol{U},o}^{-1} \circ (\mathrm{id}_{\mathbf{I}}, \boldsymbol{m}) \circ z_{\boldsymbol{U},o}^{-1}$$

is the required world line function which can be given by a simple formula:

$$\boldsymbol{t} \mapsto o + \boldsymbol{m}(z_{\boldsymbol{U},o}^{-1}(\boldsymbol{t})) + \boldsymbol{u} z_{\boldsymbol{U},o}(\boldsymbol{t}).$$

4.4.2. The previous formulae are not satisfactory yet because $z_{\boldsymbol{U}}$ and $z_{\boldsymbol{U},o}$ are defined by $\boldsymbol{U}$ and $(\boldsymbol{U}, o)$ together with the world line function r to be found; we have to determine them — or their inverse — from $\boldsymbol{U}$, $(\boldsymbol{U}, o)$ and the motion m or $\boldsymbol{m}$.

Equalities in 4.1.1 and 4.1.4 result in

$$\left(z_{\boldsymbol{U}}^{-1}\right)^{\cdot} = \frac{1}{\sqrt{1 - |\dot{m}|^2}}, \qquad \left(z_{\boldsymbol{U},o}^{-1}\right)^{\cdot} = \frac{1}{\sqrt{1 - |\dot{\boldsymbol{m}}|^2}}.$$

$\dot{m}$ and $\dot{\boldsymbol{m}}$ are given functions, hence $z_{\boldsymbol{U}}^{-1}$ and $z_{\boldsymbol{U},o}^{-1}$ can be obtained by a simple integration.

4.4.3. Let us consider the basic observer with the zero as reference origin in the arithmetic spacetime model (see Exercise 3.5). A motion is given by a function $\boldsymbol{m} : \mathbb{R} \rightarrowtail \mathbb{R}^3$. Let $h : \mathbb{R} \rightarrowtail \mathbb{R}$ be a primitive function of $\frac{1}{\sqrt{1-|\dot{\boldsymbol{m}}|^2}}$. Then $\mathbb{R} \rightarrowtail \mathbb{R} \times \mathbb{R}^3$, $t \mapsto \big(h(t), \boldsymbol{m}(h(t))\big)$ is the world line function regained from the motion $\boldsymbol{m}$.

4.5. Relative accelerations

Let r be a world line function and let $\boldsymbol{U}$ be a global inertial observer with constant velocity value $\boldsymbol{u}$. Then $r_{\boldsymbol{U}}$ is twice differentiable and a differentiation of the equality in 4.2.1 yields

$$\ddot{r}_{\boldsymbol{U}} = \left(\frac{1}{(\boldsymbol{u} \cdot \dot{r})^2} \left(\mathbf{g} + \frac{\dot{r} \otimes \boldsymbol{u}}{-\boldsymbol{u} \cdot \dot{r}} \right) \cdot \ddot{r} \right) \circ z_{\boldsymbol{U}}.$$

If a reference origin o is chosen as well, $\ddot{r}_{\boldsymbol{U},o}$ is given by a similar formula, with $z_{\boldsymbol{U},o}$ instead of $z_{\boldsymbol{U}}$.

We see that, in contradistinction to the non-relativistic case, the relative acceleration does not equal the absolute one. Of course, the relative acceleration takes values in $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I} \otimes \mathbf{I}}$, the absolute acceleration takes values in $\frac{\mathbf{E}_{\dot{r}(t)}}{\mathbf{I} \otimes \mathbf{I}}$.

4.6. Some particular motions

4.6.1. Take the inertial world line function

$$r(\boldsymbol{t}) = x_o + \boldsymbol{u}_o \boldsymbol{t} \qquad (\boldsymbol{t} \in \mathbf{I}) \quad (*).$$

Let $\boldsymbol{U}$ be a global inertial observer with constant velocity value $\boldsymbol{u}$.

Then

$$\dot{z}_{\boldsymbol{U}} = \frac{1}{-\boldsymbol{u} \cdot \boldsymbol{u}_o}$$

(see 4.1.1) from which we get immediately

$$z_{\boldsymbol{U}}(t) = \frac{t - t_o}{-\boldsymbol{u} \cdot \boldsymbol{u}_o}$$

for some $t_o \in \mathrm{I}_{\boldsymbol{U}}$. Consequently (see 3.4.1(v)),

$$\begin{aligned} r_{\boldsymbol{U}}(t) &= \left(x_o + \boldsymbol{u}_o \frac{t - t_o}{-\boldsymbol{u} \cdot \boldsymbol{u}_o} \right) + \boldsymbol{u} \otimes \mathbf{I} = \big(x_o + \boldsymbol{u} \otimes \mathbf{I} \big) + \frac{\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{u}_o}{-\boldsymbol{u} \cdot \boldsymbol{u}_o}(t - t_o) = \\ &= q_{x_o} + \boldsymbol{v}_{\boldsymbol{u}_o \boldsymbol{u}}(t - t_o) \qquad (t \in \mathrm{I}_{\boldsymbol{U}}), \end{aligned}$$

where $q_{x_o} := x_o + \boldsymbol{u} \otimes \mathbf{I}$ is the $\boldsymbol{U}$-space point that x_o is incident with.

This is a uniform and rectilinear motion.

Conversely, suppose that we are given a uniform and rectilinear motion relative to the inertial observer, i.e. there are a $q \in \mathrm{E}_{\boldsymbol{U}}$, a $t_o \in \mathrm{I}_{\boldsymbol{U}}$ and a $\boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ such that

$$r_{\boldsymbol{U}}(t) = q + \boldsymbol{v}(t - t_o) \qquad (t \in \mathrm{I}_{\boldsymbol{U}}).$$

Then letting x_o denote the unique world point in the intersection of q and t_o and putting $\boldsymbol{u}_o := \frac{\boldsymbol{u}+\boldsymbol{v}}{\sqrt{1-|\boldsymbol{v}|^2}}$, the world line function of form $(*)$ gives rise to the given motion.

4.6.2. Take the uniformly accelerated world line function

$$r(\boldsymbol{t}) = x_o + \boldsymbol{u}_o \frac{\mathrm{sh}\alpha\boldsymbol{t}}{\alpha} + \boldsymbol{a}_o \frac{\mathrm{ch}\alpha\boldsymbol{t} - 1}{\alpha^2} \qquad (\boldsymbol{t} \in \mathbf{I})$$

where $\alpha := |\boldsymbol{a}_o|$; then

$$\dot{r}(\boldsymbol{t}) = \boldsymbol{u}_o \mathrm{ch}\alpha\boldsymbol{t} + \boldsymbol{a}_o \frac{\mathrm{sh}\alpha\boldsymbol{t}}{\alpha}.$$

Let $\boldsymbol{U}$ be the global inertial observer with constant velocity value $\boldsymbol{u}$. The formulae will be more tractable if we choose a reference origin o. Then

$$\begin{aligned} z^{-1}_{\boldsymbol{U},o}(\boldsymbol{t}) &= -\boldsymbol{u} \cdot (r(\boldsymbol{t}) - o) = \\ &= -\boldsymbol{u} \cdot (x_o - o) - \boldsymbol{u} \cdot \boldsymbol{u}_o \frac{\mathrm{sh}\alpha\boldsymbol{t}}{\alpha} - \boldsymbol{u} \cdot \boldsymbol{a}_o \frac{\mathrm{ch}\alpha\boldsymbol{t} - 1}{\alpha^2} \qquad (\boldsymbol{t} \in \mathbf{I}). \end{aligned}$$

Let us consider the special case $\boldsymbol{u} \cdot \boldsymbol{a}_o = 0$; then

$$z_{\boldsymbol{U},o}(\boldsymbol{t}) = \frac{\mathrm{arsh}\alpha \frac{\boldsymbol{t}-\boldsymbol{t}_o}{-\boldsymbol{u}\cdot\boldsymbol{u}_o}}{\alpha} \qquad (\boldsymbol{t} \in \mathbf{I}),$$

where $\boldsymbol{t}_o := -\boldsymbol{u} \cdot (x_o - o)$. Thus

$$\begin{aligned} r_{\boldsymbol{U},o}(\boldsymbol{t}) &= \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \left(x_o - o + \boldsymbol{u}_o \frac{\boldsymbol{t} - \boldsymbol{t}_o}{-\boldsymbol{u} \cdot \boldsymbol{u}_o} + \boldsymbol{a}_o \frac{\sqrt{1 + \alpha^2 \frac{(\boldsymbol{t}-\boldsymbol{t}_o)^2}{(\boldsymbol{u}\cdot\boldsymbol{u}_o)^2}} - 1}{\alpha^2} \right) = \\ &= \boldsymbol{q}_o + \boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{u}}(\boldsymbol{t} - \boldsymbol{t}_o) + \boldsymbol{b}_o \frac{\sqrt{1 + \beta^2(\boldsymbol{t} - \boldsymbol{t}_o)^2} - 1}{\beta^2} \qquad (\boldsymbol{t} \in \mathbf{I}), \end{aligned}$$

where

$$\boldsymbol{q}_o := \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (x_o - o), \qquad \boldsymbol{b}_o := \boldsymbol{a}_o\big(1 - |\boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{u}}|^2\big), \qquad \beta^2 := \alpha^2\big(1 - |\boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{u}}|^2\big).$$

4.7. Light speed

4.7.1. We wish to determine the motion of a light signal with respect to an inertial observer. The procedure will be similar to that in Sections 4.1 and 4.2.

We introduce the notation

$$\mathrm{V}(0) := \left\{ \boldsymbol{w} \in \frac{\mathbf{M}}{\mathbf{I}} \,\middle|\, \boldsymbol{w}^2 = 0,\ \boldsymbol{w} \otimes \mathbf{I}^+ \subset \mathrm{L}^{\rightarrow} \right\}.$$

The elements of $\mathrm{V}(0)$ are future-directed lightlike vectors of cotype $\mathbf{I}$. Though the notation is similar to $\mathrm{V}(1)$, observe a significant difference: if two elements in $\mathrm{V}(1)$ are parallel then they are equal; on the other hand, if $\boldsymbol{w}$ is in $\mathrm{V}(0)$ then $\alpha\boldsymbol{w}$, too, is in $\mathrm{V}(0)$ for all $\alpha \in \mathbb{R}^+$.

Let $\boldsymbol{U}$ be a global inertial observer with constant velocity value $\boldsymbol{u}$. Let us consider a light signal F, i.e. a straight line directed by a vector in $\mathrm{V}(0)$. The motion of the light signal with respect to the observer is described by

$$f_{\boldsymbol{U}} : \mathrm{I}_{\boldsymbol{U}} \to \mathrm{E}_{\boldsymbol{U}}, \qquad t \mapsto (\mathrm{F} \star t) + \boldsymbol{u} \otimes \mathbf{I}$$

where $\mathrm{F} \star t$ denotes the unique element in the intersection of the straight line F and the hyperplane t.

If $t, s \in \mathrm{E}_{\boldsymbol{U}}$ then $t - s = -\boldsymbol{u} \cdot (\mathrm{F} \star t - \mathrm{F} \star s)$ by the definition of the affine structure of $\mathrm{I}_{\boldsymbol{U}}$ (see 3.4.2); then we easily find that

$$\mathrm{F} \star t - \mathrm{F} \star s = \boldsymbol{w} \frac{t - s}{-\boldsymbol{u} \cdot \boldsymbol{w}},$$

hence

$$f_{\boldsymbol{U}}(t) - f_{\boldsymbol{U}}(s) = \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (\mathrm{F} \star t - \mathrm{F} \star s) = \left(\frac{\boldsymbol{w}}{-\boldsymbol{u} \cdot \boldsymbol{w}} - \boldsymbol{u} \right) (t - s).$$

Thus the light signal moves uniformly on a straight line relative to the observer.

4.7.2. Definition. Let $\boldsymbol{w} \in \mathrm{V}(0)$ and $\boldsymbol{u} \in \mathrm{V}(1)$. Then

$$\boldsymbol{v}_{\boldsymbol{wu}} := \frac{\boldsymbol{w}}{-\boldsymbol{u}\cdot\boldsymbol{w}} - \boldsymbol{u}$$

is the *relative velocity of* $\boldsymbol{w}$ *with respect to* $\boldsymbol{u}$.

Proposition. $\boldsymbol{v}_{\boldsymbol{wu}}$ is an element of $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ and

$$|\boldsymbol{v}_{\boldsymbol{wu}}| = 1. \qquad \blacksquare$$

Observe that given an arbitrary $\boldsymbol{u}$, $\boldsymbol{w}$ and $\alpha\boldsymbol{w}$ have the same relative velocities with respect to $\boldsymbol{u}$.

We do not define the relative velocity of $\boldsymbol{u}$ with respect to $\boldsymbol{w}$.

According to the previous proposition, the magnitude of the relative velocity is the same number, namely 1, for every light signal and every inertial observer.

Light signals propagate isotropically with respect to all inertial observers.

4.7.3. Recall that relative velocity values have magnitude but two relative velocity values need not have an angle between themselves; relative velocities with respect to the same element of $\mathrm{V}(1)$ do form an angle.

Now we look for the relation between certain angles formed by relative velocity values. The physical situation is similar to that in I.6.2.3. A car is going on a straight road and it is raining. The raindrops hit the road and the car at different angles relative to the direction of the road. What is the relation between the two angles? Now we can treat another question, too, considering instead of raindrops light beams (continuous sequences of light signals) arriving from the sun.

Let $\boldsymbol{u}$ and $\boldsymbol{u}'$ be different elements of $\mathrm{V}(1)$ (representing the absolute velocity values of the road and of the car, respectively). If $\boldsymbol{w}$ is an element of $\mathrm{V}(1)\cup\mathrm{V}(0)$ (repesenting the absolute velocity value of the raindrops or the absolut direction of the light beam), $\boldsymbol{w} \neq \boldsymbol{u}$, $\boldsymbol{w} \neq \boldsymbol{u}'$, then

$$\theta(\boldsymbol{w}) := \arccos\frac{\boldsymbol{v}_{\boldsymbol{wu}}\cdot\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}}{|\boldsymbol{v}_{\boldsymbol{wu}}|\,|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|}, \qquad \theta'(\boldsymbol{w}) := \arccos\frac{\boldsymbol{v}_{\boldsymbol{wu}'}\cdot(-\boldsymbol{v}_{\boldsymbol{uu}'})}{|\boldsymbol{v}_{\boldsymbol{wu}'}|\,|\boldsymbol{v}_{\boldsymbol{uu}'}|}$$

are the angles formed by the relative velocity values in question. A simple calculation verifies that

$$\cos\theta(\boldsymbol{w}) = \frac{\frac{|\boldsymbol{v}_{\boldsymbol{wu}'}|}{|\boldsymbol{v}_{\boldsymbol{wu}}|}\cos\theta'(\boldsymbol{w}) + \frac{|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|}{|\boldsymbol{v}_{\boldsymbol{wu}}|}}{1 + |\boldsymbol{v}_{\boldsymbol{wu}'}|\,|\boldsymbol{v}_{\boldsymbol{uu}'}|\cos\theta'(\boldsymbol{w})}.$$

If $\boldsymbol{w} \in \mathrm{V}(0)$ then $|\boldsymbol{v}_{\boldsymbol{wu}}| = |\boldsymbol{v}_{\boldsymbol{wu}'}| = 1$ and

$$\cos\theta(\boldsymbol{w}) = \frac{\cos\theta'(\boldsymbol{w}) + |\boldsymbol{v}_{\boldsymbol{uu}'}|}{1 + |\boldsymbol{v}_{\boldsymbol{uu}'}|\cos\theta'(\boldsymbol{w})}.$$

This formula is known as the *aberration of light*: two different inertial observers see the same light beam under different angles with respect to their relative velocities; the angles are related by the above formula.

5. Some observations

5.1. Physically equal vectors in different spaces

5.1.1. It is an important fact that the spaces of different global inertial observers are affine spaces over different vector spaces. Thus it has no meaning, in general, that a straight line segment (a vector) in the space of an inertial observer coincides with a straight line segment (with a vector) in the space of another inertial observer.

Let us consider two different inertial observers $\boldsymbol{U}$ and $\boldsymbol{U}'$ with constant velocity values $\boldsymbol{u}$ and $\boldsymbol{u}'$, respectively. The spaces of $\boldsymbol{U}$ and $\boldsymbol{U}'$ are affine spaces over $\mathbf{E}_{\boldsymbol{u}}$ and $\mathbf{E}_{\boldsymbol{u}'}$, respectively. We know that $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$ is a two-dimensional subspace, orthogonal to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ and to $\boldsymbol{v}_{\boldsymbol{uu}'}$.

If a vector between two points in the $\boldsymbol{U}$-space lies in $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}$ then we can find two points in the $\boldsymbol{U}'$-space having the same vector connecting them. We have troubles only with vector outside this two-dimensional subspace.

To relate other vectors, too, we start from the rational agreement that "if you move with respect to me in some direction in my space then I move with respect to you in the opposite direction in your space". This suggests that the vector $\lambda\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ in $\mathbf{E}_{\boldsymbol{u}}$ and the vector $-\lambda\boldsymbol{v}_{\boldsymbol{uu}'}$ in $\mathbf{E}_{\boldsymbol{u}'}$ could be considered the same for all $\lambda \in \mathbf{I}$. More generally, every vector in $\mathbf{E}_{\boldsymbol{u}}$ has the form

$$\lambda\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} + \boldsymbol{q} \qquad (\lambda \in \mathbf{I},\ \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'})$$

and every vector in $\mathbf{E}_{\boldsymbol{u}'}$ has the form

$$\lambda'\boldsymbol{v}_{\boldsymbol{uu}'} + \boldsymbol{q}' \qquad (\lambda' \in \mathbf{I},\ \boldsymbol{q}' \in \mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}'}).$$

The observers agree that two such vectors are considered to be the same if and only if

$$\lambda' = -\lambda, \qquad \boldsymbol{q}' = \boldsymbol{q}.$$

We have a nice tool to express this agreement: the Lorentz boost.

Definition. The vectors $\boldsymbol{q}'$ in $\mathbf{E}_{\boldsymbol{u}'}$ and $\boldsymbol{q}$ in $\mathbf{E}_{\boldsymbol{u}}$ are called *physically equal* if and only if

$$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{q} = \boldsymbol{q}'. \qquad \blacksquare$$

We emphasize that the equality of vectors in different observer spaces makes no original sense, in general; we agreed to define it conveniently.

5.1.2. To be physically equal in different observer spaces, according to our convention, is a symmetric relation, but *is not a transitive relation.*

Indeed, if $\boldsymbol{q}' = \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{q}$ then $\boldsymbol{q} = \boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}') \cdot \boldsymbol{q}'$: the relation is symmetric.

However, if $\boldsymbol{u}$, $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are not coplanar, then there are $\boldsymbol{q}$, $\boldsymbol{q}'$ and $\boldsymbol{q}''$ in such a way that

$$\boldsymbol{q}' = \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{q}, \qquad \boldsymbol{q}'' = \boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}') \cdot \boldsymbol{q}',$$
$$\boldsymbol{q}'' \text{ is not parallel to } \boldsymbol{L}(\boldsymbol{u}'', \boldsymbol{u}) \cdot \boldsymbol{q}$$

(Exercise 1.9.6); $\boldsymbol{q}'$ is physically equal to $\boldsymbol{q}$, $\boldsymbol{q}''$ is physically equal to $\boldsymbol{q}'$, but $\boldsymbol{q}''$ is not physically equal to $\boldsymbol{q}$: the relation is not transitive.

In particular, "if a straight line in your space is parallel to a straight line in my space and a line in his space is parallel to your line then his line need not be parallel to mine".

This is a rather embarrassing situation but there is no escape. The truth of the common sense that the relative velocity of an observer with respect to another is the opposite of the other relative velocity and the transitivity of parallelism exclude each other.

5.2. Observations concerning spaces

5.2.1. In 5.1.1 an agreement is settled what the equality — in particular the parallelism — of vectors in different observer spaces means.

Now the question arises whether a straight line segment in the space of an inertial observer is observed by another observer to be a straight line segment parallel to the original one. The question and the answer are formulated correctly as follows (cf. I.7.1.2).

Let $\boldsymbol{U}_{\mathrm{o}}$ and $\boldsymbol{U}$ be global inertial observers with constant velocity values $\boldsymbol{u}_{\mathrm{o}}$ and $\boldsymbol{u}$, respectively. Let H_{o} be a subset (a geometrical figure) in the $\boldsymbol{U}_{\mathrm{o}}$-space. The corresponding figure observed by $\boldsymbol{U}$ at the $\boldsymbol{U}$-instant t — called the *trace* of H_{o} at t in $\mathbf{E}_{\boldsymbol{U}}$ — is the set of $\boldsymbol{U}$-space points that coincide at t with the points of H_{o} :

$$\{q \star t + \boldsymbol{u} \otimes \mathbf{I} |\ q \in \mathrm{H}_{\mathrm{o}}\}$$

where $q \star t$ denotes the unique world point in the intersection of the line q and the hyperplane t.

Introducing the map

$$P_t : \mathrm{E}_{\boldsymbol{U}_\circ} \to \mathrm{E}_{\boldsymbol{U}}, \qquad q \mapsto q \star t + \boldsymbol{u} \otimes \mathbf{I}$$

we see that the trace of H_o at t equals $P_t[\mathrm{H}_\mathrm{o}]$. It is quite easy to see (recall the definition of subtraction in the observer spaces) that

$$\begin{aligned} P_t(q_2) - P_t(q_1) = q_2 \star t - q_1 \star t = q_2 - q_1 + \boldsymbol{u}_\mathrm{o} \frac{\boldsymbol{u} \cdot (q_2 - q_1)}{-\boldsymbol{u}_\mathrm{o} \cdot \boldsymbol{u}} = \\ = \boldsymbol{P}_{\boldsymbol{u}\boldsymbol{u}_\circ} \cdot (q_2 - q_1), \end{aligned}$$

where $\boldsymbol{P}_{\boldsymbol{u}\boldsymbol{u}_\circ}$ is the projection onto $\mathbf{E}_{\boldsymbol{u}}$ along $\boldsymbol{u}_\mathrm{o} \otimes \mathbf{I}$ (see Exercise 1.9.7).

Since the restriction of $\boldsymbol{P}_{\boldsymbol{u}\boldsymbol{u}_\circ}$ onto $\mathbf{E}_{\boldsymbol{u}_\circ}$, denoted by $\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_\circ}$, is a linear bijection between $\mathbf{E}_{\boldsymbol{u}_\circ}$ and $\mathbf{E}_{\boldsymbol{u}}$, P_t is an affine bijection over $\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_\circ}$.

5.2.2. We can easily find that

$$\begin{aligned} &\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_\circ} \cdot \boldsymbol{q} = \boldsymbol{q} \quad \text{if} \quad \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}_\circ} \quad \text{i.e. if } \boldsymbol{q} \text{ is orthogonal to } \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_\circ}, \\ &\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_\circ} \cdot \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_\circ} = -\sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_\circ}|^2}\, \boldsymbol{v}_{\boldsymbol{u}_\circ\boldsymbol{u}}. \end{aligned}$$

The linear bijection $\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_\circ}$ resembles the restriction onto $\mathbf{E}_{\boldsymbol{u}_\circ}$ of the Lorentz boost $\boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}_\mathrm{o})$; an essential difference is that it maps $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_\circ}$ into $-\boldsymbol{v}_{\boldsymbol{u}_\circ\boldsymbol{u}}$ *multiplied by a real number less than* 1. Consequently, $\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_\circ}$ *is not an orthogonal map*; it does not preserve either lengths or angles which is illustrated as follows:

5.2.3. Every figure in the $\boldsymbol{U}_o$-space is of the form $q_o + \mathbf{H}_o$, where $q_o \in \mathrm{E}_{\boldsymbol{U}_o}$ and $\mathbf{H}_o \subset \mathbf{E}_{\boldsymbol{u}_o}$; then $P_t[q_o + \mathbf{H}_o] = P_t(q_o) + \boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o}[\mathbf{H}_o]$. Consequently, the observed figure and the original one are not congruent, in general.

If L_o is a straight line segment in the $\boldsymbol{U}_o$-space, then its trace is a straight line segment, too. However, the observed segment and the original one are not parallel, in general: if L_o is directed by the vector $\boldsymbol{e}_o$, $\mathrm{L}_o = q_o + \mathbb{R}\boldsymbol{e}_o$, then $P_t[\mathrm{L}_o]$ is directed by $\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_o$.

$\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_o$ is parallel to $\boldsymbol{e}_o$, by definition, if there is a real number λ such that

$$\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_o = \lambda \boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}_o) \cdot \boldsymbol{e}_o$$

which occurs if and only if $\boldsymbol{e}_o$ is in $\mathbf{E}_{\boldsymbol{u}} \cap \mathbf{E}_{\boldsymbol{u}_o}$ or in $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o} \otimes \mathbf{I}$.

Thus a straight line segment L_o in the $\boldsymbol{U}_o$-space is observed by $\boldsymbol{U}$ to be parallel to L_o if and only if L_o is

— either orthogonal to $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$

— or parallel to $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$.

5.2.4. Let L_1 and L_2 be crossing straight lines in the $\boldsymbol{U}_o$-space. Then $\boldsymbol{U}$ observes at every instant that they are crossing straight lines. However, the angle formed by L_1 and L_2 and the angle formed by the observed straight lines differ, in general.

Let L_1 and L_2 be directed by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively. If θ_o denotes the angle formed by L_1 and L_2 then

$$\cos\theta_o = \frac{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2}{|\boldsymbol{e}_1|\,|\boldsymbol{e}_2|}.$$

For the angle θ observed by $\boldsymbol{U}$ we have

$$\cos\theta = \frac{(\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_1) \cdot (\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_2)}{|\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_1|\,|\boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_2|} = \frac{\cos\theta_o - \alpha_1\alpha_2}{\sqrt{1-\alpha_1^2}\sqrt{1-\alpha_2^2}},$$

where

$$\alpha_1 := \frac{\boldsymbol{u} \cdot \boldsymbol{e}_1}{-(\boldsymbol{u}_o \cdot \boldsymbol{u})|\boldsymbol{e}_1|} = \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \frac{\boldsymbol{e}_1}{|\boldsymbol{e}_1|},$$

$$\alpha_2 := \frac{\boldsymbol{u} \cdot \boldsymbol{e}_2}{-(\boldsymbol{u}_o \cdot \boldsymbol{u})|\boldsymbol{e}_2|} = \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \frac{\boldsymbol{e}_2}{|\boldsymbol{e}_2|}.$$

Thus θ and θ_o are equal if and only if $\alpha_1 = \alpha_2 = 0$, i.e. if and only if both $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthogonal to the relative velocity $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$.

5.3. The Lorentz contraction

5.3.1. A straight line segment orthogonal to the relative velocity $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$ in the $\boldsymbol{U}_o$-space is observed by $\boldsymbol{U}$ as a straight line segment parallel to the original one and having the same length.

A straight line segment parallel to the relative velocity $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$ in the $\boldsymbol{U}_o$-space is observed by $\boldsymbol{U}$ as a shorter straight line segment parallel to the original one. This is the famous *Lorentz contraction* which will be detailed as follows.

A straight line segment in the $\boldsymbol{U}_o$-space can be represented by one of its extremities and the vector between its extremities. Since parallel segments are observed in a similar way, we can consider only the vector $\boldsymbol{e}_o \in \mathbf{E}_{\boldsymbol{u}_o}$ between the extremities.

The observation of $\boldsymbol{e}_o$ by $\boldsymbol{U}$ yields $\boldsymbol{e} := \boldsymbol{A}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_o$. A simple calculation shows that

$$|\boldsymbol{e}|^2 = |\boldsymbol{e}_o|^2 - \frac{(\boldsymbol{u} \cdot \boldsymbol{e}_o)^2}{(\boldsymbol{u} \cdot \boldsymbol{u}_o)^2} = |\boldsymbol{e}_o|^2 - (\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o} \cdot \boldsymbol{e}_o)^2.$$

The observed length, in general, is smaller than the proper one.

More closely, the observed length equals the original one if and only if the segment is orthogonal to the relative velocity; otherwise the observed length is smaller than the original one. The observed length is the smallest if the segment is parallel to the relative velocity:

$$|\boldsymbol{e}| = |\boldsymbol{e}_o|\sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|^2} \qquad \text{if } \boldsymbol{e}_o \text{ is parallel to } \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}.$$

5.3.2. One often says that the travelling length is smaller than the proper (or rest) length: "a moving rod is contracted, becomes shorter".

We emphasize that the Lorentz contraction formula does not state any real physical contraction at all. We can assert only that an observer moving relative to a rod observes the rod shorter than the observer having the rod in its own space.

Let us imagine two rods having the same proper length and resting in the spaces of different observers: both observers will observe the *other* rod to be shorter than its own one.

A number of paradoxes can arise from this situation: "I say that your rod is shorter then mine, you say that my rod is shorter than yours; which of us is right?" Keeping in mind that only illusory and no physical contractions are in question, we can accept the correct answer: both of us are right.

5.3.3. Suppose you do not believe that the contraction is illusory and you want determine experimentally which of us is right. The experiment seems extremely simple: you catch my rod (which is moving relative to you) and having stopped it you put it close to your one and then you will see which of them is shorter.

We consider an ideal case: you seize the moving rod all at once so that it stops instantaneously.

Let us translate the situation into our mathematical language. My rod is described by a line segment in the $\boldsymbol{U}_o$-space:

$$\mathrm{L}_o = q_o + [0,1]\boldsymbol{e}_o$$

and $\boldsymbol{e}_o$ is taken to be parallel to $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$.

Then the rod has the length (the proper length) $\boldsymbol{d}_o := |\boldsymbol{e}_o|$ with respect to $\boldsymbol{U}_o$ and the length (the travelling length) $\boldsymbol{d} := \sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|^2}\boldsymbol{d}_o$ observed by $\boldsymbol{U}$.

At a $\boldsymbol{U}$-instant t the history of each point of the rod will be changed into an inertial history with the velocity value $\boldsymbol{u}$; then you get

$$\mathrm{L} = \{q \star t + \boldsymbol{u} \otimes \mathbf{I} |\ q \in \mathrm{L}_o\} = q + [0,1]\boldsymbol{e},$$

where

$$q := q_o \star t + \boldsymbol{u} \otimes \mathbf{I}, \qquad \boldsymbol{e} := \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{u}_o) \cdot \boldsymbol{e}_o.$$

The segment (the seized rod) L is in the $\boldsymbol{U}$-space and has the length $|\boldsymbol{e}| = \boldsymbol{d}$, the length of L_o observed by $\boldsymbol{U}$.

5.3.4. You can relax: you showed that my rod is "really" shorter than yours. But then you think that I can execute a similar experiment to show that your rod is "really" shorter than mine. Again the same disturbing situation.

To solve the seeming contradiction, note that in your experiment your rod continues to exist without any effect on it, while my rod is affected by your seizure, and in my experiment your rod is affected. The seizure means a physical change in the rod which causes contraction.

Let us analyze the problem more thoroughly.

(i) The rod resting in the $\boldsymbol{U}_o$-space moves relative to the observer $\boldsymbol{U}$ which finds that the length of the rod is $\boldsymbol{d}$. At a $\boldsymbol{U}$-instant t the observer $\boldsymbol{U}$ seizes the rod, stops it, and discovers that this rod has the same length $\boldsymbol{d}$. According to $\boldsymbol{U}$, *the rod did not change length in the seizure,* in other words, the observer $\boldsymbol{U}$ sees the rod as *rigid* and this is well understandable from its point of view because the rod is stopped *at an instant* with respect to $\boldsymbol{U}$, i.e. every point of the rod stops *simultaneously* with respect to $\boldsymbol{U}$.

(ii) The rod rests in the $\boldsymbol{U}_o$-space. As $\boldsymbol{U}$ seizes the rod, the observer $\boldsymbol{U}_o$ sees that the rod begins to move but not instantaneously with respect to $\boldsymbol{U}_o$: the points of the rod begin the movement at different $\boldsymbol{U}_o$-instants! First the backward extremity (from the point of view of the relative velocity of $\boldsymbol{U}$ with respect to $\boldsymbol{U}_o$) starts and then successively the other points, at last the forward extremity. Evidently, $\boldsymbol{U}_o$ sees the rod is not rigid, it contracts during the *time interval* of seizure.

(iii) The rod experiences that it moves relative to $\boldsymbol{U}$ which begins to stop it in such a way that first the forward extremity (from the point of view of the relative velocity of the rod (i.e. of $\boldsymbol{U}_o$) with respect to $\boldsymbol{U}$) stops and then successively the other points and at last the backward extremity. The rod experiences contraction during the procedure of seizure.

We have examined three standpoints. Two of them concern inertial observers and the third concerns a non-inertial object.

5.3.5. The ideal case that every point of the rod changes its velocity abruptly at a $\boldsymbol{U}$-instant can be replaced by the more realistic one that every point of the rod changes its velocity from $\boldsymbol{u}_o$ to $\boldsymbol{u}$ during a $\boldsymbol{U}$-time interval, as the following Figure shows.

5.3.6. Note that we started with the problem that "you seize the moving rod all at once so that it stops instantaneously", we considered "instantaneously" with respect to $\boldsymbol{U}$ without calling attention to the extremely important fact that "instantaneously" has no unique meaning.

The reader is asked to analyze the problem that the rod is caught by $\boldsymbol{U}$ instantaneously with respect to $\boldsymbol{U}_o$ (i.e. every point of the rod is stopped by $\boldsymbol{U}$ simultaneously with respect to $\boldsymbol{U}_o$).

5.4. The tunnel paradox

5.4.1. Consider a train and a tunnel. The proper length of the train is greater than the proper length of the tunnel. The travelling train enters the tunnel.

The observer resting with respect to the tunnel observes Lorentz contraction on the train, thus it sees that, if the velocity of the train is high enough, the train is entirely in the tunnel during a time interval.

On the contrary, the observer resting with respect to the train observes Lorentz contraction on the tunnel, thus it experiences that the train is never entirely in the tunnel.

Which of them is right? We know that both. However, it seems to be a very strange situation, because the observer resting with respect to the tunnel says that "when the train is entirely inside I close both gates of the tunnel, thus I confine the train in the tunnel, I am right and the observer in the train is wrong".

5.4.2. On the basis of our previous examination we can remove the paradox easily.

In the assertion above "when" means that the gates become closed *simultaneously with respect to the tunnel.*

From the point of view of the train the gates will not be closed simultaneously: first the forward gate closes and later the backward gate. *When* the forward gate closes, the forepart of the train is in the tunnel (the back part is still outside), *when* the backward gate closes, the back part of the train is in the tunnel (the forepart is already outside).

"I confined the train in the tunnel" means that the closed gates hinder the train from leaving the tunnel. But how do they do this? The train is moving; it must be stopped to be confined definitely in the tunnel: some apparatus in the tunnel brakes the train or the train hits against the front gate which is so strongly closed that stops the train. In any case, as we have seen, stopping means a real contraction of the train, consequently, it finds room in the tunnel; however, then the train *ceases to be inertial* in all its existence, that is why the assertion "I am never in the tunnel entirely" (true for an inertial train) will be false.

5.5. No measuring rods

5.5.1. In the special relativistic spacetime model the absolute rigid rod is not a meaningful notion. We have seen in 5.3.4 that the same rod seems to be rigid to an inertial observer and not rigid to another observer.

If C_1 and C_2 are world lines and $\boldsymbol{U}$ is a global inertial observer with constant velocity value, then the *vector* and the *distance observed* by $\boldsymbol{U}$ between C_1 and C_2 at the $\boldsymbol{U}$-instant t are

$$\mathrm{C}_2 \star t - \mathrm{C}_1 \star t \qquad \text{and} \qquad |\mathrm{C}_2 \star t - \mathrm{C}_1 \star t|,$$

respectively, provided neither of $\mathrm{C}_2 \cap t$ and $\mathrm{C}_1 \cap t$ is void.

Absolute vectors and absolute distances between the world lines do not exist. Evidently, in general, different observers observe different vectors and distances between the world lines. By no means can we define an absolutely rigid rod.

5.5.2. As a consequence, measuring rods are useless for determining the distance between two points, the length of a line etc. in observer spaces: it is questionable whether one can take a rod, carry it to the figure to be measured, put it consecutively at convenient places in such a way that its length does not change during these procedures.

Spacetime measurements in the non-relativistic case are based on clocks (saying the absolute time) and measuring rods (that are absolutely rigid).

Spacetime measurements in the special relativistic case are based on clocks (saying their proper times) and light signals.

Recall, for instance, that defining simultaneity we need the proper times passing in observer space points and the distance between observer space points; this latter is measured by light signals and proper time intervals (radar).

5.6. The time dilation

5.6.1. Let $\boldsymbol{U}$ be a global inertial observer with constant velocity value $\boldsymbol{u}$.

$\boldsymbol{U}$-time is an affine space over $\mathbf{I}$. Since $\mathbf{I}$ is oriented, later and earlier makes sense between $\boldsymbol{U}$-instants: t is earlier than s (s is later than t) if $s - t$ is positive.

A unique $\boldsymbol{U}$-instant $\tau_{\boldsymbol{U}}(x)$ is assigned to every world point x. Consequently, we can decide which of two arbitrary world points is later according to $\boldsymbol{U}$.

Definition. The *time observed by* $\boldsymbol{U}$ between the world points x and y is

$$\mathbf{t}_{\boldsymbol{U}}(x, y) := \tau_{\boldsymbol{U}}(y) - \tau_{\boldsymbol{U}}(x) = -\boldsymbol{u} \cdot (y - x).$$

The world point y is *later* than the world point x (x is *earlier* than y) according to $\boldsymbol{U}$ if the time observed between x and y is positive. ■

Neither of x and y is later according to $\boldsymbol{U}$ if and only if they are simultaneous according to $\boldsymbol{U}$.

5.6.2. Fix two different world points x and y.

If they are spacelike separated then there are global inertial observers $\boldsymbol{U}_{\mathrm{o}}$, $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ such that y is simultaneous with x according to $\boldsymbol{U}_{\mathrm{o}}$, y is later than x according to $\boldsymbol{U}_1$ and y is earlier than x according to $\boldsymbol{U}_2$.

If y is future-like with respect to x (they are lightlike or timelike separated) then y is later than x according to all inertial observers.

5.6.3. Suppose $y \in x + \mathrm{T}^{\rightarrow}$. Then $\mathbf{t}(x, y) = |y - x|$ is the inertial time between x and y,

$$\boldsymbol{u}_{\mathrm{o}} := \frac{y - x}{|y - x|} \in \mathrm{V}(1),$$

and $y = x + \boldsymbol{u}_{\mathrm{o}} \mathbf{t}(x, y)$.

Consequently, if $\boldsymbol{U}$ is the global inertial obvserver with the constant velocity value $\boldsymbol{u}$ then

$$\mathbf{t}_{\boldsymbol{U}}(x, y) = -(\boldsymbol{u} \cdot \boldsymbol{u}_{\mathrm{o}}) \mathbf{t}(x, y) = \frac{\mathbf{t}(x, y)}{\sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_{\mathrm{o}}}|^2}}.$$

$\mathbf{t}_{\boldsymbol{U}}(x, y) = \mathbf{t}(x, y)$ if and only if $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{o}}$ i.e. if and only if x and y are incident with the same $\boldsymbol{U}$-space point. In any other cases $\mathbf{t}_{\boldsymbol{U}}(x, y)$ is greater than $\mathbf{t}(x, y)$.

5.6.4. Let us illustrate our result as follows.

Let us consider an inertial (point-like) clock. Let x and y be the occurrences that the clock says 11 and 12, respectively. The observer, relative to which the clock is at rest, observes 1 hour between the occurrences. Another observer, moving relative to the clock, observes more than 1 hour between the occurrences. This is the famous *time dilation.*

The clock and the observer move with respect to each other. In usual formulations one considers that the observer is at rest and the clock is moving and one says that “a moving clock works more slowly than a clock at rest”.

We emphasize that the time dilation formula does not state any real physical dilation of time at all. We can assert only that an observer moving relative to a clock observes the clock working more slowly than the observer having the clock in its own space.

In other words, an observer observes that the time of another observer passes more slowly.

Let us take two different inertial observers. Then both will observe that the *other's* time passes more slowly.

5.7. The twin paradox

5.7.1. Let us consider two twins, Peter and Paul. Both are launched in separate missiles. Peter says that Paul is moving relative to him, hence Paul's time passes more slowly and he observes that when he (Peter) is forty then his brother (Paul) is only twenty. On the other hand, Paul says that Peter is moving relative to him and he observes that when he (Paul) is forty then his brother (Peter) is only twenty. Which of them is right?

Keeping in mind that only illusory and no physical time dilations are in question, we can be convinced that both are right. How can it be possible?

The paradox is based on our everyday concept of absolute time, i.e. that "when" has an absolute meaning. However, the first "when" means simultaneity with respect to Peter and the second "when" means simultaneity with respect to Paul; we know well that these simultaneities are different.

5.7.2. Suppose the brothers do not believe that time dilation is illusory and they want an experimental test: let them meet and then a simple inspection will determine which of them is older.

However, the time dilation formula concerns *inertial observers.* It is excusable that both missiles are considered to be inertial. But if they remain inertial then the brothers never meet. If the brothers meet then at least one of them *ceases to be inertial.* Anyhow, the mutually equivalent situation of the brothers breaks. It will not be true that both are right saying "when I am forty then my brother is only twenty".

Let the brothers meet. Both existed somehow between the two occurrences, the departure and the arrival, and their proper times passed during their existence. The times elapsed depend on their existence and need not be equal. It can happen that Peter is older than Paul; e.g. if Peter remains inertial (the inertial time between two world points is always greater than a time passed on a non-inertial world line, see 2.2.3). This difference of proper times is an absolute fact and has nothing to do with the illusory time dilation.

It is important to distinguish between illusory time dilation concerning two inertial observers and really different times passed along two world lines between two world points.

5.8. Experiments concerning time

5.8.1. We have an experimental proof for time dilation.

Cosmic rays produce muons in the ionosphere. Some of those muons come to the earth. Detecting the magnitude v of their velocity with respect to the earth and knowing the height d of the place where they are produced, we can calculate the time of their travel (uniform and rectilinear motion seems a good approximation). It turns out that the time of travel d/v exceeds the lifetime T of muons (muons are not stable particles, they decay having a well-defined average lifetime). Thus the earth observes as if the muons lived more than their lifetime.

The muon in question exists inertially thus it "feels" the inertial time t_o of its travel which is less than its lifetime; the earth observes a longer time (time dilation):

$$\frac{d}{v} = \frac{t_o}{\sqrt{1-v^2}} > T > t_o.$$

It is interesting that we can give another explanation, too. The muon observes the distance $d\sqrt{1-v^2}$ between its birth place and the earth (Lorentz contraction), hence it travels for $t_o = \frac{d\sqrt{1-v^2}}{v}$.

5.8.2. Let us suppose that, simultaneously with the muon in the ionosphere (muon I), a muon is produced and remains resting on the earth (muon E). According to what has been said, muon E decays before muon I arrives at the earth: muon E "sees" that time passes more slowly for muon I.

Of course, muon I "sees" as well that time passes more slowly for muon E. Then one could suspect a contradiction (paradox): according to muon I, muon E would be alive at the end of the travel of muon I.

There is no contradiction: *simultaneously* in the present context means simultaneously according to the earth, i.e. to muon E, and muon I is produced simultaneously according to muon E. Then, according to muon I, the other muon is born earlier; consequently, muon I, though sees time passing more slowly for muon E, will observe that the life of muon E ends before muon I meets the earth.

5.8.3. Experiments show that instable particles revolving in an accelerator have longer lifetime. This supports that different times can pass between two world points along different world lines, as it will be explained.

Suppose two muons are produced at the same time and at the same place (i.e. a single world point corresponds to their birth) and one of them (muon R) remains resting beside the accelerator, the other (muon A) is constrained to revolve in the accelerator. The muons meet several times. Then muon R decays, but muon A continues to revolve and meets again the void place of muon R; we observe (resting with muon R) as if muon A had a longer life time. Nevertheless, both muons have the same proper lifetime T.

The world line of muon R is inertial while muon A has a noninertial world line; the two world lines intersect each other several times. Different times t_{R} and t_{A} pass along the different world lines of the muons between their two successive meetings. Inertial time is always greater than a non-inertial: $t_{\mathrm{A}} < t_{\mathrm{R}}$. That is why there can be a natural number n such that $nt_{\mathrm{A}} < T < nt_{\mathrm{R}}$, i.e. muon R does not last until the n-th meeting but muon A survives it.

5.9. Exercises

1. Prove that the addition formula 4.3.2 of relative velocities remains valid for $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$, $\boldsymbol{u}'' \in \mathrm{V}(0)$.

2. Take the motion treated in 4.6.2. Demonstrate that

$$\lim_{\mathbf{t}\to\infty} \dot{r}_{\boldsymbol{U}}(\boldsymbol{t}) = \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_\text{o}} + \frac{\boldsymbol{b}_\text{o}}{\beta}, \qquad \lim_{\mathbf{t}\to\infty} |\dot{r}_{\boldsymbol{U}}(\boldsymbol{t})| = 1.$$

3. Consider the uniformly accelerated world line treated in 4.6.2. Try to describe the corresponding motion relative to an inertial observer with the constant velocity value $\boldsymbol{u}$ which is not $\mathbf{g}$-orthogonal to $\boldsymbol{a}_\text{o}$.

4. Let x and y be different world points simultaneous with respect to an observer (x : a plane lands in London at 12.00; y : a train leaves Paris at 12.00). Then there is an observer which observes that x is later than y and there is an observer that observes that x is earlier then y.

5. We have a clock that can measure a proper time period of $10^{-8}\boldsymbol{s}$. At which relative velocity magnitude can we measure a time dilation in a minute? (Keep in mind that $1 \equiv (2{,}9979\ldots)10^8\boldsymbol{m}/\boldsymbol{s}$.)

6. Let $\boldsymbol{u} \in \mathrm{V}(1)$, $\boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $|\boldsymbol{v}| < 1$. Let $\boldsymbol{t}_\text{o} \in \mathbf{I}^+$. Consider the world line function r that passes through the world point x_o and

$$(i) \qquad \dot{r}(\boldsymbol{t}) = \frac{\boldsymbol{u}+\boldsymbol{v}\sin(\boldsymbol{t}/\boldsymbol{t}_\text{o})}{\sqrt{1-|\boldsymbol{v}|^2\sin^2(\boldsymbol{t}/\boldsymbol{t}_\text{o})}} \qquad (\boldsymbol{t} \in \mathbf{I});$$

$$(ii) \qquad \dot{r}(\boldsymbol{t}) = \boldsymbol{u}\sqrt{1+|\boldsymbol{v}|^2\sin^2(\boldsymbol{t}/\boldsymbol{t}_\text{o})} + \boldsymbol{v}\sin(\boldsymbol{t}/\boldsymbol{t}_\text{o}) \qquad (\boldsymbol{t} \in \mathbf{I}).$$

Prove that the inertial world line $x_\text{o} + \boldsymbol{u}\otimes\mathbf{I}$ and the world line Ran r intersect each other in $x_\text{o} + 2\pi n\boldsymbol{t}_\text{o}$ for all integers n.

Evidently, $\boldsymbol{t}_\text{o}$ is the time passed along Ran r between two consecutive intersections. Estimate the time passed between two consecutive intersections along the inertial world line.

6. Some special non-inertial observers*

6.1. General observers

In most of the textbooks one says that special relativity concerns only inertial observers, non-inertial observers require the authority of general relativity. We emphasize that this is not true.

The difference between special relativity and general relativity does not lie in observers which is evident from our point of view: spacetime models are defined without the notion of observers; on the contrary, observers are defined by means of spacetime models.

Non-inertial observers are right objects in the special relativistic spacetime model. Inertial observers and non-inertial observers differ only in the level of mathematical tools they require. Inertial observers remain in the nice and simple framework of affine spaces while the deep treatment of non-inertial observers needs the same mathematical tools as the treatment of general relativistic spacetime models: the theory of pseudo-Riemannian manifolds.

Fortunately, to describe some special and important aspects of non-inertial special relativistic observers, we can avoid the theory of manifolds; nevertheless, we shall meet some complications.

6.2. Simultaneities

6.2.1. The general notion of observers was given in 3.1.1.

The space of any observer has a simple and natural meaning but time can be defined in a satisfactory way only for inertial observers. (In fact we have considered global inertial observers for the sake of simplicity; globality permits avoiding some complications connected with domains of functions.)

Why the synchronization procedure, i.e. the method of establishing simultaneity by light signals is not completely satisfactory for a non-inertial observer?

Let us take two space points q and q' of a non-inertial observer $\boldsymbol{U}$. A light signal starting at the world point x^- incident with q meets q' at y; the reflected light signal meets q at x^+. Then the world point x incident with q would be considered simultaneous with y if the proper time passed between x^- and x equals the proper time passed between x and x^+.

Unfortunately, the simultaneity defined by light signals starting from q' does not necessarily coincide with the simultaneity defined by light signals starting from q (see Exercise 6.9.12).

Let us accept that simultaneity defined by light signals works well "infinitesimally". This means that the $\boldsymbol{U}$-line passing through the world point x, in a neighbourhood of x, can be approximated by a straight line directed by $\boldsymbol{U}(x)$.

Thus we can say that in a neighbourhood of x the world points approximately simultaneous with x according to $\boldsymbol{U}$ are the elements of $x + \mathbf{E}_{\boldsymbol{U}(x)}$. The smaller the neighbourhood is the better the approximation we get. A clear reasoning leads us then to the idea that world points simultaneous with each other according to $\boldsymbol{U}$ would constitute a hypersurface whose tangent space at every x equals $\mathbf{E}_{\boldsymbol{U}(x)}$. Such a definition of simultaneity does not depend on the $\boldsymbol{U}$-space point ($\boldsymbol{U}$-line) from which light signals start. However, it may happen that there is no such hypersurface at all (see 6.7.6)! And even if such hypersurfaces exist, it may happen that the proper times passed between two such hypersurfaces along different $\boldsymbol{U}$-lines are different (see 6.6.5), thus the simultaneity is not satisfactory in all respects.

6.2.2. In general, there is no *natural* simultaneity with respect to a non-inertial observer; consequently, there is no natural time of such an observer. Of course, a non-inertial observer can choose some sort of *artificial* simultaneity (e.g. chooses one of its space points and makes the synchronization procedure by light signals relative to this space point; on the earth one makes such a synchronization relative to Greenwich).

Now we shall study what a simultaneity should mean.

First we deal with world surfaces which are necessary for simultaneities.

Definition. A *world surface* is a connected three-dimensional smooth submanifold in M whose every tangent space is a spacelike linear subspace of $\mathbf{M}$. ■

If F is a world surface, then for $x \in \mathrm{F}$ there is a unique $\boldsymbol{u}(x) \in \mathrm{V}(1)$ such that $\mathrm{T}_x(\mathrm{F}) = \mathbf{E}_{\boldsymbol{u}(x)}$.

We can prove, similarly to the corresponding assertion for world lines, that if F is a world surface and $x \in \mathrm{F}$, then $\mathrm{F} \setminus \{x\} \subset \mathrm{S}$.

Consequently, if F is a world surface and C is a world line then $\mathrm{C} \cap \mathrm{F}$ is either void or contains a single element which we shall denote by $\mathrm{C} \star \mathrm{F}$.

6.2.3. Obviously, a simultaneity must be a relation between world points, clearly having the following properties:

— every world point is simultaneous with itself;

— if x is simultaneous with y, then y is simultaneous with x;

— if x is simultaneous with y and y is simultaneous with z, then x is simultaneous with z.

In other words, a simultaneity is to be an equivalence relation.

Evidently, we require some other conditions, too. For instance, timelike separated world points cannot be simultaneous, i.e. simultaneous world points must be spacelike separated.

Moreover, we expect that simultaneity is continuous or differentiable in some sense.

Definition. A *simultaneity* $\mathcal{S}$ on a connected open subset G of M is an equivalence relation on G such that

(i) the equivalence classes are world surfaces;

(ii) $\mathcal{S}$ is smooth in the following sense: to every $x \in \mathrm{G}$ there is a unique $\boldsymbol{U}_{\mathcal{S}}(x) \in \mathrm{V}(1)$ such that the tangent space of the equivalence class (world surface) at x equals $\mathbf{E}_{\boldsymbol{U}_{\mathcal{S}}(x)}$; then $\boldsymbol{U}_{\mathcal{S}} : \mathrm{M} \rightarrowtail \mathrm{V}(1) \subset \frac{\mathbf{M}}{\mathbf{I}}$ is required to be smooth.

The *time* corresponding to the simultaneity $\mathcal{S}$ is the set of the equivalence classes of $\mathcal{S}$. ■

The time corresponding to $\mathcal{S}$ is often called $\mathcal{S}$-time and is denoted by $\mathrm{I}_{\mathcal{S}}$; its elements are the $\mathcal{S}$*-instants* and $\tau_{\mathcal{S}}(x)$ will stand for the $\mathcal{S}$-instant (world surface) containing the world point x.

Evidently, simultaneities exist: e.g. for all $\boldsymbol{u} \in \mathrm{V}(1)$, the simultaneity defined by the corresponding global inertial observer: x and y are simultaneous if and only if $\boldsymbol{u} \cdot (x - y) = \mathbf{0}$.

6.2.4. If $\mathcal{S}$ is a simultaneity then $\boldsymbol{U}_{\mathcal{S}}$ is an observer; in other words, a unique observer corresponds to every simultaneity. On the other hand, there are observers to which no simultaneity corresponds in a natural way.

Definition. An observer $\boldsymbol{U}$ is called *regular* if there is a (necessarily unique) simultaneity $\mathcal{S}$ such that $\boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U}$. ■

We mention again that there are non-regular observers (see 6.7.6).

The simultaneity due to a regular observer $\boldsymbol{U}$ is called $\boldsymbol{U}$*-simultaneity* and the corresponding time, denoted by $\mathrm{I}_{\boldsymbol{U}}$, is called $\boldsymbol{U}$*-time.* The elements of $\mathrm{I}_{\boldsymbol{U}}$ are world surfaces; they are called $\boldsymbol{U}$*-surfaces* (regarded as subsets of M) or $\boldsymbol{U}$*-instants* (regarded as elements of $\mathrm{I}_{\boldsymbol{U}}$). The $\boldsymbol{U}$-surface containing the world point x is denoted by $\tau_{\boldsymbol{U}}(x)$.

Let $\boldsymbol{U}$ be an arbitrary observer. A simultaneity on the domain of $\boldsymbol{U}$ which does not equal $\boldsymbol{U}$-simultaneity is called *artificial with respect to* $\boldsymbol{U}$ and then we say *artificial time, artificial instants.*

6.2.5. Definition. Let $\mathcal{S}$ be a simultaneity, $t, s \in \mathrm{I}_{\mathcal{S}}$. We say that s is *later* than t (t is *earlier* than s) if there are $x \in t$ and $y \in s$ such that y is later than x. ■

It cannot occur that both of t and s are earlier than the other. Indeed, let s, t and x, y be as in the definition. Then for all $y' \in s$, $x' \in t$ we have $y' - x' = (y' - y) + (y - x) + (x - x')$. Because of the properties of world surfaces, $y' - y$ and $x - x'$ are spacelike vectors. Thus, in view of Exercise V.4.22.2, if $y - x \in \mathrm{T}^{\rightarrow}$ (t is earlier than s) then $y' - x' \notin \mathrm{T}^{\leftarrow}$ (s is not earlier than t).

We easily find that "later" is an ordering (a reflexive, antisymmetric and transitive relation) on $\mathrm{I}_{\mathcal{S}}$. However, it need not be total: there can be t and s in $\mathrm{I}_{\mathcal{S}}$ such that neither of them is later than the other:

We say that the simultaneity $\mathcal{S}$ is *well posed* if the relation "later" on $\mathrm{I}_{\mathcal{S}}$ is a total ordering.

It can be shown that every world point x_o in the domain of $\mathcal{S}$ has a neighbourhood such that the restriction of $\mathcal{S}$ onto this neighbourhood is well posed. If C_o is the $\boldsymbol{U}_{\mathcal{S}}$-line passing through x_o, then $\{x \in \mathrm{M} | \, \mathrm{C}_o \cap \tau_{\mathcal{S}}(x) \neq \emptyset\}$ is such a neighbourhood.

6.2.6. An observer $\boldsymbol{U}$, together with a simultaneity $\mathcal{S}$ on $\mathrm{Dom}\, \boldsymbol{U}$ split a part of spacetime (the domain of $\boldsymbol{U}$) into $\mathcal{S}$-time and $\boldsymbol{U}$-space:

$$H_{\boldsymbol{U},\mathcal{S}} : \ \mathrm{M} \rightarrowtail \mathrm{I}_{\mathcal{S}} \times \mathrm{E}_{\boldsymbol{U}}, \qquad x \mapsto \big(\tau_{\mathcal{S}}(x), C_{\boldsymbol{U}}(x)\big).$$

6.3. Distances in observer spaces

6.3.1. How distances are measured in an observer space? Let $\boldsymbol{U}$ be an observer and suppose a simultaneity $\mathcal{S}$ is given on the domain of $\boldsymbol{U}$. We should like to determine the distance between two $\boldsymbol{U}$-space points q and q' at an $\mathcal{S}$-instant t.

First we make the following heuristic considerations. Let us put $x := q \star t$ and suppose q' is "near" to q. According to the "infinitesimal" simultaneity which is reasonable from the point of view of the observer, $y' := q' \star (x + \mathbf{E}_{\boldsymbol{U}(x)})$ is the world point on q' that is approximately simultaneous with x in a natural way. Then $\boldsymbol{d} := |y' - x|$ is the approximate value of the distance to be determined.

The world point $x' := q' \star t$ is simultaneous with x according to $\mathcal{S}$. Since $y' \approx x' + \boldsymbol{U}(x') \frac{\boldsymbol{U}(x)\cdot(x'-x)}{-\boldsymbol{U}(x)\cdot \boldsymbol{U}(x')} \approx x' + \boldsymbol{U}(x)\big(\boldsymbol{U}(x) \cdot (x' - x)\big)$, we see that $\boldsymbol{d} \approx |\boldsymbol{\pi}_{\boldsymbol{U}(x)} \cdot (x' - x)|$.

We have got a formula for "infinitesimal" distances from which we can define the length of a curve in a natural way by an integration. The distance between two observer space points will be defined to be the least length of curves connecting the space points.

Before going further, the reader is asked to study Section VI.7.

6.3.2. Definition. Let $\boldsymbol{U}$ be an observer. A subset L of the observer space $\mathrm{E}_{\boldsymbol{U}}$ is called a *curve* if there is a simultaneity $\mathcal{S}$ on Dom $\boldsymbol{U}$ such that $\mathrm{L}_t := \{q \star t \mid q \in \mathrm{L},\ q \cap t \neq \emptyset\}$ is either void or a curve in M for all $t \in \mathrm{I}_{\mathcal{S}}$. ∎

Note that in fact L_t is contained in the hypersurface t.

We say that the curve L *connects* the $\boldsymbol{U}$-space points q_1 and q_2 if L_t connects $q_1 \star t$ and $q_2 \star t$ for all $t \in \mathrm{I}_{\mathcal{S}}$, provided that $q_1 \cap t$ and $q_2 \cap t$ are not void.

6.3.3. Definition. Let L be a curve in $\mathbf{E}_{\boldsymbol{U}}$. Then

$$\ell_t(\mathrm{L}) := \ell_{\boldsymbol{U}}(\mathrm{L}_t) := \int_{\mathrm{L}_t} |\boldsymbol{\pi}_{\boldsymbol{U}(\cdot)} \mathrm{dL}_t|$$

is called the *length* of L at the $\mathcal{S}$-time point t.

The *distance* between the $\boldsymbol{U}$-space points q and q' at t is

$$\begin{aligned}\boldsymbol{d}_t(q,q') &:= \\ &:= \inf\{\ell_t(\mathrm{L}) \mid \mathrm{L} \text{ is a curve connecting } q \text{ and } q'\}. \quad ∎\end{aligned}$$

It is worth describing explicitly that if p_t is a parametrization of L_t then

$$\begin{aligned}\ell_t(\mathrm{L}) &= \int_{\mathrm{Dom}\ p_t} |\boldsymbol{\pi}_{\boldsymbol{U}(p(a))} \cdot \dot{p}_t(a)| \mathrm{d}a = \\ &= \int_{\mathrm{Dom}\ p_t} \sqrt{|\dot{p}_t(a)|^2 + \big(\boldsymbol{U}(p(a)) \cdot \dot{p}_t(a)\big)^2} \mathrm{d}a.\end{aligned}$$

Note the special case when $\boldsymbol{U}$ is regular and $\mathcal{S}$ is the $\boldsymbol{U}$-simultaneity; then $\boldsymbol{\pi}_{\boldsymbol{U}(x)} \cdot \boldsymbol{x} = \boldsymbol{x}$ if $x \in t$ and $\boldsymbol{x}$ is a tangent vector of t at x. Consequently,

$$\ell_t(\mathrm{L}) = \int_{\mathrm{L}_t} |\mathrm{dL}_t| = \int_{\mathrm{Dom}\ p_t} |\dot{p}_t(a)|\mathrm{d}a \qquad \text{for a regular observer.}$$

In particular, if $\boldsymbol{U}$ is inertial and $\boldsymbol{U}$-time is used then, for all t, $\boldsymbol{d}_t(q, q')$ equals the distance $|q' - q|$ defined earlier.

Keep in mind the following important remark: suppose the $\mathcal{S}$-instant t is a hyperplane; then there is a unique $\boldsymbol{u}_o \in \mathrm{V}(1)$ such that t is directed by $\mathbf{E}_{\boldsymbol{u}_o}$. The distance between the $\boldsymbol{U}$-space points at t does not equal, in general, the distance observed by the inertial observer with the velocity value $\boldsymbol{u}_o$. Recall, e.g. the case that $\boldsymbol{U}$ is an inertial observer with the velocity value $\boldsymbol{u}$ (see Section 5.3).

6.3.4. Definition. The observer $\boldsymbol{U}$ is called *rigid* if there is a simultaneity $\mathcal{S}$ on Dom $\boldsymbol{U}$ in such a way that if
— L is an arbitrary curve in $\mathrm{E}_{\boldsymbol{U}}$,
— $t, t' \in \mathrm{I}_{\mathcal{S}}$ and $q \cap t \neq \emptyset$, $q \cap t' \neq \emptyset$ for all $q \in \mathrm{L}$,
then $\ell_t(\mathrm{L}) = \ell_{t'}(\mathrm{L})$. ■

Note that rigidity of observers is a highly complicated notion in the special relativistic spacetime model, in contradistinction to the non-relativistic case.

6.3.5. The following assertions can be proved by means of the tools of smooth manifolds.

(*i*) Our definition of a curve in $\mathrm{E}_{\boldsymbol{U}}$ involves a simultaneity; nevertheless, it does not depend on simultaneity: if there is a simultaneity with the required conditions then these conditions are satisfied for all other simultaneities as well.

(*ii*) The distance between two $\boldsymbol{U}$-space points at an $\mathcal{S}$-instant is defined by an infimum; this infimum is in fact a minimum, i.e. for each $\mathcal{S}$-instant t there is a curve connecting the points whose length at t equals the distance between the $\boldsymbol{U}$-space points at t.

(*iii*) Our definition of rigidity involves a simultaneity; nevertheless, it does not depend on simultaneity: if there is a simultaneity with respect to which the observer is rigid, then the observer is rigid with respect to all other simultaneities as well.

6.4. A method of finding the observer space

6.4.1. To find the space of an observer, i.e. the $\boldsymbol{U}$-lines, we have to find the solutions of the differential equation

$$(x : \mathbf{I} \rightarrowtail \mathrm{M})? \qquad \dot{x} = \boldsymbol{U}(x).$$

A frequently applicable method is to transform the differential equation by

$$\boldsymbol{h}_{\boldsymbol{u}_o,o}:\ \mathrm{M}\to\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}_o},\qquad x\mapsto\big(-\boldsymbol{u}_o\cdot(x-o),\ \boldsymbol{\pi}_{\boldsymbol{u}_o}\cdot(x-o)\big)$$

according to VI.6.3, where $\boldsymbol{u}_o$ is a suitably chosen element of V(1).

The transformed differential equation will have the form

$$\big((\boldsymbol{t},\boldsymbol{q}):\ \mathbf{I}\rightarrowtail\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}_o}\big)?\qquad(\boldsymbol{t},\boldsymbol{q})^{\cdot}=\big(-\boldsymbol{u}_o\cdot\boldsymbol{U}(o+\boldsymbol{u}_o\boldsymbol{t}+\boldsymbol{q}),\boldsymbol{\pi}_{\boldsymbol{u}_o}\cdot\boldsymbol{U}(o+\boldsymbol{u}_o\boldsymbol{t}+\boldsymbol{q})\big),$$

i.e.

$$(\boldsymbol{t}:\ \mathbf{I}\rightarrowtail\mathbf{I})?\qquad\dot{\boldsymbol{t}}=-\boldsymbol{u}_o\cdot\boldsymbol{U}(o+\boldsymbol{u}_o\boldsymbol{t}+\boldsymbol{q}),$$

$$(\boldsymbol{q}:\ \mathbf{I}\rightarrowtail\mathbf{E}_{\boldsymbol{u}_o})?\qquad\dot{\boldsymbol{q}}=\boldsymbol{\pi}_{\boldsymbol{u}_o}\cdot\boldsymbol{U}(o+\boldsymbol{u}_o t+\boldsymbol{q}).$$

Let $\boldsymbol{s}\mapsto\boldsymbol{t}(\boldsymbol{s})$ and $\boldsymbol{s}\mapsto\boldsymbol{q}(\boldsymbol{s})$ denote the solutions of these differential equations with the initial conditions $\boldsymbol{t}(\mathbf{0})=\mathbf{0}$, $\boldsymbol{q}(\mathbf{0})=\boldsymbol{q}_o$, where $\boldsymbol{q}_o$ is an arbitrary element in $\mathbf{E}_{\boldsymbol{u}_o}$ such that $o+\boldsymbol{q}_o$ is in the domain of $\boldsymbol{U}$. Then

$$\boldsymbol{s}\mapsto o+\boldsymbol{u}_o\boldsymbol{t}(\boldsymbol{s})+\boldsymbol{q}(\boldsymbol{s})$$

is the world line function giving the $\boldsymbol{U}$-line passing through $o+\boldsymbol{q}_o$.

It is worth using more precise notations: let x be an element of $(o+\mathbf{E}_{\boldsymbol{u}_o})\cap(\mathrm{Dom}\,\boldsymbol{U})$; then $\boldsymbol{s}\mapsto\boldsymbol{t}_x(\boldsymbol{s})$ and $\boldsymbol{s}\mapsto\boldsymbol{q}_x(\boldsymbol{s})$ will denote the solutions of the differential equations with the initial conditions $\boldsymbol{t}_x(\mathbf{0})=\mathbf{0}$, $\boldsymbol{q}_x(\mathbf{0})=x-o$. Then

$$\mathbf{I}\rightarrowtail\mathrm{M},\qquad\boldsymbol{s}\mapsto r_x(\boldsymbol{s})=o+\boldsymbol{u}_o\boldsymbol{t}_x(\boldsymbol{s})+\boldsymbol{q}_x(\boldsymbol{s})$$

is the world line function giving the $\boldsymbol{U}$-space point that x is incident with.

6.4.2. Consider the global inertial observer $\boldsymbol{U}_o$ with constant velocity value $\boldsymbol{u}_o$ and $\boldsymbol{U}_o$-time as an artificial time for the observer $\boldsymbol{U}$. Then, according to 4.1.1, $\boldsymbol{s}\mapsto\boldsymbol{t}_x(\boldsymbol{s})$ gives $\boldsymbol{U}_o$-time as a function of the proper time of the $\boldsymbol{U}$-line passing through x; in other words, $\boldsymbol{t}_x(\boldsymbol{s})$ is the $\boldsymbol{U}_o$-time passed between $t_o:=o+\mathbf{E}_{\boldsymbol{u}_o}$ and $t_x(\boldsymbol{s}):=r_x(\boldsymbol{s})+\mathbf{E}_{\boldsymbol{u}_o}:\ \boldsymbol{t}_x(\boldsymbol{s})=t_x(\boldsymbol{s})-t_o$.

This function is strictly monotone increasing; its inverse, denoted by $\mathbf{I}\rightarrowtail\mathbf{I}$, $\boldsymbol{t}\mapsto\boldsymbol{s}_x(\boldsymbol{t})$, gives the proper time between the $\boldsymbol{U}_o$-instants t_o and $t_o+\boldsymbol{t}$ passed in the $\boldsymbol{U}$-space point that x is incident with.

6.5. Uniformly accelerated observer I

6.5.1. In the special relativistic spacetime model the definition of a uniformly accelerated observer is not so straightforward as in the non-relativistic case. We know that here the acceleration of a uniformly accelerated world line function is not constant, thus a uniformly accelerated observer will not be an observer with constant acceleration field. Anyhow, we wish to find an observer whose lines are uniformly accelerated.

Omitting the thorny way of searching, let us take an observer satisfying the requirements and study its properties.

Let $o \in \mathrm{M}$, $\boldsymbol{u}_o \in \mathrm{V}(1)$ and $\boldsymbol{0} \neq \boldsymbol{a}_o \in \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I}\otimes\mathbf{I}}$ and define the global observer

$$\boldsymbol{U}(x) := \boldsymbol{u}_o\sqrt{1 + |\boldsymbol{a}_o|^2\big(\boldsymbol{u}_o \cdot (x - o)\big)^2} - \boldsymbol{a}_o\big(\boldsymbol{u}_o \cdot (x - o)\big) \qquad (x \in \mathrm{M}).$$

Note that $\boldsymbol{u}_o = \boldsymbol{U}(o)$.

The observer has the acceleration field

$$\boldsymbol{A}_{\boldsymbol{U}}(x) = \boldsymbol{a}_o\sqrt{1 + |\boldsymbol{a}_o|^2\big(\boldsymbol{u}_o \cdot (x - o)\big)^2} - \boldsymbol{u}_o|\boldsymbol{a}_o|^2\big(\boldsymbol{u}_o \cdot (x - o)\big) \qquad (x \in \mathrm{M}),$$

thus $\boldsymbol{a}_o = \boldsymbol{A}_{\boldsymbol{U}}(o)$.

It is trivial that

$$\boldsymbol{U}(x + \boldsymbol{q}) = \boldsymbol{U}(x), \qquad \boldsymbol{A}_{\boldsymbol{U}}(x + \boldsymbol{q}) = \boldsymbol{A}_{\boldsymbol{U}}(x) \qquad (x \in \mathrm{M},\ \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}_o}),$$

i.e. $\boldsymbol{U}$ and $\boldsymbol{A}_{\boldsymbol{U}}$ are constant on the hyperplanes directed by $\mathbf{E}_{\boldsymbol{u}_o}$.

As a consequence, the translation of a $\boldsymbol{U}$-line by a vector in $\mathbf{E}_{\boldsymbol{u}_o}$ is a $\boldsymbol{U}$-line, too.

6.5.2. Transforming the differential equation of the observer according to 6.4, we get

$$\dot{\boldsymbol{t}} = \sqrt{1 + |\boldsymbol{a}_o|^2\boldsymbol{t}^2},$$
$$\dot{\boldsymbol{q}} = \boldsymbol{a}_o\boldsymbol{t}.$$

The first equation, with the initial value $\boldsymbol{t}(\boldsymbol{0}) = \boldsymbol{0}$, has the solution

$$\boldsymbol{t}(\boldsymbol{s}) = \frac{\mathrm{sh}|\boldsymbol{a}_o|\boldsymbol{s}}{|\boldsymbol{a}_o|} \qquad (\boldsymbol{s} \in \mathbf{I}).$$

Then the second equation becomes very simple and we find its solutions in the form

$$\boldsymbol{q}(\boldsymbol{s}) = \boldsymbol{a}_o\frac{\mathrm{ch}|\boldsymbol{a}_o|\boldsymbol{s} - 1}{|\boldsymbol{a}_o|^2} + \boldsymbol{q}_o \qquad (\boldsymbol{s} \in \mathbf{I}).$$

Hence we obtain that the $\boldsymbol{U}$-line passing through $x \in o + \mathbf{E}_{\boldsymbol{u}_o}$ is given by the world line function

$$r_x(\boldsymbol{s}) = x + \boldsymbol{u}_o\frac{\mathrm{sh}|\boldsymbol{a}_o|\boldsymbol{s}}{|\boldsymbol{a}_o|} + \boldsymbol{a}_o\frac{\mathrm{ch}|\boldsymbol{a}_o|\boldsymbol{s} - 1}{|\boldsymbol{a}_o|^2} \qquad (\boldsymbol{s} \in \mathbf{I}).$$

It is not hard to see that every $\boldsymbol{U}$-line meets the hyperplane $o + \mathbf{E}_{\boldsymbol{u}_o}$, hence every $\boldsymbol{U}$-line can be given by such a world line function: all $\boldsymbol{U}$-lines are uniformly accelerated.

6.5.3. Because $\boldsymbol{U}$ and $\boldsymbol{A}_{\boldsymbol{U}}$ are constant on the hyperplanes directed by $\mathbf{E}_{\boldsymbol{u}_o}$, all $\boldsymbol{U}$-lines have the same velocity and the same acceleration on the hyperplanes directed by $\mathbf{E}_{\boldsymbol{u}_o}$.

Using the notations of 6.4, we see that

$$\boldsymbol{s}_x(\boldsymbol{t}) = \frac{\operatorname{arsh}|\boldsymbol{a}_o|\boldsymbol{t}}{|\boldsymbol{a}_o|} =: \boldsymbol{s}(\boldsymbol{t}) \qquad (\boldsymbol{t} \in \mathbf{I})$$

for all x in $o + \mathbf{E}_{\boldsymbol{u}_o}$. Thus, given two $\boldsymbol{U}_o$-instants, the same time passes along all $\boldsymbol{U}$-lines between them.

Because of these properties of $\boldsymbol{U}$-lines it seems suitable to associate with $\boldsymbol{U}$ the $\boldsymbol{U}_o$-simultaneity and the $\boldsymbol{U}_o$-time as an artificial time.

6.5.4. Let us examine whether this observer is regular.

Evidently, $o + \mathbf{E}_{\boldsymbol{u}_o}$ is a world surface $\mathbf{g}$-orthogonal to $\boldsymbol{U}$.

Introduce the notation

$$h(x) :=$$

$$:= \boldsymbol{a}_o \cdot (x - o) - \sqrt{1 + |\boldsymbol{a}_o|^2 \big(\boldsymbol{u}_o \cdot (x - o)\big)^2} + \operatorname{arcth}\sqrt{1 + |\boldsymbol{a}_o|^2 \big(\boldsymbol{u}_o \cdot (x - o)\big)^2}$$

for $x \in \mathrm{M}$, $x \notin o + \mathbf{E}_{\boldsymbol{u}_o}$ and put for $\lambda \in \mathbb{R}$

$$t_\lambda := \begin{cases} \{x \in \mathrm{M} |\ h(x) = \ln\lambda,\ -\boldsymbol{u}_o \cdot (x - o) > \mathbf{0}\} & \text{if } \lambda > 0 \\ o + \mathbf{E}_{\boldsymbol{u}_o} & \text{if } \lambda = 0 \\ \{x \in \mathrm{M} |\ h(x) = \ln(-\lambda),\ -\boldsymbol{u}_o \cdot (x - o) < 0\} & \text{if } \lambda < 0. \end{cases}$$

Evidently, h is a differentiable function outside $o+\mathbf{E}_{\boldsymbol{u}_o}$ and

$$\mathrm{D}h(x)=\boldsymbol{a}_o+\frac{\boldsymbol{u}_o\sqrt{1+|\boldsymbol{a}_o|^2\big(\boldsymbol{u}_o\cdot(x-o)\big)^2}}{-\boldsymbol{u}_o\cdot(x-o)}=\frac{\boldsymbol{U}(x)}{-\boldsymbol{u}_o\cdot(x-o)}.$$

As a consequence, for all $\lambda\in\mathbb{R}$, t_λ is a three-dimensional submanifold whose tangent space at x equals $\mathrm{Ker}\ \mathrm{D}h(x)=\mathbf{E}_{\boldsymbol{U}(x)}$. This means that t_λ-s are $\boldsymbol{U}$-surfaces, $\boldsymbol{U}$ is regular and

$$\mathrm{I}_{\boldsymbol{U}}=\{t_\lambda|\ \lambda\in\mathbb{R}\}.$$

6.5.5. If q_1 and q_2 are $\boldsymbol{U}$-lines, then $q_2\star t-q_1\star t$ is the same for all $\boldsymbol{U}_o$-instants t. In other words, the vector and the distance observed by the inertial observer $\boldsymbol{U}_o$ between two $\boldsymbol{U}$-space points is the same for all $\boldsymbol{U}_o$-instants. We can say that $\boldsymbol{U}_o$ observes $\boldsymbol{U}$ to be rigid and rotation-free. Is $\boldsymbol{U}$ rigid and rotation-free?

We have not defined when an observer is rotation-free, thus we can answer only the question regarding rigidity as defined in 6.3.4.

This observer $\boldsymbol{U}$ is not rigid. Let us take a $\boldsymbol{U}_o$-instant t. For all $x\in t$ we have $-\boldsymbol{u}_o\cdot(x-o)=t-t_o$ (where $t_o:=o+\mathbf{E}_{\boldsymbol{u}_o}$), thus

$$\boldsymbol{U}(x)=\boldsymbol{a}_o(t-t_o)+\boldsymbol{u}_o\sqrt{1+|\boldsymbol{a}_o|^2(t-t_o)^2}=:\boldsymbol{u}_t \qquad (x\in t,\ t\in\mathrm{I}_{\boldsymbol{U}_o})$$

($\boldsymbol{U}$ is constant on the $\boldsymbol{U}_o$-instants).

The formula in 6.5.3 gives us the proper time $\boldsymbol{s}(\boldsymbol{t})$ passed in every $\boldsymbol{U}$-space point between the $\boldsymbol{U}_o$-instants (artificial time points) t_o and $t:=t_o+\boldsymbol{t}$:

$$\boldsymbol{t}=\frac{\mathrm{sh}|\boldsymbol{a}_o|\boldsymbol{s}(\boldsymbol{t})}{|\boldsymbol{a}_o|}.$$

Let L_o be a curve in $o + \mathbf{E}_{\boldsymbol{u}_o}$; then the set of $\boldsymbol{U}$-space points that meet L_o, $\mathrm{L} := \{q \in \mathrm{E}_{\boldsymbol{U}} |\ q \cap \mathrm{L}_o \neq \emptyset\}$ is a curve in the observer space. Indeed, if p_o is a parametrization of L_o, then

$$p_t := p_o + \boldsymbol{u}_o \boldsymbol{t} + \frac{\boldsymbol{a}_o}{|\boldsymbol{a}_o|^2}\left(\sqrt{1 + |\boldsymbol{a}_o|^2 \boldsymbol{t}^2} - 1\right)$$

is a parametrization of L_t (see 6.3.2) for $t = t_o + \boldsymbol{t} \in \mathrm{I}_{\boldsymbol{U}_o}$.

Then

$$\dot{p}_t = \dot{p}_o$$

and

$$\boldsymbol{U}\big(p_t(a)\big) \cdot \dot{p}_t(a) = -\boldsymbol{a}_o \cdot \dot{p}_o(a)\boldsymbol{t} \qquad (a \in \mathrm{Dom}\ p_o);$$

consequently,

$$|\dot{p}_t|^2 + \big((\boldsymbol{U} \circ p_t) \cdot \dot{p}_t\big)^2 = |\dot{p}_o|^2 + \big(\boldsymbol{a}_o \cdot \dot{p}_o\big)^2 \boldsymbol{t}^2,$$

which shows that the length of curves depends on the artificial time points: the observer is not rigid.

6.5.6. The length of curves in $\boldsymbol{U}$-space, consequently the distance between $\boldsymbol{U}$-space points, in general, decreases prior to t_o and increases after t_o, as $\boldsymbol{U}_o$-time passes. This is well understandable from a heuristic point of view. Though we defined Lorentz contraction between two inertial observers, we can say e.g. that after t_o the space points of $\boldsymbol{U}$ move faster and faster with respect to $\boldsymbol{U}_o$, thus their distances seem more and more contracted with respect to $\boldsymbol{U}_o$; that is, their distances must increase continually in order that the distances observed by $\boldsymbol{U}_o$ be constant.

6.6. Uniformly accelerated observer II

6.6.1. Let $o \in \mathrm{M}$, $\boldsymbol{u}_o \in \mathrm{V}(1)$ and $\mathbf{0} \neq \boldsymbol{a}_o \in \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I}\otimes\mathbf{I}}$, put

$$\boldsymbol{B}(x) := \big(\boldsymbol{a}_o \cdot (x - o)\big)\boldsymbol{u}_o - \big(\boldsymbol{u}_o \cdot (x - o)\big)\boldsymbol{a}_o$$

for $x \in \mathrm{M}$ and define the non-global observer by

$$\mathrm{Dom}\ \boldsymbol{U} := \{x \in \mathbf{M} |\ \boldsymbol{B}(x) \text{ is future-directed timelike}\}$$

$$\boldsymbol{U}(x) := \frac{\boldsymbol{B}(x)}{|\boldsymbol{B}(x)|} \qquad (x \in \mathrm{Dom}\ \boldsymbol{U}).$$

Note that $\boldsymbol{B}(x)$ is future-directed timelike if and only if

$$\mathbf{0} > \big(\boldsymbol{B}(x)\big)^2 = -\big(\boldsymbol{a}_o \cdot (x - o)\big)^2 + \big(\boldsymbol{u}_o \cdot (x - o)\big)^2 |\boldsymbol{a}_o|^2,$$
$$\mathbf{0} > \boldsymbol{u}_o \cdot \boldsymbol{B}(x) = -\boldsymbol{a}_o \cdot (x - o).$$

Then we find that a world point x for which $x-o$ lies in the plane generated by $\boldsymbol{u}_o$ and $\boldsymbol{a}_o$ is in the domain of $\boldsymbol{U}$ if and only if $x-o$ is spacelike and $\boldsymbol{a}_o\cdot(x-o) > \mathbf{0}$.

6.6.2. If $\boldsymbol{q}$ is a world vector $\mathbf{g}$-orthogonal to both $\boldsymbol{u}_o$ and $\boldsymbol{a}_o$ then

$$\mathrm{Dom}\,\boldsymbol{U} + \boldsymbol{q} = \mathrm{Dom}\,\boldsymbol{U}$$

and

$$\boldsymbol{U}(x+\boldsymbol{q}) = \boldsymbol{U}(x) \qquad (x \in \mathrm{Dom}\,\boldsymbol{U},\ \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}_o},\ \boldsymbol{a}_o \cdot \boldsymbol{q} = 0).$$

The observer has the acceleration field

$$\boldsymbol{A}_{\boldsymbol{U}}(x) = \frac{\big(\boldsymbol{a}_o \cdot (x-o)\big)\boldsymbol{a}_o - |\boldsymbol{a}_o|^2\big(\boldsymbol{u}_o \cdot (x-o)\big)\boldsymbol{u}_o}{|\boldsymbol{B}(x)|^2} \qquad (x \in \mathrm{Dom}\,\boldsymbol{U}).$$

Then we easily find that

$\boldsymbol{U}(x) = \boldsymbol{u}_o$	if and only if $x-o$ is parallel to $\boldsymbol{a}_o$,
$\boldsymbol{A}_{\boldsymbol{U}}(x) \neq \boldsymbol{a}_o$	for all $x \in \mathrm{Dom}\,\boldsymbol{U}$,
$\boldsymbol{A}_{\boldsymbol{U}}(x) = \frac{\boldsymbol{a}_o}{\boldsymbol{a}_o\cdot(x-o)}$	if and only if $x-o$ is parallel to $\boldsymbol{a}_o$.

6.6.3. Let us introduce the notation

$$\boldsymbol{n}_o := \frac{\boldsymbol{a}_o}{|\boldsymbol{a}_o|}.$$

If $\lambda \in \mathbb{R}$ then

$$\boldsymbol{u}_\lambda := \frac{\boldsymbol{u}_o + (\mathrm{th}\lambda)\boldsymbol{n}_o}{\sqrt{1-(\mathrm{th}\lambda)^2}} = \boldsymbol{u}_o\mathrm{ch}\lambda + \boldsymbol{n}_o\mathrm{sh}\lambda$$

is in $\mathrm{V}(1)$ and we easily find that

$$\boldsymbol{U}(x) = \boldsymbol{u}_\lambda \qquad (x \in \mathrm{Dom}\,\boldsymbol{U},\ x-o \in \mathbf{E}_{\boldsymbol{u}_\lambda}).$$

Thus $t_\lambda := \big(o + \mathbf{E}_{\boldsymbol{u}_\lambda}\big) \cap (\mathrm{Dom}\,\boldsymbol{U})$ is a $\boldsymbol{U}$-surface. To every $x \in \mathrm{Dom}\,\boldsymbol{U}$ there is such a $\boldsymbol{U}$-surface containing x, given by

$$\lambda_x := \mathrm{arth}\left(\frac{-\boldsymbol{u}_o\cdot(x-o)}{\boldsymbol{n}_o\cdot(x-o)}\right).$$

This means that $\boldsymbol{U}$ is regular, and

$$\mathrm{I}_{\boldsymbol{U}} := \{t_\lambda |\ \lambda \in \mathbb{R}\}.$$

6.6.4. To find the $\boldsymbol{U}$-lines we use the method outlined in 6.4.

Transforming the differential equation $\dot{x} = \boldsymbol{U}(x)$ by $\boldsymbol{h}_{\boldsymbol{u}_o,o}$ we get

$$\dot{\boldsymbol{t}} = \frac{\boldsymbol{n}_o \cdot \boldsymbol{q}}{\sqrt{(\boldsymbol{n}_o \cdot \boldsymbol{q})^2 - \boldsymbol{t}^2}}, \tag{$*$}$$

$$\dot{\boldsymbol{q}} = \frac{\boldsymbol{n}_o \boldsymbol{t}}{\sqrt{(\boldsymbol{n}_o \cdot \boldsymbol{q})^2 - \boldsymbol{t}^2}}. \tag{$**$}$$

Equation $(**)$ implies $\boldsymbol{n}_o \cdot \dot{\boldsymbol{q}} = \frac{\boldsymbol{t}}{\sqrt{(\boldsymbol{n}_o \cdot \boldsymbol{q})^2 - \boldsymbol{t}^2}}$ which, together with equation $(*)$, results in

$$(\boldsymbol{n}_o \cdot \boldsymbol{q})(\boldsymbol{n}_o \cdot \dot{\boldsymbol{q}}) = \boldsymbol{t}\dot{\boldsymbol{t}}$$

implying

$$(\boldsymbol{n}_o \cdot \boldsymbol{q})^2 - \boldsymbol{t}^2 = \text{const} =: \frac{1}{\alpha^2}.$$

Then differentiating equation $(*)$ we obtain

$$\ddot{\boldsymbol{t}} = \alpha^2 \boldsymbol{t}$$

from which — taking the initial values $\boldsymbol{t}(\mathbf{0}) = \mathbf{0}$, $\dot{\boldsymbol{t}}(\mathbf{0}) = 1$ — we infer

$$\boldsymbol{t}(\boldsymbol{s}) = \frac{\operatorname{sh}\alpha\boldsymbol{s}}{\alpha}.$$

As a consequence, equation $(**)$ takes an extremely simple form, and we find its solutions easily:

$$\boldsymbol{q}(\boldsymbol{s}) = \boldsymbol{n}_o \frac{\operatorname{ch}\alpha\boldsymbol{s} - 1}{\alpha} + \boldsymbol{q}_o.$$

Hence we obtain that the $\boldsymbol{U}$-line passing through $x \in \left(o + \mathbf{E}_{\boldsymbol{u}_o}\right) \cap (\operatorname{Dom} \boldsymbol{U})$ is given by the world line function

$$r_x(\boldsymbol{s}) = x + \boldsymbol{u}_o \frac{\operatorname{sh}|\boldsymbol{a}_x|\boldsymbol{s}}{|\boldsymbol{a}_x|} + \boldsymbol{a}_x \frac{\operatorname{ch}|\boldsymbol{a}_x|\boldsymbol{s} - 1}{|\boldsymbol{a}_x|^2} \qquad (\boldsymbol{s} \in \mathbf{I})$$

where

$$\boldsymbol{a}_x := \frac{\boldsymbol{a}_o}{\boldsymbol{a}_o \cdot (x - o)} = \frac{\boldsymbol{n}_o}{\boldsymbol{n}_o \cdot (x - o)}.$$

It is not hard to see that every $\boldsymbol{U}$-line meets the hyperplane $o + \mathbf{E}_{\boldsymbol{u}_o}$, hence every $\boldsymbol{U}$-line can be given by such a world line function; all $\boldsymbol{U}$-lines are uniformly accelerated.

6.6.5. The present observer $\boldsymbol{U}$ serves as an example to show that the observer is regular, but different times pass in different $\boldsymbol{U}$-space points (along different $\boldsymbol{U}$-lines) between two $\boldsymbol{U}$-instants.

Let us consider the $\boldsymbol{U}$-line passing through $x \in o + \mathbf{E}_{\boldsymbol{u}_o}$, described by the world line function r_x given previously; a simple calculation yields that $r_x(\boldsymbol{s})$ is in the $\boldsymbol{U}$-surface t_λ if and only if $\boldsymbol{s} = \frac{\lambda}{|\boldsymbol{a}_x|}$. In other words,

$$\boldsymbol{s}_x(\lambda) := \frac{\lambda}{|\boldsymbol{a}_x|} = \big(\boldsymbol{n}_o \cdot (x - o)\big)\lambda$$

which clearly depends on x, is the time passed between the $\boldsymbol{U}$-time points t_0 and t_λ in the $\boldsymbol{U}$-space point that x is incident with.

6.6.6. Now we shall show that this observer is rigid.

Let L_o be a curve in $o + \mathbf{E}_{\boldsymbol{u}_o}$; then the set of $\boldsymbol{U}$-space points that meet L_o, $\mathrm{L} := \{q \in \mathrm{E}_{\boldsymbol{U}} \,|\, q \cap \mathrm{L}_o \neq \emptyset\}$ is a curve in the observer space. Indeed, if p_o is a parametrization of L_o, then, according to the previous result on proper times,

$$p_{t_\lambda} := p_o + (\boldsymbol{n}_o \cdot p_o - o)\big(\boldsymbol{u}_o \mathrm{sh}\lambda + \boldsymbol{n}_o(\mathrm{ch}\lambda - 1)\big)$$

is a parametrization of L_{t_λ} for all $t_\lambda \in \mathrm{I}_{\boldsymbol{U}}$.

Then

$$\dot{p}_{t_\lambda} = \dot{p}_o + (\boldsymbol{n}_o \cdot \dot{p}_o)\big(\boldsymbol{u}_o \mathrm{sh}\lambda + \boldsymbol{n}_o(\mathrm{ch}\lambda - 1)\big)$$

and

$$|\dot{p}_{t_\lambda}| = |\dot{p}_o| \qquad (t_\lambda \in \mathrm{I}_{\boldsymbol{U}}).$$

Since $\boldsymbol{U}$ is regular and $\boldsymbol{U}$-time is considered, $\boldsymbol{U}(p_{t_\lambda}(a)) \cdot \dot{p}_{t_\lambda}(a) = \mathbf{0}$ for all $\lambda \in \mathbb{R}$ and $a \in \mathrm{Dom}\, p_o = \mathrm{Dom}\, p_{t_\lambda}$, this means that $\ell_{t_\lambda}(\mathrm{L}) = \ell_o(\mathrm{L})$ for all

$t_\lambda \in \mathrm{I}_{\boldsymbol{U}}$. It is not hard to see that every curve in the observer space can be obtained from a curve in $o + \mathbf{E}_{\boldsymbol{u}_o}$ by the previous method; consequently, the observer is rigid.

6.6.7. Two uniformly accelerated observers have been treated. Neither of them possesses all the good properties of the uniformly accelerated observer in the non-relativistic spacetime model. It is an open question whether we can find a special relativistic observer $\boldsymbol{U}$ such that

(i) all $\boldsymbol{U}$-lines are uniformly accelerated,

(ii) $\boldsymbol{U}$ and $\boldsymbol{A_U}$ are constant on each instant (world surface) of an (artificial) time,

(iii) $\boldsymbol{U}$ is rigid.

The observer in 6.4 does not satisfy (iii); the observer in 6.5 does not satisfy (ii).

6.7. Uniformly rotating observer I

6.7.1. In defining the uniformly rotating observer we encounter problems similar to those in the previous section and, in the same manner, we find two possibilities but neither of them possesses all the good properties of the non-relativistic uniformly rotating observer.

Let $o \in \mathrm{M}$, $\boldsymbol{u}_o \in \mathrm{V}(1)$ and let $\Omega : \mathbf{E}_{\boldsymbol{u}_o} \to \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I}}$ be a non-zero antisymmetric linear map and define the global observer

$$\boldsymbol{U}(x) := \Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x - o) + \boldsymbol{u}_o \sqrt{1 + |\Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x - o)|^2} \qquad (x \in \mathrm{M}).$$

Note that

$$\boldsymbol{u}_o = \boldsymbol{U}(o),$$

and

$$\boldsymbol{U}(x + \boldsymbol{q}) = \boldsymbol{U}(x) \qquad (x \in \mathrm{M},\ \boldsymbol{q} \in \mathrm{Ker}\ \Omega).$$

The observer has the acceleration field

$$\boldsymbol{A_U}(x) = \Omega \cdot \Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x - o) \qquad (x \in \mathrm{M}).$$

6.7.2. To find the $\boldsymbol{U}$-lines we apply the well-proved method: transforming the differential equation $\dot{x} = \boldsymbol{U}(x)$ by $\boldsymbol{h}_{\boldsymbol{u}_o,o}$ we get

$$\dot{\boldsymbol{t}} = \sqrt{1 + |\Omega \cdot \boldsymbol{q}|^2},$$
$$\dot{\boldsymbol{q}} = \Omega \cdot \boldsymbol{q}.$$

The second equation can be solved immediately:

$$\boldsymbol{q}(\boldsymbol{s}) = \mathrm{e}^{\boldsymbol{s}\Omega} \cdot \boldsymbol{q}_{\mathrm{o}} \qquad (\boldsymbol{s} \in \mathbf{I}).$$

Then the first equation becomes $\dot{\boldsymbol{t}} = \sqrt{1 + |\Omega \cdot \boldsymbol{q}_{\mathrm{o}}|^2}$ having the solution — with the initial value $\boldsymbol{t}(\mathbf{0}) = \mathbf{0}$ —

$$\boldsymbol{t}(\boldsymbol{s}) = \boldsymbol{s}\sqrt{1 + |\Omega \cdot \boldsymbol{q}_{\mathrm{o}}|^2} \qquad (\boldsymbol{s} \in \mathbf{I}).$$

Thus the $\boldsymbol{U}$-line passing through $x \in o + \mathbf{E}_{\boldsymbol{u}_{\mathrm{o}}}$ (the $\boldsymbol{U}$-space point that x is incident with) is given by the world line function

$$r_x(\boldsymbol{s}) = o + \boldsymbol{u}_{\mathrm{o}}\boldsymbol{s}\sqrt{1 + |\Omega \cdot (x - o)|^2} + \mathrm{e}^{\boldsymbol{s}\Omega} \cdot (x - o) \qquad (\boldsymbol{s} \in \mathbf{I}).$$

It is not hard to see that every $\boldsymbol{U}$-line meets the hyperplane $o + \mathbf{E}_{\boldsymbol{u}_{\mathrm{o}}}$, thus every $\boldsymbol{U}$-line is of this form.

6.7.3. Note that the $\boldsymbol{U}$-line passing through $o + \boldsymbol{e}$, where $\boldsymbol{e}$ is in Ker Ω, is a straight line directed by $\boldsymbol{u}_{\mathrm{o}}$; then the set of $\boldsymbol{U}$-space points

$$\{o + \boldsymbol{e} + \boldsymbol{u}_{\mathrm{o}} \otimes \mathbf{I} |\ \boldsymbol{e} \in \mathrm{Ker}\ \Omega\}$$

can be interpreted as the *axis of rotation.*

If x is in $o + \mathbf{E}_{\boldsymbol{u}_{\mathrm{o}}}$, then $x - o$ can be decomposed into a sum $\boldsymbol{e}_x + \boldsymbol{q}_x$ where $\boldsymbol{e}_x$ is in Ker Ω and $\boldsymbol{q}_x$ is orthogonal to Ker Ω. Then the $\boldsymbol{U}$-line above can be written in the form

$$r_x(\boldsymbol{s}) = o + \boldsymbol{e}_x + \boldsymbol{u}_{\mathrm{o}}\boldsymbol{s}\sqrt{1 + \omega^2|\boldsymbol{q}_x|^2} + \mathrm{e}^{\boldsymbol{s}\Omega} \cdot \boldsymbol{q}_x, \qquad (*)$$

where ω is the magnitude of Ω (see Exercise V.3.21.1).

Hence all the $\boldsymbol{U}$-lines are composed of an inertial line (with a proper time "accelerated" relative to the proper time of the points of the axis) and a uniform rotation.

Let $\boldsymbol{U}_{\mathrm{o}}$ denote the global inertial observer that has the velocity value $\boldsymbol{u}_{\mathrm{o}}$. Let us consider $\boldsymbol{U}_{\mathrm{o}}$-time.

Put $t_{\mathrm{o}} := o + \mathbf{E}_{\boldsymbol{u}_{\mathrm{o}}}$. Then

$$\boldsymbol{s}_x(\boldsymbol{t}) = \frac{\boldsymbol{t}}{\sqrt{1 + |\Omega \cdot (x - o)|^2}} = \frac{\boldsymbol{t}}{\sqrt{1 + |\Omega \cdot \boldsymbol{q}_x|^2}}$$

time passes between the $\boldsymbol{U}_{\mathrm{o}}$-instants t_{o} and $t_{\mathrm{o}} + \boldsymbol{t}$ in the $\boldsymbol{U}$-space point that x is incident with.

The distance observed by $\boldsymbol{U}_{\mathrm{o}}$ at the $\boldsymbol{U}_{\mathrm{o}}$-instant $t_{\mathrm{o}} + \boldsymbol{t}$ between the $\boldsymbol{U}$-space point that x is incident with and $o + \boldsymbol{e}_x + \boldsymbol{u}_{\mathrm{o}} \otimes \mathbf{I}$ (the axis of rotation) equals

$$|r_x(s_x(\boldsymbol{t})) - (o + \boldsymbol{e}_x + \boldsymbol{u}_{\mathrm{o}}\boldsymbol{t})| = |\boldsymbol{q}_x|,$$

which is independent of $\boldsymbol{t}$.

6.7.4. Because of the term $\boldsymbol{s} \mapsto \mathrm{e}^{\boldsymbol{s}\Omega} \cdot (x - o)$ in $(*)$ we can state that the time period T of rotation is the same for all $\boldsymbol{U}$-space points (out of the axis of rotation), concerning their proper times: $T = \frac{2\pi}{\omega}$.

On the other hand, concerning $\boldsymbol{U}_{\mathrm{o}}$-time, the time period of rotation of a $\boldsymbol{U}$-space point having the $\boldsymbol{U}_{\mathrm{o}}$-distance $\boldsymbol{d} > \boldsymbol{0}$ from the axis of rotation equals $T_{\mathrm{o}}(\boldsymbol{d}) := \frac{2\pi}{\omega}\sqrt{1 + \omega^2\boldsymbol{d}^2}$; it increases from $\frac{2\pi}{\omega}$ to infinity as $\boldsymbol{d}$ increases from zero to infinity.

The following Figure illustrates the situation. Two $\boldsymbol{U}$-line segments are represented; the proper time passed along both segments equals $\frac{2\pi}{\omega}$.

Another figure shows the plane in the $\boldsymbol{U}_{\rm o}$-space, orthogonal to Ker Ω, and illustrates the angles of rotation of $\boldsymbol{U}$-space points during a $\boldsymbol{U}_{\rm o}$-time interval $\frac{2\pi}{\omega}$.

6.7.5. This observer is not rigid.

Let $\mathrm{L_o}$ be a curve in $o + \mathbf{E}_{\boldsymbol{u}_{\rm o}}$; then the set of $\boldsymbol{U}$-space points that meet $\mathrm{L_o}$, $L := \{q \in \mathrm{E}_{\boldsymbol{U}} \,|\, q \cap \mathrm{L_o} \neq \emptyset\}$ is a curve in the observer space. Indeed, if $p_{\rm o}$ is a parametrization of $\mathrm{L_o}$, then

$$p_t(a) := o + \boldsymbol{u}_{\rm o}\boldsymbol{t} + \exp\left(\frac{\boldsymbol{t}\Omega}{\sqrt{1 + |\Omega\cdot(p_{\rm o}(a) - o)|^2}}\right)\cdot(p_{\rm o}(a) - o)$$

$$(a \in \mathrm{Dom}\; p_{\rm o})$$

is a parametrization of L_t for $t = t_{\rm o} + \boldsymbol{t} \in \mathrm{I}_{\boldsymbol{U}_{\rm o}}$. Then

$$\dot p_t = \exp\left(\frac{\boldsymbol{t}\Omega}{\sqrt{1 + |\Omega\cdot(p_{\rm o} - o)|^2}}\right)\cdot\left(\frac{\boldsymbol{t}(\Omega\cdot p_{\rm o})\cdot(\Omega\cdot\dot p_{\rm o})}{\left(1 + |\Omega\cdot(p_{\rm o} - o)|^2\right)^{3/2}}\cdot\Omega\cdot(p_{\rm o} - o) + \dot p_{\rm o}\right).$$

We easily find that

$$\boldsymbol{U}\circ p_t = \exp\left(\frac{\boldsymbol{t}\Omega}{\sqrt{1 + |\Omega\cdot(p_{\rm o} - o)|^2}}\right)\cdot\Omega\cdot(p_{\rm o} - o) + \boldsymbol{u}_{\rm o}\sqrt{1 + |\Omega\cdot(p_{\rm o} - o)|^2}.$$

Then, using
$$(\mathrm{e}^{\alpha\Omega} \cdot \boldsymbol{q}_1) \cdot (\mathrm{e}^{\alpha\Omega} \cdot \boldsymbol{q}_2) = \boldsymbol{q}_1 \cdot \boldsymbol{q}_2$$
for all $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbf{E}_{\boldsymbol{u}_o}$ and $\alpha \in \mathbb{R}$, the reader can demonstrate without difficulty that $|\dot{p}_t|^2 + \big((\boldsymbol{U} \circ p_t) \cdot \dot{p}_t\big)^2$ depends on t : the observer is not rigid.

6.7.6. This observer is not regular. It is easy to show that there is no world surface $\mathbf{g}$-orthogonal to $\boldsymbol{U}$ and passing through o.

Suppose such a world surface F exists. Then F has $\mathbf{E}_{\boldsymbol{u}_o}$ as its tangent space at o.

For all $\boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}_o}$,
$$f(a) := o + a\boldsymbol{q} \qquad (a \in \mathbb{R})$$
is a function such that $f(0) = o$ and
$$(\boldsymbol{U} \circ f) \cdot \dot{f} = \left(\Omega \cdot \boldsymbol{q} + \boldsymbol{u}_o\sqrt{1 + |\Omega \cdot \boldsymbol{q}|^2}\right) \cdot \boldsymbol{q} = \mathbf{0}.$$

The curve (in fact a straight line) Ran f passes through $o \in$ F and all of its tangent vectors are $\mathbf{g}$-orthogonal to the corresponding values of $\boldsymbol{U}$ which would imply that Ran $f \subset$ F. Since $\boldsymbol{q}$ is arbitrary in $\mathbf{E}_{\boldsymbol{u}_o}$, this means that $o + \mathbf{E}_{\boldsymbol{u}_o} =$ F; in particular, every tangent space of F equals $\mathbf{E}_{\boldsymbol{u}_o}$. However, if $x \in o + \mathbf{E}_{\boldsymbol{u}_o} =$ F and $x - o$ is not in Ker Ω then $\boldsymbol{U}(x) \neq \boldsymbol{u}_o$; thus the tangent space of F at x is not $\mathbf{g}$-orthogonal to $\boldsymbol{U}(x)$: a contradiction.

6.8. Uniformly rotating observer II

6.8.1. Let $o \in$ M, $\boldsymbol{u}_o \in$ V(1) and $\Omega : \mathbf{E}_{\boldsymbol{u}o} \to \frac{\mathbf{E}_{\boldsymbol{u}_o}}{\mathbf{I}}$ be a non-zero antisymmetric linear map and define the non-global observer
$$\mathrm{Dom}\ \boldsymbol{U} := \left\{x \in \mathrm{M} \mid |\Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x - o)|^2 < 1\right\},$$
$$\boldsymbol{U}(x) := \frac{\boldsymbol{u}_o + \Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x - o)}{\sqrt{1 - |\Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x - o)|^2}} \qquad (x \in \mathrm{Dom}\ \boldsymbol{U}).$$

If $\boldsymbol{q}$ is in Ker Ω, then
$$\mathrm{Dom}\ \boldsymbol{U} + \boldsymbol{q} = \mathrm{Dom}\ \boldsymbol{U}$$
and
$$\boldsymbol{U}(x + \boldsymbol{q}) = \boldsymbol{U}(x) \qquad (x \in \mathrm{Dom}\ \boldsymbol{U},\ \boldsymbol{q} \in \mathrm{Ker}\ \Omega).$$

The observer has the acceleration field

$$\boldsymbol{A_U}(x) = \frac{\Omega \cdot \Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_\circ} \cdot (x - o)}{1 - |\Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_\circ} \cdot (x - o)|^2} \qquad (x \in \mathrm{Dom}\, \boldsymbol{U}).$$

6.8.2. To find the $\boldsymbol{U}$-lines, we again use the known transformation and we obtain

$$\dot{\boldsymbol{t}} = \frac{1}{\sqrt{1 - |\Omega \cdot \boldsymbol{q}|^2}},$$
$$\dot{\boldsymbol{q}} = \frac{\Omega \cdot \boldsymbol{q}}{\sqrt{1 - |\Omega \cdot \boldsymbol{q}|^2}}.$$

Now we apply a new trick: "dividing" the second equation by the first one we get a very simple differential equation which has the following correct meaning. Consider the initial conditions

$$\boldsymbol{t}(\mathbf{0}) = \mathbf{0}, \qquad \boldsymbol{q}(\mathbf{0}) = x - o,$$

where $x \in \big(o + \mathbf{E}_{\boldsymbol{u}_\circ}\big) \cap (\mathrm{Dom}\, \boldsymbol{U})$. The formula for the derivative of inverse function results in — with the notations of 6.4 —

$$\frac{\mathrm{d}\boldsymbol{s}_x(t)}{\mathrm{d}\boldsymbol{t}} = \sqrt{1 - |\Omega \cdot \boldsymbol{q}(s_x(t))|^2}. \qquad (*)$$

Then introducing the function $\boldsymbol{t} \mapsto \mathbf{q}(\boldsymbol{t}) := \boldsymbol{q}(\boldsymbol{s}_x(\boldsymbol{t}))$ we get the differential equation

$$\frac{\mathrm{d}\mathbf{q}(\boldsymbol{t})}{\mathrm{d}\boldsymbol{t}} = \dot{\boldsymbol{q}}(\boldsymbol{s}_x(\boldsymbol{t}))\frac{\mathrm{d}\boldsymbol{s}_x(\boldsymbol{t}))}{\mathrm{d}\boldsymbol{t}} = \Omega \cdot \mathbf{q}(\boldsymbol{t})$$

which has the solution

$$\mathbf{q}(\boldsymbol{t}) = \mathrm{e}^{\boldsymbol{t}\Omega} \cdot (x - o) \qquad (\boldsymbol{t} \in \mathbf{I}).$$

Consequently $|\Omega \cdot \boldsymbol{q}(\boldsymbol{s}_x(\boldsymbol{t}))| = |\Omega \cdot (x - o)|$, thus equation $(*)$ becomes trivial having the solution — with the initial condition $\boldsymbol{s}_x(\mathbf{0}) = \mathbf{0}$ —

$$\boldsymbol{s}_x(\boldsymbol{t}) = \boldsymbol{t}\sqrt{1 - |\Omega \cdot (x - o)|^2}.$$

Finally we obtain

$$\boldsymbol{t}_x(\boldsymbol{s}) = \frac{\boldsymbol{s}}{\sqrt{1 - |\Omega \cdot (x - o)|^2}},$$
$$\boldsymbol{q}_x(\boldsymbol{s}) = \exp\left(\frac{\boldsymbol{s}}{\sqrt{1 - |\Omega \cdot (x - o)|^2}}\right) \cdot (x - o)$$

from which we regain the world line function giving the $\boldsymbol{U}$-line passing through $x \in o + \mathbf{E}_{\boldsymbol{u}_o}$:

$$r_x(\boldsymbol{s}) = o + \boldsymbol{u}_o \frac{\boldsymbol{s}}{\sqrt{1 - |\Omega \cdot (x - o)|^2}} + \exp\left(\frac{\boldsymbol{s}\Omega}{\sqrt{1 - |\Omega \cdot (x - o)|^2}}\right) \cdot (x - o)$$
$$(\boldsymbol{s} \in \mathbf{I}).$$

It is not hard to see that every $\boldsymbol{U}$-line meets the hyperplane $o + \mathbf{E}_{\boldsymbol{u}_o}$, thus every $\boldsymbol{U}$-line is of this form.

6.8.3. Note that the $\boldsymbol{U}$-line passing through $o + \boldsymbol{e}$, where $\boldsymbol{e}$ is in Ker Ω, is a straight line directed by $\boldsymbol{u}_o$; then the set of $\boldsymbol{U}$-space points

$$\{o + \boldsymbol{e} + \boldsymbol{u}_o \otimes \mathbf{I} | \ \boldsymbol{e} \in \text{Ker } \Omega\}$$

is interpreted as the *axis of rotation.*

If x is in $o + \mathbf{E}_{\boldsymbol{u}_o}$, then $x - o$ can be decomposed into a sum $\boldsymbol{e}_x + \boldsymbol{q}_x$, where $\boldsymbol{e}_x$ is in Ker Ω and $\boldsymbol{q}_x$ is orthogonal to Ker Ω. Then the above given world line function can be written in the form

$$r_x(\boldsymbol{s}) = o + \boldsymbol{e}_x + \boldsymbol{u}_o \frac{\boldsymbol{s}}{\sqrt{1 - \omega^2 |\boldsymbol{q}_x|^2}} + \exp\left(\frac{\boldsymbol{s}\Omega}{\sqrt{1 - \omega^2 |\boldsymbol{q}_x|^2}}\right) \cdot \boldsymbol{q}_x, \qquad (**)$$

where ω is the magnitude of Ω.

Hence all the $\boldsymbol{U}$-lines are composed of an inertial line (with a proper time "accelerated" relative to the proper time of the points of the axis) and a uniform rotation.

Let $\boldsymbol{U}_o$ be the global inertial observer that has the velocity value $\boldsymbol{u}_o$. Let us consider $\boldsymbol{U}_o$-time putting $t_o := o + \mathbf{E}_{\boldsymbol{u}_o}$. Then

$$\boldsymbol{s}_x(\boldsymbol{t}) = \boldsymbol{t}\sqrt{1 - |\Omega \cdot (x - o)|^2}$$

is the time passed between the $\boldsymbol{U}_o$-instants t_o and $t_o + \boldsymbol{t}$ in the $\boldsymbol{U}$-space point that x is incident with.

The distance observed by $\boldsymbol{U}_o$ at the $\boldsymbol{U}_o$-instant $t_o + \boldsymbol{t}$ between the $\boldsymbol{U}$-space point that x is incident with and $o + \boldsymbol{e}_x + \boldsymbol{u}_o \otimes \mathbf{I}$ (the axis of rotation) equals

$$\left| r_x(\boldsymbol{s}_x(\boldsymbol{t})) - (o + \boldsymbol{e}_x + \boldsymbol{u}_o \boldsymbol{t}) \right| = |\boldsymbol{q}_x|$$

which is independent of $\boldsymbol{t}$.

6.8.4. Because of the term $\boldsymbol{s} \mapsto \mathrm{e}^{\boldsymbol{t}_x(\boldsymbol{s})\Omega} \cdot (x - o)$ in $(**)$ we can state that the time period T_{o} of rotation is the same for all $\boldsymbol{U}$-space points (out of the axis of rotation), concerning the $\boldsymbol{U}_{\mathrm{o}}$-time: $T_{\mathrm{o}} = \frac{2\boldsymbol{\pi}}{\omega}$.

On the other hand, concerning the proper times of $\boldsymbol{U}$-space points, the time period of rotation of a $\boldsymbol{U}$-space point having the $\boldsymbol{U}_{\mathrm{o}}$-distance $\mathbf{0} < \boldsymbol{d} < \frac{1}{\omega}$ from the axis of rotation equals $T(\boldsymbol{d}) := \frac{2\pi}{\omega\sqrt{1-\omega^2\boldsymbol{d}^2}}$; it increases from $\frac{2\boldsymbol{\pi}}{\omega}$ to infinity as $\boldsymbol{d}$ increases from zero to $\frac{1}{\omega}$.

The following figure illustrates the situation. Two $\boldsymbol{U}$-line segments are represented; the proper time passed along both segments equals $\frac{2\boldsymbol{\pi}}{\omega}$.

6.8.5. This observer is rigid.

Let L_{o} be a curve in $o + \mathbf{E}_{\boldsymbol{u}_{\mathrm{o}}}$; then the set of $\boldsymbol{U}$-space points that meet L_{o}, $\mathrm{L} := \{q \in \mathbf{E}_{\boldsymbol{U}} \,|\, q \cap \mathrm{L}_{\mathrm{o}} \neq \emptyset\}$ is a curve in the observer space. Indeed, if p_{o} is a parametrization of L_{o} then

$$p_t := o + \boldsymbol{u}_{\mathrm{o}}\boldsymbol{t} + \mathrm{e}^{\boldsymbol{t}\Omega} \cdot (p_{\mathrm{o}} - o)$$

is a parametrization of L_t for $t = t_o + \boldsymbol{t} \in \mathrm{I}_{\boldsymbol{U}_o}$. Then

$$\dot{p}_t = \mathrm{e}^{\boldsymbol{t}\Omega} \cdot \dot{p}_o$$

and we easily find that

$$\boldsymbol{U} \circ p_t = \frac{\boldsymbol{u}_o + \Omega \cdot \mathrm{e}^{\boldsymbol{t}\Omega} \cdot (p_o - o)}{\sqrt{1 - |\Omega \cdot (p_o - o)|^2}}$$

and

$$(\boldsymbol{U} \circ p_t) \cdot \dot{p}_t = \frac{\big(\Omega \cdot (p_o - o)\big) \cdot \dot{p}_o}{\sqrt{1 - |\Omega \cdot (p_o - o)|^2}}.$$

Consequently,

$$|\dot{p}_t|^2 + \big((\boldsymbol{U} \circ p_t) \cdot \dot{p}_t\big)^2 = |\dot{p}_o|^2 + \frac{\big(\dot{p}_o \cdot \Omega \cdot (p_o - o)\big)^2}{1 - |\Omega \cdot (p_o - o)|^2}$$

is independent of t : the observer is rigid.

6.8.6. This observer furnishes a good instance that the laws of Euclidean geometry do not hold necessarily in the space of a non-inertial observer.

Since the observer is rigid, all the lengths in $\boldsymbol{U}$-space can be calculated by curves in $\big(o + \mathbf{E}_{\boldsymbol{u}_o}\big) \cap (\mathrm{Dom}\ \boldsymbol{U})$ which can be reduced to curves in

$$\begin{aligned} \mathbf{E}_{\boldsymbol{u}_o} &\cap (\mathrm{Dom}\ \boldsymbol{U} - o) = \\ &= \mathrm{Ker}\ \Omega + \{\boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}_o} \mid \boldsymbol{q} \text{ is orthogonal to Ker } \Omega,\ |\boldsymbol{q}| < \frac{1}{\omega}\} =: \mathbf{E}_\Omega. \end{aligned}$$

If L_o is a curve in $\big(o + \mathbf{E}_{\boldsymbol{u}_o}\big) \cap (\mathrm{Dom}\ \boldsymbol{U})$ then $\mathbf{L} := \mathrm{L}_o - o$ is a curve in $\mathbf{E}_\Omega$; if p_o is a parametrization of L_o then $\boldsymbol{p} := p_o - o$ is a parametrization of $\mathbf{L}$.

$\mathbf{E}_\Omega$ is a subset of the Euclidean vector space $\mathbf{E}_{\boldsymbol{u}_o}$ in which distances and curve lengths have a well-defined meaning; however, now curves in $\mathbf{E}_\Omega$ will represent curves in $\boldsymbol{U}$-space and their lengths will be calculated in this sense. Thus, to avoid misunderstanding, we shall say $\boldsymbol{U}$-length and $\boldsymbol{U}$-distance, indicating it in notations, too.

A curve $\mathbf{L}$ in $\mathbf{E}_\Omega$ has the $\boldsymbol{U}$-length

$$\ell_{\boldsymbol{U}}(\mathbf{L}) = \int_{\mathrm{Dom}\ \boldsymbol{p}} \sqrt{|\dot{\boldsymbol{p}}(a)|^2 + \frac{\big(\dot{\boldsymbol{p}}(a) \cdot \Omega \cdot \boldsymbol{p}(a)\big)^2}{1 - |\Omega \cdot \boldsymbol{p}(a)|^2}}\mathrm{d}a. \qquad (***)$$

Take arbitrary elements $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbf{E}_\Omega$. Then we easily find for the straight line segment connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, $]\boldsymbol{x}, \boldsymbol{y}[:= \{\boldsymbol{x} + a(\boldsymbol{y} - \boldsymbol{x}) \mid 0 < a < 1\}$ that

$$\ell_{\boldsymbol{U}}(]\boldsymbol{x},\boldsymbol{y}[) \geq |\boldsymbol{y}-\boldsymbol{x}|$$

and equality holds if and only if $\boldsymbol{x}\cdot\Omega\cdot\boldsymbol{y}=\mathbf{0}$ which is equivalent to the fact that the straight line passing through $\boldsymbol{x}$ and $\boldsymbol{y}$ meets the kernel of Ω.

Suppose the straight line passing through $\boldsymbol{x}$ and $\boldsymbol{y}$ meets Ker Ω and $\mathbf{L}$ is a broken line connecting $\boldsymbol{x}$ and $\boldsymbol{y}$; then the previous inequality implies

$$\ell_{\boldsymbol{U}}(\mathbf{L}) \geq |\boldsymbol{y}-\boldsymbol{x}| = \ell_{\boldsymbol{U}}(]\boldsymbol{x},\boldsymbol{y}[).$$

As a consequence, the inequality above will be valid for an arbitrary $\mathbf{L}$ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ because $\ell(\mathbf{L})$ is obtained as the supremum of $\boldsymbol{U}$-lengths of broken lines approximating the curve $\mathbf{L}$. Since the $\boldsymbol{U}$-distance $\boldsymbol{d}_{\boldsymbol{U}}(\boldsymbol{x},\boldsymbol{y})$ between $\boldsymbol{x}$ and $\boldsymbol{y}$ is the infimum of curve lengths connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ we see that

$$\boldsymbol{d}_{\boldsymbol{U}}(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{y}-\boldsymbol{x}|$$

if the straight line passing through $\boldsymbol{x}$ and $\boldsymbol{y}$ intersects Ker Ω.

Let $\boldsymbol{d}$ be an element of $\mathbf{I}$, $\mathbf{0}<\boldsymbol{d}<\frac{1}{\omega}$, and put

$$\mathrm{C}_{\boldsymbol{d}} := \{\boldsymbol{q}\in\mathbf{E}_\Omega \mid \boldsymbol{q} \text{ is orthogonal to Ker } \Omega,\ |\boldsymbol{q}|=\boldsymbol{d}\}.$$

Evidently, if $\boldsymbol{q}\in\mathrm{C}_{\boldsymbol{d}}$ then $-\boldsymbol{q}\in\mathrm{C}_{\boldsymbol{d}}$ as well. Moreover, according to our previous result, the $\boldsymbol{U}$-distance between $\boldsymbol{q}$ and $-\boldsymbol{q}$ equals $|\boldsymbol{q}-(-\boldsymbol{q})|=2\boldsymbol{d}$.

This means that $\mathrm{C}_{\boldsymbol{d}}$ represents a circle of radius $\boldsymbol{d}$ in the observer space. Let us calculate the circumference of this circle.

Choosing the parametrization

$$\boldsymbol{p}:]-\pi,\pi] \to \mathrm{C}_{\boldsymbol{d}}, \qquad a \mapsto \exp\left(a\frac{\Omega}{\omega}\right)\cdot\boldsymbol{q}_o$$

where $\boldsymbol{q}_o$ is an arbitrarily fixed element of $\mathrm{C}_{\boldsymbol{d}}$, we find

$$\dot{\boldsymbol{p}} = \frac{\Omega}{\omega}\cdot\boldsymbol{p}, \qquad |\dot{\boldsymbol{p}}|^2 = \boldsymbol{d}^2,$$
$$|\Omega\cdot\boldsymbol{p}|^2 = \omega^2\boldsymbol{d}^2, \qquad \left(\dot{\boldsymbol{p}}\cdot\Omega\cdot\boldsymbol{p}\right)^2 = \omega^2\boldsymbol{d}^4.$$

Applying formula $(***)$ we obtain

$$\ell_{\boldsymbol{U}}(\mathrm{C}_{\boldsymbol{d}}) = \frac{2\pi\boldsymbol{d}}{\sqrt{1-\omega^2\boldsymbol{d}^2}}.$$

The circumference of the circle of radius $\boldsymbol{d}$ is longer than $2\pi\boldsymbol{d}$.

6.9. Exercises

1. Let $\boldsymbol{U}$ be a global inertial observer with velocity value $\boldsymbol{u}$. Take a velocity value $\boldsymbol{u}_\mathrm{o} \neq \boldsymbol{u}$ and the artificial time consisting of the hyperplanes directed by $\mathbf{E}_{\boldsymbol{u}_\mathrm{o}}$ (i.e. $\mathrm{I}_{\boldsymbol{U}_\mathrm{o}}$). Demonstrate that the distances in $\boldsymbol{U}$-space calculated at $\mathrm{I}_{\boldsymbol{U}_\mathrm{o}}$-instants according to definition 6.3.3 equal the distances defined earlier in $\boldsymbol{U}$-space.

2. Suppose the artificial time points are hyperplanes or hyperplane sections and the observer $\boldsymbol{U}$ is constant on them. Take such an artificial time point t; then there is a $\boldsymbol{u}_t \in \mathrm{V}(1)$ such that $\boldsymbol{U}(x) = \boldsymbol{u}_t$ for all x in t. Suppose t is directed by $\mathbf{E}$. Then

$$|\boldsymbol{\pi}_{\boldsymbol{u}_t} \cdot (\boldsymbol{q}_1 + \boldsymbol{q}_2)| \leq |\boldsymbol{\pi}_{\boldsymbol{u}_t} \cdot \boldsymbol{q}_1| + |\boldsymbol{\pi}_{\boldsymbol{u}_t} \cdot \boldsymbol{q}_2|$$

for all $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbf{E}$. As a consequence, straight lines realize the distance between the points of t, thus

$$\boldsymbol{d}_t(q_1, q_2) = |\boldsymbol{\pi}_{\boldsymbol{u}_t}(q_2 \star t - q_1 \star t)| \qquad (q_1, q_2 \in \mathrm{E}_{\boldsymbol{U}}).$$

3. Let $\boldsymbol{U}$ be the uniformly accelerated observer treated in 6.5. Then

$$\boldsymbol{v}_{\boldsymbol{U}(x)\boldsymbol{u}_\mathrm{o}} = \frac{\boldsymbol{a}_\mathrm{o}\big((-\boldsymbol{u}_\mathrm{o} \cdot (x - o))\big)}{\sqrt{1 + |\boldsymbol{a}_\mathrm{o}|^2\big(\boldsymbol{u}_\mathrm{o} \cdot (x - o)\big)^2}} \qquad (x \in \mathrm{M}).$$

4. Let $\boldsymbol{U}$ be as before. Verify that every $\boldsymbol{U}$-line is obtained from a chosen one by a translation with a vector in $\mathbf{E}_{\boldsymbol{u}_\mathrm{o}}$. In other words, $\mathrm{E}_{\boldsymbol{U}}$ endowed with the subtraction

$$q' - q := x' - x \qquad (x' \in q,\ x \in q,\ x' - x \in \mathbf{E}_{\boldsymbol{u}_\mathrm{o}})$$

is an affine space over $\mathbf{E}_{\boldsymbol{u}_\mathrm{o}}$.

5. Let $\boldsymbol{U}$ be the uniformly accelerated observer treated in 6.6. Then the $\boldsymbol{U}$-line passing through $x \in o + \mathbf{E}_{\boldsymbol{u}_\mathrm{o}}$ intersects t_λ if and only if $\boldsymbol{a}_\mathrm{o} \cdot (x - o) < \ln|\lambda|$.

6. Show that

$$\mathrm{Dom}\ \boldsymbol{U} = \{o + \alpha\boldsymbol{u}_\mathrm{o} + \beta\boldsymbol{a}_\mathrm{o} + \boldsymbol{q} \mid \beta > \mathbf{0},\ \beta^2|\boldsymbol{a}_\mathrm{o}|^2 > \alpha^2,\ \boldsymbol{u}_\mathrm{o} \cdot \boldsymbol{q} = \mathbf{0},\ \boldsymbol{a}_\mathrm{o} \cdot \boldsymbol{q} = \mathbf{0}\}$$

for the uniformly accelerated observer treated in 6.6.

7. Let $\boldsymbol{U}$ be as before. Then

$$\boldsymbol{v}_{\boldsymbol{U}(x)\boldsymbol{u}_\mathrm{o}} = \frac{\boldsymbol{a}_\mathrm{o}\big(-\boldsymbol{u}_\mathrm{o} \cdot (x - o)\big)}{\boldsymbol{a}_\mathrm{o} \cdot (x - o)} \qquad (x \in \mathrm{Dom}\ \boldsymbol{U}).$$

8. Show that the distance observed by the inertial observer $\boldsymbol{U}_\mathrm{o}$ with velocity value $\boldsymbol{u}_\mathrm{o}$ between the space points of the uniformly accelerated observer treated in 6.6. is not constant in $\boldsymbol{U}_\mathrm{o}$-time. Give an explanation similar to that in 6.5.6.

9. Verify that

$$\mathrm{Dom}\ \boldsymbol{U} = o + \boldsymbol{u}_o \otimes \mathbf{I} + \mathrm{Ker}\ \Omega + \left\{ \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}_o} \mid \boldsymbol{q} \text{ is orthogonal to } \mathrm{Ker}\ \Omega,\ |\boldsymbol{q}| < \frac{1}{\omega} \right\}$$

for the uniformly rotating observer treated in 6.8.

10. Demonstrate that

$$\boldsymbol{v}_{\boldsymbol{U}(x)\boldsymbol{u}_o} = \frac{\Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x-o)}{\sqrt{1 + |\Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x-o)|^2}} \qquad (x \in \mathrm{M})$$

and

$$\boldsymbol{v}_{\boldsymbol{U}(x)\boldsymbol{u}_o} = \Omega \cdot \boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x-o) \qquad (x \in \mathrm{Dom}\ \boldsymbol{U})$$

where $\boldsymbol{U}$ is the uniformly rotating observer treated in 6.7. and 6.8, respectively.

11. The uniformly rotating observer treated in 6.8. is not regular.

12. Let o be a world point and consider the observer

$$\boldsymbol{U}(x) := \frac{x-o}{|x-o|} \qquad (x \in o + \mathrm{T}^{\to}).$$

Prove that

$$\mathrm{E}_{\boldsymbol{U}} = \{o + \boldsymbol{u} \otimes \mathbf{I}^+ \mid \boldsymbol{u} \in \mathrm{V}(1)\}.$$

$\boldsymbol{U}$ is regular and

$$\{\mathrm{V}(1)\boldsymbol{t} \mid \boldsymbol{t} \in \mathbf{I}^+\}$$

is the set of $\boldsymbol{U}$-surfaces ($\boldsymbol{U}$-instants).

Show that if this observer defined simultaneity like an inertial observer (light signals and mirrors, see 3.2.) then simultaneity would depend on the $\boldsymbol{U}$-space point of the light source.

13. Let $o \in \mathrm{M}$, $\boldsymbol{u}_o \in \mathrm{V}(1)$, $\boldsymbol{h} \in \mathbf{I}^*$ and define the observer

$$\mathrm{Dom}\ \boldsymbol{U} := \{x \in \mathrm{M} \mid \boldsymbol{h}^2 |\boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x-o)|^2 < 1\},$$

$$\boldsymbol{U}(x) := \frac{\boldsymbol{u}_o + \boldsymbol{h}\boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x-o)}{\sqrt{1 - \boldsymbol{h}^2 |\boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x-o)|^2}} \qquad (x \in \mathrm{Dom}\ \boldsymbol{U}).$$

Applying the method given in 6.4. find that the $\boldsymbol{U}$-line passing through $x \in \left(o + \mathbf{E}_{\boldsymbol{u}_o}\right) \cap (\mathrm{Dom}\ \boldsymbol{U})$ is given by the world line function

$$r_x(\boldsymbol{s}) = o + \boldsymbol{u}_o \boldsymbol{t}_x(\boldsymbol{s}) + \mathrm{e}^{\boldsymbol{h}\boldsymbol{t}_x(\boldsymbol{s})}(x-o)$$

where $\boldsymbol{s} \mapsto \boldsymbol{t}_x(\boldsymbol{s})$ is the solution of the differential equation

$$(\boldsymbol{t} : \mathbf{I} \rightarrowtail \mathbf{I})? \qquad \dot{\boldsymbol{t}} = \frac{1}{\sqrt{1 - \boldsymbol{h}^2 |x-o|^2 \mathrm{e}^{2\boldsymbol{h}\boldsymbol{t}}}}$$

with the initial condition $\boldsymbol{t}(\mathbf{0}) = \mathbf{0}$.

14. Let $o \in \mathrm{M}$, $\boldsymbol{u}_o \in \mathrm{V}(1)$, $\boldsymbol{h} \in \mathbf{I}^*$ and define the observer

$$\boldsymbol{U}(x) := \boldsymbol{u}_o\sqrt{1 + \boldsymbol{h}^2|\boldsymbol{\pi}_{\boldsymbol{u}_o} \cdot (x-o)|^2} + \boldsymbol{h}\boldsymbol{\pi}_{\boldsymbol{u}_o}(x-o) \qquad (x \in \mathrm{M}).$$

Applying the method given in 6.4 find that the $\boldsymbol{U}$-line passing through $x \in o + \mathbf{E}_{\boldsymbol{u}_o}$ is given by the world line function

$$r_x(\boldsymbol{s}) = o + \boldsymbol{u}_o\boldsymbol{t}_x(\boldsymbol{s}) + \mathrm{e}^{\boldsymbol{h}\boldsymbol{s}}(x-o)$$

where $\boldsymbol{s} \mapsto \boldsymbol{t}_x(\boldsymbol{s})$ is the function for which $\boldsymbol{t}_x(\mathbf{0}) = \mathbf{0}$ holds and has the derivative $\boldsymbol{s} \mapsto \sqrt{1 + \boldsymbol{h}^2|x-o|^2\mathrm{e}^{2\boldsymbol{h}\boldsymbol{s}}}$.

15. Compare the observers of the previous two exercises with the non-relativistic observer in Exercise I.5.4.9.

7. Vector splittings

7.1. Splitting of vectors

7.1.1. For $\boldsymbol{u} \in \mathrm{V}(1)$ we have already defined

$$\boldsymbol{\tau}_{\boldsymbol{u}} : \mathbf{M} \to \mathbf{I}, \qquad \boldsymbol{x} \mapsto -\boldsymbol{u} \cdot \boldsymbol{x}$$

and

$$\boldsymbol{\pi}_{\boldsymbol{u}} : \mathbf{M} \to \mathbf{E}_{\boldsymbol{u}}, \qquad \boldsymbol{x} \mapsto \boldsymbol{x} - (\boldsymbol{\tau}_{\boldsymbol{u}} \cdot \boldsymbol{x})\boldsymbol{u} = \boldsymbol{x} + (\boldsymbol{u} \cdot \boldsymbol{x})\boldsymbol{u}$$

i.e. with the usual identifications,

$$\boldsymbol{\tau}_{\boldsymbol{u}} = -\boldsymbol{u}, \qquad \boldsymbol{\pi}_{\boldsymbol{u}} = \mathbf{g} + \boldsymbol{u} \otimes \boldsymbol{u}$$

(see 1.3.2) and the linear bijection $\boldsymbol{h}_{\boldsymbol{u}} := (\boldsymbol{\tau}_{\boldsymbol{u}}, \boldsymbol{\pi}_{\boldsymbol{u}}) : \mathbf{M} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$ having the inverse

$$(\boldsymbol{t}, \boldsymbol{q}) \mapsto \boldsymbol{u}\boldsymbol{t} + \boldsymbol{q}$$

(see 1.3.5).

Definition. $\boldsymbol{\tau}_{\boldsymbol{u}} \cdot \boldsymbol{x} = -\boldsymbol{u} \cdot \boldsymbol{x}$ and $\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x}$ are called the $\boldsymbol{u}$-*timelike component* and the $\boldsymbol{u}$-*spacelike component* of the vector $\boldsymbol{x}$. $(-\boldsymbol{u} \cdot \boldsymbol{x}, \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x})$ is the $\boldsymbol{u}$-split form of $\boldsymbol{x}$. $\boldsymbol{h}_{\boldsymbol{u}} = (\boldsymbol{\tau}_{\boldsymbol{u}}, \boldsymbol{\pi}_{\boldsymbol{u}})$ is the *splitting* of $\mathbf{M}$ corresponding to $\boldsymbol{u}$, or the $\boldsymbol{u}$-*splitting* of $\mathbf{M}$. ■

Note that

$$\boldsymbol{x} \cdot \boldsymbol{y} = -(\boldsymbol{u} \cdot \boldsymbol{x})(\boldsymbol{u} \cdot \boldsymbol{y}) + (\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x}) \cdot (\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{y}),$$

in particular,

$$\boldsymbol{x}^2 = -(\boldsymbol{u}\cdot\boldsymbol{x})^2 + |\boldsymbol{\pi_u}\cdot\boldsymbol{x}|^2$$

for all $\boldsymbol{x},\boldsymbol{y} \in \mathbf{M}$. In other words,

$$\text{if} \quad \boldsymbol{h_u}\cdot\boldsymbol{x} = (\boldsymbol{t},\boldsymbol{q}) \quad \text{then} \quad \boldsymbol{x}^2 = -\boldsymbol{t}^2 + |\boldsymbol{q}|^2.$$

7.1.2. If $\mathbf{A}$ is a measure line, $\mathbf{A}\otimes\mathbf{M}$ $\left(\frac{\mathbf{M}}{\mathbf{A}}\right)$ is split into $(\mathbf{A}\otimes\mathbf{I})\times(\mathbf{A}\otimes\mathbf{E_u})$ $\left(\frac{\mathbf{I}}{\mathbf{A}}\times\frac{\mathbf{E_u}}{\mathbf{A}}\right)$ by $\boldsymbol{h_u}$; thus the $\boldsymbol{u}$-timelike component and the $\boldsymbol{u}$-spacelike component of a vector of type $\mathbf{A}$ (cotype $\mathbf{A}$) are in $\mathbf{A}\otimes\mathbf{I}$ $\left(\frac{\mathbf{I}}{\mathbf{A}}\right)$ and in $\mathbf{A}\otimes\mathbf{E_u}$ $\left(\frac{\mathbf{E_u}}{\mathbf{A}}\right)$, respectively.

In particular, $\boldsymbol{h_u}$ splits $\frac{\mathbf{M}}{\mathbf{I}}$ into $\mathbb{R}\times\frac{\mathbf{E_u}}{\mathbf{I}}$ and for all $\boldsymbol{u}'\in \mathrm{V}(1)$

$$\boldsymbol{h_u}\cdot\boldsymbol{u}' = \big(-\boldsymbol{u}\cdot\boldsymbol{u}',\, \boldsymbol{u}' + (\boldsymbol{u}\cdot\boldsymbol{u}')\boldsymbol{u}\big) = \frac{1}{\sqrt{1-|\boldsymbol{v_{u'u}}|^2}}(1,\boldsymbol{v_{u'u}}).$$

7.1.3. In contradistinction to the non-relativistic case, here not only the $\boldsymbol{u}$-spacelike component but also the $\boldsymbol{u}$-timelike component of vectors depend on $\boldsymbol{u}$. The transformation rule that shows how the $\boldsymbol{u}$-components of a vector vary with $\boldsymbol{u}$, is much more complicated here than in the non-relativistic case.

Proposition. Let $\boldsymbol{u},\boldsymbol{u}'\in\mathrm{V}(1)$. Then for all $(\boldsymbol{t},\boldsymbol{q})\in\mathbf{I}\times\mathbf{E_u}$ we have

$$\begin{aligned}
\left(\boldsymbol{h_{u'}}\cdot\boldsymbol{h_u}^{-1}\right)\cdot(\boldsymbol{t},\boldsymbol{q}) &= \big((-\boldsymbol{u}'\cdot\boldsymbol{u})t - \boldsymbol{u}'\cdot\boldsymbol{q},\, (\boldsymbol{u}+(\boldsymbol{u}'\cdot\boldsymbol{u})\boldsymbol{u})\boldsymbol{t} + \boldsymbol{q} + (\boldsymbol{u}'\cdot\boldsymbol{q})\boldsymbol{u}'\big) \\
&= \left(\frac{1}{\sqrt{1-|\boldsymbol{v_{u'u}}|^2}}(\boldsymbol{t}-\boldsymbol{v_{u'u}}\cdot\boldsymbol{q}),\right. \\
&\left.\frac{1}{\sqrt{1-|\boldsymbol{v_{u'u}}|^2}}\left(\boldsymbol{v_{uu'}}\boldsymbol{t} - \frac{\boldsymbol{v_{uu'}}+\boldsymbol{v_{u'u}}\sqrt{1-|\boldsymbol{v_{u'u}}|^2}}{|\boldsymbol{v_{u'u}}|^2}(\boldsymbol{v_{u'u}}\cdot\boldsymbol{q})\right) + \boldsymbol{q}\right).
\end{aligned}$$

Proof. The first equality is quite simple. The second one is derived with the aid of the formulae in 4.3.2 and the relation $\boldsymbol{u}'\cdot\boldsymbol{q} = -(\boldsymbol{u}'\cdot\boldsymbol{u})\left(\frac{\boldsymbol{u}'}{-\boldsymbol{u}'\cdot\boldsymbol{u}} - \boldsymbol{u}\right)\cdot\boldsymbol{q}$ which is true because $\boldsymbol{u}\cdot\boldsymbol{q} = \mathbf{0}$. ∎

Note that both $\boldsymbol{v_{u'u}}$ and $\boldsymbol{v_{uu'}}$ appear in that formula.

7.1.4. The previous formula is not a good transformation rule: we want to compare the $\boldsymbol{u}'$-components of a vector with its $\boldsymbol{u}$-components $(\boldsymbol{t},\boldsymbol{q})$. However, the $\boldsymbol{u}'$-components and the $\boldsymbol{u}$-components are in different spaces: $(\boldsymbol{t},\boldsymbol{q})$ is in

$\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}}$ and $\left(\boldsymbol{h}_{\boldsymbol{u}'}\cdot\boldsymbol{h}_{\boldsymbol{u}}^{-1}\right)\cdot(\boldsymbol{t},\boldsymbol{q})$ is in $\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}'}$, they cannot be compared directly. To obtain a convenient formula, we have to relate $\mathbf{E}_{\boldsymbol{u}'}$ and $\mathbf{E}_{\boldsymbol{u}}$; we have agreed that such a relation is established by the corresponding Lorentz boost. Thus, leaving invariant the first component, we shall transform the second component of $\left(\boldsymbol{h}_{\boldsymbol{u}'}\cdot\boldsymbol{h}_{\boldsymbol{u}}^{-1}\right)\cdot(\boldsymbol{t},\boldsymbol{q})$ by $\boldsymbol{L}(\boldsymbol{u},\boldsymbol{u}')$.

Definition. Let $\boldsymbol{u},\boldsymbol{u}'\in\mathrm{V}(1)$. Then

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}:=\left(\mathrm{id}_{\mathbf{I}}\times\boldsymbol{L}(\boldsymbol{u},\boldsymbol{u}')|_{\mathbf{E}_{\boldsymbol{u}'}}\right)\cdot\left(\boldsymbol{h}_{\boldsymbol{u}'}\cdot\boldsymbol{h}_{\boldsymbol{u}}^{-1}\right)$$

is called the *vector transformation rule* from $\boldsymbol{u}$-splitting into $\boldsymbol{u}'$-splitting. ■

Proposition. For all $(\boldsymbol{t},\boldsymbol{q})\in\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}}$ we have

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}\cdot(\boldsymbol{t},\boldsymbol{q})=\left(\frac{1}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}(\boldsymbol{t}-\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\cdot\boldsymbol{q}),\right.$$
$$\left.\frac{1}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}\left(-\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\left(\boldsymbol{t}-\frac{1-\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}{|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\cdot\boldsymbol{q})\right)\right)+\boldsymbol{q}\right). \quad ■$$

In connection with this formula we mention the following frequently useful relation:

$$\frac{1-\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}{|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}=\frac{1}{1+\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}.$$

7.1.5. The previous formula is a bit fearsome. We can make it more apparent decomposing $\boldsymbol{q}$ into a sum of vectors parallel and orthogonal to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$:

$$(\boldsymbol{t},\boldsymbol{q})=(\boldsymbol{t},\boldsymbol{q}_{\|})+(\boldsymbol{0},\boldsymbol{q}_{\perp})$$

where $\boldsymbol{q}_{\|}$ is parallel to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, i.e. there is a $\lambda\in\mathbf{I}$ such that $\boldsymbol{q}_{\|}=\lambda\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ and $\boldsymbol{q}_{\perp}$ is orthogonal to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, i.e. $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\cdot\boldsymbol{q}_{\perp}=\boldsymbol{0}$.

Then we easily find that

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}\cdot(\boldsymbol{0},\boldsymbol{q}_{\perp})=(\boldsymbol{0},\boldsymbol{q}_{\perp}),$$
$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}\cdot(\boldsymbol{t},\boldsymbol{q}_{\|})=\frac{1}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}\left(\boldsymbol{t}-\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\cdot\boldsymbol{q}_{\|},\ -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\boldsymbol{t}+\boldsymbol{q}_{\|}\right).$$

7.1.6. The last formula — in a slightly different form — appears in the literature as the formula of Lorentz transformation. To get the usual form we put $\boldsymbol{v} := \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$; let $(\boldsymbol{t}, \boldsymbol{q})$ denote the $\boldsymbol{u}$-components of a vector and let $(\boldsymbol{t}', \boldsymbol{q}')$ denote its $\boldsymbol{u}'$-components mapped by the Lorentz boost $\boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}')$ into $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$; then supposing $\boldsymbol{q}$ is parallel to $\boldsymbol{v}$ we have

$$\boldsymbol{t}' = \frac{1}{\sqrt{1 - |\boldsymbol{v}|^2}}(\boldsymbol{t} - \boldsymbol{v} \cdot \boldsymbol{q}), \qquad \boldsymbol{q}' = \frac{1}{\sqrt{1 - |\boldsymbol{v}|^2}}(-\boldsymbol{v}\boldsymbol{t} + \boldsymbol{q}).$$

This (or its equivalent in the arithmetic spacetime model) is the usual "Lorentz transformation" formula.

We emphasize that $\boldsymbol{q}'$ *is not* the $\boldsymbol{u}'$-spacelike component of the vector having the $\boldsymbol{u}$-components $(\boldsymbol{t}, \boldsymbol{q})$; it is the Lorentz-boosted $\boldsymbol{u}'$-spacelike component.

Lorentz transformations (see Section 9) are transformations of vectors, i.e. mappings from $\mathbf{M}$ into $\mathbf{M}$; the transformation rule is a mapping from $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$ into $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$. Transformation rules and Lorentz transformations are different mathematical objects. Of course, there is some connection between them. We easily find that

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} = \boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}') \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1}$$

where $\boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}')$ is the Lorentz boost from $\boldsymbol{u}'$ into $\boldsymbol{u}$.

In the split spacetime model $\mathbf{M}$ and $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$ coincide: *the special structure of the split spacetime model (and the arithmetic spacetime model) involves the possibility of confusing transformation rules with Lorentz transformations.*

7.1.7. Using a matrix form of the linear maps $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$ (see IV.3.7) we can write

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} = \kappa(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) \begin{pmatrix} 1 & -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \\ -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \boldsymbol{D}(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) \end{pmatrix},$$

where

$$\kappa(\boldsymbol{v}) := \frac{1}{\sqrt{1 - |\boldsymbol{v}|^2}},$$
$$\boldsymbol{D}(\boldsymbol{v}) := \frac{1}{\kappa(\boldsymbol{v})}\left(\mathrm{id}_{\mathbf{E}_{\boldsymbol{u}}} + \frac{\kappa(\boldsymbol{v})^2}{\kappa(\boldsymbol{v}) + 1}\boldsymbol{v} \otimes \boldsymbol{v}\right)$$

for $\boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $|\boldsymbol{v}| < 1$.

7.2. Splitting of covectors

7.2.1. For $\boldsymbol{u} \in \mathrm{V}(1)$, $\mathbf{M}^*$ is split by the transpose of the inverse of $\boldsymbol{h_u}$:

$$\boldsymbol{r_u} := \left(\boldsymbol{h_u}^{-1}\right)^* : \ \mathbf{M}^* \to (\mathbf{I} \times \mathbf{E_u})^* \equiv \mathbf{I}^* \times \mathbf{E_u}^*.$$

Then for all $\boldsymbol{k} \in \mathbf{M}^*$, $(\boldsymbol{t}, \boldsymbol{q}) \in \mathbf{E_u}$ we have

$$(\boldsymbol{r_u} \cdot \boldsymbol{k}) \cdot (\boldsymbol{t}, \boldsymbol{q}) = \boldsymbol{k} \cdot \boldsymbol{h_u}^{-1} \cdot (\boldsymbol{t}, \boldsymbol{q}) = \boldsymbol{k} \cdot (\boldsymbol{ut} + \boldsymbol{q}) = (\boldsymbol{k} \cdot \boldsymbol{u})\boldsymbol{t} + \boldsymbol{k} \cdot \boldsymbol{q}.$$

Of course, instead of $\boldsymbol{k}$ in $\boldsymbol{k} \cdot \boldsymbol{q}$ we can write $\boldsymbol{k}|_{\mathbf{E_u}} = \mathbf{i}_{\boldsymbol{u}}^* \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \mathbf{i}_{\boldsymbol{u}} \in \mathbf{E}_{\boldsymbol{u}}^*$. Then we can state that

$$\boldsymbol{r_u} \cdot \boldsymbol{k} = (\boldsymbol{k} \cdot \boldsymbol{u},\, \boldsymbol{k} \cdot \mathbf{i}_{\boldsymbol{u}}) = (\boldsymbol{u} \cdot \boldsymbol{k}, \mathbf{i}_{\boldsymbol{u}}^* \cdot \boldsymbol{k}) \qquad (\boldsymbol{k} \in \mathbf{M}^*).$$

This form is suitable for a comparison with the non-relativistic case. However, we can get a form more convenient from the point of view of applications. Applying the usual identifications we have $\mathbf{i}_{\boldsymbol{u}}^* = \boldsymbol{\pi_u}$ (see 1.3.6), thus

$$\boldsymbol{r_u} \cdot \boldsymbol{k} = (\boldsymbol{u} \cdot \boldsymbol{k},\, \boldsymbol{\pi_u} \cdot \boldsymbol{k}) \qquad (\boldsymbol{k} \in \mathbf{M}^*).$$

Recall the identification $\mathbf{M}^* \equiv \frac{\mathbf{M}}{\mathbf{I} \otimes \mathbf{I}}$ which implies that $\boldsymbol{k}$ can be split as a vector of cotype $\mathbf{I} \otimes \mathbf{I}$, too:

$$\boldsymbol{h_u} \cdot \boldsymbol{k} = (-\boldsymbol{u} \cdot \boldsymbol{k}, \boldsymbol{\pi_u} \cdot \boldsymbol{k}) \qquad (\boldsymbol{k} \in \mathbf{M}^*).$$

The two splittings are nearly the same. In the literature (in a somewhat different setting) the split form of $\boldsymbol{k} \in \mathbf{M}^*$ by $\boldsymbol{r_u}$ and $\boldsymbol{h_u}$ are called the *covariant* and the *contravariant* components of $\boldsymbol{k}$, respectively.

Of course, in view of $\mathbf{M} \equiv \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{M}^*$, also the elements of $\mathbf{M}$ can be split by $\boldsymbol{r_u}$: a vector, too, has covariant and contravariant components.

Introducing the notation

$$\boldsymbol{j_u} : \ \mathbf{I} \times \mathbf{E_u} \to \mathbf{I} \times \mathbf{E_u}, \qquad (\boldsymbol{t}, \boldsymbol{q}) \mapsto (-\boldsymbol{t}, \boldsymbol{q})$$

we have (with the usual identifications)

$$\boldsymbol{r_u} = \boldsymbol{j_u} \cdot \boldsymbol{h_u}.$$

Note that $\boldsymbol{r_u}^{-1} = \boldsymbol{h_u}^{-1} \cdot \boldsymbol{j_u}$, i.e.

$$\boldsymbol{r_u}^{-1} \cdot (\boldsymbol{e}, \boldsymbol{p}) = -\boldsymbol{eu} + \boldsymbol{p} \qquad (\boldsymbol{e} \in \mathbf{I}^*,\, \boldsymbol{p} \in \mathbf{E}_{\boldsymbol{u}}^*).$$

7.2.2. The *covector transformation rule* is defined to be

$$\boldsymbol{R}_{\boldsymbol{u}'\boldsymbol{u}} := \left(\mathrm{id}_{\mathbf{I}} \times \boldsymbol{L}(\boldsymbol{u},\boldsymbol{u}')|_{\mathbf{E}_{\boldsymbol{u}}}\right) \cdot \boldsymbol{r}_{\boldsymbol{u}'} \cdot \boldsymbol{r}_{\boldsymbol{u}}^{-1}.$$

It can be easily deduced from the vector transformation rule that, apart from a negative sign, they are the same. Indeed,

$$\left(\mathrm{id}_{\mathbf{I}} \times \boldsymbol{L}(\boldsymbol{u},\boldsymbol{u}')|_{\mathbf{E}_{\boldsymbol{u}'}}\right)\boldsymbol{j}_{\boldsymbol{u}'} = \boldsymbol{j}_{\boldsymbol{u}} \cdot \left(\mathrm{id}_{\mathbf{I}} \times \boldsymbol{L}(\boldsymbol{u},\boldsymbol{u}')|_{\mathbf{E}_{\boldsymbol{u}'}},\right.$$

thus

$$\boldsymbol{R}_{\boldsymbol{u}'\boldsymbol{u}} = \boldsymbol{j}_{\boldsymbol{u}} \cdot \boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{j}_{\boldsymbol{u}}.$$

Consequently, if $(\boldsymbol{e},\boldsymbol{p}) \in \mathbf{I}^* \times \mathbf{E}_{\boldsymbol{u}}^*$ and $\boldsymbol{p}$ is parallel to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ then

$$\boldsymbol{R}_{\boldsymbol{u}'\boldsymbol{u}} \cdot (\boldsymbol{e},\boldsymbol{p}) = \frac{1}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}\left(\boldsymbol{e} + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{p},\, \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}\boldsymbol{e} + \boldsymbol{p}\right).$$

7.2.3. It is worth mentioning that $\mathbf{E}_{\boldsymbol{u}}^*$ can be considered to be a linear subspace of $\mathbf{M}^*$, since $\mathbf{E}_{\boldsymbol{u}}^* \equiv \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}\otimes\mathbf{I}} \subset \frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}} \equiv \mathbf{M}^*$ and

$$\mathbf{E}_{\boldsymbol{u}}^* = \{\boldsymbol{k} \in \mathbf{M}^* \mid \boldsymbol{k} \cdot \boldsymbol{u} = \boldsymbol{0}\},$$

in other words, $\mathbf{E}_{\boldsymbol{u}}^*$ is the annullator of $\boldsymbol{u} \otimes \mathbf{I}$.

In the non-relativistic case $\mathbf{E}^*$ is not a linear subspace of $\mathbf{M}^*$. For all $\boldsymbol{u} \in \mathrm{V}(1)$ there is a linear subspace $\mathbf{E}^* \cdot \boldsymbol{\pi}_{\boldsymbol{u}}$ of $\mathbf{M}^*$, the annullator of $\boldsymbol{u} \otimes \mathbf{I}$, but it is not the dual of any linear subspace in $\mathbf{M}$.

Observe that the special relativistic vector transformation rule which is nearly the same as the covector tramsformation rule resembles a combination of the non-relativistic vector and covector transformation rules.

We emphasize that in the special relativistic case *there is no absolute spacelike vector and there is no absolute timelike covector,* in contradistinction to the non-relativistic case.

7.3. Splitting of vector fields

7.3.1. In applications vector fields $\mathrm{M} \rightarrowtail \mathbf{M}$ and covector fields $\mathrm{M} \rightarrowtail \mathbf{M}^*$ appear frequently. Evidently, a covector field can be considered a vector field of cotype $\mathbf{I}\otimes\mathbf{I}$. Their splitting according to global inertial observers can be treated analogously to the non-relativistic case (see I.8.5).

Let $\boldsymbol{U}$ be a global inertial observer with the velocity value $\boldsymbol{u}$ and let $\boldsymbol{K} : \mathrm{M} \rightarrowtail \mathbf{M}^*$ be a covector field. At every world point x the value $\boldsymbol{K}(x)$ is split according to the velocity value $\boldsymbol{u}$ so we get the *half $\boldsymbol{U}$-split form* of the field:

$$(-V_{\boldsymbol{u}}, \boldsymbol{A}_{\boldsymbol{u}}) := \boldsymbol{r}_{\boldsymbol{u}} \cdot \boldsymbol{K} :\ \mathrm{M} \rightarrowtail \mathbf{I}^* \times \mathbf{E}_{\boldsymbol{u}}^*, \qquad x \mapsto= \left(\boldsymbol{u} \cdot \boldsymbol{K}(x),\, \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{K}(x)\right).$$

Furthermore, the observer splits spacetime as well, thus instead of world points $\boldsymbol{U}$-instants and $\boldsymbol{U}$-space points will be introduced to get the *completely split form* of the field:

$$(-V_{\boldsymbol{U}}, \boldsymbol{A}_{\boldsymbol{U}}) := (-V_{\boldsymbol{u}}, \boldsymbol{A}_{\boldsymbol{u}}) \circ \boldsymbol{H}_{\boldsymbol{U}}^{-1} = \boldsymbol{r}_{\boldsymbol{u}} \cdot \boldsymbol{K} \circ \boldsymbol{H}_{\boldsymbol{U}}^{-1} : \ \mathrm{I}_{\boldsymbol{U}} \times \mathrm{E}_{\boldsymbol{U}} \rightarrowtail \mathbf{I}^* \times \mathbf{E}_{\boldsymbol{u}}^*,$$
$$(t, q) \mapsto \big(\boldsymbol{u} \cdot \boldsymbol{K}(q \star t), \, \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{K}(q \star t)\big),$$

where $q \star t$ denotes the single element in the intersection of q and t.

7.3.2. Potentials are covector fields. We can introduce the scalar potential and the vector potential according to an observer by the previous split forms. Regarding the transformation rule concerning scalar potentials and vector potentials we can repeat essentially what we said in I.8.5.3; of course, the transformation rule will be significantly more complicated.

An important difference between the non-relativistic spacetime model and the special relativistic one is that here *there are no absolute scalar potentials* because there are no absolute timelike covectors. This forecasts that the description of gravitation in the relativistic case will differ significantly from its description in the non-relativistic case where absolute scalar potentials are used.

7.3.3. In contradistinction to the non-relativistic case, force fields are split differently according to different observers.

Let us take a force field $\boldsymbol{f} : \mathrm{M} \times \mathrm{V}(1) \rightarrowtail \frac{\mathbf{M}^*}{\mathbf{I}}$. Because of the property $\boldsymbol{f}(x, \boldsymbol{u}') \cdot \boldsymbol{u}' = \mathbf{0}$ for all $(x, \boldsymbol{u}') \in \mathrm{Dom}\, \boldsymbol{f}$, the $\boldsymbol{u}$-spacelike component and the $\boldsymbol{u}$-timelike component of $\boldsymbol{f}$ are not independent. Using the formula in 7.1.1 we get

$$\mathbf{0} = \boldsymbol{f}(x, \boldsymbol{u}') \cdot \boldsymbol{u}' = -\big(\boldsymbol{u} \cdot \boldsymbol{f}(x, \boldsymbol{u}')\big)(\boldsymbol{u} \cdot \boldsymbol{u}') + \big(\boldsymbol{\pi}_{\boldsymbol{u}} \cdot f(x, \boldsymbol{u}')\big) \cdot (\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{u}'),$$

which yields

$$-\boldsymbol{u} \cdot \boldsymbol{f}(x, \boldsymbol{u}') = \big(\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{f}(x, \boldsymbol{u}')\big) \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}.$$

7.3.4. Splittings of vector fields according to rigid observers in the non-relativistic case can be treated in the mathematical framework of affine spaces. However, splittings according to general observers require the theory of manifolds.

In the special relativistic case splittings according to non-inertial observers can be treated only in the framework of manifolds and they do not appear here.

7.4. Exercises

1. Show that $\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{x} = (\boldsymbol{u} \wedge \boldsymbol{x}) \cdot \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathrm{V}(1)$, $x \in \mathbf{M}$.

2. Take the arithmetic spacetime model. Give the completely split form of the vector field

$$(\xi^0, \boldsymbol{\xi}) \mapsto \left(\xi^1 + \xi^2, \cos(\xi^0 - \xi^3), 0, 0\right)$$

according to the global inertial observer with the velocity value $\frac{1}{\sqrt{1-v^2}}(1, v, 0, 0)$.

8. Tensor splittings

8.1. Splitting of tensors

8.1.1. The various tensors — elements of $\mathbf{M} \otimes \mathbf{M}$, $\mathbf{M} \otimes \mathbf{M}^*$, etc. — are split according to $\boldsymbol{u} \in \mathrm{V}(1)$ by the maps $\boldsymbol{h_u} \otimes \boldsymbol{h_u}$, $\boldsymbol{h_u} \otimes \boldsymbol{r_u}$, etc. as in the non-relativistic case. However, now it suffices to deal with $\boldsymbol{h_u} \otimes \boldsymbol{h_u}$ because the identification $\mathbf{M}^* \equiv \frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$ and $\boldsymbol{r_u} = \boldsymbol{j_u} \cdot \boldsymbol{h_u}$ (see 7.2.1) allow us to derive the other splittings from this one.

With the usual identifications we have

$$\boldsymbol{h_u} \otimes \boldsymbol{h_u} : \ \mathbf{M}\otimes\mathbf{M} \to (\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}})\otimes(\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}}) \equiv (\mathbf{I}\otimes\mathbf{I})\times(\mathbf{I}\otimes\mathbf{E}_{\boldsymbol{u}})\times(\mathbf{E}_{\boldsymbol{u}}\otimes\mathbf{I})\times(\mathbf{E}_{\boldsymbol{u}}\otimes\mathbf{E}_{\boldsymbol{u}}),$$

and for $\boldsymbol{T} \in \mathbf{M} \otimes \mathbf{M}$:

$$(\boldsymbol{h_u} \otimes \boldsymbol{h_u}) \cdot \boldsymbol{T} = \boldsymbol{h_u} \cdot \boldsymbol{T} \cdot \boldsymbol{h}_{\boldsymbol{u}}^* = \boldsymbol{h_u} \cdot \boldsymbol{T} \cdot \boldsymbol{r}_{\boldsymbol{u}}^{-1} = \begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{u} & -\boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \\ -\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{T} \cdot \boldsymbol{u} & \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{T} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \end{pmatrix} =$$

$$= \begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{u} & -\boldsymbol{u} \cdot \boldsymbol{T} - \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{u}) \\ -\boldsymbol{T} \cdot \boldsymbol{u} - \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{u}) & \boldsymbol{T} + \boldsymbol{u} \otimes (\boldsymbol{u} \cdot \boldsymbol{T}) + (\boldsymbol{T} \cdot \boldsymbol{u}) \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{u}) \end{pmatrix},$$

for $\boldsymbol{L} \in \mathbf{M} \otimes \mathbf{M}^*$:

$$(\boldsymbol{h_u} \otimes \boldsymbol{r_u}) \cdot \boldsymbol{L} = \boldsymbol{h_u} \cdot \boldsymbol{L} \cdot \boldsymbol{r}_{\boldsymbol{u}}^* = \boldsymbol{h_u} \cdot \boldsymbol{L} \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1} = \begin{pmatrix} -\boldsymbol{u} \cdot \boldsymbol{L} \cdot \boldsymbol{u} & -\boldsymbol{u} \cdot \boldsymbol{L} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \\ \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{L} \cdot \boldsymbol{u} & \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{L} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \end{pmatrix},$$

for $\boldsymbol{P} \in \mathbf{M}^* \otimes \mathbf{M}$:

$$(\boldsymbol{r_u} \otimes \boldsymbol{h_u}) \cdot \boldsymbol{P} = \boldsymbol{r_u} \cdot \boldsymbol{P} \cdot \boldsymbol{h}_{\boldsymbol{u}}^* = \boldsymbol{r_u} \cdot \boldsymbol{P} \cdot \boldsymbol{r}_{\boldsymbol{u}}^{-1} = \begin{pmatrix} -\boldsymbol{u} \cdot \boldsymbol{P} \cdot \boldsymbol{u} & \boldsymbol{u} \cdot \boldsymbol{P} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \\ -\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{P} \cdot \boldsymbol{u} & \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{P} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \end{pmatrix},$$

for $\boldsymbol{F} \in \mathbf{M}^* \otimes \mathbf{M}^*$:

$$(\boldsymbol{r_u} \otimes \boldsymbol{r_u}) \cdot \boldsymbol{F} = \boldsymbol{r_u} \cdot \boldsymbol{F} \cdot \boldsymbol{r}_{\boldsymbol{u}}^* = \boldsymbol{r_u} \cdot \boldsymbol{F} \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1} = \begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{F} \cdot \boldsymbol{u} & \boldsymbol{u} \cdot \boldsymbol{F} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \\ \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{F} \cdot \boldsymbol{u} & \boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{F} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \end{pmatrix}.$$

8.1.2. The splittings corresponding to different velocity values $\boldsymbol{u}$ and $\boldsymbol{u}'$ are different. The tensor transformation rule that shows how the splittings depend on velocity values is rather complicated, much more complicated than in the non-relativistic case. We shall study it only for antisymmetric tensors.

8.2. Splitting of antisymmetric tensors

8.2.1. If $\boldsymbol{T}$ is an antisymmetric tensor, i.e. $\boldsymbol{T} \in \mathbf{M} \wedge \mathbf{M}$, then $\boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{u} = \mathbf{0}$, $\boldsymbol{u} \cdot \boldsymbol{T} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* = -(\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{T} \cdot \boldsymbol{u})^*$ and $\boldsymbol{\pi}_{\boldsymbol{u}} \cdot \boldsymbol{T} \cdot \boldsymbol{\pi}_{\boldsymbol{u}}^* \in \mathbf{E}_{\boldsymbol{u}} \wedge \mathbf{E}_{\boldsymbol{u}}$ which means (of course) that the $\boldsymbol{u}$-split form of $\boldsymbol{T}$ is antisymmetric as well. Thus $\boldsymbol{u}$-splitting maps the elements of $\mathbf{M} \wedge \mathbf{M}$ into elements of form

$$\begin{pmatrix} \mathbf{0} & -\boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{A} \end{pmatrix} \equiv (\boldsymbol{a}, \boldsymbol{A})$$

where $\boldsymbol{a} \in \mathbf{E}_{\boldsymbol{u}} \otimes \mathbf{I} \equiv \mathbf{I} \otimes \mathbf{E}_{\boldsymbol{u}}$, $\boldsymbol{A} \in \mathbf{E}_{\boldsymbol{u}} \wedge \mathbf{E}_{\boldsymbol{u}}$.

The corresponding formula in 8.1.1 gives for $\boldsymbol{T} \in \mathbf{M} \wedge \mathbf{M}$

$$\boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{T} \cdot \boldsymbol{h}_{\boldsymbol{u}}^* = \big(-\boldsymbol{T} \cdot \boldsymbol{u}, \boldsymbol{T} + (\boldsymbol{T} \cdot \boldsymbol{u}) \wedge \boldsymbol{u}\big).$$

Definition. $-\boldsymbol{T} \cdot \boldsymbol{u}$ and $\boldsymbol{T} + (\boldsymbol{T} \cdot \boldsymbol{u}) \wedge \boldsymbol{u}$ are called the *$\boldsymbol{u}$-timelike component* and the *$\boldsymbol{u}$-spacelike component* of the antisymmetric tensor $\boldsymbol{T}$.

8.2.2. The following *transformation rule* shows how splittings depend on velocity values.

Proposition. Let $\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)$. Then

$$\begin{aligned}
&\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \big(\boldsymbol{a}, \boldsymbol{A}\big) \cdot \boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}^* = \\
&= \left(\frac{1}{\sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}} \left(\boldsymbol{a} + \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \frac{1 - \sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}{|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2} (\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \boldsymbol{a}) + \boldsymbol{A} \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \right), \right. \\
&\qquad \left. \frac{1}{\sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}} \left(-\boldsymbol{a} - \boldsymbol{A} \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \frac{1 - \sqrt{1 - |\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}{|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2} \right) \wedge \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} + \boldsymbol{A} \right).
\end{aligned}$$

Proof. Using the matrix forms we have

$$\begin{aligned}
&\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} \cdot \big(\boldsymbol{a}, \boldsymbol{A}\big) \cdot \boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}^* = \\
&\qquad = \kappa(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}})^2 \begin{pmatrix} 1 & -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \\ -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \boldsymbol{D}(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) \end{pmatrix} \begin{pmatrix} \mathbf{0} & -\boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} 1 & -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \\ -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \boldsymbol{D}(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) \end{pmatrix},
\end{aligned}$$

from which we can get the desired formula.

8.2.3. The previous fearsome formula becomes nicer if we write $(\boldsymbol{a},\boldsymbol{A})$ as the sum of components parallel and orthogonal to the relative velocity:

$$\boldsymbol{a}=\boldsymbol{a}_{\|}+\boldsymbol{a}_{\perp},\qquad \boldsymbol{A}=\boldsymbol{A}_{\|}+\boldsymbol{A}_{\perp}$$

where $\boldsymbol{a}_{\|}$ is parallel to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, $\boldsymbol{a}_{\perp}$ is orthogonal to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, and the kernel of $\boldsymbol{A}_{\|}$ is parallel to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, i.e. $\boldsymbol{A}_{\|}\cdot\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}=\mathbf{0}$ and the kernel of $\boldsymbol{A}_{\perp}$ is orthogonal to $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$, i.e. $(\boldsymbol{A}_{\perp}\cdot\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}})\wedge\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}=-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2\boldsymbol{A}_{\perp}$ (see Exercise V.3.21.1). Then we easily find

$$\begin{aligned}\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}\cdot\big(\boldsymbol{a}_{\|},\boldsymbol{A}_{\|}\big)\cdot\boldsymbol{H}^*_{\boldsymbol{u}'\boldsymbol{u}}&=\big(\boldsymbol{a}_{\|},\boldsymbol{A}_{\|}\big),\\ \boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}\cdot\big(\boldsymbol{a}_{\perp},\boldsymbol{A}_{\perp}\big)\cdot\boldsymbol{H}^*_{\boldsymbol{u}'\boldsymbol{u}}&=\frac{1}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}|^2}}\big(\boldsymbol{a}_{\perp}+\boldsymbol{A}_{\perp}\cdot\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}},\ -\boldsymbol{a}_{\perp}\wedge\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}+\boldsymbol{A}_{\perp}\big).\end{aligned}$$

8.2.4. The splitting and the transformation rule of antisymmetric cotensors i.e. elements of $\mathbf{M}^*\wedge\mathbf{M}^*$ are the same, apart from a negative sign. The details are left to the reader.

It is interesting that here, in contradistinction to the non-relativistic case, $\mathbf{M}\wedge\mathbf{M}^*$ makes sense because of the identification $\mathbf{M}^*\equiv\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$. The mixed tensor $\boldsymbol{H}\in\mathbf{M}\wedge\mathbf{M}^*$ has the $\boldsymbol{u}$-split form

$$\boldsymbol{h}_{\boldsymbol{u}}\cdot\boldsymbol{H}\cdot\boldsymbol{h}_{\boldsymbol{u}}^{-1}=\begin{pmatrix}\mathbf{0} & \boldsymbol{H}\cdot\boldsymbol{u}\\ \boldsymbol{H}\cdot\boldsymbol{u} & \boldsymbol{H}+(\boldsymbol{H}\cdot\boldsymbol{u})\wedge\boldsymbol{u}\end{pmatrix}$$

which, as a matrix, is not antisymmetric. It need not be antisymmetric, because the symmmetric or antisymmetric properties of matrices refer to these properties of linear maps regarding duals without any identifications (see IV.1.5 and V.4.19).

8.3. Splitting of tensor fields

8.3.1. The splitting of various tensor fields according to inertial observers can be treated analogously to the non-relativistic case.

Let $\boldsymbol{U}$ be a global inertial observer with velocity value $\boldsymbol{u}$. The antisymmetric cotensor field $\boldsymbol{F}$ has the *half split form* according to $\boldsymbol{U}$

$$\begin{aligned}\big(\boldsymbol{E}_{\boldsymbol{u}},-\boldsymbol{B}_{\boldsymbol{u}}\big):=\boldsymbol{r}_{\boldsymbol{u}}\cdot\boldsymbol{F}\cdot\boldsymbol{r}^*_{\boldsymbol{u}}:\ \mathrm{M}&\rightarrowtail(\mathbf{E}^*_{\boldsymbol{u}}\otimes\mathbf{I}^*)\times(\mathbf{E}^*_{\boldsymbol{u}}\wedge\mathbf{E}^*_{\boldsymbol{u}}),\\ x&\mapsto\big(\boldsymbol{F}(x)\cdot\boldsymbol{u},\ \boldsymbol{F}(x)+\big(\boldsymbol{F}(x)\cdot\boldsymbol{u}\big)\wedge\boldsymbol{u}\big)\end{aligned}$$

and the *completely split form*

$$\begin{aligned}\big(\boldsymbol{E_U}, -\boldsymbol{B_U}\big) &:= \big(\boldsymbol{E_u}, -\boldsymbol{B_u}\big) \circ H_{\boldsymbol{U}}^{-1} = \\ &= \boldsymbol{r_u} \cdot (\boldsymbol{F} \circ H_{\boldsymbol{U}}^{-1}) \cdot \boldsymbol{r}_{\boldsymbol{u}}^* : \ \mathrm{I} \times \mathrm{E}_{\boldsymbol{U}} \rightarrowtail (\mathbf{E}_{\boldsymbol{u}}^* \otimes \mathbf{I}) \times (\mathbf{E}_{\boldsymbol{u}}^* \wedge \mathbf{E}_{\boldsymbol{u}}^*), \\ &\qquad (t, q) \mapsto \big(\boldsymbol{F}(q \star t) \cdot \boldsymbol{u},\ \boldsymbol{F}(q \star t) + \big(\boldsymbol{F}(q \star t) \cdot \boldsymbol{u}\big) \wedge \boldsymbol{u}\big).\end{aligned}$$

8.3.2. The electromagnetic field is described by an antisymmetric cotensor field $\boldsymbol{F}$ which is the exterior derivative of a potential $\boldsymbol{K}$, $\boldsymbol{F} = \mathrm{D} \wedge \boldsymbol{K}$. The electric field and the magnetic field relative to the inertial observer $\boldsymbol{U}$ are the corresponding components of the completely split form of $\boldsymbol{F}$.

The relation between the completely split form $(-V_{\boldsymbol{U}}, \boldsymbol{A_U})$ of $\boldsymbol{K}$ and the completly split form $\big(\boldsymbol{E_U}, -\boldsymbol{B_U}\big)$ of $\boldsymbol{F}$ is exactly the same as in the non-relativistic case:

$$\boldsymbol{E_U} = -\partial_0 \boldsymbol{A_U} - \nabla V_{\boldsymbol{U}}, \qquad \boldsymbol{B_U} = -\nabla \wedge \boldsymbol{A_U}.$$

Since the force field defined by the potential $\boldsymbol{K}$ equals

$$\boldsymbol{f}(x, \boldsymbol{u}') = \boldsymbol{F}(x) \cdot \boldsymbol{u}' \qquad\qquad (x \in \mathrm{Dom}\ \boldsymbol{K},\ \boldsymbol{u}' \in \mathrm{V}(1)),$$

where $\boldsymbol{F} := \mathrm{D} \wedge \boldsymbol{K}$, we can state again that a masspoint in the world point x having the velocity value $\boldsymbol{u}'$ "feels" only the $\boldsymbol{u}'$-timelike component of the field; a masspoint always "feels" the electric field according to its instantaneous velocity value.

Because of the more complicated tranformation rule in the special relativistic case the Lorentz force is expressed by the $\boldsymbol{U}$-electric field and the $\boldsymbol{U}$-magnetic field more complicatedly than in the non-relativistic case.

8.4. Exercises

1. Let $\boldsymbol{x} \in \mathbf{M}$, $\boldsymbol{T} \in \mathbf{M} \otimes \mathbf{M}$ and $\boldsymbol{L} \in \mathbf{M} \otimes \mathbf{M}^*$. Give the $\boldsymbol{u}$-split form of $\boldsymbol{T} \cdot \boldsymbol{x}$ and $\boldsymbol{L} \cdot \boldsymbol{x}$ using the $\boldsymbol{u}$-split form of $\boldsymbol{x}$, $\boldsymbol{T}$ and $\boldsymbol{L}$.
2. Let $\boldsymbol{T}$ and $\boldsymbol{L}$ as before. Give the $\boldsymbol{u}$-split form of $\boldsymbol{T} \cdot \boldsymbol{L}$ and $\boldsymbol{L} \cdot \boldsymbol{T}$ using the $\boldsymbol{u}$-split form of $\boldsymbol{T}$ and $\boldsymbol{L}$.
3. Recall the non-degenarate bilinear form (see V.4.15)

$$(\mathbf{M} \wedge \mathbf{M}^*) \times (\mathbf{M} \wedge \mathbf{M}^*) \to \mathbb{R}, \qquad (\boldsymbol{F}, \boldsymbol{H}) \mapsto \boldsymbol{F} \bullet \boldsymbol{H} := -\frac{1}{2}\,\mathrm{Tr}\,\boldsymbol{F} \cdot \boldsymbol{H}.$$

Express $\boldsymbol{F} \cdot \boldsymbol{H}$ using the $\boldsymbol{u}$-timelike and the $\boldsymbol{u}$-spacelike components of $\boldsymbol{F}$ and $\boldsymbol{H}$.

9. Reference frames

9.1. The notion of a reference frame

9.1.1. We can repeat word by word what we said in I. 7.1.1 with the single exception that instead of (absolute) time now we have to consider an (artificial) time derived from a simultaneity, to arrive at the following notion.

Recall that an observer $\boldsymbol{U}$ together with a simultaneity $\mathcal{S}S$ establishes the splitting $H_{\boldsymbol{U},\mathcal{S}} = (\tau_{\mathcal{S}}, C_{\boldsymbol{U}}) : \mathrm{M} \rightarrowtail \mathrm{I}_{\mathcal{S}} \times \mathrm{E}_{\boldsymbol{U}}$.

Definition. A *reference system* is a quartet $(\boldsymbol{U}, \mathcal{S}, T_{\mathcal{S}}, S_{\boldsymbol{U}})$ where

(i) $\boldsymbol{U}$ is an observer,

(ii) $\mathcal{S}$ is a simultaneity on the domain of $\boldsymbol{U}$,

(iii) $T_{\mathcal{S}} : \mathrm{I}_{\mathcal{S}} \rightarrowtail \mathbb{R}$ is a strictly monotone increasing mapping,

(iv) $S_{\boldsymbol{U}} : E_{\boldsymbol{U}} \rightarrowtail \mathbb{R}^3$ is a mapping

such that $(T_{\mathcal{S}} \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U},\mathcal{S}} = (T_{\mathcal{S}} \circ \tau_{\mathcal{S}},\ S_{\boldsymbol{U}} \circ C_{\boldsymbol{U}}) : \mathrm{M} \rightarrowtail \mathbb{R} \times \mathbb{R}^3$ is an orientation-preserving coordinatization. ■

We call $T_{\mathcal{S}}$ and $S_{\boldsymbol{U}}$ the *coordinatization* of $\mathcal{S}$-time and $\boldsymbol{U}$-space, respectively, in spite of the fact that we introduced the notion of coordinatization only for affine spaces and, in general, neither $\mathrm{I}_{\mathcal{S}}$ nor $\mathrm{E}_{\boldsymbol{U}}$ is an affine space. (We mention that in any case $\mathrm{I}_{\mathcal{S}}$ and $\mathrm{E}_{\boldsymbol{U}}$ can be endowed with a smooth structure and in the framework of smooth structures $T_{\mathcal{S}}$ and $S_{\boldsymbol{U}}$ do become a coordinatization.)

Note that condition (iii) involves that $T_{\mathcal{S}}$ is defined on a subset of $\mathrm{I}_{\mathcal{S}}$ where the ordering "later" is total; consequently, the coordinatization of spacetime is defined on a subset of Dom $\boldsymbol{U}$ where the simultaneity is well posed.

9.1.2. Definition. A coordinatization $K : \mathrm{M} \rightarrowtail \mathbb{R} \times \mathbb{R}^3$ is called a *reference frame* if there is a reference system $(\boldsymbol{U}, \mathcal{S}, T_{\mathcal{S}}, S_{\boldsymbol{U}})$ such that $K = (T_{\mathcal{S}} \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U},\mathcal{S}}$.

$\boldsymbol{U}$, $\mathcal{S}$, $T_{\mathcal{S}}$ and $S_{\boldsymbol{U}}$ are called *the observer, the simultaneity, the $\mathcal{S}$-time coordinatization and the $\boldsymbol{U}$-space coordinatization* corresponding to the reference frame. ■

As usual, we number the coordinates of $\mathbb{R} \times \mathbb{R}^3$ from zero to three. Accordingly, we find convenient to use the notation $K = (\kappa^0, \boldsymbol{\kappa}) : \mathrm{M} \rightarrowtail \mathbb{R} \times \mathbb{R}^3$ for the coordinatizations of spacetime. Then the equality

$$\mathrm{D}\boldsymbol{\kappa}(x) \cdot \partial_0 K^{-1}(K(x)) = \mathbf{0}$$

well-known and used in the non-relativistic case will hold now as well, since its deduction rests only on the affine structure of M.

If K is a reference frame then

$$\kappa^0 = T_{\mathcal{S}} \circ \tau_{\mathcal{S}}, \qquad \boldsymbol{\kappa} = S_{\boldsymbol{U}} \circ C_{\boldsymbol{U}}.$$

9.1.3. Proposition. A coordinatization $K = (\kappa^0, \boldsymbol{\kappa}) : \mathrm{M} \rightarrowtail \mathbb{R} \times \mathbb{R}^3$ is a reference frame if and only if
(i) K is orientation preserving,
(ii) $\partial_0 K^{-1}(K(x))$ is a future-directed timelike vector,
(iii) $-\big(\mathrm{D}\kappa^0\big)(x)$ is a future-directed timelike vector
for all $x \in \mathrm{Dom}\ K$.

Then
$$\boldsymbol{U}(x) = \frac{\partial_0 K^{-1}(K(x))}{|\partial_0 K^{-1}(K(x))|} \qquad (x \in \mathrm{Dom}\ K), \tag{1}$$
is the corresponding observer and the corresponding simultaneity $\mathcal{S}$ is determined as follows:
$$x \text{ is simultaneous with } y \text{ if and only if } \kappa^0(x) = \kappa^0(y) \tag{2}$$
moreover,
$$T_{\mathcal{S}}(t) = \kappa^0(x) \qquad (t \in \mathrm{I}_{\mathcal{S}},\ x \in t), \tag{3}$$
$$S_{\boldsymbol{U}}(q) = \boldsymbol{\kappa}(x) \qquad (q \in \mathrm{E}_{\boldsymbol{U}},\ x \in q). \tag{4}$$

Proof. If K is a reference frame, $K = (T_{\mathcal{S}} \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U},\mathcal{S}}$, then ($i$) is trivial.

κ^0 is constant on the $\mathcal{S}$-instants. In other words, $\mathcal{S}$-instants — more precisely their part in $\mathrm{Dom}\ K$ — have the form $\{x \in \mathrm{Dom}\ K \,|\, \kappa^0(x) = \alpha\}$. Then $\big\{\boldsymbol{x} \in \mathbf{M} \,|\, \big(\mathrm{D}\kappa^0\big)(x) \cdot \boldsymbol{x} = \mathbf{0}\big\}$ is the tangent space of the corresponding world surface passing through x. Since this tangent space is spacelike, $\big(\mathrm{D}\kappa^0\big)(x)$ must be timelike. If $y - x \in \mathrm{T}^{\rightarrow}$, then the properties of $\tau_{\mathcal{S}}$ and $T_{\mathcal{S}}$ imply that $\kappa^0(y) - \kappa^0(x) > 0$; then $\big(\mathrm{D}\kappa^0\big)(x) \cdot (y - x) + \mathrm{ordo}(y - x) > 0$ results in that $\big(\mathrm{D}\kappa^0\big)(x) \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathrm{T}^{\rightarrow}$, proving ($iii$).

As concerns (ii), note that a world line function r satisfies $\dot{r}(\boldsymbol{s}) = \boldsymbol{U}(r(\boldsymbol{s}))$ and takes values in the domain of K if and only if $K(r(\boldsymbol{s})) = \big(\kappa^0(r(\boldsymbol{s})), \boldsymbol{\xi}\big)$ i.e. $r(\boldsymbol{s}) = K^{-1}\big(\kappa^0(r(\boldsymbol{s}), \boldsymbol{\xi}\big)$ for a $\boldsymbol{\xi} \in \mathbb{R}^3$ and for all $\boldsymbol{s} \in \mathrm{Dom}\ r$. As a consequence, we have
$$\boldsymbol{U}(r(\boldsymbol{s})) = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{s}} K^{-1}\big(\kappa^0(r(\boldsymbol{s})), \boldsymbol{\xi}\big) = \partial_0 K^{-1}(\kappa^0(r(\boldsymbol{s})), \boldsymbol{\xi})) \cdot \big(\mathrm{D}\kappa^0\big)(r(\boldsymbol{s})) \cdot \dot{r}(\boldsymbol{s})$$
which, together with condition (iii), implies that $\boldsymbol{U}(x)$ is a positive multiple of $\partial_0 K^{-1}(K(x))$ for all $x \in \mathrm{Dom}\ K$, proving (ii) and equality (1).

Suppose now that $K = (\kappa^0, \boldsymbol{\kappa})$ is a coordinatization that fulfils conditions (ii)–(iii).

Then condition (ii) implies that $\boldsymbol{U}$ *defined* by equality (1) is an observer.

According to (iii), the simultaneity $\mathcal{S}$ is well *defined* by (2) (i.e. the subsets of form $\{x \in \mathrm{Dom}\ K \,|\, \kappa^0(x) = \alpha\}$ are world surfaces and $\mathcal{S}$ is smooth). Consequently, $T_{\mathcal{S}}$ is well *defined* by formula (3) and it is strictly monotone increasing.

If r is a world line such that $\dot{r}(\mathbf{s}) = \boldsymbol{U}(r(\mathbf{s}))$ then

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{s}}(\boldsymbol{\kappa}(r(\mathbf{s})) = \mathrm{D}\boldsymbol{\kappa}(r(\mathbf{s})) \cdot \boldsymbol{U}(r(t)) = \mathrm{D}\boldsymbol{\kappa}(r(\mathbf{s})) \cdot \frac{\partial_0 K^{-1}(K(r(\mathbf{s})))}{|\partial_0 K^{-1}(K(r(\mathbf{s})))|} = \mathbf{0}$$

which means that $\boldsymbol{\kappa} \circ r$ is a constant mapping, in other words, $\boldsymbol{\kappa}$ is constant on the $\boldsymbol{U}$-lines; hence $S_{\boldsymbol{U}}$ is well *defined* by formula (4).

Finally, it is evident that $K = (T_{\mathcal{S}} \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U},\mathcal{S}}$. ∎

9.2. Lorentz reference frames

9.2.1. Now we are interested in what kind of affine coordinatization of spacetime can be a reference frame.

Let us take an affine coordinatization $K : \mathrm{M} \to \mathbb{R}^4$. Then there are

— an $o \in \mathrm{M}$,

— an ordered basis $(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ of $\mathbf{M}$

such that

$$K(x) = \big(\boldsymbol{k}^i \cdot (x - o) |\, i = 0, 1, 2, 3\big) \qquad (x \in \mathrm{M}),$$

$$K^{-1}(\xi) = \sum_{i=0}^{3} \xi^i x_i \qquad (\xi \in \mathbb{R}^4),$$

where $(\boldsymbol{k}^0, \boldsymbol{k}^1, \boldsymbol{k}^2, \boldsymbol{k}^3)$ is the dual of the basis in question.

Proposition. The affine coordinatization K is a reference frame if and only if

(i) $(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ is a positively oriented basis,

(ii) $\boldsymbol{x}_0$ is a future-directed timelike vector,

(iii) $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are spacelike vectors spanning a spacelike linear subspace of $\mathbf{M}$.

Then the corresponding observer is global and inertial, having the constant value

$$\boldsymbol{u} := \frac{\boldsymbol{x}_0}{|\boldsymbol{x}_0|},$$

and the simultaneity is given by the hyperplanes directed by the spacelike subspace that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ span.

Proof. We show that the present conditions (i)–(iii) correspond to the conditions listed in Proposition 8.1.3.

(i) The coordinatization is orientation-preserving if and only if the corresponding basis is positively oriented;

(ii) $\partial_0 K^{-1}(K(x)) = \boldsymbol{x}_0$;

(iii) $-(\mathrm{D}\kappa^0)(x) = -\boldsymbol{k}^0$ for all $x \in \mathrm{M}$. Since $\boldsymbol{k}^0 \cdot \boldsymbol{x}_\alpha = 0$ $(\alpha = 1,2,3)$, $-\boldsymbol{k}^0$ is timelike if and only if $\boldsymbol{x}_\alpha$-s span a spacelike linear subspace; then, since $\boldsymbol{k}^0 \cdot \boldsymbol{x}_0 = 1 > 0$ and since $\boldsymbol{x}_0$ is future-directed timelike, $-\boldsymbol{k}^0$ must be future-directed.

9.2.2. Let G denote the Lorentz form on $\mathbb{R}^4$ treated in V.4.19 and recall that a linear map $\boldsymbol{L} : \mathbf{M} \to \mathbb{R}^4$ is called $\mathbf{g}$-G-orthogonal if there is an $\boldsymbol{s} \in \mathbf{I}$ such that $\mathrm{G}(\boldsymbol{L}\cdot\boldsymbol{x}\cdot\boldsymbol{L}\cdot\boldsymbol{y}) = \frac{\mathbf{g}(\boldsymbol{x},\boldsymbol{y})}{\boldsymbol{s}^2}$ for all $\boldsymbol{x},\boldsymbol{y} \in \mathbf{M}$.

Definition. A reference frame K is called *Lorentzian* if
— K is affine,
— $\boldsymbol{K} : \mathbf{M} \to \mathbb{R}^4$ is $\mathbf{g}$-G-orthogonal. ■

From the previous result we get immediately the following:

Proposition. A reference frame K is Lorentzian if and only if there are
(i) an $o \in \mathrm{M}$,
(ii) a positively oriented $\mathbf{g}$-orthogonal basis $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, normed to an $\boldsymbol{s}$, of $\mathbf{M}$ such that $\boldsymbol{e}_0$ is future-directed timelike,
and

$$K(x) = \left(\frac{\boldsymbol{e}_i \cdot (x - o)}{\boldsymbol{e}_i^2} \,\middle|\, i = 0,1,2,3 \right) \qquad (x \in \mathrm{M}). \quad ■$$

We shall use the following names for a Lorentz reference frame: o is its *origin,* $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is its *spacetime basis*; moreover, $\boldsymbol{s} := |\boldsymbol{e}_0|$ is its *time and distance unit,* $\boldsymbol{u} := \frac{\boldsymbol{e}_0}{\boldsymbol{s}}$ is its *velocity value* and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is its *space basis.*

9.2.3. Let K be a Lorentz reference frame and use the previous notations.
We see from 1.6 that the Lorentz reference frame establishes an isomorphism between the spacetime model $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ and the arithmetic spacetime model. More precisely, the coordinatization K and the mapping $B : \mathbf{I} \to \mathbb{R}$, $\boldsymbol{t} \mapsto \frac{\boldsymbol{t}}{\boldsymbol{s}}$ constitute an isomorphism.
This isomorphism transforms vectors, covectors and tensors, cotensors etc. into vectors, covectors etc. of the arithmetic spacetime model.
In particular,

$$\boldsymbol{K} : \mathbf{M} \to \mathbb{R}^4, \qquad \boldsymbol{x} \mapsto \left(\frac{\boldsymbol{e}_i \cdot \boldsymbol{x}}{\boldsymbol{e}_i^2} \,\middle|\, i = 0,1,2,3 \right)$$

is the coordinatization of vectors and

$$\left(\boldsymbol{K}^{-1}\right)^* : \mathbf{M}^* \to \mathbb{R}^4, \qquad \boldsymbol{k} \mapsto \left(\boldsymbol{k} \cdot \boldsymbol{e}_i \middle|\, i = 0,1,2,3\right),$$

is the coordinatization of covectors.
We can generalize the coordinatization for vectors (covectors) of type or cotype $\mathbf{A}$, i.e. for elements in $\mathbf{M} \otimes \mathbf{A}$ or $\frac{\mathbf{M}}{\mathbf{A}}$ $\left(\mathbf{M}^* \otimes \mathbf{A}, \frac{\mathbf{M}^*}{\mathbf{A}}\right)$, too, where $\mathbf{A}$

is a measure line. For instance, elements of $\frac{\mathbf{M}}{\mathbf{I}}$ or $\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$ are coordinatized by the basis $\left(\frac{\boldsymbol{e}_i}{\boldsymbol{s}}\,\middle|\; i=0,1,2,3\right)$ and by the basis $\left(\frac{\boldsymbol{e}_i}{\boldsymbol{s}^2}\,\middle|\; i=0,1,2,3\right)$, respectively:

$$\begin{aligned} \frac{\mathbf{M}}{\mathbf{I}} &\to \mathbb{R}^4, \qquad \boldsymbol{w} \mapsto \boldsymbol{s}\left(\frac{\boldsymbol{e}_i\cdot\boldsymbol{w}}{\boldsymbol{e}_i^2}\,\middle|\; i=0,1,2,3\right), \\ \frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}} &\to \mathbb{R}^4, \qquad \boldsymbol{p} \mapsto \boldsymbol{s}^2\left(\frac{\boldsymbol{e}_i\cdot\boldsymbol{p}}{\boldsymbol{e}_i^2}\,\middle|\; i=0,1,2,3\right). \end{aligned}$$

9.3. Equivalent reference frames

9.3.1. We can repeat, according to the sense, what we said in I.7.5.1.

Recall the notion of automorphisms of the spacetime model (see 1.6.1). An automorphism is a transformation that leaves invariant (preserves) the structure of the spacetime model. Strict automorphisms do not change time periods and distances.

It is quite natural that two objects transformed into each other by a strict automorphism of the spacetime model are considered equivalent (i.e. the same from a physical point of view).

Recalling that $\mathcal{O}(\mathbf{g})$ denotes the set of $\mathbf{g}$-orthogonal linear maps in $\mathbf{M}$ (see V.2.7) let us introduce the notation

$$\begin{aligned} &\mathcal{P}^{+\to} := \\ &\{L : \mathrm{M} \to \mathrm{M} \mid L \text{ is affine, } \boldsymbol{L} \in \mathcal{O}(\mathbf{g}),\ \boldsymbol{L} \text{ is orientation- and arrow-preserving }\} \end{aligned}$$

and let us call the elements of $\mathcal{P}^{+\to}$ *proper Poincaré transformations.* We shall study these transformations in the next paragraph. For the moment it suffices to know the quite evident fact that $(L, \mathrm{id}_{\mathbf{I}})$ is a strict automorphism if and only if L is a proper Poincaré transformation.

9.3.2. Definition. The reference frames K and K' are called *equivalent* if there is a proper Poincaré transformation L such that

$$K' \circ L = K.$$

Two reference systems are *equivalent* if the corresponding reference frames are equivalent.

Proposition. Let $(\boldsymbol{U}, \mathcal{S}, T_{\mathcal{S}}, S_{\boldsymbol{U}})$ and $(\boldsymbol{U}', \mathcal{S}', T_{\mathcal{S}'}, S_{\boldsymbol{U}'})$ be the reference systems corresponding to the reference frames K and K', respectively. If K and K' are equivalent, $K' \circ L = K$, then

(i) $\boldsymbol{L} \cdot \boldsymbol{U} = \boldsymbol{U}' \circ L$,

(ii) $(T_{\mathcal{S}'}^{-1} \circ T_{\mathcal{S}}) \circ \tau_{\mathcal{S}} = \tau_{\mathcal{S}'} \circ L$

(iii) $(S_{\boldsymbol{U}'}^{-1} \circ S_{\boldsymbol{U}}) \circ C_{\boldsymbol{U}} = C_{\boldsymbol{U}'} \circ L$

Proof. For (i) we can argue as in I.10.5.3, using $(\boldsymbol{L} \cdot \boldsymbol{x})^2 = \boldsymbol{x}^2$ for all $\boldsymbol{x} \in \mathbf{M}$. As concerns (ii) and (iii), we can copy the reasoning of (iii) in I.10.5.3.

9.3.3. Now we shall see that our definition of equivalence of reference frames is in accordance with the intuitive notion expounded in I.10.5.1.

Proposition. Two Lorentz reference frames are equivalent if and only if they have the same unit of time (and distance).

Proof. Let the Lorentz reference frames K and K' be defined by the origins o and o' and the spacetime bases $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ and $(\boldsymbol{e}_0', \boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3')$, respectively.

Then $L := K'^{-1} \circ K : \mathrm{M} \to \mathrm{M}$ is the affine bijection determined by

$$L(o) = o', \qquad \boldsymbol{L} \cdot \boldsymbol{e}_i = \boldsymbol{e}_i' \qquad (i = 0, 1, 2, 3).$$

Evidently, L is orientation-preserving. Moreover, $\boldsymbol{L} \in \mathcal{O}(\mathbf{g})$ if and only if $|\boldsymbol{e}_0| = |\boldsymbol{e}_0'|$ and it is arrow-preserving if and only if $\boldsymbol{e}_0$ and $\boldsymbol{e}_0'$ have the same arrow.

9.4. Curve lengths calculated in coordinates

9.4.1. In 6.3.3 we dealt with lengths of curves in the space of an observer $\boldsymbol{U}$ at instants of a simultaneity $\mathcal{S}$. It is an interesting question how to calculate these lengths in coordinates corresponding to a reference frame $K = (T_{\mathcal{S}} \times S_{\boldsymbol{U}}) \circ H_{\boldsymbol{U},\mathcal{S}}$.

We shall use the notation $P := K^{-1}$ (P is the parametrization corresponding to the coordinatization K).

Let L and L_t be as in 6.3.3 and let ξ^0 be the coordinate of $t \in \mathrm{I}_{\mathcal{S}}$, i.e. $\xi^0 = \tau_{\mathcal{S}}(t)$.

A parametrization p_t of L_t has the coordinatized form

$$a \mapsto K(p_t(a)) =: \big(\xi^0, (p^\alpha(a) |\ \alpha = 1, 2, 3)\big) =: \big(\xi^0, \boldsymbol{p}(a)\big)$$

from which we deduce

$$p_t = P(\xi^0, \boldsymbol{p}),$$
$$\dot{p}_t = \partial_\alpha P(\xi^0, p)\dot{\boldsymbol{p}}^\alpha \qquad \text{(Einstein summation).}$$

Furthermore, we know (see 9.1.3)

$$\boldsymbol{U}(P) = \frac{\partial_0 P}{|\partial_0 P|}.$$

Consequently,

$$\begin{aligned}|\boldsymbol{\pi}_{\boldsymbol{U}(p_t)} \cdot \dot{p}_t|^2 &= |\dot{p}_t|^2 + |\boldsymbol{U}(p_t)\cdot \dot{p}_t|^2 = \\ &= \left(\partial_\alpha P \cdot \partial_\beta P + \frac{(\partial_0 P \cdot \partial_\alpha P)(\partial_0 P \cdot \partial_\beta P)}{|\partial_0 P|^2}\right)(\xi^0, \boldsymbol{p})\dot{p}^\alpha \dot{p}^\beta .\end{aligned}$$

Let us put

$$\mathbf{g}_{ik} := \partial_i P \cdot \partial_k P \qquad (i,k = 0,1,2,3).$$

Taking into account that $\mathbf{g}_{00}$ is negative, we see that

$$\mathbf{b}_{\alpha\beta} := \mathbf{g}_{\alpha\beta} - \frac{\mathbf{g}_{0\alpha}\mathbf{g}_{0\beta}}{\mathbf{g}_{00}} \qquad (\alpha, \beta = 1,2,3)$$

is the "metric tensor" in the $\boldsymbol{U}$-space, i.e. a curve in the $\boldsymbol{U}$-space parametrized by $\boldsymbol{p}$ at an $\mathcal{S}$-instant coordinatized by ξ^0 has length

$$\int \sqrt{\mathbf{b}_{\alpha\beta}(\xi^0, \boldsymbol{p}(a))\dot{p}^\alpha(a)\dot{p}^\beta(a)}\mathrm{d}a.$$

9.4.2. Note that $\mathbf{g}_{ik}$ is a function from $\mathbb{R}^4$ into $\mathbf{I}\otimes\mathbf{I}$.

We know that $\big(\partial_i P(\xi) \,|\, i = 0,1,2,3\big)$ is a basis in $\mathbf{M}$ (the local basis at $P(\xi)$ (see VI.5.6)). Thus, according to V.4.21, $\big(\mathbf{g}_{ik}(\xi) \,|\, i,k = 0,1,2,3\big)$ is the coordinatized form of $\mathbf{g}$ corresponding to this basis. More precisely, we get those formulae choosing an $\boldsymbol{s} \in \mathbf{I}^+$ and putting

$$g_{ik}(\xi) := \frac{\mathbf{g}_{ik}(\xi)}{\boldsymbol{s}^2}.$$

9.5. Exercises

1. Let $\boldsymbol{U}$ be the observer corresponding to the reference frame K. Demonstrate that the coordinatized form of $\boldsymbol{U}$ according to K is the constant mapping $(1, \mathbf{0})$. (By definition, $(DK \cdot \boldsymbol{U}) \circ K^{-1}$ is the coordinatized form of $\boldsymbol{U}$ according to K, see VI.5).

2. Take the uniformly accelerated observer $\boldsymbol{U}$ treated in 6.5. Fix $\boldsymbol{s} \in \mathbf{I}^+$ and define a Lorentz reference frame with an arbitrary origin o and with a spacetime

basis such that $\boldsymbol{e}_0 := \boldsymbol{s}\boldsymbol{U}(o)$, $\boldsymbol{e}_1 := \boldsymbol{s}\frac{\boldsymbol{a}_o}{|\boldsymbol{a}_o|}$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are arbitrary. Demonstrate that $\boldsymbol{U}$ will have the coordinatized form

$$(\xi^0, \xi^1, \xi^2, \xi^3) \mapsto \left(\sqrt{1 + (\alpha\xi^0)^2}, \alpha\xi^0, 0, 0\right)$$

where α is the number for which $|\boldsymbol{a}_o| = \alpha\frac{1}{\boldsymbol{s}}$ holds.

The $\boldsymbol{U}$-line passing through $o + \sum_{i=0}^{3} \xi^i \boldsymbol{e}_i$ becomes

$$\left\{ \left(\frac{1}{\alpha}\mathrm{sh}\alpha s,\ \xi^1 + \frac{1}{\alpha}(\mathrm{ch}\alpha s - 1),\ \xi^2,\ \xi^3 \right) \middle|\ s \in \mathbb{R} \right\}$$

3. Take the uniformly accelerated observer $\boldsymbol{U}$ treated in 6.6. Fix an $\boldsymbol{s} \in \mathbf{I}^+$ and define a Lorentz reference frame with an arbitrary origin o and with a spacetime basis such that $\boldsymbol{e}_0 := \boldsymbol{s}\boldsymbol{U}(o)$, $\boldsymbol{e}_1 := \boldsymbol{s}\frac{\boldsymbol{a}_o}{|\boldsymbol{a}_o|}$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are arbitrary. Demonstrate that $\boldsymbol{U}$ will have the coordinatized form

$$\{(\xi^0, \xi^1, \xi^2, \xi^3) \in \mathbb{R}^4 \mid \xi^1 > |\xi^0|\} \to \mathbb{R}^4,$$
$$(\xi^0, \xi^1, \xi^2, \xi^3) \mapsto \frac{1}{\sqrt{-(\xi^0)^2 + (\xi^1)^2}}(\xi^1, \xi^0, 0, 0).$$

The $\boldsymbol{U}$-line passing through $o + \sum_{i=0}^{3} \xi^i \boldsymbol{e}_i$ becomes

$$\left\{ \left(\frac{1}{\xi^1}\mathrm{sh}\xi^1 s,\ \xi^1 + \frac{1}{\xi^1}(\mathrm{ch}\xi^1 s - 1),\ \xi^2,\ \xi^3 \right) \middle|\ s \in \mathbb{R} \right\}.$$

4. Take the uniformly rotating observer $\boldsymbol{U}$ treated in 6.7. Fix an $\boldsymbol{s} \in \mathbf{I}^+$ and define a Lorentz reference frame with o, $\boldsymbol{e}_0 := \boldsymbol{s}\boldsymbol{U}(o)$, $\boldsymbol{e}_3$ positively oriented in Ker Ω, $|\boldsymbol{e}_3| = \boldsymbol{s}$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ arbitrary. Demonstrate that $\boldsymbol{U}$ will have the coordinatized form

$$(\xi^0, \xi^1, \xi^2, \xi^3) \mapsto \left(\sqrt{1 + \omega^2\big((\xi^1)^2 + (\xi^2)^2}, -\omega\xi^2, \omega\xi^1, 0\right)$$

where ω is the number for which $|\Omega| = \omega\frac{1}{\boldsymbol{s}}$ holds.

The $\boldsymbol{U}$-line passing through $o + \sum_{i=0}^{3} \xi^i \boldsymbol{e}_i$ becomes

$$\left\{ \left(s\sqrt{1 + \omega^2\big((\xi^1)^2 + (\xi^2)^2},\ \xi^1\cos\omega s - \xi^2\sin\omega s,\ \xi^1\sin\omega s + \xi^2\cos\omega s,\ \xi^3 \right) \middle|\ s \in \mathbb{R} \right\}.$$

5. Take the uniformly rotating observer $\boldsymbol{U}$ treated in 6.8. Fix an $\boldsymbol{s} \in \mathbf{I}^+$ and define a Lorentz reference frame with o, $\boldsymbol{e}_0 := \boldsymbol{s}\boldsymbol{U}(o)$, $\boldsymbol{e}_3$ positively oriented in Ker Ω, $|\boldsymbol{e}_3| = \boldsymbol{s}$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ arbitrary. Demonstrate that $\boldsymbol{U}$ will have the coordinatized form

$$\left\{(\xi^0,\xi^1,\xi^2,\xi^3) \in \mathbb{R}^4 \mid \omega^2\left((\xi^1)^2 + (\xi^2)^2\right) < 1\right\} \to \mathbb{R}^4,$$
$$(\xi^0,\xi^1,\xi^2,\xi^3) \mapsto \frac{1}{\sqrt{1-\omega^2\left((\xi^1)^2+(\xi^2)^2\right)}}(1, -\omega\xi^2, \omega\xi^1, 0)$$

where ω is the number for which $|\Omega| = \omega\frac{1}{\boldsymbol{s}}$ holds.

The $\boldsymbol{U}$-line passing through $o + \sum_{i=0}^{3} \xi^i \boldsymbol{e}_i$ becomes

$$\left\{ \left(t(s),\, \xi^1 \cos\omega t(s) - \xi^2 \sin\omega t(s),\, \xi^1 \sin\omega t(s) + \xi^2 \cos\omega t(s),\, \xi^3\right) \,\middle|\, s \in \mathbb{R}\right\},$$

where

$$t(s) := \frac{s}{\sqrt{1-\omega^2\left((\xi^1)^2+(\xi^2)^2\right)}}.$$

6. Find necessary and sufficient conditions that two affine reference frames be equivalent.

7. Take the uniformly accelerated observer treated in 6.5, consider $\boldsymbol{U}_o$-simultaneity and find a convenient reference frame for them.

8. A reference frame defined for a uniformly accelerated observer cannot be equivalent to a reference frame defined for a uniformly rotating observer.

10. Spacetime groups*

10.1. The Lorentz group

10.1.1. We shall deal with linear maps from $\mathbf{M}$ into $\mathbf{M}$, permanently using the identification $\mathrm{Lin}(\mathbf{M}) \equiv \mathbf{M} \otimes \mathbf{M}^*$.

Recall the notion of $\mathbf{g}$-adjoints, $\mathbf{g}$-orthogonal maps, $\mathbf{g}$-antisymmetric maps (V.1.5, V.2.7).

Definition.

$$\mathcal{L} := \{\boldsymbol{L} \in \mathbf{M} \otimes \mathbf{M}^* \mid \boldsymbol{L}^{\cdot}\boldsymbol{L} = \mathrm{id}_{\mathbf{M}}\} = \mathcal{O}(\mathbf{g})$$

is called the *Lorentz group*; its elements are the *Lorentz transformations.*

If $\boldsymbol{L}$ is a Lorentz transformation then

$$\mathrm{ar}\boldsymbol{L} := \begin{cases} +1 & \text{if } \boldsymbol{L} \text{ is arrow-preserving} \\ -1 & \text{if } \boldsymbol{L} \text{ is arrow-reversing} \end{cases}$$

is the *arrow* of $\boldsymbol{L}$ and

$$\operatorname{sign}\boldsymbol{L} := \begin{cases} +1 & \text{if } \boldsymbol{L}|_{\mathbf{E}_{\boldsymbol{u}}} \text{ is orientation-preserving} \\ -1 & \text{if } \boldsymbol{L}|_{\mathbf{E}_{\boldsymbol{u}}} \text{ is orientation-reversing} \end{cases}$$

is the *sign* of $\boldsymbol{L}$ where $\boldsymbol{u}$ is an arbitrary element of $\mathrm{V}(1)$.

Let us put

$$\begin{aligned} \mathcal{L}^{+\rightarrow} &:= \{\boldsymbol{L} \in \mathcal{L} \mid \operatorname{sign}\boldsymbol{L} = \operatorname{ar}\boldsymbol{L} = 1\}, \\ \mathcal{L}^{+\leftarrow} &:= \{\boldsymbol{L} \in \mathcal{L} \mid \operatorname{sign}\boldsymbol{L} = -\operatorname{ar}\boldsymbol{L} = 1\}, \\ \mathcal{L}^{-\rightarrow} &:= \{\boldsymbol{L} \in \mathcal{L} \mid \operatorname{sign}\boldsymbol{L} = -\operatorname{ar}\boldsymbol{L} = -1\}, \\ \mathcal{L}^{-\leftarrow} &:= \{\boldsymbol{L} \in \mathcal{L} \mid \operatorname{sign}\boldsymbol{L} = \operatorname{ar}\boldsymbol{L} = -1\}. \end{aligned}$$

$\mathcal{L}^{+\rightarrow}$ is called the *proper Lorentz group*.

10.1.2. (*i*) From VII.5 we infer that the Lorentz group is a six-dimensional Lie group having the Lie algebra

$$\mathbf{La}(\mathcal{L}) = \mathrm{A}(\mathbf{g}) = \{\boldsymbol{H} \in \mathbf{M} \otimes \mathbf{M}^* \mid \boldsymbol{H}^{=} - \boldsymbol{H}\}.$$

(*ii*) S, T and L, the set of spacelike vectors, the set of timelike vectors and the set of lightlike vectors are invariant under Lorentz transformations. The arrow of a Lorentz transformation $\boldsymbol{L}$ is $+1$ if and only if $\mathrm{T}^{\rightarrow}$, the set of future-directed timelike vectors, is invariant for $\boldsymbol{L}$.

(*iii*) The sign of Lorentz transformations is correctly defined. Indeed, if $\boldsymbol{u} \in \mathrm{V}(1)$ then $\boldsymbol{L}$ maps $\mathbf{E}_{\boldsymbol{u}}$ onto $\mathbf{E}_{(\operatorname{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u}}$; these two linear subspaces are oriented according to 1.3.4. It is not hard to see that if the restriction of $\boldsymbol{L}$ onto $\mathbf{E}_{\boldsymbol{u}}$ is orientation-preserving for some $\boldsymbol{u}$ then it is orientation-preserving for all $\boldsymbol{u}$.

(*iv*) The mappings $\mathcal{L} \to \{-1, 1\}$, $\boldsymbol{L} \mapsto \operatorname{ar}\boldsymbol{L}$ and $\mathcal{L} \to \{-1, 1\}$, $\boldsymbol{L} \mapsto \operatorname{sign}\boldsymbol{L}$ are continuous group homomorphisms. As a consequence, the Lorentz group is disconnected. We shall see in 10.2.4 that the proper Lorentz group $\mathcal{L}^{+\rightarrow}$ is connected. It is quite trivial that if $\boldsymbol{L} \in \mathcal{L}^{+\leftarrow}$ then $\boldsymbol{L} \cdot \mathcal{L}^{+\rightarrow} = \mathcal{L}^{+\leftarrow}$ and similar assertions hold for $\mathcal{L}^{-\rightarrow}$ and $\mathcal{L}^{-\leftarrow}$ as well. Consequently, the Lorentz group has four connected components, the four subsets given in Definition 10.1.1.

From these four components only $\mathcal{L}^{+\rightarrow}$ — the proper Lorentz group — is a subgroup; nevertheless, the union of an arbitrary component and of the proper Lorentz group is a subgroup as well.

$\mathcal{L}^{\rightarrow} := \mathcal{L}^{+\rightarrow} \cup \mathcal{L}^{-\rightarrow}$ is called the *orthochronous Lorentz group*.

(*v*) The arrow of $\boldsymbol{L}$ is $+1$ if and only if $\mathrm{T}^{\rightarrow}$, the set of future-directed timelike vectors is invariant for $\boldsymbol{L}$:

$$\begin{aligned} &\text{if } \operatorname{ar}\boldsymbol{L} = 1 && \text{then } \boldsymbol{L}[\mathrm{T}^{\rightarrow}] = \mathrm{T}^{\rightarrow}, \quad \boldsymbol{L}[\mathrm{T}^{\leftarrow}] = \mathrm{T}^{\leftarrow}, \\ &\text{if } \operatorname{ar}\boldsymbol{L} = -1 && \text{then } \boldsymbol{L}[\mathrm{T}^{\rightarrow}] = \mathrm{T}^{\leftarrow}, \quad \boldsymbol{L}[\mathrm{T}^{\leftarrow}] = \mathrm{T}^{\rightarrow}. \end{aligned}$$

Moreover, the elements of $\mathcal{L}^{+\rightarrow}$ and $\mathcal{L}^{-\leftarrow}$ preserve the orientation of $\mathbf{M}$, whereas the elements of $\mathcal{L}^{+\leftarrow}$ and $\mathcal{L}^{-\rightarrow}$ reverse the orientation.

10.1.3. $\mathbf{M}$ is of even dimensions, thus $-\mathrm{id}_{\mathbf{M}}$ is orientation-preserving. Evidently, $-\mathrm{id}_{\mathbf{M}}$ is in $\mathcal{L}^{-\leftarrow}$; it is called the *inversion of spacetime vectors.* We have that $\mathcal{L}^{-\leftarrow} = (-\mathrm{id}_{\mathbf{M}}) \cdot \mathcal{L}^{+\rightarrow}$.

We have seen previously that the elements of $\mathcal{L}^{+\leftarrow}$ invert in some sense the timelike vectors and do not invert the spacelike vectors; the elements of $\mathcal{L}^{-\rightarrow}$ invert in some sense the spacelike vectors and do not invert the timelike vectors. However, we cannot select an element of $\mathcal{L}^{+\leftarrow}$ and an element of $\mathcal{L}^{-\rightarrow}$ that we could consider to be the time inversion and the space inversion, respectively.

For each $\boldsymbol{u} \in \mathrm{V}(1)$ we can give a $\boldsymbol{u}$-timelike inversion and a $\boldsymbol{u}$-spacelike inversion as follows.

The *$\boldsymbol{u}$-timelike inversion* $\boldsymbol{T_u} \in \mathcal{L}^{+\leftarrow}$ inverts the vectors parallel to $\boldsymbol{u}$ and leaves invariant the spacelike vectors $\mathbf{g}$-orthogonal to $\boldsymbol{u}$:

$$\boldsymbol{T_u} \cdot \boldsymbol{u} := -\boldsymbol{u} \quad \text{and} \quad \boldsymbol{T_u} \cdot \boldsymbol{q} := \boldsymbol{q} \quad \text{for} \quad \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}}.$$

In general,

$$\boldsymbol{T_u} \cdot \boldsymbol{x} = \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x}) + \boldsymbol{\pi_u} \cdot \boldsymbol{x} = 2\boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x}) + \boldsymbol{x} \qquad (\boldsymbol{x} \in \mathbf{M}),$$

i.e.

$$\boldsymbol{T_u} = \mathbf{g} + 2\boldsymbol{u} \otimes \boldsymbol{u}$$

where, as usual, $\mathbf{g} := \mathrm{id}_{\mathbf{M}}$.

The *$\boldsymbol{u}$-spacelike inversion* $\boldsymbol{P_u} \in \mathcal{L}^{-\rightarrow}$ inverts the spacelike vectors $\mathbf{g}$-orthogonal to $\boldsymbol{u}$ and leaves invariant the vectors parallel to $\boldsymbol{u}$:

$$\boldsymbol{P_u} \cdot \boldsymbol{u} := \boldsymbol{u} \quad \text{and} \quad \boldsymbol{P_u} \cdot \boldsymbol{q} := -\boldsymbol{q} \quad \text{for} \quad \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}}.$$

In general,

$$\boldsymbol{P_u} \cdot \boldsymbol{x} = -\boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x}) - \boldsymbol{\pi_u} \cdot \boldsymbol{x} = -2\boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{x}) - \boldsymbol{x} \qquad (\boldsymbol{x} \in \mathbf{M}),$$

i.e.

$$\boldsymbol{P_u} = -\mathbf{g} - 2\boldsymbol{u} \otimes \boldsymbol{u}.$$

We easily deduce the following equalities:

$$\boldsymbol{T_u}^{-1} = \boldsymbol{T_u}, \qquad \boldsymbol{P_u}^{-1} = \boldsymbol{P_u},$$
$$-\boldsymbol{T_u} = \boldsymbol{P_u},$$
$$\boldsymbol{T_u} \cdot \boldsymbol{P_u} = \boldsymbol{P_u} \cdot \boldsymbol{T_u} = -\mathbf{g}.$$

10.1.4. For $\boldsymbol{u} \in \mathrm{V}(1)$ let us consider the Euclidean vector space $(\mathbf{E}_{\boldsymbol{u}}, \mathbf{I}, \mathbf{b}_{\boldsymbol{u}})$ where $\mathbf{b}_{\boldsymbol{u}}$ is the restriction of $\mathbf{g}$ onto $\mathbf{E}_{\boldsymbol{u}} \times \mathbf{E}_{\boldsymbol{u}}$. The $\mathbf{b}_{\boldsymbol{u}}$-orthogonal group, $\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})$, called also the group of $\boldsymbol{u}$*-spacelike orthogonal transformations,* can be identified with a subgroup of the Lorentz group:

$$\mathcal{O}(\mathbf{b}_{\boldsymbol{u}}) \equiv \{\boldsymbol{L} \in \mathcal{L}^{\rightarrow} \mid \boldsymbol{L}\cdot\boldsymbol{u} = \boldsymbol{u}\}\,.$$

The Lorentz group is an analogue of the Galilean group and we have already seen a number of their common properties. However, as concerns their relation to three-dimensional orthogonal groups, they differ significantly.

In the non-relativistic case there is a *single* three-dimensional orthogonal group in question, $\mathcal{O}(\mathbf{b})$, and it can be injected into the Galilean group in different ways according to different velocity values. Moreover, $\boldsymbol{L} \mapsto \boldsymbol{L}|_{\mathbf{E}}$ is a surjective group homomorphism from the Galilean group onto the three-dimensional orthogonal group.

In the relativistic case there are a lot of three-dimensional orthogonal groups, being subgroups of the Lorentz group; one corresponds to each velocity value. Note that, for all $\boldsymbol{u}$, $\boldsymbol{L} \mapsto \boldsymbol{L}|_{\mathbf{E}_{\boldsymbol{u}}}$ is not a surjective group homomorphism from $\mathcal{L}$ onto $\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})$; indeed, $\mathbf{E}_{\boldsymbol{u}}$ is invariant for $\boldsymbol{L}$ if and only if $\boldsymbol{L}\cdot\boldsymbol{u} = (\mathrm{ar}\boldsymbol{L})\boldsymbol{u}$.

As a consequence, there is not either a "special Lorentz group" or a "$\boldsymbol{u}$-special Lorentz group" which would be the kernel of the group homomorphism $\boldsymbol{L} \mapsto \boldsymbol{L}|_{\mathbf{E}_{\boldsymbol{u}}}$.

10.1.5. The problem is that, in general, $\mathbf{E}_{\boldsymbol{u}}$ is not invariant for a Lorentz transformation $\boldsymbol{L}$; more closely, $\boldsymbol{L}$ maps $\mathbf{E}_{\boldsymbol{u}}$ onto $\mathbf{E}_{(\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u}}$ for all $\boldsymbol{u} \in \mathrm{V}(1)$. Let us try to rule out this uneasiness with the aid of the corresponding Lorentz boost $\boldsymbol{L}(\boldsymbol{u}, (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u})$ which maps $\mathbf{E}_{(\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u}}$ onto $\mathbf{E}_{\boldsymbol{u}}$ in a "handsome" way. A simple calculation yields the following result.

Proposition. For all Lorentz transformations $\boldsymbol{L}$ and for all $\boldsymbol{u} \in \mathrm{V}(1)$,

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{L},\boldsymbol{u}) &:= (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}(\boldsymbol{u}, (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u})\cdot\boldsymbol{L} = \\ &= (\mathrm{ar}\boldsymbol{L})\boldsymbol{L} + \frac{\big(\boldsymbol{u} + (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u}\big)\otimes\big(((\mathrm{ar}\boldsymbol{L})\boldsymbol{L})^{-1}\cdot\boldsymbol{u} + \boldsymbol{u}\big)}{1 - \boldsymbol{u}\cdot(\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u}} - 2\boldsymbol{u}\otimes\boldsymbol{u}\end{aligned}$$

is an element of $\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})$. ■

This suggests the idea that an orthochronous Lorentz transformation $\boldsymbol{L}$ should be considered "special" if $\boldsymbol{R}(\boldsymbol{L},\boldsymbol{u})|_{\mathbf{E}_{\boldsymbol{u}}} = \mathrm{id}_{\mathbf{E}_{\boldsymbol{u}}}$; then $\boldsymbol{L}(\boldsymbol{u}, (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u})\cdot\boldsymbol{L} = \mathbf{g}$ and consequently $\boldsymbol{L}$ is a Lorentz boost.

Thus Lorentz boosts can be regarded as counterparts of special Galilean transformations. That is why we call them *special Lorentz transformations* as well. However, it is very important that the special Lorentz transformations (Lorentz boosts) *do not form a subgroup* (see 1.3.9).

Note that our result can be formulated as follows: given an arbitrary $\boldsymbol{u} \in \mathrm{V}(1)$, every Lorentz transformation $\boldsymbol{L}$ can be decomposed into the product of a special

Lorentz transformation and a $\boldsymbol{u}$-spacelike orthogonal transformation, multiplied by the arrow of $\boldsymbol{L}$:

$$\boldsymbol{L} = (\mathrm{ar}\boldsymbol{L})\boldsymbol{L}((\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u},\boldsymbol{u})\cdot\boldsymbol{R}(\boldsymbol{L},\boldsymbol{u}).$$

10.1.6. It is worth mentioning that the product of the $\boldsymbol{u}'$-timelike ($\boldsymbol{u}'$-spacelike) inversion and the $\boldsymbol{u}$-timelike ($\boldsymbol{u}$-spacelike) inversion is a special Lorentz transformation. Since

$$\boldsymbol{T}_{\boldsymbol{u}}^{-1} = \boldsymbol{T}_{\boldsymbol{u}} = -\boldsymbol{P}_{\boldsymbol{u}} = \mathbf{g} + 2\boldsymbol{u}\otimes\boldsymbol{u},$$

we find — because of $-\boldsymbol{u} - 2(\boldsymbol{u}\cdot\boldsymbol{u}')\boldsymbol{u}' = \boldsymbol{u} - \frac{2\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2}}$ — that $\boldsymbol{T}_{\boldsymbol{u}'}\cdot\boldsymbol{T}_{\boldsymbol{u}}^{-1} = \boldsymbol{P}_{\boldsymbol{u}'}\cdot\boldsymbol{P}_{\boldsymbol{u}}^{-1}$ is the Lorentz boost from $\boldsymbol{u}$ to $\boldsymbol{u} - \frac{2\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}}{\sqrt{1-|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}|^2}}$.

10.1.7. (*i*) Take an $\boldsymbol{u}\in \mathrm{V}(1)$ and a $\boldsymbol{0}\neq\boldsymbol{H}\in\mathrm{A}(\mathbf{g})$ for which $\boldsymbol{H}\cdot\boldsymbol{u}=\boldsymbol{0}$ holds. Then $\boldsymbol{H}^3 = -|\boldsymbol{H}|^2\boldsymbol{H}$ (V.4.18(*i*)) and we can repeat the proof of I.11.1.8 to have

$$\mathrm{e}^{\boldsymbol{H}} = \left(\mathbf{g} + \frac{\boldsymbol{H}^2}{|\boldsymbol{H}|^2}\right) + \frac{\boldsymbol{H}^2}{|\boldsymbol{H}|^2}\cos|\boldsymbol{H}| + \frac{\boldsymbol{H}}{|\boldsymbol{H}|}\sin|\boldsymbol{H}|$$

which is an element of $\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})$.

(*ii*) Take an $\boldsymbol{u}\in \mathrm{V}(1)$ and a $\boldsymbol{0}\neq\boldsymbol{H}\in\mathrm{A}(\mathbf{g})$ whose kernel lies in $\mathbf{E}_{\boldsymbol{u}}$. Then $\boldsymbol{H}^3 = |\boldsymbol{H}|^2\boldsymbol{H}$ (V.4.18(*ii*)) and we can prove as in I.11.1.8 that

$$\mathrm{e}^{\boldsymbol{H}} = \left(\mathbf{g} - \frac{\boldsymbol{H}^2}{|\boldsymbol{H}|^2}\right) + \frac{\boldsymbol{H}^2}{|\boldsymbol{H}|^2}\mathrm{ch}|\boldsymbol{H}| + \frac{\boldsymbol{H}}{|\boldsymbol{H}|}\mathrm{sh}|\boldsymbol{H}|.$$

We can demonstrate this is a Lorentz boost. Recall that there is an $\boldsymbol{n}\in\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $|\boldsymbol{n}|=1$ such that $\boldsymbol{H}=\alpha\boldsymbol{u}\wedge\boldsymbol{n}$, where $\alpha := |\boldsymbol{H}|$. Then $\boldsymbol{H}^2 = \alpha^2(\boldsymbol{n}\otimes\boldsymbol{n} - \boldsymbol{u}\otimes\boldsymbol{u})$ and executing some calculations we obtain:

Proposition. Let $\boldsymbol{u}\in\mathrm{V}(1)$, $\boldsymbol{n}\in\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $|\boldsymbol{n}|=1$, and $\alpha\in\mathbb{R}$. Then

$$\exp\big(\alpha(\boldsymbol{u}\wedge\boldsymbol{n})\big) = \boldsymbol{L}(\boldsymbol{u}\mathrm{ch}\alpha + \boldsymbol{n}\mathrm{sh}\alpha,\ \boldsymbol{u}).$$

10.1.8. Originally the Lorentz tranformations are defined to be linear maps from $\mathbf{M}$ into $\mathbf{M}$. In the usual way, we can consider them to be linear maps from $\frac{\mathbf{M}}{\mathbf{I}}$ into $\frac{\mathbf{M}}{\mathbf{I}}$ as we already did in the preceding paragraphs as well.

V(1) is invariant under orthochronous Lorentz transformations. However, contrary to the non-relativistic case, here V(1) is not an affine subspace, hence we cannot say anything similar to those in I.11.3.8.

This, too, indicates that the structure of the Lorentz group is more complicated than the structure of the Galilean group.

10.2. The $\boldsymbol{u}$-split Lorentz group

10.2.1. The Lorentz transformations, being elements of $\mathbf{M} \otimes \mathbf{M}^*$, are split by velocity values according to 8.1. These splittings are significantly more complicated than the splittings of Galilean transformations.

Let us start with the splittings of Lorentz boosts. The map $\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}$ defined in 7.1.4 is such a splitting:

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} = \boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{L}(\boldsymbol{u}, \boldsymbol{u}') \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1} \qquad (\boldsymbol{u}, \boldsymbol{u}' \in \mathrm{V}(1)).$$

For a $\boldsymbol{u} \in \mathrm{V}(1)$, it is convenient to introduce the notations

$$\mathrm{B}_{\boldsymbol{u}} := \left\{ \boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}} \;\middle|\; |\boldsymbol{v}| < 1 \right\}$$

and

$$\kappa(\boldsymbol{v}) := \frac{1}{\sqrt{1 - |\boldsymbol{v}|^2}},$$

$$\boldsymbol{D}(\boldsymbol{v}) := \frac{1}{\kappa(\boldsymbol{v})} \left(\mathrm{id}_{\mathbf{E}_{\boldsymbol{u}}} + \frac{\kappa(\boldsymbol{v})^2}{\kappa(\boldsymbol{v}) + 1} \boldsymbol{v} \otimes \boldsymbol{v} \right)$$

for $\boldsymbol{v} \in \mathrm{B}_{\boldsymbol{u}}$. It is worth mentioning the relation

$$\frac{\kappa(\boldsymbol{v}^2)}{\kappa(\boldsymbol{v}) + 1} = \frac{\kappa(\boldsymbol{v}) - 1}{|\boldsymbol{v}|^2}.$$

Applying the usual matrix forms we have

$$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}} = \kappa(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) \begin{pmatrix} 1 & -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \\ -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \boldsymbol{D}(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) \end{pmatrix}.$$

A simple calculation yields that

$$\boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u}) \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1} = \boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}^{-1} = \kappa(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) \begin{pmatrix} 1 & \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \\ \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} & \boldsymbol{D}(\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}) \end{pmatrix}.$$

10.2.2. Now taking an arbitrary Lorentz transformation $\boldsymbol{L}$ and a $\boldsymbol{u} \in \mathrm{V}(1)$, we make the following manipulation:

$$\boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{L} \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1} = \left(\boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{L}(\boldsymbol{u}, (\mathrm{ar}\boldsymbol{L})\boldsymbol{L} \cdot \boldsymbol{u})^{-1} \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1} \right) \cdot \left(\boldsymbol{h}_{\boldsymbol{u}} \cdot \boldsymbol{L}(\boldsymbol{u}, (\mathrm{ar}\boldsymbol{L})\boldsymbol{L} \cdot \boldsymbol{u}) \cdot \boldsymbol{L} \cdot \boldsymbol{h}_{\boldsymbol{u}}^{-1} \right).$$

The first factor on the right-hand side equals $\boldsymbol{H}^{-1}_{(\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u},\boldsymbol{u}}$. As concerns the second factor, we find that

$$\begin{aligned}\big(\boldsymbol{h}_{\boldsymbol{u}}\cdot\boldsymbol{L}(\boldsymbol{u},(\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u})\cdot\boldsymbol{L}\cdot\boldsymbol{h}_{\boldsymbol{u}}^{-1}\big)(\boldsymbol{t},\boldsymbol{q}) &= \big(\boldsymbol{h}_{\boldsymbol{u}}\cdot\boldsymbol{L}(\boldsymbol{u},(\mathrm{ar}\boldsymbol{L})\boldsymbol{L}\cdot\boldsymbol{u})\cdot\boldsymbol{L}\big)(\boldsymbol{u}\boldsymbol{t}+\boldsymbol{q}) = \\ = \boldsymbol{h}_{\boldsymbol{u}}\cdot\big((\mathrm{ar}\boldsymbol{L})\boldsymbol{u}\boldsymbol{t}+\boldsymbol{R}(\boldsymbol{L},\boldsymbol{u})\cdot\boldsymbol{q}\big) &= \big((\mathrm{ar}\boldsymbol{L})\boldsymbol{t},\boldsymbol{R}(\boldsymbol{L},\boldsymbol{u})\cdot\boldsymbol{q}\big)\end{aligned}$$

for all $(\boldsymbol{t},\boldsymbol{q})\in\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}}$, i.e. the second factor has the matrix form

$$\begin{pmatrix}\mathrm{ar}\boldsymbol{L} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{R}(\boldsymbol{L},\boldsymbol{u})\end{pmatrix}.$$

As a consequence, we see that the following definition describes the $\boldsymbol{u}$-split form of Lorentz transformations.

Definition. The *$\boldsymbol{u}$-split Lorentz group* is

$$\left\{\kappa(\boldsymbol{v})\begin{pmatrix}1 & \boldsymbol{v}\\ \boldsymbol{v} & \boldsymbol{D}(\boldsymbol{v})\end{pmatrix}\begin{pmatrix}\alpha & \mathbf{0}\\ \mathbf{0} & \boldsymbol{R}\end{pmatrix}\,\middle|\,\alpha\in\{-1,1\},\,\boldsymbol{v}\in\mathrm{B}_{\boldsymbol{u}},\,\boldsymbol{R}\in\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})\right\}.$$

Its elements are called *$\boldsymbol{u}$-split Lorentz transformations.* ■

The $\boldsymbol{u}$-split Lorentz transformations can be regarded as linear maps $\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}}\to\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}}$; the one in the definition makes the correspondence

$$(\boldsymbol{t},\boldsymbol{q})\mapsto\kappa(\boldsymbol{v})\big(\alpha\boldsymbol{t}+\boldsymbol{v}\cdot\boldsymbol{R}\cdot\boldsymbol{q},\alpha\boldsymbol{v}\boldsymbol{t}+\boldsymbol{D}(\boldsymbol{v})\cdot\boldsymbol{R}\cdot\boldsymbol{q}\big).$$

The $\boldsymbol{u}$-split Lorentz group is a six-dimensional Lie group having the Lie algebra

$$\left\{\begin{pmatrix}\mathbf{0} & \boldsymbol{v}\\ \boldsymbol{v} & \boldsymbol{A}\end{pmatrix}\,\middle|\,\boldsymbol{v}\in\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}},\,\boldsymbol{A}\in\mathrm{A}(\mathbf{b}_{\boldsymbol{u}})\right\}.$$

10.2.3. The splitting according to $\boldsymbol{u}$ establishes a Lie-group isomorphism between the Lorentz group and the $\boldsymbol{u}$-split Lorentz group. The isomorphisms corresponding to different $\boldsymbol{u}'$ and $\boldsymbol{u}$ are different.

The difference of splittings can be seen by the usual transformation rule which is rather complicated; since here we need not it, we do not give the details.

10.2.4. The $\boldsymbol{u}$-splitting sends the proper Lorentz group onto

$$\left\{\kappa(\boldsymbol{v})\begin{pmatrix}1 & \boldsymbol{v}\\ \boldsymbol{v} & \boldsymbol{D}(\boldsymbol{v})\end{pmatrix}\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{R}\end{pmatrix}\,\middle|\,\boldsymbol{v}\in\mathrm{B}_{\boldsymbol{u}},\,\boldsymbol{R}\in\mathcal{SO}(\mathbf{b}_{\boldsymbol{u}})\right\}$$

which is evidently a connected set. Since the $\boldsymbol{u}$-splitting is a Lie group isomorphisms, $\mathcal{L}^{+\to}$ is connected as well.

10.2.5. We easily verify that

$$\left\{ \kappa(\boldsymbol{v}) \begin{pmatrix} 1 & \boldsymbol{v} \\ \boldsymbol{v} & \boldsymbol{D}(\boldsymbol{v}) \end{pmatrix} \,\middle|\, \boldsymbol{v} \in \mathrm{B}_{\boldsymbol{u}} \right\}$$

is not a subgroup of the $\boldsymbol{u}$-split Lorentz group; this reflects the well-known fact that the Lorentz boosts do not form a subgroup.

10.2.6. The Lie algebra of the Lorentz group, too, consists of elements of $\mathbf{M} \otimes \mathbf{M}^*$, thus they are split by velocity values in the same way as the Lorentz transformations; evidently, their split form will be different.

If $\boldsymbol{H}$ is in the Lie algebra of the Lorentz group — i.e. $\boldsymbol{H}$ is a $\mathbf{g}$-antisymmetric tensor — and $\boldsymbol{u} \in \mathrm{V}(1)$, then

$$\boldsymbol{h_u} \cdot \boldsymbol{H} \cdot \boldsymbol{h_u}^{-1} = \begin{pmatrix} \mathbf{0} & \boldsymbol{H} \cdot \boldsymbol{u} \\ \boldsymbol{H} \cdot \boldsymbol{u} & \boldsymbol{H} - \boldsymbol{u} \wedge \boldsymbol{H} \cdot \boldsymbol{u} \end{pmatrix}.$$

The splitting according to $\boldsymbol{u}$ establishes a Lie algebra isomorphism between the Lie algebra of the Lorentz group and the Lie algebra of the $\boldsymbol{u}$-split Lorentz group. The isomorphisms corresponding to different $\boldsymbol{u}'$ and $\boldsymbol{u}$ are different.

10.3. Exercises

1. The Lorentz group is not transitive, i.e. for all $\boldsymbol{x} \in \mathbf{M}$, $\{\boldsymbol{L} \cdot \boldsymbol{x} \mid \boldsymbol{L} \in \mathcal{L}\} \neq \mathbf{M}$. What are the orbits of the Lorentz group?
2. The subgroup generated by the Lorentz boosts equals the proper Lorentz group.
3. Prove that the Lie algebra of $\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})$ equals $\{\boldsymbol{H} \in \mathrm{A}(\mathbf{g}) \mid \boldsymbol{H} \cdot \boldsymbol{u} = \mathbf{0}\}$ which can be identified with $\mathrm{A}(\mathbf{b}_{\boldsymbol{u}})$.
4. What is the subgroup generated by $\{\boldsymbol{T_u} \mid \boldsymbol{u} \in \mathrm{V}(1)\}$?
5. Prove that

$$\boldsymbol{h_u} \cdot \boldsymbol{T}_{\boldsymbol{u}'} \cdot \boldsymbol{h_u}^{-1} = \boldsymbol{H}_{\boldsymbol{u},\boldsymbol{u}''} \cdot \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \mathrm{id}_{\mathbf{E}_{\boldsymbol{u}}} \end{pmatrix},$$

$$\boldsymbol{h_u} \cdot \boldsymbol{P}_{\boldsymbol{u}'} \circ \boldsymbol{h_u}^{-1} = \boldsymbol{H}_{\boldsymbol{u},\boldsymbol{u}''} \cdot \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -\mathrm{id}_{\mathbf{E}_{\boldsymbol{u}}} \end{pmatrix}$$

where $\boldsymbol{u}'' := \boldsymbol{u} - 2\kappa(\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'})\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$.

10.4. The Poincaré group

10.4.1. Now we shall deal with affine maps $L : \mathrm{M} \to \mathrm{M}$; as usual, the linear map under L is denoted by $\boldsymbol{L}$.

Definition.

$$\mathcal{P} := \{L :\ \mathrm{M} \to \mathrm{M} |\ L \text{ is affine},\ \boldsymbol{L} \in \mathcal{L}\}$$

is called the Poincaré group; its elements are the *Poincaré transformations.*
If L is a Poincaré transformation then

$$\mathrm{ar} L := \mathrm{ar} \boldsymbol{L}, \qquad \mathrm{sign} L := \mathrm{sign} \boldsymbol{L}.$$

$\mathcal{P}^{+\to}$, $\mathcal{P}^{+\leftarrow}$, $\mathcal{P}^{-\to}$ and $\mathcal{P}^{-\leftarrow}$ are the subsets of $\mathcal{P}$ consisting of elements whose underlying linear maps belong to $\mathcal{L}^{+\to}$, $\mathcal{L}^{+\leftarrow}$, $\mathcal{L}^{-\to}$ and $\mathcal{L}^{-\leftarrow}$, respectively.
$\mathcal{P}^{+\to}$ is called the *proper Poincaré group.* ■

According to VII.3.2(ii) we can state the following.

Proposition. The Poincaré group is a ten-dimensional Lie group; its Lie algebra consists of the affine maps $H : \mathrm{M} \to \mathbf{M}$ whose underlying linear map is in the Lie algebra of the Lorentz group:

$$\mathbf{La}(\mathcal{P}) = \{H \in \mathrm{Aff}(\mathrm{M}, \mathbf{M}) |\ \boldsymbol{H} \in \mathrm{A}(\mathbf{g})\}\,. \qquad ■$$

The proper Poincaré group is a connected subgroup of the Poincaré group. As regards $\mathcal{P}^{+\leftarrow}$, etc. we can repeat what we said about the components of the Lorentz group.
$\mathcal{P}^{\to} := \mathcal{P}^{+\to} \cup \mathcal{P}^{-\to}$ is called the *orthochronous Poincaré group.*

10.4.2. We can say that the elements of $\mathcal{P}^{-\leftarrow}$ invert spacetime in some sense but there is no element that we could call the spacetime inversion.

For every $o \in \mathbf{M}$ we can give the *o-centered spacetime inversion* in the well-known way (cf. I.11.6.2):

$$I_o(x) := o - (x - o) \qquad (x \in \mathrm{M}).$$

Similarly, we can say that in some sense the elements of $\mathcal{P}^{-\to}$ contain spacelike inversion and do not contain timelike inversion; the elements of $\mathcal{P}^{+\leftarrow}$ contain timelike inversion and do not contain spacelike inversion. However, *the* space inversion and *the* time inversion do not exist.

For every $o \in \mathrm{M}$ and $\boldsymbol{u} \in \mathrm{V}(1)$ we can give the *o-centered* $\boldsymbol{u}$*-timelike inversion* and the *o-centered* $\boldsymbol{u}$*-spacelike inversion* as follows:

$$\begin{aligned} T_{\boldsymbol{u},o}(x) &:= o + \boldsymbol{T_u} \cdot (x - o), \\ P_{\boldsymbol{u},o}(x) &:= o + \boldsymbol{P_u} \cdot (x - o) \end{aligned} \qquad (x \in \mathrm{M}).$$

10.4.3. The Poincaré transformations are mappings of spacetime. They play a fundamental role because the proper Poincaré transformations can be considered to be the strict automorphisms of the spacetime model. The following statement is quite trivial.

Proposition. $(F, \mathrm{id}_{\mathbf{I}})$ is a strict automorphism of the special relativistic space time model $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ if and only if F is a proper Poincaré transformation.

10.4.4. The Lorentz group is not a subgroup of the Poincaré group. The mapping $\mathcal{P} \to \mathcal{L}$, $L \mapsto \boldsymbol{L}$ is a surjective Lie group homomorphism whose kernel is $\mathcal{T}n(\mathbf{M})$, the translation group of $\mathbf{M}$,

$$\mathcal{T}n(\mathbf{M}) = \{T_{\boldsymbol{x}} |\ \boldsymbol{x} \in \mathbf{M}\} = \{L \in \mathcal{P} |\ \boldsymbol{L} = \mathrm{id}_{\mathbf{M}}\}.$$

As we know, its Lie algebra is $\mathbf{M}$ regarded as the set of constant maps from M into $\mathbf{M}$ (VII.3.3).

For every $o \in \mathrm{M}$,

$$\mathcal{L}_o := \{L \in \mathcal{P} |\ L(o) = o\},$$

called the group of *o-centered Lorentz transformations,* is a subgroup of the Poincaré group; the restriction of the homomorphism $L \mapsto \boldsymbol{L}$ onto $\mathcal{L}_o$ is a bijection between $\mathcal{L}_o$ and $\mathcal{L}$.

In other words, given $o \in \mathrm{M}$, we can assign to every Lorentz transformation $\boldsymbol{L}$ the Poincaré transformation

$$x \mapsto o + \boldsymbol{L} \cdot (x - o),$$

called the *o-centered Lorentz transformation by* $\boldsymbol{L}$.

10.4.5. For every $\boldsymbol{u} \in \mathrm{V}(1)$ we can define the subgroup of $\boldsymbol{u}$-timelike translations

$$\mathcal{T}n(\mathbf{I})_{\boldsymbol{u}} := \{T_{\boldsymbol{u}\boldsymbol{t}} |\ \boldsymbol{t} \in \mathbf{I}\} \subset \mathcal{T}n(\mathbf{M})$$

and the subgroup of $\boldsymbol{u}$-spacelike translations

$$\mathcal{T}n(\mathbf{E}_{\boldsymbol{u}}) := \{T_{\boldsymbol{q}} |\ \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}}\} \subset \mathcal{T}n(\mathbf{M})$$

10.4.6. For every $\boldsymbol{u} \in \mathrm{V}(1)$ and $o \in \mathrm{M}$,

$$\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})_o := \{L \in \mathcal{P}^{\to} |\ L(o) = o,\ \boldsymbol{L} \cdot \boldsymbol{u} = \boldsymbol{u}\},$$

called the group of *o-centered* $\boldsymbol{u}$*-spacelike orthogonal transformations,* is a subgroup of $\mathcal{P}^{\to}$.

In other words, given $(\boldsymbol{u}, o) \in \mathrm{V}(1) \times \mathrm{M}$, we can assign to every $\boldsymbol{R} \in \mathcal{O}(\mathbf{b}_{\boldsymbol{u}})$ the Poincaré transformation

$$x \mapsto o - \boldsymbol{u}\big(\boldsymbol{u} \cdot (x - o)\big) + \boldsymbol{R} \cdot \boldsymbol{\pi}_{\boldsymbol{u}} \cdot (x - o),$$

called the *o-centered $\boldsymbol{u}$-spacelike orthogonal transformation by $\boldsymbol{R}$*.

10.5. The vectorial Poincaré group

10.5.1. Recall that for an arbitrary world point o, the vectorization of M with origin o, $O_o : \mathrm{M} \to \mathbf{M}$, $x \mapsto x - o$ is an affine bijection.

With the aid of such a vectorization we can "vectorize" the Poincaré group as well: if L is a Poincaré transformation then $O_o \circ L \circ O_o^{-1}$ is an affine transformation of $\mathbf{M}$, represented by the matrix (see VI.2.4(*ii*) and Exercise V I.2.5.2)

$$\begin{pmatrix} 1 & \mathbf{0} \\ L(o) - o & \boldsymbol{L} \end{pmatrix}.$$

The Lie algebra of the Poincaré group consists of affine maps $H : \mathrm{M} \to \mathbf{M}$ where $\mathbf{M}$ is considered to be a *vector space* (the *sum* of such maps is a part of the Lie algebra structure). Thus the vectorization $H \circ O_o^{-1}$ is an affine map $\mathbf{M} \to \mathbf{M}$ where the range is regarded as a vector space. Then it is represented by the matrix (see VI.2.4(*iii*))

$$\begin{pmatrix} 0 & \mathbf{0} \\ H(o) & \boldsymbol{H} \end{pmatrix}.$$

10.5.2. Definition. The *vectorial Poincaré group* is

$$\left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{L} \end{pmatrix} \middle| \ \boldsymbol{a} \in \mathbf{M},\ \boldsymbol{L} \in \mathcal{L} \right\}. \qquad \blacksquare$$

The vectorial Poincaré group is a ten-dimensional Lie group, its Lie algebra is the vectorization of the Lie algebra of the Poincaré group:

$$\left\{ \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{H} \end{pmatrix} \middle| \ \boldsymbol{a} \in \mathbf{M},\ \boldsymbol{H} \in \mathbf{La}(\mathcal{L}) \right\}.$$

An advantage of this block matrix representation is that the commutator of two Lie algebra elements can be computed as the difference of their two products.

10.5.3. A vectorization of the Poincaré group is a Lie group isomorphism between the Poincaré group and the vectorial Poincaré group. The following transformation rule shows how the vectorizations depend on the world points serving as origins of the vectorization. Let o and o' be two world points; then

$$O_{o'} \circ O_o^{-1} = T_{o-o'} = \begin{pmatrix} 1 & 0 \\ (o - o') & \mathrm{id}_{\mathbf{M}} \end{pmatrix}$$

and

$$T_{o-o'} \cdot \begin{pmatrix} 1 & 0 \\ \boldsymbol{a} & \boldsymbol{L} \end{pmatrix} \cdot T_{o-o'}^{-1} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{a} + (\boldsymbol{L} - \mathrm{id}_{\mathbf{M}}) \cdot (o' - o) & \boldsymbol{L} \end{pmatrix} \qquad (\boldsymbol{a} \in \mathbf{M},\, \boldsymbol{L} \in \mathcal{L}).$$

As concerns the corresponding Lie algebra isomorphisms, we have

$$\begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{H} \end{pmatrix} \cdot T_{o-o'}^{-1} = \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{a} + \boldsymbol{H} \cdot (o' - o) & \boldsymbol{H} \end{pmatrix} \qquad (\boldsymbol{a} \in \mathbf{M},\, \boldsymbol{H} \in \mathbf{La}(\mathcal{N})).$$

10.6. The $\boldsymbol{u}$-split Poincaré group

10.6.1. With the aid of the splitting corresponding to $\boldsymbol{u} \in \mathrm{V}(1)$, we send the transformations of $\mathbf{M}$ into the transformations of $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$. Composing a vectorization and a splitting, we convert Poincaré transformations into affine transformations of $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$.

Embedding the affine transformations of $\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}$ into the linear transformations of $\mathbb{R} \times (\mathbf{I} \times \mathbf{E}_{\boldsymbol{u}})$ (see VI.2.4(*ii*)) and using the customary matrix representation of such linear maps, we introduce the following notion.

Definition. The *$\boldsymbol{u}$-split Poincaré group* is

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t} & \kappa(\boldsymbol{v}) & \kappa(\boldsymbol{v})\boldsymbol{v} \\ \boldsymbol{q} & \kappa(\boldsymbol{v})\boldsymbol{v} & \kappa(\boldsymbol{v})\boldsymbol{D}(\boldsymbol{v}) \end{pmatrix} \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{0} & \alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{R} \end{pmatrix} \;\middle|\; \begin{matrix} \alpha \in \{-1, 1\},\, \boldsymbol{t} \in \mathbf{I},\, \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \\ \boldsymbol{v} \in \mathrm{B}_{\boldsymbol{u}},\, \boldsymbol{R} \in \mathcal{O}(\mathbf{b}_{\boldsymbol{u}}) \end{matrix} \right\}. \quad \blacksquare$$

The $\boldsymbol{u}$-split Poincaré group is a ten-dimensional Lie group having the Lie algebra

$$\left\{ \begin{pmatrix} 0 & 0 & \mathbf{0} \\ \boldsymbol{t} & 0 & \boldsymbol{v} \\ \boldsymbol{q} & \boldsymbol{v} & \boldsymbol{A} \end{pmatrix} \;\middle|\; \boldsymbol{t} \in \mathbf{I},\, \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}},\, \boldsymbol{v} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}},\, \boldsymbol{A} \in \mathrm{A}(\mathbf{b}_{\boldsymbol{u}}) \right\}.$$

Keep in mind that the group multiplication of $\boldsymbol{u}$-split Poincaré transformations coincides with the usual matrix multiplication and the commutator of Lie algebra elements is the difference of their two products.

For $\boldsymbol{u} \in \mathrm{V}(1)$ and $o \in \mathrm{M}$ put

$$h_{\boldsymbol{u},o} := \boldsymbol{h}_{\boldsymbol{u}} \circ \mathrm{O}_o : \; \mathrm{M} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}.$$

Then $L \mapsto h_{\boldsymbol{u},o} \circ L \circ h_{\boldsymbol{u},o}^{-1}$ is a Lie group isomorphism between the Poincaré group and the $\boldsymbol{u}$-split Poincaré group. Evidently, for different elements of $\mathrm{V}(1) \times$

M, the isomorphisms are different. The transformation rule that shows how the isomorphism depends on $(\boldsymbol{u}, o)$ is rather complicated.

Though the Poincaré group and the $\boldsymbol{u}$-split Poincaré group are isomorphic (they have the same Lie group structure), they are not "identical" : there is no "canonical" isomorphism between them that we could use to identify them.

The $\boldsymbol{u}$-split Poincaré group is the Poincaré group of the $\boldsymbol{u}$-split special relativistic spacetime model $(\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}}, \mathbf{I}, \mathbf{g}_{\boldsymbol{u}})$ (see 1.7). The spacetime model $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ and the corresponding $\boldsymbol{u}$-split spacetime model are isomorphic, but they cannot be identified as we pointed out in 1.6.3. Due to the additional structures of the $\boldsymbol{u}$-split spacetime model, the $\boldsymbol{u}$-split Poincaré group has a number of additional structures that the Poincaré group has not.

10.6.2. The $\boldsymbol{u}$-split Poincaré group has the following subgroups:

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \boldsymbol{t} & 1 & \mathbf{0} \\ 0 & \mathbf{0} & \mathrm{id}_{\mathbf{E}_{\boldsymbol{u}}} \end{pmatrix} \,\middle|\, \boldsymbol{t} \in \mathbf{I} \right\}, \qquad \left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \boldsymbol{q} & \mathbf{0} & \mathrm{id}_{\mathbf{E}_{\boldsymbol{u}}} \end{pmatrix} \,\middle|\, \boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}} \right\},$$

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & \boldsymbol{R} \end{pmatrix} \,\middle|\, \boldsymbol{R} \in \mathcal{O}(\mathbf{b}_{\boldsymbol{u}}) \right\}.$$

In contradistinction to the non-relativistic case,

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{0} & \kappa(\boldsymbol{v}) & \kappa(\boldsymbol{v})\boldsymbol{v} \\ \mathbf{0} & \kappa(\boldsymbol{v})\boldsymbol{v} & \kappa(\boldsymbol{v})\boldsymbol{D}(\boldsymbol{v}) \end{pmatrix} \,\middle|\, \boldsymbol{v} \in \mathrm{B}_{\boldsymbol{u}} \right\}$$

is not a subgroup of the $\boldsymbol{u}$-split Poincaré group.

The listed $\boldsymbol{u}$-split Poincaré transformations correspond (by the isomorphism established by $(\boldsymbol{u}, o) \in \mathrm{V}(1)\times\mathrm{M}$) to the following Poincaré transformations:

$$\begin{aligned}
&x \mapsto x + \boldsymbol{u}\boldsymbol{t} && (\boldsymbol{t} \in \mathbf{I}),\\
&x \mapsto x + \boldsymbol{q} && (\boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}}),\\
&x \mapsto o - \boldsymbol{u}\big(\boldsymbol{u}\cdot(x-o)\big) + \boldsymbol{R}\cdot\boldsymbol{\pi}_{\boldsymbol{u}}\cdot(x-o) && (\boldsymbol{R} \in \mathcal{O}(\mathbf{b}_{\boldsymbol{u}})),\\
&x \mapsto o + \boldsymbol{L}\big(\kappa(\boldsymbol{v})(\boldsymbol{u}+\boldsymbol{v}), \boldsymbol{u}\big)\cdot\boldsymbol{\pi}_{\boldsymbol{u}}\cdot(x-o) && (\boldsymbol{v} \in \mathrm{B}_{\boldsymbol{u}}).
\end{aligned}$$

10.6.3. Taking a linear bijection $\mathbf{I} \to \mathbb{R}$ and an orthogonal linear bijection $\mathbf{E}_{\boldsymbol{u}} \to \mathbb{R}^3$, we can transfer the $\boldsymbol{u}$-split Poincaré group into an affine transformation group of $\mathbb{R}\times\mathbb{R}^3$, called the *arithmetic Poincaré group* which is the Poincaré group of the arithmetic spacetime model.

The arithmetic Poincaré transformations can be given in the same form as the $\boldsymbol{u}$-split Poincaré transformations: $\mathbb{R}$, $\mathbb{R}^3$, $\mathcal{O}(3)$ and the open unit ball in $\mathbb{R}^3$ are to be substituted for $\mathbf{I}$, $\mathbf{E}_{\boldsymbol{u}}$, $\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})$ and $\mathrm{B}_{\boldsymbol{u}}$, respectively.

In conventional treatments one always considers the arithmetic Poincaré group and one speaks about *the* time translation subgroup, *the* space translation subgroup, *the* space rotation subgroup, *the* time inversion etc. which in applications can result in misunderstandings.

Since time and space do not exist and only observer time and observer space make sense, the Poincaré group has no such subgroups; it contains $\boldsymbol{u}$-timelike translations, $\boldsymbol{u}$-spacelike translations, o-centered $\boldsymbol{u}$-spacelike rotations etc.

10.7. Exercises

1. Let L be a Poincaré transformation for which $\boldsymbol{L} = -\mathrm{id}_{\mathbf{M}}$. Then there is a unique $o \in \mathrm{M}$ such that L is the o-centered spacetime inversion.

2. Prove that for all $o \in \mathrm{M}$,

$$O_o \circ \mathcal{L}_o \circ O_o^{-1} = \left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{L} \end{pmatrix} \middle| \ \boldsymbol{L} \in \mathcal{L} \right\}.$$

3. Find $h_{\boldsymbol{u},o} \cdot T_{\boldsymbol{u},o} \cdot h_{\boldsymbol{u},o}^{-1}$ and $h_{\boldsymbol{u},o} \cdot P_{\boldsymbol{u},o} \cdot h_{\boldsymbol{u},o}^{-1}$.

4. Prove that the subgroup generated by $\{T_{\boldsymbol{u},o} | \, \boldsymbol{u} \in \mathrm{V}(1),\, o \in \mathrm{M}\}$ equals $\mathcal{P}^{+\to} \cup \mathcal{P}^{+\leftarrow}$.

5. Prove that the derived Lie algebra of the Poincaré group equals the Lie algebra of the Poincaré group, i.e. $[\mathbf{La}(\mathcal{P}), \mathbf{La}(\mathcal{P})] = \mathbf{La}(\mathcal{P})$.

6. Let L be a Poincaré transformation. Consider the real number $\mathrm{ar}L$ to be a linear map $\mathbf{I} \to \mathbf{I}$, $\boldsymbol{t} \mapsto (\mathrm{ar}L)\boldsymbol{t}$.

If r is a world line function then $L \circ r \circ (\mathrm{ar}L)^{-1}$ is a world line function, too.

If C is a world line then $L[\mathrm{C}]$ is a world line, too; moreover, if $\mathrm{C} = \mathrm{Ran}\ r$ then $L[\mathrm{C}] = \mathrm{Ran}\ \left(L \circ r \circ (\mathrm{ar}L)^{-1}\right)$.

11. Relation between the special relativistic spacetime model and the non-relativistic spacetime model

11.1. One often asserts that non-relativistic physics is the limit of special relativistic physics as the light speed tends to infinity.

Can we give such an exact statement concerning our spacetime models? The answer is no.

We have two different mathematical structures. There is no natural way of introducing a convergence notion even on a class of mathematical structures of the same kind (e.g. on the class of groups and to say that a sequence of groups converges to a given one) and it is quite impossible to introduce a convergence notion on a class consisting of structures of different kinds (e.g. to say that a sequence of groups converges to an algebra).

There is no reasonable limit procedure in which a sequence of special relativistic spacetime models converges to a non-relativistic spacetime model.

11.2. The following considerations show the real meaning of the usual statements.

Let us fix a special relativistic global inertial observer with the velocity value $\boldsymbol{u}$.

Let us rename $\mathbf{I}$ to $\mathbf{D}$, calling it the measure line of distances. Let us introduce for time periods a new measure line, denoted by $\mathbf{I}$. Let us choose a positive element c of $\frac{\mathbf{D}}{\mathbf{I}}$; it makes the correspondence $\mathbf{I} \to \mathbf{D}$, $\boldsymbol{t} \mapsto c\boldsymbol{t}$.

If $\boldsymbol{u}' \in \mathrm{V}(1)$ then $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{D}}$ and $\boldsymbol{v} := c\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ will be considered the relative velocity with respect to the observer. Evidently, $|\boldsymbol{v}| < c$, thus c is the light speed in the new system of measure lines.

Substituting $\frac{\boldsymbol{v}}{c}$ for $\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$ and $c\boldsymbol{t}$ for $\boldsymbol{t}$ in the formula 7.1.4 and letting c tend to infinity — which has an exact meaning because elements of finite dimensional vector spaces are involved in that formula — we get the corresponding non-relativistic transformation rule in I.8.2.4.

Similar statements hold for other formulae that concern relative velocities; e.g. for the addition formula of relative velocities, for the formula of light aberration etc.

However, such a statement, in general, will not be valid for formulae that do not concern relative velocities: e.g. the uniformly accelerated observer treated in 6.6 has no limit as c tends to infinity.

III. FUNDAMENTAL NOTIONS OF GENERAL RELATIVISTIC SPACETIME MODELS

1. As we have already mentioned, the non-relativistic spacetime model is suitable for describing "sluggish" mechanical phenomena. To describe "brisk" mechanical phenomena and electromagnetic phenomena we have to use the special relativistic spacetime model. Of course, the special relativistic spacetime model is good for "sluggish" mechanical phenomena, too, but their relativistic description is much more complicated than the non-relativistic one and gives practically the same results.

To avoid misunderstandings, we emphasize that the mechanical effects of electromagnetic phenomena (e.g. the history of charged masspoints in a given electromagnetic field) can be well described non-relativistically as well, provided that the mechanical phenomena remain "sluggish" (the relative velocities of masspoints remain much smaller than the light speed). The non-relativistic spacetime model is not suitable for the description of the electromagnetic phenomena in vacuum: how do charges produce electromagnetic field, how does an electromagnetic radiation propagate etc.

Gravitational actions are well described in the non-relativistic spacetime model by absolute scalar potentials. Such potentials do not exist in the special relativistic spacetime model. Other potentials or force fields do not give convenient (experimentally verified) models of gravitational actions.

The problem that faces us is that gravitational actions in "brisk" mechanical phenomena and electromagnetic phenomena cannot be described and, of course, gravitational phenomena (how do masses produce gravitational fields) cannot be treated in the framework of the special relativistic spacetime model.

There is only one way out: if we want to describe gravitational phenomena as well, then we have to construct a new spacetime model. However, it is not straightforward at all, how we shall do this.

2. Recall what we said about our experience regarding the structure of our space and time: in our space we find straight lines represented by light signals or stretched threads. We know, however, that a thread stretched in the gravitational field of the earth is not straight, it bends; if the thread is short enough and the stretching is strong enough then the curvature of the thread is negligible. However, for longer threads — imagine a thread (wire) across a river — the curvature can be significant.

It seems, that a light signal is better for realizing a straight line. Indeed, in terrestrial distances we do not experience that a light signal is not straight. However, the distances on the earth are small for a light signal. It may happen that light signals turn out to be curved in cosmic distances. Of course, to prove or disprove this possibility we meet great difficulties. A minor problem is that cosmic distances are hardly manageable.

To state that a line is straight or not we have to know what the straight lines are. Straight lines in terrestrial distances are defined in the most convenient way by the trajectories of light signals. Have we a better way to define straight lines in cosmic size? Can we define straight lines in this way? Can we define straight lines at all?

We have to recognize that it makes no sense that a single line in itself is straight or is not straight. We have to relate the trajectories of more light signals and to test whether they satisfy the conditions we expect the set of straight lines have. For instance, if two different light signals meet in more than one point, the trajectories of the signals cannot be straight lines. Unfortunately, it is rather difficult to execute such examinations in cosmic size.

Nevertheless, we have experimental evidence that shows that gravitation influences the propagation of light. The angle between two light beams arriving from two stars have been measured in different circumstances: first the light beams travel “freely”, far from gravitational action; second, they travel near the Sun i.e. under a strong gravitational action. The angles are significantly different.

Light travels along different trajectories in two circumstances. Evidently, the trajectories cannot be straight lines in both cases.

The affine structure of spacetime in the special relativistic model has been based on the straight propagation of light. Thus if we want to construct a spacetime model suitable for the treatment of gravitational phenomena, we have to reject the affine structure.

We have to get accustomed to the strange fact: in general, the notion of a straight line makes no sense. It is worth repeating why. Every notion in our mathematical model must have a physical background. A straight line would be realized by a light beam: we have no better possibility. However, in strong gravitational fields (in cosmic size) the set of light beams maybe does not satisfy the usual conditions imposed on the set of straight lines. One usually says that gravitation "curves" spacetime. The properties of a curved spacetime can be illustrated as follows: it may happen that two light beams starting simultaneously from the same source in different directions meet again somewhere (this is a "spacelike curvature") or that two light beams starting from the same source in the same direction in different instants meet again somewhere (this is a "timelike curvature").

3. According to the idea of Einstein, spacetime models must reflect gravitational actions, a *spacetime model is to be a model of a gravitational action*; the absence of gravitation is modelled by the special relativistic spacetime model.

The theory of gravitation, a deep and large theory, lies out of the scope of this book. That is why only the framework of general relativistic spacetime models will be outlined.

In constructing a general relativistic spacetime model, we do not adhere to the affine structure and we require only that spacetime is a four-dimensional smooth manifold M.

A four-dimensional smooth manifold M is an abstract mathematical structure similar to a four-dimensional smooth submanifold in an affine space; it has the following fundamental properties: every $x \in \mathrm{M}$ has a neighbourhood which can be parametrized by $p : \mathbb{R}^4 \rightarrowtail \mathrm{M}$; if p and q are parametrizations then $q^{-1} \circ p$ is smooth. Then to each point x of M a four-dimensional vector space $\mathrm{T}_x(\mathrm{M})$, the tangent space at x, is assigned; every differentiable curve passing through x has its tangent vector in $\mathrm{T}_x(\mathrm{M})$. A neighbourhood of zero of $\mathrm{T}_x(\mathrm{M})$ approximates a neighbourhood of x in M. Smooth submanifolds of an affine space (thus affine spaces themselves) are smooth manifolds.

Our experience that gravitational action in small size does not contradict the notion of a straight line suggests that a general relativistic spacetime model in small size can be "similar" to a special relativistic spacetime model. That is why we accept that there is a measure line $\mathbf{I}$, and a Lorentz form $\mathbf{g}_x : \mathrm{T}_x(\mathrm{M}) \times \mathrm{T}_x(\mathrm{M}) \to \mathbf{I} \otimes \mathbf{I}$ is given for all $x \in \mathrm{M}$ in such a way that $x \mapsto \mathbf{g}_x$ is smooth in a conveniently defined sense. The assignment $x \mapsto \mathbf{g}_x$ is called a *Lorentz field* and is denoted by $\mathbf{g}$. Moreover, we assume that every $\mathbf{g}_x$ is endowed with an arrow orientation which, too, depends on x in a conveniently defined smooth way.

Definition. *A general relativistic spacetime model* is a triplet $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ where

— M is a four-dimensional smooth manifold (called *spacetime* or *world*),

— $\mathbf{I}$ is a one-dimensional oriented vector space (the measure line of spacetime lengths),

— $\mathbf{g}$ is an arrow-oriented Lorentz field on M.

Evidently, a special relativistic spacetime model is a general relativistic spacetime model: M is an affine space (then every tangent space equals $\mathbf{M}$) and $\mathbf{g}_x$ is the same for all $x \in \mathrm{M}$.

4. Take a general relativistic spacetime model $(\mathrm{M}, \mathbf{I}, \mathbf{g})$. Then S_x, T_x and L_x, the set of spacelike tangent vectors etc. in $\mathrm{T}_x(\mathrm{M})$ are defined by $\mathbf{g}_x$ for all world points x and they have the following meaning:

— a *world line* (the history of a masspoint) is a curve in M whose tangent vectors are timelike (i.e. the tangent vector of a world line C at x is in T_x);

— a *light signal* is a curve in M whose tangent vectors are lightlike.

Let us give an illustration of a general relativistic spacetime model. Let the plane of the page represent the spacetime M, and at the same time, every tangent space is represented by the plane of the page as well. Then we draw the future light cone to every world point.

Illustrating the non-relativistic and the special relativistic spacetime models we have got accustomed to the fact that the Euclidean structure of the plane has to be neglected: the angles and distances in the plane of the page do not reflect, in general, objects of the spacetime model. Now we have to neglect the affine structure of the plane as well: the straight lines of the plane, in general, do not correspond to objects of the spacetime.

We call attention to the fact that in our illustration the spacetime manifold and its tangent spaces which are different sets, are represented by the same plane. The straight lines representing light cones in the previous figure are lines in tangent spaces, they do not lie in the spacetime manifold.

The following figures show a world line and a light signal in the general relativistic spacetime model.

5. As we have said, a general relativistic spacetime model is to be a model of a gravitational action. The theory of gravitation has the task to expound how a gravitational action is modelled by a spacetime model. We know that a special relativistic spacetime model corresponds to the lack of gravitation.

There are different special relativistic spacetime models; however, all of them correspond to the same physical situation: the lack of gravitation. This is reflected in the fact that all special relativistic spacetime models are isomorphic.

It may happen that two general relativistic spacetime models correspond to the same gravitational action; we expect that they must be isomorphic. Now we give the notion of isomorphism.

Definition. The general relativistic spacetime model $(\mathrm{M}, \mathbf{I}, \mathbf{g})$ is *isomorphic to* $(\mathrm{M}', \mathbf{I}', \mathbf{g}')$ if there are

— a diffeomorphism $F : \mathrm{M} \to \mathrm{M}'$,

— an orientation preserving linear bijection $\boldsymbol{Z} : \mathbf{I} \to \mathbf{I}'$

such that

$$\mathbf{g}'_{F(x)} \circ \big(\mathrm{D}F(x) \times \mathrm{D}F(x)\big) = (\boldsymbol{Z} \otimes \boldsymbol{Z}) \circ \mathbf{g}_x \qquad (x \in \mathrm{M}). \quad \blacksquare$$

The phrase F is diffeomorphism means that F is a bijection and both F and F^{-1} are smooth. The derivative of F at x, $\mathrm{D}F(x)$, is a linear map from $\mathrm{T}_x(\mathrm{M})$ into $\mathrm{T}_{F(x)}(\mathrm{M}')$.

6. As examples we give a certain kind of general relativistic spacetime models where the spacetime manifold is a submanifold of an affine space, hence we can use the well-known mathematical tools treated in this book.

Take a special relativistic spacetime model $(\mathrm{M}, \mathbf{I}, \mathbf{g})$, select an open subset $\mathrm{M}^{\boldsymbol{A}}$ of M; $\mathrm{M}^{\boldsymbol{A}}$ is an open submanifold of M and $\mathrm{T}_x(\mathrm{M}^{\boldsymbol{A}}) = \mathbf{M}$ for all $x \in \mathrm{M}$. Give a smooth map $\boldsymbol{A} : \mathrm{M} \rightarrowtail \mathcal{G}\ell(\mathbf{M})$ (i.e. $\boldsymbol{A}(x)$ is a linear bijection $\mathbf{M} \to \mathbf{M}$ for all $x \in \mathrm{M}$). For all $x \in \mathbf{M}^{\boldsymbol{A}}$ we define the Lorentz form $\mathbf{g}_x^{\boldsymbol{A}}$ by

$$\mathbf{g}_x^{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) := \mathbf{g}\big(\boldsymbol{A}(x) \cdot \boldsymbol{x},\ \boldsymbol{A}(x) \cdot \boldsymbol{y}\big) \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathbf{M}).$$

The Lorentz form $\mathbf{g}^{\boldsymbol{A}}$ is endowed with an arrow orientation as follows: let $\mathrm{T}^{\rightarrow}$ be the future-directed timelike cone of $\mathbf{g}$; then the future-directed timelike cone of $\mathbf{g}_x^{\boldsymbol{A}}$ is defined to be $\boldsymbol{A}(x)^{-1}[\mathrm{T}^{\rightarrow}]$.

Then $(\mathrm{M}^{\boldsymbol{A}}, \mathbf{I}, \mathbf{g}^{\boldsymbol{A}})$ is a general relativistic spacetime model.

PART TWO

MATHEMATICAL TOOLS

IV. TENSORIAL OPERATIONS

In this section $\mathbb{K}$ denotes the field of complex numbers or the field of real numbers, and all vector spaces are given over $\mathbb{K}$.

Tensors and operations with tensors are essential mathematical tools in physics; the simplest physical notions — e.g. meter/secundum — require tensorial operations. Those being familiar with tensors will find no difficulty in reading this book.

0. Identifications

Identifications make easy to handle tensors.

Let $\mathbf{X}$ and $\mathbf{Y}$ be vector spaces over the same field. If there is a linear injection $\mathbf{i}: \mathbf{X} \to \mathbf{Y}$ which we find natural ("canonical") from some point of view, we *identify* $\boldsymbol{x}$ and $\mathbf{i}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbf{X}$, i.e. we omit $\mathbf{i}$ from the notations considering $\mathbf{X}$ to be a linear subspace of $\mathbf{Y}$. Then we write

$$\mathbf{X} \subset\!\!\to \mathbf{Y}, \qquad \boldsymbol{x} \equiv \mathbf{i}(\boldsymbol{x}),$$

and if $\mathbf{i}$ is a bijection,

$$\mathbf{X} \equiv \mathbf{Y}, \qquad \boldsymbol{x} \equiv \mathbf{i}(\boldsymbol{x}).$$

In practice, instead of $\boldsymbol{x} \equiv \mathbf{i}(\boldsymbol{x})$ an appropriate formula appears that allows us to consider $\mathbf{i}$ to be natural.

Of course, "natural" and "canonical" are not mathematical notions and it depends on us whether we accept or reject an identification. There are commonly accepted identifications and there are some cases in which some people find a given identification convenient and others do not.

Later, using a lot of identifications, the reader will have the opportunity to see their importance.

1. Duality

1.1. Let $\mathbf{V}$ and $\mathbf{U}$ be vector spaces. Then $\mathrm{Lin}(\mathbf{V},\mathbf{U})$ denotes the vector space of linear maps $\mathbf{V}\to\mathbf{U}$; $\mathrm{Lin}(\mathbf{V}) := \mathrm{Lin}(\mathbf{V},\mathbf{V})$.

The value of $\boldsymbol{L}\in\mathrm{Lin}(\mathbf{V},\mathbf{U})$ at $\boldsymbol{v}\in\mathbf{V}$ is denoted by $\boldsymbol{L}\cdot\boldsymbol{v}$.

The composition of linear maps is denoted by a dot as well: for $\boldsymbol{L}\in\mathrm{Lin}(\mathbf{V},\mathbf{U})$, $\boldsymbol{K}\in\mathrm{Lin}(\mathbf{U},\mathbf{W})$ we write $\boldsymbol{K}\cdot\boldsymbol{L}$.

$\mathbf{V}^* := \mathrm{Lin}(\mathbf{V},\mathbb{K})$ is the *dual* of $\mathbf{V}$. The elements of $\mathbf{V}^*$ are often called *linear functionals* or *covectors.*

The dual *separates* the elements of the vector space which means that if $\boldsymbol{v}\in\mathbf{V}$, and $\boldsymbol{p}\cdot\boldsymbol{v}=0$ for all $\boldsymbol{p}\in\mathbf{V}^*$, then $\boldsymbol{v}=0$ or, equivalently, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are different elements of $\mathbf{V}$, then there is a $\boldsymbol{p}\in\mathbf{V}^*$ such that $\boldsymbol{p}\cdot\boldsymbol{v}_1\neq\boldsymbol{p}\cdot\boldsymbol{v}_2$.

If $\{\boldsymbol{v}_i|\ i\in I\}$ is a basis of $\mathbf{V}$ then there is a set $\{\boldsymbol{p}^i|\ i\in I\}$ in $\mathbf{V}^*$, called the *dual of the basis,* such that

$$\boldsymbol{p}^i\cdot\boldsymbol{v}_j=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}\qquad (i,j\in I).$$

If $\mathbf{V}$ is finite dimensional, then the dual of a basis is a basis in $\mathbf{V}^*$, hence $\dim(\mathbf{V}^*)=\dim\mathbf{V}$.

Let N denote the (finite) dimension of $\mathbf{V}$. If $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_N\}$ is a basis of $\mathbf{V}$ and $\{\boldsymbol{p}^1,\cdots,\boldsymbol{p}^N\}$ is its dual, then for all $\boldsymbol{v}\in\mathbf{V}$ and $\boldsymbol{p}\in\mathbf{V}^*$ we have

$$\boldsymbol{v}=\sum_{i=1}^N(\boldsymbol{p}^i\cdot\boldsymbol{v})\boldsymbol{v}_i,$$
$$\boldsymbol{p}=\sum_{i=1}^N(\boldsymbol{p}\cdot\boldsymbol{v}_i)\boldsymbol{p}^i.$$

1.2. To every element $\boldsymbol{v}$ of $\mathbf{V}$ we can associate a linear map $\mathbf{i}(\boldsymbol{v}):\mathbf{V}^*\to\mathbb{K}$, $\boldsymbol{p}\mapsto\boldsymbol{p}\cdot\boldsymbol{v}$, i.e. an element of $\mathbf{V}^{**}$. The correspondence $\mathbf{V}\to\mathbf{V}^{**}$, $\boldsymbol{v}\mapsto\mathbf{i}(\boldsymbol{v})$ is a linear injection which seems so natural and simple that we find it convenient to identify $\boldsymbol{v}$ and $\mathbf{i}(\boldsymbol{v})$ for all $\boldsymbol{v}\in\mathbf{V}$:

$$\mathbf{V}\subset\!\!\to\mathbf{V}^{**},\qquad v\equiv\mathbf{i}(\boldsymbol{v}),$$

i.e.

$$\boldsymbol{v}\cdot\boldsymbol{p}\equiv\boldsymbol{p}\cdot\boldsymbol{v}\qquad\qquad (\boldsymbol{v}\in\mathbf{V},\ \boldsymbol{p}\in\mathbf{V}^*).$$

If $\mathbf{V}$ is finite dimensional then this correspondence is a linear bijection between $\mathbf{V}$ and $\mathbf{V}^{**}$, i.e. the whole dual of $\mathbf{V}^*$ can be identified with $\mathbf{V}$:

$$\mathbf{V}\equiv\mathbf{V}^{**},\qquad \boldsymbol{v}\cdot\boldsymbol{p}\equiv\boldsymbol{p}\cdot\boldsymbol{v}.$$

1.3. The Cartesian product $\mathbf{V} \times \mathbf{U}$ of the vector spaces $\mathbf{V}$ and $\mathbf{U}$ is a vector space with the pointwise addition and pointwise multiplication by numbers:

$$(\boldsymbol{v}_1, \boldsymbol{u}_1) + (\boldsymbol{v}_2, \boldsymbol{u}_2) := (\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{u}_1 + \boldsymbol{u}_2),$$
$$\alpha(\boldsymbol{v}, \boldsymbol{u}) := (\alpha \boldsymbol{v}, \alpha \boldsymbol{u})$$

for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbf{V}$, $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbf{U}$ and $\alpha \in \mathbb{K}$.

We have the identification

$$\mathbf{V}^* \times \mathbf{U}^* \equiv (\mathbf{V} \times \mathbf{U})^*, \qquad (\boldsymbol{p}, \boldsymbol{q}) \cdot (\boldsymbol{v}, \boldsymbol{u}) \equiv \boldsymbol{p} \cdot \boldsymbol{v} + \boldsymbol{q} \cdot \boldsymbol{u}.$$

$$((\boldsymbol{p}, \boldsymbol{q}) \in \mathbf{V}^* \times \mathbf{U}^*, \qquad (\boldsymbol{v}, \boldsymbol{u}) \in \mathbf{V} \times \mathbf{U}).$$

1.4. The *transpose* of $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V}, \mathbf{U})$ is the linear map

$$\boldsymbol{L}^* : \mathbf{U}^* \to \mathbf{V}^*, \qquad \boldsymbol{f} \mapsto \boldsymbol{f} \circ \boldsymbol{L},$$

i.e.

$$(\boldsymbol{L}^* \cdot \boldsymbol{f}) \cdot \boldsymbol{v} = \boldsymbol{f} \cdot (\boldsymbol{L} \cdot \boldsymbol{v})$$

or, with the identification introduced in 1.2,

$$\boldsymbol{v} \cdot \boldsymbol{L}^* \cdot \boldsymbol{f} = \boldsymbol{f} \cdot \boldsymbol{L} \cdot \boldsymbol{v} \qquad (\boldsymbol{f} \in \mathbf{U}^*,\ \boldsymbol{v} \in \mathbf{V}).$$

If $\boldsymbol{L}, \boldsymbol{K} \in \mathrm{Lin}(\mathbf{V}, \mathbf{U})$, $\alpha \in \mathbb{K}$, then

$$(\boldsymbol{L} + \boldsymbol{K})^* = \boldsymbol{L}^* + \boldsymbol{K}^*,$$
$$(\alpha \boldsymbol{L})^* = \alpha \boldsymbol{L}^*.$$

If $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V}, \mathbf{U})$, $\boldsymbol{K} \in \mathrm{Lin}(\mathbf{U}, \mathbf{W})$, then

$$(\boldsymbol{K} \cdot \boldsymbol{L})^* = \boldsymbol{L}^* \cdot \boldsymbol{K}^*.$$

If $\mathbf{V}$ and $\mathbf{U}$ are finite dimensional, then
— $\boldsymbol{L}$ is injective if and only if $\boldsymbol{L}^*$ is surjective,
— $\boldsymbol{L}$ is surjective if and only if $\boldsymbol{L}^*$ is injective.

Moreover, in this case — because of the identification $\mathbf{V}^{**} \equiv \mathbf{V}$, $\mathbf{U}^{**} \equiv \mathbf{U}$ — we have

$$\boldsymbol{L}^{**} = \boldsymbol{L}.$$

If $\boldsymbol{L}$ is bijective, then

$$(\boldsymbol{L}^{-1})^* = (\boldsymbol{L}^*)^{-1}.$$

1.5. Let $\mathbf{V}$ be a finite dimensional vector space and $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V}, \mathbf{V}^*)$. Then $\boldsymbol{L}^*$ is a linear map from $\mathbf{V}^{**}$ into $\mathbf{V}^*$, i.e. because of the identification $\mathbf{V}^{**} \equiv \mathbf{V}$ we have $\boldsymbol{L}^* \in \mathrm{Lin}(\mathbf{V}, \mathbf{V}^*)$.

The linear map $\boldsymbol{L} : \mathbf{V} \to \mathbf{V}^*$ is called *symmetric or antisymmetric* if $\boldsymbol{L} = \boldsymbol{L}^*$ or $\boldsymbol{L} = -\boldsymbol{L}^*$, respectively.

In general, the *symmetric and antisymmetric parts* of $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V}, \mathbf{V}^*)$ are

$$\frac{\boldsymbol{L} + \boldsymbol{L}^*}{2} \qquad \text{and} \qquad \frac{\boldsymbol{L} - \boldsymbol{L}^*}{2},$$

respectively.

Similar definitions work well for linear maps $\mathbf{V}^* \to \mathbf{V}$.

On the other hand, the notions of symmetricity, symmetric part etc. make no sense for linear maps $\mathbf{V} \to \mathbf{V}$ *and* $\mathbf{V}^* \to \mathbf{V}^*$.

1.6. $\mathbb{K}^N$, the set of ordered N-tuples of numbers, is a well-known vector space. It is known as well that the linear maps from $\mathbb{K}^N$ into $\mathbb{K}^M$ are identified with the matrices of M rows and N columns, in other words, $\mathrm{Lin}(\mathbb{K}^N, \mathbb{K}^M) \equiv \mathbb{K}^{M \times N}$. As a consequence, we have the identification

$$\left(\mathbb{K}^N\right)^* = \mathrm{Lin}(\mathbb{K}^N, \mathbb{K}) \equiv \mathbb{K}^{1 \times N} = \mathbb{K}^N$$

$$\boldsymbol{p} \cdot \boldsymbol{x} \equiv \sum_{i=1}^{N} p_i x^i \qquad (\boldsymbol{p}, \boldsymbol{x} \in \mathbb{K}^N).$$

We adhered to the trick used in physical applications according to which $\left(\mathbb{K}^N\right)^*$ is *identified* with $\mathbb{K}^N$ in such a way that they are *distinguished* in notations as follows.

The components of the elements of $\mathbb{K}^N$ are indexed by superscripts:

$$\boldsymbol{x} = (x^1, \dots, x^N) \in \mathbb{K}^N,$$

and the components of the elements of $\left(\mathbb{K}^N\right)^* \equiv \mathbb{K}^N$ are indexed by subscripts:

$$\boldsymbol{p} = (p_1, \dots, p_N) \in \left(\mathbb{K}^N\right)^*.$$

The identification in question, called the *standard identification,* means that to every $(x^1, \dots, x^N) \in \mathbb{K}^N$ we assign $(x_1, \dots, x_N) \in \left(\mathbb{K}^N\right)^*$ in such a way that $x_i = x^i$ for all $i = 1, \dots, N$.

Moreover, for the sake of simplicity, we often shall not write that the indices run from 1 to N (or to M), denoting the elements in the form (x^i) and (x_i), respectively.

The fundamental rule is that a summation can be carried out only for indices in opposite positions: up and down. Accordingly, the matrix entries are indexed corresponding to the domain and range of the matrix as a linear map:

$$\begin{aligned}
&\left(L^i{}_k\right) : \mathbb{K}^N \to \mathbb{K}^M, \\
&\left(L_{ik}\right) : \mathbb{K}^N \to \left(\mathbb{K}^M\right)^*, \\
&\left(L_i{}^k\right) : \left(\mathbb{K}^N\right)^* \to \left(\mathbb{K}^M\right)^*, \\
&\left(L^{ik}\right) : \left(\mathbb{K}^N\right)^* \to \mathbb{K}^M.
\end{aligned}$$

This trick works well until actual vectors are not involved; this notation does not show for instance whether the ordered pair of numbers $(1,2)$ is an element of $\mathbb{R}^2$ or $\left(\mathbb{R}^2\right)^*$, and whether the matrix

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

maps from $\mathbb{R}^2$ into $\mathbb{R}^2$ or from $\mathbb{R}^2$ into $\left(\mathbb{R}^2\right)^*$ etc.

The set of vectors $\boldsymbol{\chi}_1 := (1,0,\dots,0)$, $\boldsymbol{\chi}_2 := (0,1,...,0)$, $\dots$, $\boldsymbol{\chi}_N := (0,0,\dots,1)$ is called the *standard basis* of $\mathbb{K}^N$. In the mentioned identification $\left(\mathbb{K}^N\right)^* \equiv \mathbb{K}^N$ the dual of the standard basis is the standard basis itself.

According to this identification the transpose of a matrix as a linear map is the usual matrix transpose.

The above notation shows well that symmetricity, symmetric part etc. make sense only for matrices $\left(L_{ik}\right)$ and $\left(L^{ik}\right)$.

1.7. The symbol $\mathrm{Bilin}(\mathbf{U}\times\mathbf{V},\mathbb{K})$ stands for the vector space of bilinear maps $\mathbf{U}\times\mathbf{V}\to\mathbb{K}$, often called *bilinear forms.*

We have that

$$\mathbf{i} : \mathrm{Lin}(\mathbf{V},\mathbf{U}) \to \mathrm{Bilin}(\mathbf{U}^*\times\mathbf{V},\mathbb{K})$$

defined by

$$(\mathbf{i}(\boldsymbol{L}))\,(\boldsymbol{f},\boldsymbol{v}) := \boldsymbol{f}\cdot\boldsymbol{L}\cdot\boldsymbol{v}$$

$$(\boldsymbol{L}\in\mathrm{Lin}(\mathbf{V},\mathbf{U})\ ,\ \boldsymbol{f}\in\mathbf{U}^*\ ,\ \boldsymbol{v}\in\mathbf{V})$$

is a linear injection which we use for the identification

$$\mathrm{Lin}(\mathbf{V},\mathbf{U}) \subset\!\!\to \mathrm{Bilin}(\mathbf{U}^*\times\mathbf{V},\mathbb{K}), \qquad \boldsymbol{f}\cdot\boldsymbol{L}\cdot\boldsymbol{v} \equiv \boldsymbol{L}(\boldsymbol{f},\boldsymbol{v}).$$

If the vector spaces $\mathbf{U}$ and $\mathbf{V}$ have finite dimension then $\mathbf{i}$ is a bijection, hence $\equiv$ stands instead of $\subset\!\!\to$.

The reader is asked to examine this identification in the case of matrices i.e. for $\mathrm{Lin}(\mathbb{K}^N,\mathbb{K}^M)$.

1.8. A bilinear form $\mathbf{b} : \mathbf{V} \times \mathbf{V} \to \mathbb{K}$ is called *symmetric* or *antisymmetric* if $\mathbf{b}(\boldsymbol{v}, \boldsymbol{u}) = \mathbf{b}(\boldsymbol{u}, \boldsymbol{v})$ or $\mathbf{b}(\boldsymbol{v}, \boldsymbol{u}) = -\mathbf{b}(\boldsymbol{u}, \boldsymbol{v})$, respectively, for all $\boldsymbol{v}, \boldsymbol{u} \in \mathbf{V}$.

Similar definitions are accepted for bilinear forms $\mathbf{V}^* \times \mathbf{V}^* \to \mathbb{K}$.

Observe that for finite dimensional $\mathbf{V}$ the notions introduced here and in 1.5 coincide in the identification $\mathrm{Lin}(\mathbf{V}, \mathbf{V}^*) \equiv \mathrm{Bilin}(\mathbf{V}^* \times \mathbf{V}^*, \mathbb{K})$.

2. Coordinatization

2.1. Let $\mathbf{V}$ be an N-dimensional vector space over $\mathbb{K}$.

An element $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)$ of $\mathbf{V}^N$ is called an *ordered basis* of $\mathbf{V}$ if the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$ is a basis of $\mathbf{V}$.

An ordered basis of $\mathbf{V}$ induces a linear bijection $\boldsymbol{K} : \mathbf{V} \to \mathbb{K}^N$ defined by $\boldsymbol{K} \cdot \boldsymbol{v}_i := \boldsymbol{\chi}_i$ $(i = 1, \ldots, N)$ where $(\boldsymbol{\chi}_1, \ldots \boldsymbol{\chi}_N)$ is the ordered standard basis of $\mathbb{K}^N$. $\boldsymbol{K}$ is called the *coordinatization* of $\mathbf{V}$ corresponding to the given ordered basis. The inverse of the coordinatization, $\boldsymbol{P} := \boldsymbol{K}^{-1}$, is called the *parametrization* of $\mathbf{V}$ corresponding to the given ordered basis. It is quite evident that

$$\boldsymbol{P} \cdot (x^i) = \sum_{i=1}^{N} x^i \boldsymbol{v}_i \qquad \left((x^i) \in \mathbb{K}^N\right).$$

Thus, in view of 1.1 we have

$$\boldsymbol{K} \cdot \boldsymbol{v} = \left(\boldsymbol{p}^i \cdot \boldsymbol{v} \middle| \, i = 1, \ldots, N\right) \qquad (\boldsymbol{v} \in \mathbf{V})$$

where $(\boldsymbol{p}^1, \ldots, \boldsymbol{p}^N)$ is the ordered dual basis of $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)$.

Obviously, every linear bijection $\boldsymbol{K} : \mathbf{V} \to \mathbb{K}^N$ is a coordinatization in the above sense: the one corresponding to the ordered basis $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)$ where $\boldsymbol{v}_i := \boldsymbol{K}^{-1} \cdot \boldsymbol{\chi}_i$ $(i = 1, \ldots, N)$.

2.2. A coordinatization of $\mathbf{V}$ determines a coordinatization of $\mathbf{V}^*$, that is induced by the corresponding ordered dual basis. Using the previous notations and denoting the coordinatization in question by $\boldsymbol{C} : \mathbf{V}^* \to (\mathbb{K}^N)^*$ we have

$$\boldsymbol{C} \cdot \boldsymbol{p} = (\boldsymbol{p} \cdot \boldsymbol{v}_i | \, i = 1, \ldots, N) \qquad (\boldsymbol{p} \in \mathbf{V}^*).$$

It is not hard to see that

$$\boldsymbol{C} = \left(\boldsymbol{K}^{-1}\right)^* = \boldsymbol{P}^*.$$

2.3. In the coordinatization $\boldsymbol{K}$, a linear map $\boldsymbol{L} : \mathbf{V} \to \mathbf{V}$ is represented by the matrix

$$\boldsymbol{K} \cdot \boldsymbol{L} \cdot \boldsymbol{K}^{-1} = \boldsymbol{K} \cdot \boldsymbol{L} \cdot \boldsymbol{P} = \left(\boldsymbol{p}^i \cdot \boldsymbol{L} \cdot \boldsymbol{v}_k \middle| \, i, k = 1, \ldots, N\right).$$

To deduce this equality argue as follows:

$$\sum_{k=1}^{N} \left(\boldsymbol{K} \cdot \boldsymbol{L} \cdot \boldsymbol{K}^{-1}\right)^{i}{}_{k} x^{k} = \left(\boldsymbol{K} \cdot \boldsymbol{L} \cdot \boldsymbol{K}^{-1} \cdot x\right)^{i} =$$

$$= \boldsymbol{p}^{i} \cdot \boldsymbol{L} \cdot \sum_{k=1}^{N} x^{k} \boldsymbol{v}_{k} = \sum_{k=1}^{N} (\boldsymbol{p}^{i} \cdot \boldsymbol{L} \cdot \boldsymbol{v}_{k}) x^{k}.$$

The linear map $\boldsymbol{T} : \mathbf{V} \to \mathbf{V}^*$ is represented by the matrix

$$\left(\boldsymbol{K}^{-1}\right)^{*} \boldsymbol{T} \cdot \boldsymbol{K}^{-1} = \boldsymbol{P}^{*} \cdot \boldsymbol{T} \cdot \boldsymbol{P} = (\boldsymbol{v}_{i} \cdot \boldsymbol{T} \cdot \boldsymbol{v}_{k} \,|\, i, k = 1, \dots, N).$$

It is left to the reader to find the matrix of linear maps $\mathbf{V}^* \to \mathbf{V}$ and $\mathbf{V}^* \to \mathbf{V}^*$.

3. Tensor products

3.1. We start with an abstract definition of tensor products that may seem strange; the properties of tensor products following from this definition will clarify its real meaning.

Definition. Let $\mathbf{V}$ and $\mathbf{U}$ be vector spaces (over the same field $\mathbb{K}$). A *tensor product* of $\mathbf{U}$ and $\mathbf{V}$ is a pair $(\mathbf{Z}, \mathbf{b})$, where

(i) $\mathbf{Z}$ is a vector space,

(ii) $\mathbf{b} : \mathbf{U} \times \mathbf{V} \to \mathbf{Z}$ is a bilinear map having the property that

— if $\mathbf{W}$ is a vector space and $\mathbf{c} : \mathbf{U} \times \mathbf{V} \to \mathbf{W}$ is a bilinear map,

— then there exists a unique linear map $\boldsymbol{L} : \mathbf{Z} \to \mathbf{W}$ such that

$$\mathbf{c} = \boldsymbol{L} \circ \mathbf{b}.$$

Proposition. The pair $(\mathbf{Z}, \mathbf{b})$ satisfying (i) and (ii) is a tensor product of $\mathbf{U}$ and $\mathbf{V}$ if and only if

1) $\mathbf{Z}$ is spanned ($\mathbf{Z}$ is the linear subspace generated) by Ran $\mathbf{b}$,

2) if $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are linearly independent elements of $\mathbf{V}$ and $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ are elements of $\mathbf{U}$ then $\sum\limits_{i=1}^{n} \mathbf{b}(\boldsymbol{u}_i, \boldsymbol{v}_i) = \mathbf{0}$ implies $\boldsymbol{u}_1 = \cdots = \boldsymbol{u}_n = \mathbf{0}$.

Proof. Exclude the trivial cases $\mathbf{V} = \mathbf{0}$ or $\mathbf{U} = \mathbf{0}$.

Suppose 1) is fulfilled. Then every element of $\mathbf{Z}$ is of the form $\sum\limits_{k=1}^{r} \alpha_k \mathbf{b}(\boldsymbol{u}_k, \boldsymbol{v}_k)$. Since $\alpha \mathbf{b}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{b}(\alpha \boldsymbol{u}, \boldsymbol{v})$, we conclude that the elements of $\mathbf{Z}$ can be written in the form $\sum\limits_{k=1}^{r} \mathbf{b}(\boldsymbol{u}_k, \boldsymbol{v}_k)$.

Suppose 2) is fulfilled, too. Take a bilinear map $\mathbf{c} : \mathbf{U} \times \mathbf{V} \to \mathbf{W}$ and define the map $\boldsymbol{L} : \mathbf{Z} \to \mathbf{W}$ by

$$\boldsymbol{L} \cdot \left(\sum_{k=1}^{r} \mathbf{b}(\boldsymbol{u}_k, \boldsymbol{v}_k) \right) := \sum_{k=1}^{r} \mathbf{c}(\boldsymbol{u}_k, \boldsymbol{v}_k).$$

If $\boldsymbol{L}$ is well-defined, then it is linear, $\boldsymbol{L} \circ \mathbf{b} = \mathbf{c}$, and it is unique with this property. To demonstrate that $\boldsymbol{L}$ is well-defined, we have to show that

$$\sum_{k=1}^{r} \mathbf{b}(\boldsymbol{u}_k, \boldsymbol{v}_k) = \sum_{j=1}^{s} \mathbf{b}(\boldsymbol{x}_j, \boldsymbol{y}_j) \quad \text{implies} \quad \sum_{k=1}^{r} \mathbf{c}(\boldsymbol{u}_k, \boldsymbol{v}_k) = \sum_{j=1}^{s} \mathbf{c}(\boldsymbol{x}_j, \boldsymbol{y}_j),$$

which is eqivalent to

$$\sum_{i=1}^{m} \mathbf{b}(\boldsymbol{u}_i, \boldsymbol{v}_i) = \mathbf{0} \qquad \text{implies} \qquad \sum_{i=1}^{m} \mathbf{c}(\boldsymbol{u}_i, \boldsymbol{v}_i) = \mathbf{0}.$$

Let us choose a largest set of linearly independent vectors from $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$; without loss of generality, we can suppose it is $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ (where, of course, $n \leq m$). If $\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i$ then $\mathbf{b}(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{n} \mathbf{b}(\alpha_i \boldsymbol{u}, \boldsymbol{v}_i)$ and a similar formula holds for $\mathbf{c}(\boldsymbol{u}, \boldsymbol{v})$ as well. Consequently, a rearrangement of the terms in the previous formulae yields that $\boldsymbol{L}$ is well-defined if

$$\sum_{i=1}^{n} \mathbf{b}(\boldsymbol{u}_i, \boldsymbol{v}_i) = \mathbf{0} \qquad \text{implies} \qquad \sum_{i=1}^{n} \mathbf{c}(\boldsymbol{u}_i, \boldsymbol{v}_i) = \mathbf{0}$$

whenever $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent which follows from condition 2).

We have proved that conditions 1) and 2) are sufficient for a tensor product.

Since $\boldsymbol{L} \circ \mathbf{b} = \mathbf{r}$ can define $\boldsymbol{L}$ only on the linear subspace spanned by the range of $\mathbf{b}$, condition 1) is necessary for the uniqueness of $\boldsymbol{L}$.

If condition 2) is not satisfied then we can find a bilinear map $\mathbf{r}$ such that $\boldsymbol{L} \circ \mathbf{b} \neq \mathbf{r}$ for all linear maps $\boldsymbol{L}$. Indeed, let the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be linearly independent, $\sum_{i=1}^{n} \mathbf{b}(\boldsymbol{u}_i, \boldsymbol{v}_i) = \mathbf{0}$, and at least one of the $\boldsymbol{u}_i$-s is not zero. Without loss of generality we can assume that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ (where $m \leq n$) are linearly independent and all the other $\boldsymbol{u}_i$-s are their linear combinations. Complete $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ to a basis in $\mathbf{V}$ and $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ to a basis in $\mathbf{U}$. Define the bilinear map $\mathbf{r} : \mathbf{U} \times \mathbf{V} \to \mathbb{K}$ in such a way that $\mathbf{r}(\boldsymbol{u}_1, \boldsymbol{v}_1) := 1$ and $\mathbf{r}(\boldsymbol{u}, \boldsymbol{v}) := 0$ for all other basis elements $\boldsymbol{u}$ and $\boldsymbol{v}$. Then for all linear maps $\boldsymbol{L} : \mathbf{Z} \to \mathbb{K}$ we have $\boldsymbol{L} \cdot \left(\sum_{i=1}^{n} \mathbf{b}(\boldsymbol{u}_i, \boldsymbol{v}_i) \right) = 0 \neq 1 = \sum_{i=1}^{n} \boldsymbol{r}(\boldsymbol{u}_i, \boldsymbol{v}_i)$.

3.2. In the next item the existence of tensor products will be proved. Observe that in the case $\mathbf{W} = \mathbf{Z}$, $\mathbf{c} = \mathbf{b}$, the identity map of $\mathbf{Z}$ fulfils $\mathbf{b} = \mathrm{id}_{\mathbf{Z}} \circ \mathbf{b}$; according to the definition of the tensor product this is the only possibility, i.e. if $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{Z})$ and $\mathbf{b} = \boldsymbol{L} \circ \mathbf{b}$ then $\boldsymbol{L} = \mathrm{id}_{\mathbf{Z}}$.

As a consequence, if $(\mathbf{Z}', \mathbf{b}')$ is another tensor product of $\mathbf{U}$ and $\mathbf{V}$ then there is a unique linear bijection $\boldsymbol{L} : \mathbf{Z} \to \mathbf{Z}'$ such that $\mathbf{b}' = \boldsymbol{L} \circ \mathbf{b}$. This means that the tensor products of $\mathbf{U}$ and $\mathbf{V}$ are "canonically isomorphic" or "essentially the same", hence we speak of *the* tensor product and applying a customary abuse of language we call the corresponding vector space the tensor product ($\mathbf{Z}$ in the definition) denoting it by $\mathbf{U} \otimes \mathbf{V}$, and writing

$$\mathbf{U} \times \mathbf{V} \to \mathbf{U} \otimes \mathbf{V}, \qquad (\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u} \otimes \boldsymbol{v}$$

for the corresponding bilinear map ($\mathbf{b}$ in the definition); $\boldsymbol{u} \otimes \boldsymbol{v}$ is called the *tensor product* of $\boldsymbol{u}$ and $\boldsymbol{v}$.

An actual given tensor product is called a *realization* of the tensor product and the following symbols are used: $\mathbf{U} \otimes \mathbf{V} \hookrightarrow \mathbf{W}$ or $\mathbf{U} \otimes \mathbf{V} \equiv \mathbf{W}$ denote that the tensor product of $\mathbf{U}$ and $\mathbf{V}$ is realized as a subspace of $\mathbf{W}$ or as the whole vector space $\mathbf{W}$, respectively.

It is worth repeating the results of the previous paragraph in the new notations.

Every element of $\mathbf{U} \otimes \mathbf{V}$ can be written in the form $\sum\limits_{i=1}^{n} \boldsymbol{u}_i \otimes \boldsymbol{v}_i$ where $\boldsymbol{v}_1, \dots .\boldsymbol{v}_n$ are linearly independent vectors in $\mathbf{V}$. Moreover, if the sum is zero, then $\boldsymbol{u}_1 = \dots = \boldsymbol{u}_n = \mathbf{0}$. In particular, if $\boldsymbol{u} \neq \mathbf{0}$ and $\boldsymbol{v} \neq \mathbf{0}$ then $\boldsymbol{u} \otimes \boldsymbol{v} \neq \mathbf{0}$.

3.3. For $\boldsymbol{u} \in \mathbf{U}$ and $\boldsymbol{v} \in \mathbf{V}$ we define the linear map

$$\boldsymbol{u} \otimes \boldsymbol{v} : \mathbf{V}^* \to \mathbf{U}, \qquad \boldsymbol{p} \mapsto (\boldsymbol{p} \cdot \boldsymbol{v})\boldsymbol{u}.$$

Proposition. $\mathbf{U} \times \mathbf{V} \to \mathrm{Lin}(\mathbf{V}^*, \mathbf{U})$, $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u} \otimes \boldsymbol{v}$ is a bilinear map satisfying condition 2) of Proposition 2.1. As a consequence, the linear map $\boldsymbol{u} \otimes \boldsymbol{v}$ is the tensor product of $\boldsymbol{u}$ and $\boldsymbol{v}$ (that is why we used in advance this notation) and $\mathbf{U} \otimes \mathbf{V}$ is realized as a linear subspace of $\mathrm{Lin}(\mathbf{V}^*, \mathbf{U})$ spanned by such elements.

Proof. It is trivial that $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u} \otimes \boldsymbol{v}$ is bilinear.

Suppose that $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are linearly independent vectors in $\mathbf{V}$ and $\sum\limits_{i=1}^{n} \boldsymbol{u}_i \otimes \boldsymbol{v}_i = \mathbf{0}$. Then for arbitrary $\boldsymbol{p} \in \mathbf{V}^*$ and $\boldsymbol{f} \in \mathbf{U}^*$ we have

$$0 = \boldsymbol{f} \cdot \left(\left(\sum_{i=1}^{n} \boldsymbol{u}_i \otimes \boldsymbol{v}_i \right) \cdot \boldsymbol{p} \right) = \sum_{i=1}^{n} (\boldsymbol{f} \cdot \boldsymbol{u}_i)(\boldsymbol{p} \cdot \boldsymbol{v}_i) = \boldsymbol{p} \cdot \left(\sum_{i=1}^{n} (\boldsymbol{f} \cdot \boldsymbol{u}_i) \boldsymbol{v}_i \right).$$

Since $\mathbf{V}^*$ separates the elements of $\mathbf{V}$, this means that $\sum_{i=1}^{n}(\boldsymbol{f}\cdot\boldsymbol{u}_i)\boldsymbol{v}_i = \mathbf{0}$. Because of the linear independence of $\boldsymbol{v}_i$-s this involves $\boldsymbol{f}\cdot\boldsymbol{u}_i = 0$ for all $i = 1,\dots,n$. Since $\mathbf{U}^*$ separates the elements of $\mathbf{U}$, it follows that $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \dots = \boldsymbol{u}_n = \mathbf{0}$.

3.4. Proposition. If $\{\boldsymbol{v}_i |\ i \in I\}$ is a basis in $\mathbf{V}$ and $\{\boldsymbol{u}_j |\ j \in J\}$ is a basis in $\mathbf{U}$ then $\{\boldsymbol{u}_j \otimes \boldsymbol{v}_i |\ j \in J,\ i \in I\}$ is a basis in $\mathbf{U}\otimes\mathbf{V}$. ■

According to Propositions 3.3 and 1.7 we have

$$\mathbf{U}\otimes\mathbf{V} \subset\!\!\rightarrow \mathrm{Lin}(\mathbf{V}^*,\mathbf{U}) \subset\!\!\rightarrow \mathrm{Bilin}(\mathbf{U}^*\times\mathbf{V}^*,\mathbb{K}).$$

If $\mathbf{U}$ and $\mathbf{V}$ are finite dimensional then

$$\dim(\mathbf{U}\otimes\mathbf{V}) = (\dim\mathbf{U})(\dim\mathbf{V}).$$

Moreover, in this case $\dim(\mathbf{U}\otimes\mathbf{V}) = \dim\left(\mathrm{Lin}(\mathbf{V}^*,\mathbf{U})\right)$, hence the present proposition on the bases implies that for finite dimensional vector spaces

$$\mathbf{U}\otimes\mathbf{V} \equiv \mathrm{Lin}(\mathbf{V}^*,\mathbf{U}) \equiv \mathrm{Bilin}(\mathbf{U}^*\times\mathbf{V}^*,\mathbb{K})$$

and because of $\mathbf{V}^{**}\equiv\mathbf{V}$, $\mathbf{U}^{**}\equiv\mathbf{U}$,

$$\begin{aligned}
\mathbf{U}\otimes\mathbf{V}^* &\equiv \mathrm{Lin}(\mathbf{V},\mathbf{U}) \equiv \mathrm{Bilin}(\mathbf{U}^*\times\mathbf{V},\mathbb{K}),\\
\mathbf{U}^*\otimes\mathbf{V} &\equiv \mathrm{Lin}(\mathbf{V}^*,\mathbf{U}^*) \equiv \mathrm{Bilin}(\mathbf{U}\times\mathbf{V}^*,\mathbb{K}),\\
\mathbf{U}^*\otimes\mathbf{V}^* &\equiv \mathrm{Lin}(\mathbf{V},\mathbf{U}^*) \equiv \mathrm{Bilin}(\mathbf{U}\times\mathbf{V},\mathbb{K}).
\end{aligned}$$

3.5. We have the following identifications.

(i)
$$\mathbb{K}\otimes\mathbf{V}\equiv\mathbf{V}, \qquad \alpha\otimes\boldsymbol{v}\equiv\alpha\boldsymbol{v},$$

(ii)
$$\begin{aligned}
&(\mathbf{U}\times\mathbf{V})\otimes\mathbf{W} \equiv (\mathbf{U}\otimes\mathbf{W})\times(\mathbf{V}\otimes\mathbf{W},)\\
&(\boldsymbol{u},\boldsymbol{v})\otimes\boldsymbol{w} \equiv (\boldsymbol{u}\otimes\boldsymbol{w},\boldsymbol{v}\otimes\boldsymbol{w}),
\end{aligned}$$

$$\begin{aligned}
&\mathbf{W}\otimes(\mathbf{U}\times\mathbf{V}) \equiv (\mathbf{W}\otimes\mathbf{U})\times(\mathbf{W}\otimes\mathbf{V}),\\
&\boldsymbol{w}\otimes(\boldsymbol{u},\boldsymbol{v}) \equiv (\boldsymbol{w}\otimes\boldsymbol{u},\boldsymbol{w}\otimes\boldsymbol{v}),
\end{aligned}$$

(iii) If $\mathbf{U}$ and $\mathbf{V}$ are finite dimensional then

$$\mathbf{U}^*\otimes\mathbf{V}^*\equiv(\mathbf{U}\otimes\mathbf{V})^*, \qquad (\boldsymbol{f}\otimes\boldsymbol{p}):(\boldsymbol{u}\otimes\boldsymbol{v})\equiv(\boldsymbol{f}\cdot\boldsymbol{u})(\boldsymbol{p}\cdot\boldsymbol{v}),$$
$$(\boldsymbol{f}\in\mathbf{U}^*, \quad \boldsymbol{p}\in\mathbf{V}^*, \quad \boldsymbol{u}\in\mathbf{U}, \quad \boldsymbol{v}\in\mathbf{V})$$

where we found convenient to write the symbol : for the bilinear map of duality; we shall give an explanation later.

3.6. In mathematical books the tensor product is often said to be commutative which means that we have a unique linear bijection $\mathbf{U}\otimes\mathbf{V}\to\mathbf{V}\otimes\mathbf{U}$, $\boldsymbol{u}\otimes\boldsymbol{v}\mapsto\boldsymbol{v}\otimes\boldsymbol{u}$ admitting an identification. However, we do not find convenient to use this identification because of two reasons:

1) if $\mathbf{U}=\mathbf{V}$, $\boldsymbol{u},\boldsymbol{v}\in\mathbf{V}$ and $\boldsymbol{u}\neq\boldsymbol{v}$ then, in general, $\boldsymbol{u}\otimes\boldsymbol{v}\neq\boldsymbol{v}\otimes\boldsymbol{u}$;

2) $\boldsymbol{u}\otimes\boldsymbol{v}\in\mathbf{U}\otimes\mathbf{V}\subset\!\!\to \mathrm{Lin}(\mathbf{V}^*,\mathbf{U})$, $\boldsymbol{v}\otimes\boldsymbol{u}\in\mathbf{V}\otimes\mathbf{U}\subset\!\!\to \mathrm{Lin}(\mathbf{U}^*,\mathbf{V})\subset \mathrm{Lin}(\mathbf{U}^*,\mathbf{V}^{**})$; it is not hard to see that the transpose of $\boldsymbol{u}\otimes\boldsymbol{v}$ equals $\boldsymbol{v}\otimes\boldsymbol{u}$:

$$(\boldsymbol{u}\otimes\boldsymbol{v})^*=\boldsymbol{v}\otimes\boldsymbol{u}.$$

Hence the unique linear bijection between $\mathbf{U}\otimes\mathbf{V}$ and $\mathbf{V}\otimes\mathbf{U}$ that sends $\boldsymbol{u}\otimes\boldsymbol{v}$ into $\boldsymbol{v}\otimes\boldsymbol{u}$ is the transposing map. We do not want, in general, to identify a linear map with its transpose (e.g. a matrix with its transpose).

However, if one of the vector spaces is one-dimensional, we accept the mentioned identification, i.e.

$$\mathbf{A}\otimes\mathbf{V}\equiv\mathbf{V}\otimes\mathbf{A},\quad \boldsymbol{a}\otimes\boldsymbol{v}\equiv\boldsymbol{v}\otimes\boldsymbol{a}\qquad\text{if}\quad \dim\mathbf{A}=1.$$

Moreover, in this case we agree to omit the symbol $\otimes$:

$$\boldsymbol{a}\boldsymbol{v}:=\boldsymbol{a}\otimes\boldsymbol{v}\qquad\qquad (\boldsymbol{a}\in\mathbf{A},\ \boldsymbol{v}\in\mathbf{V},\ \dim\mathbf{A}=1).$$

Note that if $\dim\mathbf{A}=1$ then every element of $\mathbf{A}\otimes\mathbf{V}$ has the form $\boldsymbol{a}\boldsymbol{v}$.

Though, in general, $\mathbf{A}\otimes\mathbf{V}\neq\mathbf{V}$, it makes sense (if $\dim\mathbf{A}=1$) that an element $\boldsymbol{z}$ of $\mathbf{A}\otimes\mathbf{V}$ is *parallel* to an element $\boldsymbol{v}$ of $\mathbf{V}$: if there is an $\boldsymbol{a}\in\mathbf{A}$ such that $\boldsymbol{z}=\boldsymbol{a}\boldsymbol{v}$.

3.7. It is well known that a linear map $\boldsymbol{L}:\mathbf{V}_1\times\mathbf{V}_2\to\mathbf{U}_1\times\mathbf{U}_2$ can be represented in a matrix form:

$$\boldsymbol{L}=\begin{pmatrix}\boldsymbol{L}_{11} & \boldsymbol{L}_{12}\\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22}\end{pmatrix}$$

where $\boldsymbol{L}_{ik}\in\mathrm{Lin}(\mathbf{V}_i,\mathbf{U}_k)\quad(i,k=1,2)$ and

$$\boldsymbol{L}\cdot(\boldsymbol{v}_1,\boldsymbol{v}_2)=(\boldsymbol{L}_{11}\cdot\boldsymbol{v}_1+\boldsymbol{L}_{12}\cdot\boldsymbol{v}_2,\ \boldsymbol{L}_{21}\cdot\boldsymbol{v}_1+\ \boldsymbol{L}_{22}\cdot\boldsymbol{v}_2)\,.$$

This corresponds to the finite dimensional identifications (see in particular 3.5(*ii*))

$$\begin{aligned}
&\mathrm{Lin}(\mathbf{V}_1\times\mathbf{V}_2,\mathbf{U}_1\times\mathbf{U}_2)\equiv\\
&\qquad\equiv(\mathbf{U}_1\times\mathbf{U}_2)\otimes(\mathbf{V}_1\times\mathbf{V}_2)^*\equiv(\mathbf{U}_1\times\mathbf{U}_2)\otimes(\mathbf{V}_1^*\times\mathbf{V}_2^*)\equiv\\
&\qquad\equiv(\mathbf{U}_1\otimes\mathbf{V}_1^*)\times(\mathbf{U}_1\otimes\mathbf{V}_2^*)\times(\mathbf{U}_2\otimes\mathbf{V}_1^*)\times(\mathbf{U}_2\otimes\mathbf{V}_2^*)\equiv\\
&\qquad\equiv\mathrm{Lin}(\mathbf{V}_1,\mathbf{U}_1)\times\mathrm{Lin}(\mathbf{V}_2,\mathbf{U}_1)\times\mathrm{Lin}(\mathbf{V}_1,\mathbf{U}_2)\times\mathrm{Lin}(\mathbf{V}_2,\mathbf{U}_2).
\end{aligned}$$

Accordingly, we find convenient to write

$$(\boldsymbol{u}_1, \boldsymbol{u}_2) \otimes (\boldsymbol{p}_1, \boldsymbol{p}_2) \equiv \begin{pmatrix} \boldsymbol{u}_1 \otimes \boldsymbol{p}_1 & \boldsymbol{u}_1 \otimes \boldsymbol{p}_2 \\ \boldsymbol{u}_2 \otimes \boldsymbol{p}_1 & \boldsymbol{u}_2 \otimes \boldsymbol{p}_2 \end{pmatrix}$$

for $(\boldsymbol{u}_1, \boldsymbol{u}_2) \in (\mathbf{U}_1, \mathbf{U}_2)$ and $(\boldsymbol{p}_1, \boldsymbol{p}_2) \in \mathbf{V}_1^* \times \mathbf{V}_2^*$.

Of course, a similar formula holds for other tensor products, e.g. for the elements of $(\mathbf{U}_1 \times \mathbf{U}_2) \otimes (\mathbf{V}_1 \times \mathbf{V}_2)$:

$$(\boldsymbol{u}_1, \boldsymbol{u}_2) \otimes (\boldsymbol{v}_1, \boldsymbol{v}_2) \equiv \begin{pmatrix} \boldsymbol{u}_1 \otimes \boldsymbol{v}_1 & \boldsymbol{u}_1 \otimes \boldsymbol{v}_2 \\ \boldsymbol{u}_2 \otimes \boldsymbol{v}_1 & \boldsymbol{u}_2 \otimes \boldsymbol{v}_2 \end{pmatrix}.$$

It is not hard to see then (cf. 3.6) that

$$\begin{pmatrix} \boldsymbol{u}_1 \otimes \boldsymbol{v}_1 & \boldsymbol{u}_1 \otimes \boldsymbol{v}_2 \\ \boldsymbol{u}_2 \otimes \boldsymbol{v}_1 & \boldsymbol{u}_2 \otimes \boldsymbol{v}_2 \end{pmatrix}^* = \begin{pmatrix} \boldsymbol{v}_1 \otimes \boldsymbol{u}_1 & \boldsymbol{v}_1 \otimes \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \otimes \boldsymbol{u}_1 & \boldsymbol{v}_2 \otimes \boldsymbol{u}_2 \end{pmatrix}.$$

3.8. If $\mathbf{A}$ is a one-dimensional vector space then $\mathrm{Lin}(\mathbf{A})$ is identified with $\mathbb{K}$: the number α corresponds to the linear map $\boldsymbol{a} \mapsto \alpha\boldsymbol{a}$. As a consequence, we have the following identification, too:

$$\mathbf{A} \otimes \mathbf{A}^* \equiv \mathrm{Lin}(\mathbf{A}) \equiv \mathbb{K}, \qquad \boldsymbol{a}\boldsymbol{h} \equiv \boldsymbol{h} \cdot \boldsymbol{a} (\equiv \boldsymbol{a} \cdot \boldsymbol{h})$$

(remember: $\boldsymbol{a}\boldsymbol{h} := \boldsymbol{a} \otimes \boldsymbol{h}$). Indeed, by definition, $\boldsymbol{a}\boldsymbol{h} : \mathbf{A} \to \mathbf{A}$, $\boldsymbol{b} \mapsto (\boldsymbol{h} \cdot \boldsymbol{b})\boldsymbol{a}$. If $\boldsymbol{a} = \mathbf{0}$ then $\boldsymbol{a}\boldsymbol{h} = \mathbf{0} = \boldsymbol{h} \cdot \boldsymbol{a}$. If $\boldsymbol{a} \neq \mathbf{0}$ then there is a unique $\frac{\boldsymbol{b}}{\boldsymbol{a}} \in \mathbb{K}$ for all $\mathbf{b} \in \mathbf{A}$ such that $\boldsymbol{b} = \frac{\boldsymbol{b}}{\boldsymbol{a}}\boldsymbol{a}$. Thus $(\boldsymbol{h} \cdot \boldsymbol{b})\boldsymbol{a} = (\boldsymbol{h} \cdot \frac{\boldsymbol{b}}{\boldsymbol{a}}\boldsymbol{a})\boldsymbol{a} = (\boldsymbol{h} \cdot \boldsymbol{a})\frac{\boldsymbol{b}}{\boldsymbol{a}}\boldsymbol{a} = (\boldsymbol{h} \cdot \boldsymbol{a})\boldsymbol{b}$ and we see that $\boldsymbol{a}\boldsymbol{h}$ $(= \boldsymbol{a} \otimes \boldsymbol{h})$ equals the multiplication by $\boldsymbol{h} \cdot \boldsymbol{a}$.

For one-dimensional vector spaces we prefer the symbol of (tensor) product to the dot for expressing the bilinear map of duality i.e. the symbol $\boldsymbol{a}\boldsymbol{h}$ to $\boldsymbol{a} \cdot \boldsymbol{h}$.

3.9. Since $\mathbf{V} \times \mathbf{V}^* \to \mathbb{K}$, $(\boldsymbol{v}, \boldsymbol{p}) \mapsto \boldsymbol{p} \cdot \boldsymbol{v}$ is a bilinear map, the definition of tensor products ensures the existence of a unique linear map

$$\mathrm{Tr} : \mathbf{V} \otimes \mathbf{V}^* \to \mathbb{K} \qquad \text{such that} \qquad \mathrm{Tr}(\boldsymbol{v} \otimes \boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{v}.$$

If $\mathbf{V}$ is finite dimensional then $\mathbf{V} \otimes \mathbf{V}^* \equiv \mathrm{Lin}(\mathbf{V})$, hence $\mathrm{Tr}\mathbf{L}$, called the *trace* of $\boldsymbol{L}$, is defined for all linear maps $\boldsymbol{L} : \mathbf{V} \to \mathbf{V}$.

Since for $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}$ and $\boldsymbol{p}, \boldsymbol{q} \in \mathbf{V}^*$ we have $(\boldsymbol{u} \otimes \boldsymbol{p}) \cdot (\boldsymbol{v} \otimes \boldsymbol{q}) = (\boldsymbol{p} \cdot \boldsymbol{v})\boldsymbol{u} \otimes \boldsymbol{q}$, we easily deduce that for all $\boldsymbol{L}, \boldsymbol{K} \in \mathrm{Lin}(\mathbf{V})$ (if $\dim \mathbf{V} < \infty$)

$$\mathrm{Tr}(\boldsymbol{L} \cdot \boldsymbol{K}) = \mathrm{Tr}(\boldsymbol{K} \cdot \boldsymbol{L}).$$

If $\{\boldsymbol{v}_i |\ i = 1, \dots, N\}$ is a basis in $\mathbf{V}$ and $\{\boldsymbol{p}^i |\ i = 1, \dots, N\}$ is its dual then for all $\boldsymbol{v} \in \mathbf{V}$ and $\boldsymbol{p} \in \mathbf{V}^*$

$$\boldsymbol{p} \cdot \boldsymbol{v} = \sum_{i=1}^{N} (\boldsymbol{p}^i \cdot \boldsymbol{v})(\boldsymbol{p} \cdot \boldsymbol{v}_i) = \sum_{i=1}^{N} \boldsymbol{p}^i \cdot (\boldsymbol{v} \otimes \boldsymbol{p}) \cdot \boldsymbol{v}_i,$$

which gives

$$\mathrm{Tr}\boldsymbol{L} = \sum_{i=1}^{N} \boldsymbol{p}^i \cdot \boldsymbol{L} \cdot \boldsymbol{v}_i \qquad (\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})).$$

Note that the trace of linear maps $\mathbf{V} \to \mathbf{V}^*$ *and* $\mathbf{V}^* \to \mathbf{V}$ *makes no sense;* on the other hand, we have (for finite dimensional $\mathbf{V}$)

$$\mathrm{Tr} : \mathrm{Lin}(\mathbf{V}^*) \equiv \mathbf{V}^* \otimes \mathbf{V} \to \mathbb{K}, \qquad \boldsymbol{p} \otimes \boldsymbol{v} \mapsto \boldsymbol{p} \cdot \boldsymbol{v}$$

and we easily see by $(\boldsymbol{v} \otimes \boldsymbol{p})^* = \boldsymbol{p} \otimes \boldsymbol{v}$ that

$$\mathrm{Tr}(\boldsymbol{L}^*) = \mathrm{Tr}\boldsymbol{L} \qquad (\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})).$$

Moreover, if $\mathbf{Z}$ is a finite dimensional vector space, we define

$$\mathrm{Tr} : \mathrm{Lin}(\mathbf{V}, \mathbf{Z} \otimes \mathbf{V}) \equiv \mathbf{Z} \otimes \mathbf{V} \otimes \mathbf{V}^* \to \mathbf{Z}, \qquad \boldsymbol{z} \otimes \boldsymbol{v} \otimes \boldsymbol{p} \mapsto (\boldsymbol{p} \cdot \boldsymbol{v})\boldsymbol{z}.$$

3.10. Let $\mathbf{V}$ be finite dimensional. Then, according to 3.5(*iii*) and $\mathbf{V}^{**} \equiv \mathbf{V}$, we have $\mathbf{V}^* \otimes \mathbf{V} \equiv (\mathbf{V} \otimes \mathbf{V}^*)^*$, $(\boldsymbol{p}' \otimes \boldsymbol{v}') : (\boldsymbol{v} \otimes \boldsymbol{p}) \equiv (\boldsymbol{p}' \cdot \boldsymbol{v})(\boldsymbol{v}' \cdot \boldsymbol{p})$.

It is not hard to see that in other words this reads

$$\mathrm{Lin}(\mathbf{V}^*) \equiv (\mathrm{Lin}(\mathbf{V}))^*, \qquad \boldsymbol{B} : \boldsymbol{L} \equiv \mathrm{Tr}(\boldsymbol{B}^* \boldsymbol{L}),$$

where $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})$, $\boldsymbol{B} \in \mathrm{Lin}(\mathbf{V}^*)$ and so $\boldsymbol{B}^* \in \mathrm{Lin}(\mathbf{V})$.

Since a single dot means the composition of linear maps, we denoted the bilinear map of duality by the symbol $:$ to avoid misunderstandings.

3.11. In accordance with our results we have

$$\mathbb{K}^M \otimes \mathbb{K}^N \equiv \mathrm{Lin}\left((\mathbb{K}^N)^*, \mathbb{K}^M\right).$$

By definition, for $\boldsymbol{y} = (y^i) \in \mathbb{K}^M$ and $\boldsymbol{x} = (x^k) \in \mathbb{K}^N$,

$$\boldsymbol{y} \otimes \boldsymbol{x} : (\mathbb{K}^N)^* \to \mathbb{K}^M, \qquad \boldsymbol{p} \mapsto (\boldsymbol{p} \cdot \boldsymbol{x})\boldsymbol{y},$$

from which we deduce that

$$(\boldsymbol{y} \otimes \boldsymbol{x})^{ik} = y^i x^k \qquad (i = 1, \dots, M,\ k = 1, \dots, N).$$

Moreover, $\mathbb{K}^N \otimes (\mathbb{K}^N)^* \equiv \mathrm{Lin}(\mathbb{K}^N, \mathbb{K}^N)$, $(\boldsymbol{x} \otimes \boldsymbol{p})^i{}_k = x^i p_k$, and so

$$\mathrm{Tr}\left(L^i{}_k \,|\, i, k = 1, \dots, N\right) = \sum_{i=1}^{N} L^i{}_i.$$

Our convention that a summation can be carried out only for a pair of indices in opposite positions shows well that the trace of matrices $\left(L^{ik}\right)$ and (L_{ik}) makes no sense.

It can be proved without difficulty that

$$\left(B^j{}_i\right) : \left(L_k{}^l\right) = \sum_{i,k=1}^{N} B^k{}_i L_k{}^i.$$

3.12. Let $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{U}, \mathbf{X})$ and $\boldsymbol{K} \in \mathrm{Lin}(\mathbf{V}, \mathbf{Y})$. Then $\mathbf{U} \times \mathbf{V} \to \mathbf{X} \otimes \mathbf{Y}$, $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{L}\boldsymbol{u} \otimes \boldsymbol{K}\boldsymbol{v}$ is a bilinear map, hence there exists a unique linear map $\boldsymbol{L} \otimes \boldsymbol{K} : \mathbf{U} \otimes \mathbf{V} \to \mathbf{X} \otimes \mathbf{Y}$ such that

$$(\boldsymbol{L} \otimes \boldsymbol{K}) \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{L}\boldsymbol{u} \otimes \boldsymbol{K}\boldsymbol{v} \qquad (\boldsymbol{u} \in \mathbf{U},\ \boldsymbol{v} \in \mathbf{V}).$$

It is a simple task to show that $(\boldsymbol{L}, \boldsymbol{K}) \mapsto \boldsymbol{L} \otimes \boldsymbol{K}$ satisfies condition (*ii*) in 3.1, hence $\boldsymbol{L} \otimes \boldsymbol{K}$ is the tensor product of $\boldsymbol{L}$ and $\boldsymbol{K}$, in other words,

$$\mathrm{Lin}(\mathbf{U}, \mathbf{X}) \otimes \mathrm{Lin}(\mathbf{V}, \mathbf{Y}) \subset\!\!\!\to \mathrm{Lin}(\mathbf{U} \otimes \mathbf{V}, \mathbf{X} \otimes \mathbf{Y}).$$

If the vector spaces are finite dimensional then $\equiv$ stands instead of $\subset\!\!\!\to$.

It is not hard to show that

$$(\boldsymbol{L} \otimes \boldsymbol{K}) \cdot (\boldsymbol{B} \otimes \boldsymbol{A}) = (\boldsymbol{L} \cdot \boldsymbol{B}) \otimes (\boldsymbol{K} \cdot \boldsymbol{A})$$

and if both $\boldsymbol{L}$ and $\boldsymbol{K}$ are bijections then $\boldsymbol{L} \otimes \boldsymbol{K}$ is a bijection and

$$(\boldsymbol{L} \otimes \boldsymbol{K})^{-1} = \boldsymbol{L}^{-1} \otimes \boldsymbol{K}^{-1}.$$

3.13. For natural numbers $n \geq 2$ the definition of n-fold tensor products of vector spaces is similar to definition in 3.1, only n-fold linear maps should be taken instead of bilinear ones. We can state the existence and essential uniqueness of n-fold tensor products similarly. We use the notation $\overset{n}{\underset{k=1}{\otimes}} \mathbf{V}_k$ and $\overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_k$ for the n-fold tensor product of vector spaces $\mathbf{V}_k$ and vectors $\boldsymbol{v}_k \in \mathbf{V}_k$ $(k = 1, \dots, n)$.

We have the identifications

$$\left(\overset{m}{\underset{k=1}{\otimes}} \mathbf{V}_k\right) \otimes \left(\overset{n}{\underset{k=m+1}{\otimes}} \mathbf{V}_k\right) \equiv \overset{n}{\underset{k=1}{\otimes}} \mathbf{V}_k,$$

$$\left(\overset{m}{\underset{k=1}{\otimes}} \boldsymbol{v}_k\right) \otimes \left(\overset{n}{\underset{k=m+1}{\otimes}} \boldsymbol{v}_k\right) \equiv \overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_k.$$

If the vector spaces are finite dimensional then $\overset{n}{\underset{k=1}{\otimes}} \mathbf{V}_k$ is identified with the vector space $\mathrm{Lin}^n(\overset{n}{\underset{k=1}{\times}} \mathbf{V}_k{}^*, \mathbb{K})$ of n-linear maps $\overset{n}{\underset{k=1}{\times}} \mathbf{V}_k{}^* \to \mathbb{K}$, called *$n$-linear forms,* such that

$$\left(\overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_k\right)(\boldsymbol{p}^1, \dots, \boldsymbol{p}^n) \equiv \prod_{k=1}^{n} (\boldsymbol{p}^k \cdot \boldsymbol{v}_k).$$

3.14. For natural numbers $n \geq 2$, the n-fold tensor product of n copies of the vector space $\mathbf{V}$ is denoted by $\overset{n}{\otimes}\,\mathbf{V}$; for convenience we put $\overset{1}{\otimes}\,\mathbf{V} := \mathbf{V}$, $\overset{0}{\otimes}\,\mathbf{V} := \mathbb{K}$. Then we have for all natural numbers n and m

$$\left(\overset{n}{\otimes}\,\mathbf{V}\right) \otimes \left(\overset{m}{\otimes}\,\mathbf{V}\right) \equiv \overset{n+m}{\otimes}\mathbf{V}.$$

We define the n-fold *symmetric* and *antisymmetric* tensor products of elements of $\mathbf{V}$ as follows:

$$\overset{n}{\underset{k=1}{\vee}} \boldsymbol{v}_k := \sum_{\pi \in \mathrm{Perm}_n} \overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_{\pi(k)},$$
$$\overset{n}{\underset{k=1}{\wedge}} \boldsymbol{v}_k := \sum_{\pi \in \mathrm{Perm}_n} (\mathrm{sign}\pi) \overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_{\pi(k)},$$

where Perm_n denotes the set of permutations of $\{1, \dots, n\}$ and $\mathrm{sign}\pi$ is the sign of the permutation $\pi :$ $\mathrm{sign}\pi = 1$ if π is even and $\mathrm{sign}\pi = -1$ if π is odd.

For instance,

$$\boldsymbol{v}_1 \vee \boldsymbol{v}_2 = \boldsymbol{v}_1 \otimes \boldsymbol{v}_2 + \boldsymbol{v}_2 \otimes \boldsymbol{v}_1, \qquad \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 = \boldsymbol{v}_1 \otimes \boldsymbol{v}_2 - \boldsymbol{v}_2 \otimes \boldsymbol{v}_1.$$

The linear subspaces of $\overset{n}{\otimes}\,\mathbf{V}$ spanned by the symmetric and antisymmetric tensor products are denoted by $\overset{n}{\vee}\,\mathbf{V}$ and $\overset{n}{\wedge}\,\mathbf{V}$, respectively.

We mention that

$$\frac{1}{n!} \overset{n}{\underset{k=1}{\vee}} \boldsymbol{v}_k \qquad \text{and} \qquad \frac{1}{n!} \overset{n}{\underset{k=1}{\wedge}} \boldsymbol{v}_k$$

are called the *symmetric* and *antisymmetric part* of $\overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_k$, respectively. It is worth mentioning that the intersection of $\overset{n}{\wedge}\mathbf{V}$ and $\overset{n}{\vee}\mathbf{V}$ is the zero subspace; moreover, for $n = 2$ the subspace of antisymmetric tensor products and that of symmetric tensor products span $\mathbf{V}\otimes\mathbf{V}$.

3.15. Let $\mathbf{V}$ be finite dimensional, $\dim\mathbf{V} = N$. Then $\mathbf{V}^{**} \equiv \mathbf{V}$, and we have the following identifications:

$$
\begin{aligned}
&\overset{n}{\otimes}\mathbf{V} \equiv \{n\text{-linear forms on } \mathbf{V}^*\}, \qquad \overset{n}{\otimes}\mathbf{V}^* \equiv \{n\text{-linear forms on } \mathbf{V}\},\\
&\overset{n}{\vee}\mathbf{V} \equiv \{\text{ symmetric } n\text{-linear forms on } \mathbf{V}^*\},\\
&\overset{n}{\vee}\mathbf{V}^* \equiv \{\text{ symmetric } n\text{-linear forms on } \mathbf{V}\},\\
&\overset{n}{\wedge}\mathbf{V} \equiv \{\text{ antisymmetric } n\text{-linear forms on } \mathbf{V}^*\},\\
&\overset{n}{\wedge}\mathbf{V}^* \equiv \{\text{ antisymmetric } n\text{-linear forms on } \mathbf{V}\}.
\end{aligned}
$$

It is worth mentioning that

$$
\begin{aligned}
&\left(\overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_k\right)(\boldsymbol{p}^1,\dots,\boldsymbol{p}^n) = \left(\overset{n}{\underset{k=1}{\otimes}} \boldsymbol{p}^k\right)(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n) = \prod_{k=1}^{n}(\boldsymbol{p}^k\cdot\boldsymbol{v}_k),\\
&\left(\overset{n}{\underset{k=1}{\vee}} \boldsymbol{v}_k\right)(\boldsymbol{p}^1,\dots,\boldsymbol{p}^n) = \left(\overset{n}{\underset{k=1}{\vee}} \boldsymbol{p}^k\right)(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)\\
&\qquad\qquad\qquad = \sum_{\pi\in\mathrm{Perm}_n}\prod_{k=1}^{n}(\boldsymbol{p}^{\pi(k)}\cdot\boldsymbol{v}_k),\\
&\left(\overset{n}{\underset{k=1}{\wedge}} \boldsymbol{v}_k\right)(\boldsymbol{p}^1,\dots,\boldsymbol{p}^n) = \left(\overset{n}{\underset{k=1}{\wedge}} \boldsymbol{p}^k\right)(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n) =\\
&\qquad\qquad\qquad = \sum_{\pi\in\mathrm{Perm}_n}\mathrm{sign}\pi\prod_{k=1}^{n}(\boldsymbol{p}^{\pi(k)}\cdot\boldsymbol{v}_k),
\end{aligned}
$$

for all $\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\in\mathbf{V}$ and $\boldsymbol{p}^1,\dots,\boldsymbol{p}^n\in\mathbf{V}^*$.

If $\{\boldsymbol{v}_i \mid i = 1,\dots,N\}$ is a basis in $\mathbf{V}$ then

$$
\begin{aligned}
&\left\{\overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_{i_k} \,\middle|\, 1\le i_k\le N,\ \ k=1,\dots,n\right\},\\
&\left\{\overset{n}{\underset{k=1}{\vee}} \boldsymbol{v}_{i_k} \,\middle|\, 1\le i_1\le i_2\le\cdots\le i_n\le N\right\},\\
&\left\{\overset{n}{\underset{k=1}{\wedge}} \boldsymbol{v}_{i_k} \,\middle|\, 1\le i_1< i_2<\cdots< i_n\le N\right\}
\end{aligned}
$$

are bases in $\overset{n}{\otimes}\mathbf{V}$, $\overset{n}{\vee}\mathbf{V}$ and $\overset{n}{\wedge}\mathbf{V}$, respectively. Accordingly,

$$\dim\left(\overset{n}{\otimes}\mathbf{V}\right) = N^n, \qquad \dim\left(\overset{n}{\vee}\mathbf{V}\right) = \binom{N+n-1}{n},$$

$$\dim\left(\overset{n}{\wedge}\mathbf{V}\right) = \begin{cases} \binom{N}{n} \text{ if } n \le N \\ 0 \text{ if } n > 0 \end{cases}$$

Similar statements are true for $\mathbf{V}^*$ instead of $\mathbf{V}$.

3.16. The reader is asked to demonstrate that for $n = 2$ the notions of symmetricity, symmetric part, etc. coincide with those introduced earlier. Moreover, using the formulae in 3.7 we have

$$(\boldsymbol{u}_1, \boldsymbol{u}_2) \wedge (\boldsymbol{v}_1, \boldsymbol{v}_2) = \begin{pmatrix} \boldsymbol{u}_1 \wedge \boldsymbol{v}_1 & \boldsymbol{u}_1 \otimes \boldsymbol{v}_2 - \boldsymbol{v}_1 \otimes \boldsymbol{u}_2 \\ \boldsymbol{u}_2 \otimes \boldsymbol{v}_1 - \boldsymbol{v}_2 \otimes \boldsymbol{u}_1 & \boldsymbol{u}_2 \wedge \boldsymbol{v}_2 \end{pmatrix}$$

for $\boldsymbol{v}_1, \boldsymbol{u}_1 \in \boldsymbol{V}_1$, $\boldsymbol{v}_2, \boldsymbol{u}_2 \in \mathbf{V}_2$, and a similar equality holds for symmetric tensor products, too.

3.17. We have the following identifications:

$$\begin{aligned} \overset{n}{\otimes}\mathbf{V}^* &\equiv \left(\overset{n}{\otimes}\mathbf{V}\right)^*, \quad \left(\overset{n}{\underset{k=1}{\otimes}}\boldsymbol{p}^k\right)\cdot\left(\overset{n}{\underset{k=1}{\otimes}}\boldsymbol{v}_k\right) \equiv \left(\overset{n}{\underset{k=1}{\otimes}}\boldsymbol{p}^k\right)(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n), \\ \overset{n}{\vee}\mathbf{V}^* &\equiv \left(\overset{n}{\vee}\mathbf{V}\right)^*, \quad \left(\overset{n}{\underset{k=1}{\vee}}\boldsymbol{p}^k\right)\cdot\left(\overset{n}{\underset{k=1}{\vee}}\boldsymbol{v}_k\right) \equiv \left(\overset{n}{\underset{k=1}{\vee}}\boldsymbol{p}^k\right)(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n), \\ \overset{n}{\wedge}\mathbf{V}^* &\equiv \left(\overset{n}{\wedge}\mathbf{V}\right)^*, \quad \left(\overset{n}{\underset{k=1}{\wedge}}\boldsymbol{p}^k\right)\cdot\left(\overset{n}{\underset{k=1}{\wedge}}\boldsymbol{v}_k\right) \equiv \left(\overset{n}{\underset{k=1}{\wedge}}\boldsymbol{p}^k\right)(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n). \end{aligned}$$

3.18. Let $\mathbf{V}$ be an N-dimensional vector space. If $\mathbf{d}$ is an n-linear (symmetric, antisymmetric) form on $\mathbf{V}$ (i.e. $\mathbf{d}$ is an element of $\overset{n}{\otimes}\mathbf{V}^*$) and $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})$, then

$$\mathbf{d} \circ \left(\overset{n}{\times}\boldsymbol{L}\right) : \mathbf{V}^n \to \mathbb{K}, \qquad (\boldsymbol{v}_1,\dots,\boldsymbol{v}_n) \mapsto \mathbf{d}(\boldsymbol{L}\cdot\boldsymbol{v}_1,\dots,\boldsymbol{L}\cdot\boldsymbol{v}_n)$$

is also an n-linear (symmetric, antisymmetric) form.

Since $\overset{N}{\wedge}\mathbf{V}^*$, the vector space of antisymmetric N-linear forms is one-dimensional, for $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})$ there is a number (an element of $\mathbb{K}$) $\det\boldsymbol{L}$, called the *determinant* of $\boldsymbol{L}$, such that

$$\mathbf{c} \circ \left(\overset{N}{\times}\boldsymbol{L}\right) = (\det\boldsymbol{L})\mathbf{c}$$

for all $\mathbf{c} \in \overset{N}{\wedge}\mathbf{V}^*$.

Proposition. For all $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_N$ in $\mathbf{V}$ we have

$$\mathop{\wedge}_{k=1}^{N} \boldsymbol{L} \cdot \boldsymbol{v}_k = (\det \boldsymbol{L}) \mathop{\wedge}_{k=1}^{N} \boldsymbol{v}_k.$$

Proof. $\mathop{\wedge}_{k=1}^{N} \boldsymbol{L} \cdot \boldsymbol{v}_k$ is an antisymmetric N-linear form on $\mathbf{V}^*$; 3.15 yields that for all $\boldsymbol{p}^1, \dots, \boldsymbol{p}^N \in \mathbf{V}^*$

$$\begin{aligned}\left(\mathop{\wedge}_{k=1}^{N} \boldsymbol{L} \cdot \boldsymbol{v}_k\right)(\boldsymbol{p}^1, \dots, \boldsymbol{p}^N) &= \left(\mathop{\wedge}_{k=1}^{N} \boldsymbol{p}^k\right)(\boldsymbol{L} \cdot \boldsymbol{v}_1, \dots, \boldsymbol{L} \cdot \boldsymbol{v}_N) = \\ =&(\det \boldsymbol{L}) \left(\mathop{\wedge}_{k=1}^{N} \boldsymbol{p}^k\right)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_N) = (\det \boldsymbol{L}) \left(\mathop{\wedge}_{k=1}^{N} \boldsymbol{v}_k\right)(\boldsymbol{p}^1, \dots, \boldsymbol{p}^N). \quad \blacksquare\end{aligned}$$

As a consequence, we have for $\boldsymbol{L}, \boldsymbol{K} \in \mathrm{Lin}(\mathbf{V})$ that

$$\det(\boldsymbol{L} \cdot \boldsymbol{K}) = (\det \boldsymbol{L})(\det \boldsymbol{K}) = \det(\boldsymbol{K} \cdot \boldsymbol{L}).$$

3.19. Let $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_N)$ be an ordered basis of $\mathbf{V}$ and let $(\boldsymbol{p}^1, \dots, \boldsymbol{p}^N)$ be the corresponding dual basis in $\mathbf{V}^*$.

We know that $\left(\mathop{\wedge}_{i=1}^{N} \boldsymbol{p}^i\right)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_N) = 1$, thus if $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})$ then

$$\begin{aligned}\det \boldsymbol{L} =&(\det \boldsymbol{L}) \left(\mathop{\wedge}_{i=1}^{N} \boldsymbol{p}^i\right)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_N) = \left(\mathop{\wedge}_{i=1}^{N} \boldsymbol{p}^i\right)(\boldsymbol{L} \cdot \boldsymbol{v}_1, \dots, \boldsymbol{L} \cdot \boldsymbol{v}_N) = \\ =& \sum_{\pi \in Perm_N} \mathrm{sign}\pi \prod_{i=1}^{N} (\boldsymbol{p}^{\pi(i)} \cdot \boldsymbol{L} \cdot \boldsymbol{v}_i).\end{aligned}$$

The last formula is the determinant of the matrix representing $\boldsymbol{L}$ in the coordinatization corresponding to the given ordered basis. Thus for all coordinatizations $\boldsymbol{K}$ of $\mathbf{V}$ we have

$$\det \boldsymbol{L} = \det(\boldsymbol{K} \cdot \boldsymbol{L} \cdot \boldsymbol{K}^{-1}).$$

3.20. Proposition. Let $\mathbf{V}$ and $\mathbf{U}$ be finite dimensional vector spaces. Suppose $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}(\mathbf{V}, \mathbf{U})$ and $\boldsymbol{B}$ is a bijection. Then

$$\det(\boldsymbol{A} \cdot \boldsymbol{B}^{-1}) = \det(\boldsymbol{B}^{-1} \cdot \boldsymbol{A}).$$

Proof. Observe that if $\mathbf{U} = \mathbf{V}$ then this equality follows from that given at the end of 3.18. However, if $\mathbf{U} \neq \mathbf{V}$, the determinant of $\mathbf{A}$ and $\boldsymbol{B}^{-1}$ make no sense.

$\mathbf{V}$ and $\mathbf{U}$ have the same dimension N since $\boldsymbol{B}$ is a bijection between them. Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be coordinatizations of $\mathbf{V}$ and $\mathbf{U}$, respectively. Then

$$\det(\mathbf{A} \cdot \mathbf{B}^{-1}) = \det(\boldsymbol{L} \cdot \boldsymbol{A} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{L}^{-1}).$$

Since $\boldsymbol{L} \cdot \boldsymbol{A} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{L}^{-1} = \boldsymbol{L} \cdot \boldsymbol{A} \cdot \boldsymbol{K}^{-1} \cdot \boldsymbol{K} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{L}^{-1}$ and both $\boldsymbol{L} \cdot \boldsymbol{A} \cdot \boldsymbol{K}^{-1}$ and $\boldsymbol{K} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{L}^{-1}$ are linear maps $\mathbb{K}^N \to \mathbb{K}^N$, hence their determinant is meaningful, we can apply the formula given at the end of 3.18 to get

$$\begin{aligned}\det(\boldsymbol{L} \cdot \boldsymbol{A} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{L}^{-1}) &= \det\left((\boldsymbol{L} \cdot \boldsymbol{A} \cdot \boldsymbol{K}^{-1}) \cdot (\boldsymbol{K} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{L}^{-1})\right) = \\ &= \det\left((\boldsymbol{K} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{L}^{-1}) \cdot (\boldsymbol{L} \cdot \boldsymbol{A} \cdot \boldsymbol{K}^{-1})\right) = \det(\boldsymbol{K} \cdot \boldsymbol{B}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{K}^{-1}) \\ &= \det(\boldsymbol{B}^{-1} \cdot \boldsymbol{A}). \qquad \blacksquare\end{aligned}$$

Our result has the following corollary: if $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})$ and $\boldsymbol{B} : \mathbf{V} \to \boldsymbol{U}$ is a linear bijection then

$$\det(\boldsymbol{B} \cdot \boldsymbol{L} \cdot \boldsymbol{B}^{-1}) = \det \boldsymbol{L}.$$

3.21. For $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})$ we define

$$\overset{0}{\otimes} \boldsymbol{L} := \mathrm{id}_{\mathbb{K}},$$

$$\overset{n}{\otimes} \boldsymbol{L} : \overset{n}{\otimes} \mathbf{V} \to \overset{n}{\otimes} \mathbf{V}, \qquad \overset{n}{\underset{k=1}{\otimes}} \boldsymbol{v}_k \mapsto \overset{n}{\underset{k=1}{\otimes}} \boldsymbol{L} \cdot \boldsymbol{v}_k.$$

It is trivial that $\overset{n}{\wedge} \mathbf{V}$ and $\overset{n}{\vee} \mathbf{V}$ are invariant for $\overset{n}{\otimes} \boldsymbol{L}$; the restrictions of $\overset{n}{\otimes} \boldsymbol{L}$ onto these linear subspaces will be denoted by $\overset{n}{\wedge} \boldsymbol{L}$ and $\overset{n}{\vee} \boldsymbol{L}$, respectively.

3.22. Exercises

1. Let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_N\}$ be a basis of $\mathbf{V}$ and $\{\boldsymbol{p}^1, \dots, \boldsymbol{p}^N\}$ its dual. Then

$$\sum_{i=1}^{N} \boldsymbol{v}_i \otimes \boldsymbol{p}^i \equiv \mathrm{id}_{\mathbf{V}}, \qquad \sum_{i=1}^{N} \boldsymbol{p}^i \otimes \boldsymbol{v}_i \equiv \mathrm{id}_{\mathbf{V}^*},$$

where the symbols on the right-hand sides stand for the identity of $\mathbf{V}$ and of $\mathbf{V}^*$, respectively.

2. The linear subspaces $\mathbf{S}$ and $\mathbf{T}$ of $\mathbf{V}$ are *complementary* if $\mathbf{S} \cap \mathbf{T} = \{0\}$ and the linear subspace spanned by $\mathbf{S} \cup \mathbf{T}$ equals $\mathbf{V}$; then for every $\boldsymbol{v}$ there are

uniquely determined elements $\boldsymbol{v}_{\mathbf{S}} \in \mathbf{S}$ and $\boldsymbol{v}_{\mathbf{T}} \in \mathbf{T}$ such that $\boldsymbol{v} = \boldsymbol{v}_{\mathbf{S}} + \boldsymbol{v}_{\mathbf{T}}$. The linear map $\mathbf{V} \to \mathbf{V}$, $\boldsymbol{v} \mapsto \boldsymbol{v}_{\mathbf{S}}$ is called the *projection onto* $\mathbf{S}$ *along* $\mathbf{T}$.

Let $\boldsymbol{v} \in \mathbf{V}$, $\boldsymbol{p} \in \mathbf{V}^*$.

(i) If $\boldsymbol{p} \cdot \boldsymbol{v} \neq 0$ then $\frac{\boldsymbol{v} \otimes \boldsymbol{p}}{\boldsymbol{p} \cdot \boldsymbol{v}}$ is the projection onto $\mathbb{K}\boldsymbol{v}$ (the linear subspace spanned by $\boldsymbol{v}$) along Ker $\boldsymbol{p}$.

(ii) If $\alpha \in \mathbb{K}$ such that $\alpha \boldsymbol{p} \cdot \boldsymbol{v} \neq 1$ then $\mathrm{id}_{\mathbf{V}} - \alpha \boldsymbol{v} \otimes \boldsymbol{p}$ is a linear bijection and

$$(\mathrm{id}_{\mathbf{V}} - \alpha \boldsymbol{v} \otimes \boldsymbol{p})^{-1} = \mathrm{id}_{\mathbf{V}} + \frac{\alpha}{1 - \alpha \boldsymbol{p} \cdot \boldsymbol{v}} \boldsymbol{v} \otimes \boldsymbol{p}.$$

3. Demonstrate that

$$\boldsymbol{L} \cdot (\boldsymbol{v} \otimes \boldsymbol{p}) = (\boldsymbol{L} \cdot \boldsymbol{v}) \otimes \boldsymbol{p}, \qquad (\boldsymbol{v} \otimes \boldsymbol{p}) \cdot \boldsymbol{L} = \boldsymbol{v} \otimes \boldsymbol{L}^* \cdot \boldsymbol{p}$$

for $\boldsymbol{v} \in \mathbf{V}$, $\boldsymbol{p} \in \mathbf{V}^*$ and $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V})$.

4. Prove that

$$\left(\bigwedge_{k=1}^{n} \boldsymbol{p}^k \right) (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \det \left(\boldsymbol{p}^k \cdot \boldsymbol{v}_i | \; k, i = 1, \ldots, n \right)$$

for $\boldsymbol{p}^1, \ldots, \boldsymbol{p}^n \in V^*$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbf{V}$.

5. Prove that if $\mathbf{V}$ is a vector space over $\mathbb{K}$ then $\mathbb{K}^N \otimes \mathbf{V} \equiv \mathbf{V}^N$, $\boldsymbol{\xi} \otimes \boldsymbol{v} \equiv (\xi^1 \boldsymbol{v}, \ldots, \xi^N \boldsymbol{v})$.

4. Tensor quotients

4.1. Let $\mathbf{U}, \mathbf{V}$ and $\mathbf{Z}$ be vector spaces (over the same field). A map $\mathbf{q} : \mathbf{V} \times (\mathbf{U} \setminus \{\mathbf{0}\}) \to \mathbf{Z}$ is called *linear quotient* if

(i) $\boldsymbol{v} \mapsto \mathbf{q}(\boldsymbol{v}, \boldsymbol{u})$ is linear for all $\boldsymbol{u} \in \mathbf{U} \setminus \{\mathbf{0}\}$,

(ii) $\mathbf{q}(\boldsymbol{v}, \alpha \boldsymbol{u}) = \frac{1}{\alpha} \mathbf{q}(\boldsymbol{v}, \boldsymbol{u})$ for all $\boldsymbol{v} \in \mathbf{V}$ and $\boldsymbol{u} \in \mathbf{U} \setminus \{\mathbf{0}\}$, $\alpha \in \mathbb{K} \setminus \{\mathbf{0}\}$.

Definition. Let $\mathbf{V}$ and $\mathbf{A}$ be vector spaces, dim $\mathbf{A} = 1$. A *tensor quotient* of $\mathbf{V}$ by $\mathbf{A}$ is a pair $(\mathbf{Z}, \mathbf{q})$ where

(i) $\mathbf{Z}$ is a vector space,

(ii) $\mathbf{q} : \mathbf{V} \times (\mathbf{A} \setminus \{\mathbf{0}\}) \to \mathbf{Z}$ is a linear quotient map having the property that

– if $\mathbf{W}$ is a vector space and $\mathbf{r} : \mathbf{V} \times (\mathbf{A} \setminus \{\mathbf{0}\} \to \mathbf{W}$ is a linear quotient map

— then there exists a unique linear map $\boldsymbol{L} : \mathbf{Z} \to \mathbf{W}$ such that

$$\mathbf{r} = \boldsymbol{L} \circ \mathbf{q}. \quad \blacksquare$$

Proposition. The pair $(\mathbf{Z}, \mathbf{q})$ is a tensor quotient of $\mathbf{V}$ by $\mathbf{A}$ if and only if

1) $\mathbf{Z} = \mathrm{Ran}\ \mathbf{q}$,

2) if $\boldsymbol{v} \in \mathbf{V}$, $\boldsymbol{a} \in \mathbf{A} \setminus \{\mathbf{0}\}$ and $\mathbf{q}(\boldsymbol{v}, \boldsymbol{a}) = \mathbf{0}$ then $\boldsymbol{v} = \mathbf{0}$.

Proof. Since $\mathbf{A}$ is one-dimensional, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{A}$, $\boldsymbol{a} \neq \mathbf{0}$ let $\frac{\boldsymbol{b}}{\boldsymbol{a}}$ denote the number for which $\frac{\boldsymbol{b}}{\boldsymbol{a}} \boldsymbol{a} = \boldsymbol{b}$. Observe that if $\boldsymbol{b} \neq \mathbf{0}$ then $\frac{\boldsymbol{a}}{\boldsymbol{b}}$ is the inverse of $\frac{\boldsymbol{b}}{\boldsymbol{a}}$.

Condition 2) in the proposition is equivalent to the following one: if $\boldsymbol{v}, \boldsymbol{u} \in \mathbf{V}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{A} \setminus \{\mathbf{0}\}$ then $\mathbf{q}(\boldsymbol{v}, \boldsymbol{a}) = \mathbf{q}(\boldsymbol{u}, \boldsymbol{b})$ implies $\boldsymbol{v} = \frac{\boldsymbol{a}}{\boldsymbol{b}}\boldsymbol{u}$. Conversely, it is trivial, that if $\mathbf{r}$ is a linear quotient map and $\boldsymbol{v} = \frac{\boldsymbol{a}}{\boldsymbol{b}}\boldsymbol{u}$ then $\mathbf{r}(\boldsymbol{v}, \boldsymbol{a}) = \mathbf{r}(\boldsymbol{u}, \boldsymbol{b})$. Moreover, $\mathbf{r}(\boldsymbol{v}, \boldsymbol{a}) + \mathbf{r}(\boldsymbol{u}, \boldsymbol{b}) = \mathbf{r}\left(\frac{\boldsymbol{b}}{\boldsymbol{a}}\boldsymbol{v} + \boldsymbol{u}, \boldsymbol{b}\right)$.

Suppose 1) is fulfilled. Then every element of $\frac{\mathbf{V}}{\mathbf{A}}$ has the form $\boldsymbol{q}(\boldsymbol{v}, \boldsymbol{a})$. If 2) is valid as well and $\mathbf{r}$ is a linear quotient map then the formula

$$\boldsymbol{L} \cdot (\boldsymbol{q}(\boldsymbol{v}, \boldsymbol{a})) := \mathbf{r}(\boldsymbol{v}, \boldsymbol{a})$$

defines a unique linear map $\boldsymbol{L}$.

If 1) is not fulfilled, the uniqueness of linear maps $\boldsymbol{L}$ for which $\mathbf{r} = \boldsymbol{L} \circ \mathbf{q}$ holds fails. If 2) is not valid one can easily construct a linear quotient map for which no linear map exists with the desired composition property.

4.2. We shall see in the next item that tensor quotients exist. In the same way as in the case of tensor products, we can see that the tensor quotients of $\mathbf{V}$ by $\mathbf{A}$ are canonically isomorphic, that is why we speak of *the* tensor product and applying a customary abuse of language we call the corresponding vector space the tensor quotient ($\mathbf{Z}$ in the definition) denoting it by $\frac{\mathbf{V}}{\mathbf{A}}$ and writing

$$\mathbf{V} \times (\mathbf{A} \setminus \{\mathbf{0}\}) \to \frac{\mathbf{V}}{\mathbf{A}}, \qquad (\boldsymbol{v}, \boldsymbol{a}) \mapsto \frac{\boldsymbol{v}}{\boldsymbol{a}}$$

for the corresponding linear quotient map ($\mathbf{q}$ in the definition); $\frac{\boldsymbol{v}}{\boldsymbol{a}}$ is called the *tensor quotient* of $\boldsymbol{v}$ by $\boldsymbol{a}$.

We use the term *realization* and the symbol $\equiv$ in the same sense as in the case of tensor products.

It is worth repeating the preceding results in the new notation: every element of $\frac{\mathbf{V}}{\mathbf{A}}$ is of the form $\frac{\boldsymbol{v}}{\boldsymbol{a}}$ and $\frac{\boldsymbol{v}}{\boldsymbol{a}} = \mathbf{0}$ if and only if $\boldsymbol{v} = \mathbf{0}$.

4.3. For $\boldsymbol{v} \in \mathbf{V}$ and $\boldsymbol{a} \in \mathbf{A} \setminus \{\mathbf{0}\}$ we define the linear map

$$\frac{\boldsymbol{v}}{\boldsymbol{a}} : \mathbf{A} \to \mathbf{V}, \qquad \boldsymbol{b} \mapsto \frac{\boldsymbol{b}}{\boldsymbol{a}}\boldsymbol{v}$$

where $\frac{\boldsymbol{b}}{\boldsymbol{a}}$ is the number for which $\frac{\boldsymbol{b}}{\boldsymbol{a}}\boldsymbol{a} = \boldsymbol{b}$ holds.

Proposition. $\mathbf{V}\times\mathbf{A}\setminus\{\mathbf{0}\}\to \mathrm{Lin}(\mathbf{A},\mathbf{V})$ is a linear quotient map which satisfies conditions (i) and (ii) of proposition 4.1. As a consequence, $\frac{\boldsymbol{v}}{\boldsymbol{a}}$ is the tensor quotient of $\boldsymbol{v}$ by $\boldsymbol{a}$ (that is why we used in advance this notation) and $\frac{\mathbf{V}}{\mathbf{A}}\equiv \mathrm{Lin}(\mathbf{A},\mathbf{V})$. ■

We have $\mathrm{Lin}(\mathbf{A})\equiv\mathbb{K}$ where $\alpha\in\mathbb{K}$ is identified with the linear map $\boldsymbol{a}\mapsto\alpha\boldsymbol{a}$. Thus, according to the previous result, $\frac{\mathbf{A}}{\mathbf{A}}\equiv\mathbb{K}$ and $\frac{\boldsymbol{b}}{\boldsymbol{a}}$ is the number for which $\frac{\boldsymbol{b}}{\boldsymbol{a}}\boldsymbol{a}=\mathbf{b}$ holds, hence our notation in 4.1 used in the present proposition as well, is in accordance with the generally accepted notation for tensor quotients.

4.4. Since for all $\boldsymbol{a}\in\mathbf{A}\setminus\{\mathbf{0}\}$ the map $\mathbf{V}\to\frac{\mathbf{V}}{\mathbf{A}}$, $\boldsymbol{v}\mapsto\frac{\boldsymbol{v}}{\boldsymbol{a}}$ is a linear bijection, if $\{\boldsymbol{v}_i|\ i\in I\}$ is a basis in $\mathbf{V}$ then $\left\{\frac{\boldsymbol{v}_i}{\boldsymbol{a}}\middle|\ i\in I\right\}$ is a basis in $\frac{\mathbf{V}}{\mathbf{A}}$, and $\dim\frac{\mathbf{V}}{\mathbf{A}}=\dim\mathbf{V}$.

4.5. Let $\mathbf{V},\mathbf{U},\mathbf{A}$ and $\mathbf{B}$ be vector spaces, $\dim\mathbf{A}=\dim\mathbf{B}=1$. We have the following identifications (recall 3.4, 3.5 and 3.8):

(i)
$$\frac{\mathbb{K}}{\mathbf{A}}\equiv\mathrm{Lin}(\mathbf{A},\mathbb{K})=\mathbf{A}^*,\qquad \frac{\alpha}{\boldsymbol{a}}\cdot\boldsymbol{b}\equiv\alpha\frac{\boldsymbol{b}}{\boldsymbol{a}};$$

(ii)
$$\frac{\mathbf{V}}{\mathbb{K}}\equiv\mathrm{Lin}(\mathbb{K},\mathbf{V}))\equiv\mathbf{V},\qquad \frac{\boldsymbol{v}}{\alpha}\equiv\frac{1}{\alpha}\boldsymbol{v};$$

(iii)
$$\frac{\mathbf{V}}{\mathbf{A}}\equiv\mathrm{Lin}(\mathbf{A},\mathbf{V})\equiv\mathbf{V}\otimes\mathbf{A}^*,\qquad \frac{\boldsymbol{v}}{\boldsymbol{a}}\equiv\boldsymbol{v}\otimes\frac{1}{\boldsymbol{a}};$$

(iv)
$$\frac{\mathbf{V}^*}{\mathbf{A}^*}\equiv\left(\frac{\mathbf{V}}{\mathbf{A}}\right)^*,\qquad \frac{\boldsymbol{p}}{\boldsymbol{h}}\cdot\frac{\boldsymbol{v}}{\boldsymbol{a}}\equiv\frac{\boldsymbol{p}\cdot\boldsymbol{v}}{\boldsymbol{h}\boldsymbol{a}};$$

(v)
$$\frac{\left(\frac{\mathbf{V}}{\mathbf{A}}\right)}{\mathbf{B}}\equiv\frac{\mathbf{V}}{\mathbf{A}\otimes\mathbf{B}},\qquad \frac{\left(\frac{\boldsymbol{v}}{\boldsymbol{a}}\right)}{\mathbf{b}}\equiv\frac{\boldsymbol{v}}{\boldsymbol{a}\boldsymbol{b}};$$

(vi)
$$\frac{\mathbf{V}}{\mathbf{A}}\otimes\frac{\mathbf{U}}{\mathbf{B}}\equiv\frac{\mathbf{V}\otimes\mathbf{U}}{\mathbf{A}\otimes B}\equiv\frac{\mathbf{V}}{\mathbf{A}\otimes\mathbf{B}}\otimes\mathbf{U}\equiv etc.$$

$$\frac{\boldsymbol{v}}{\boldsymbol{a}}\otimes\frac{\boldsymbol{u}}{\boldsymbol{b}}\equiv\frac{\boldsymbol{v}\otimes\boldsymbol{u}}{\boldsymbol{a}\boldsymbol{b}}\equiv\frac{\boldsymbol{v}}{\boldsymbol{a}\boldsymbol{b}}\otimes\boldsymbol{u}\equiv etc.$$

In particular,

$$\mathbf{A} \otimes \frac{\mathbf{V}}{\mathbf{A}} \equiv \frac{\mathbf{A} \otimes \mathbf{V}}{\mathbf{A}} \equiv \mathbf{V}, \qquad \frac{\mathbf{B}}{\mathbf{A} \otimes \mathbf{B}} \equiv \frac{\mathbb{K}}{\mathbf{A}}.$$

$$(vii) \qquad \frac{\mathbf{V} \times \mathbf{U}}{\mathbf{A}} \equiv \frac{\mathbf{V}}{\mathbf{A}} \times \frac{\mathbf{U}}{\mathbf{A}}, \qquad \frac{(\boldsymbol{v}, \boldsymbol{u})}{\boldsymbol{a}} \equiv \left(\frac{\boldsymbol{v}}{\boldsymbol{a}}, \frac{\boldsymbol{u}}{\boldsymbol{a}}\right).$$

Note that according to (v) and (vi) the rules of tensorial multiplication and division coincide with those well known for numbers.

4.6. Let $\mathbf{V}, \mathbf{U}, \mathbf{A}$ and $\mathbf{B}$ be vector spaces, $\dim \mathbf{A} = \dim \mathbf{B} = 1$. If $\boldsymbol{L} \in \mathrm{Lin}(\mathbf{V}, \mathbf{U})$ and $\mathbf{0} \neq \boldsymbol{F} \in \mathrm{Lin}(\mathbf{A}, \mathbf{B})$ then $\mathbf{V} \times (\mathbf{A} \setminus \{\mathbf{0}\}) \to \frac{\mathbf{U}}{\mathbf{B}}$, $(\boldsymbol{v}, \boldsymbol{a}) \mapsto \frac{\boldsymbol{L} \cdot \boldsymbol{v}}{\boldsymbol{F} \cdot \boldsymbol{a}}$ is linear quotient, hence there exists a unique linear map $\frac{\boldsymbol{L}}{\boldsymbol{F}} : \frac{\mathbf{V}}{\mathbf{A}} \to \frac{\mathbf{U}}{\mathbf{B}}$ such that

$$\frac{\boldsymbol{L}}{\boldsymbol{F}} \cdot \frac{\boldsymbol{v}}{\boldsymbol{a}} = \frac{\boldsymbol{L} \cdot \boldsymbol{v}}{\boldsymbol{F} \cdot \boldsymbol{a}} \qquad (\boldsymbol{v} \in \boldsymbol{V},\ \boldsymbol{a} \in \mathbf{A} \setminus \{\mathbf{0}\}).$$

It is not hard to see that $\frac{\boldsymbol{L}}{\boldsymbol{F}}$ is really the quotient of $\boldsymbol{L}$ by $\boldsymbol{F}$, in other words,

$$\frac{\mathrm{Lin}(\mathbf{V}, \mathbf{U})}{\mathrm{Lin}(\mathbf{A}, \mathbf{B})} \equiv \mathrm{Lin}\left(\frac{\mathbf{V}}{\mathbf{A}}, \frac{\mathbf{U}}{\mathbf{B}}\right).$$

5. Tensorial operations and orientation

In this section $\mathbf{V}$ denotes an N-dimensional real vector space and $\mathbf{A}$ denotes a one-dimensional real vector space.

5.1. Recall that an element $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_N)$ of $\mathbf{V}^N$ is called an ordered basis of $\mathbf{V}$ if the set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_N\}$ is a basis in $\mathbf{V}$.

Definition. Two ordered bases $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_N)$ and $(\boldsymbol{v}'_1, \dots, \boldsymbol{v}'_N)$ of $\mathbf{V}$ are called *equally oriented* if the linear map defined by $\boldsymbol{v}_i \mapsto \boldsymbol{v}'_i$ $(i = 1, \dots, N)$ has positive determinant. An equivalence class of equally oriented bases is called an *orientation* of $\mathbf{V}$. $\mathbf{V}$ is *oriented* if an orientation of $\mathbf{V}$ is given; the bases in the chosen equivalence class are called *positively oriented,* the other ones are called *negatively ortiented.* (More precisely, an oriented vector space is a pair $(\mathbf{V}, \mathrm{o})$ where $\mathbf{V}$ is a vector space and o is one of the equivalence classes of bases.)

A linear bijection between oriented vector spaces is *orientation-preserving* or *orientation-reversing* if it sends positively oriented bases into positively oriented ones or into negatively oriented ones, respectively. ∎

It is worth mentioning that there are two equivalence classes of equally oriented bases.

Observe that the two bases in the definition are equally oriented if and only if $\overset{N}{\underset{i=1}{\wedge}} \boldsymbol{v}'_i$ is a positive multiple of $\overset{N}{\underset{i=1}{\wedge}} \boldsymbol{v}_i$ (see Proposition 3.18).

If $\mathbf{V}$ is oriented, we orient $\mathbf{V}^*$ by the dual of positively oriented bases of $\mathbf{V}$.

If $\mathbf{U}$ and $\mathbf{V}$ are oriented vector spaces, $\mathbf{U} \times \mathbf{V}$ is oriented by joining positively oriented bases; more closely, if $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_M)$ and $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_N)$ are positively oriented bases in $\mathbf{U}$ and in $\mathbf{V}$, respectively, then $((\boldsymbol{u}_1, \mathbf{0}), \dots, (\boldsymbol{u}_M, \mathbf{0})\ ,\ (\mathbf{0}, \boldsymbol{v}_1), \dots, (\mathbf{0}, \boldsymbol{v}_N))$ is defined to be a positively oriented basis in $\mathbf{U} \times \mathbf{V}$.

The reader is asked to verify that the orientation of the dual and the Cartesian products is welldefined.

5.2. Two bases $\boldsymbol{a}$ and $\boldsymbol{a}'$ of the one-dimensional vector space $\mathbf{A}$ are equally oriented if and only if $\boldsymbol{a}'$ is a positive multiple of $\boldsymbol{a}$, in other words, $\frac{\boldsymbol{a}'}{\boldsymbol{a}}$ is a positive number.

If $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_N)$ and $(\boldsymbol{v}'_1, \dots, \boldsymbol{v}'_N)$ are equally oriented ordered bases of $\mathbf{V}$, $\boldsymbol{a}$ and $\boldsymbol{a}'$ are equally oriented bases of $\mathbf{A}$, then $(\boldsymbol{a}\boldsymbol{v}_1, \dots, \boldsymbol{a}\boldsymbol{v}_N)$ and $(\boldsymbol{a}'\boldsymbol{v}'_1, \dots, \boldsymbol{a}'\boldsymbol{v}'_N)$ are equally oriented bases of $\mathbf{A} \otimes \mathbf{V}$. Indeed, according to our convention $\mathbf{A} \otimes \mathbf{V} \equiv \mathbf{V} \otimes \mathbf{A}$, we have $\overset{N}{\underset{i=1}{\wedge}} (\boldsymbol{a}'\boldsymbol{v}'_i) \equiv (\boldsymbol{a}')^N \overset{N}{\underset{i=1}{\wedge}} \boldsymbol{v}'_i$ which is evidently a positive multiple of $\boldsymbol{a}^N \overset{N}{\underset{i=1}{\wedge}} \boldsymbol{v}_i$.

As a consequence, an orientation of $\mathbf{V}$ and an orientation of $\mathbf{A}$ determine a unique orientation of $\mathbf{A} \otimes \mathbf{V}$; we consider $\mathbf{A} \otimes \mathbf{V}$ to be oriented by this orientation.

We can argue similarly to show that an orientation of $\mathbf{V}$ and an orientation of $\mathbf{A}$ determine a unique orientation of $\frac{\mathbf{V}}{\mathbf{A}}$; we take this orientation of the tensor quotient.

5.3. A non-zero element $\boldsymbol{a}$ of the oriented one-dimensional vector space $\mathbf{A}$ is called *positive,* in symbols $\mathbf{0} < \boldsymbol{a}$, if the corresponding basis is positively oriented.

Moreover, we write $\boldsymbol{a} \le \boldsymbol{b}$ if $\mathbf{0} \le \boldsymbol{b} - \boldsymbol{a}$. It is easily shown that in this way we defined a total ordering on $\mathbf{A}$ for which

(i) if $\boldsymbol{a} \le \boldsymbol{b}$ and $\boldsymbol{c} \le \boldsymbol{d}$ then $\boldsymbol{a} + \boldsymbol{c} \le \boldsymbol{b} + \boldsymbol{d}$,

(ii) if $\boldsymbol{a} \le \boldsymbol{b}$ and $\alpha \in \mathbb{R}^+$ then $\alpha\boldsymbol{a} \le \alpha\boldsymbol{b}$.

We introduce the notations

$$\mathbf{A}^+ := \{\boldsymbol{a} \in \mathbf{A} |\ \mathbf{0} < \boldsymbol{a}\}, \qquad \mathbf{A}^+_{\mathbf{0}} := \mathbf{A}^+ \cup \{\mathbf{0}\}.$$

Furthermore, the absolute value of $\boldsymbol{a} \in \mathbf{A}$ is

$$|\ \boldsymbol{a}|\ := \begin{cases} \boldsymbol{a} & \text{if } \boldsymbol{a} \in \mathbf{A}^+ \\ \mathbf{0} & \text{if } \boldsymbol{a} = \mathbf{0} \\ -\boldsymbol{a} & \text{if } \boldsymbol{a} \notin \mathbf{A}^+. \end{cases}$$

5.4. Even if $\mathbf{A}$ is not oriented, $\mathbf{A} \otimes \mathbf{A}$ has a "canonical" orientation in which the elements of the form $\boldsymbol{a} \otimes \boldsymbol{a}$ are positive. If $\mathbf{A}$ is oriented, the orientation of $\mathbf{A} \otimes \mathbf{A}$ induced by the orientation of $\mathbf{A}$ coincide with the canonical one. Then

$$\mathbf{A}_{\mathbf{0}}^{+} \to (\mathbf{A} \otimes \mathbf{A})_{\mathbf{0}}^{+}, \qquad \boldsymbol{a} \mapsto \boldsymbol{a} \otimes \boldsymbol{a} \tag{$*$}$$

is a bijection. Indeed, $\boldsymbol{a} \otimes \boldsymbol{a} = \mathbf{0}$ if and only if $\boldsymbol{a} = \mathbf{0}$. The elements of $(\mathbf{A} \otimes \mathbf{A})^{+}$ has the form $\boldsymbol{a} \otimes \boldsymbol{b}$ where $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{A}^{+}$; since $\mathbf{b} = \lambda \boldsymbol{a}$ for some positive number λ, we have $\boldsymbol{a} \otimes \boldsymbol{b} = \left(\sqrt{\lambda}\boldsymbol{a}\right) \otimes \left(\sqrt{\lambda}\boldsymbol{a}\right)$, i.e. the above mapping is surjective. If $\mathbf{0} \neq \boldsymbol{a} \otimes \boldsymbol{a} = \boldsymbol{b} \otimes \boldsymbol{b}$ then $\boldsymbol{a} \otimes \boldsymbol{a} = \lambda^2 \boldsymbol{a} \otimes \boldsymbol{a}$ which implies that $\lambda^2 = 1$, thus $\lambda = 1$, $\boldsymbol{a} = \boldsymbol{b}$: the mapping in question is injective.

In spite of our earlier agreement, in deducing the present result, we preferred not to omit the symbol of tensorial multiplication. However, in applications of the present result we keep our agreement; in particular, we write

$$\boldsymbol{a}^2 := \boldsymbol{a}\boldsymbol{a} \;\; (:= \boldsymbol{a} \otimes \boldsymbol{a}).$$

The inverse of the mapping $*$ is denoted by the symbol $\sqrt{}$ and is called the square root mapping.

Note that

$$\sqrt{\boldsymbol{a}^2} = |\,\boldsymbol{a}| \qquad (\boldsymbol{a} \in \mathbf{A}).$$

V. PSEUDO-EUCLIDEAN VECTOR SPACES

1. Pseudo-Euclidean vector spaces

1.1. Definition. A *pseudo-Euclidean* vector space is a triplet $(\mathbf{V}, \mathbf{B}, \mathbf{h})$ where

(i) $\mathbf{V}$ is a non-zero finite dimensional real vector space,
(ii) $\mathbf{B}$ is a one-dimensional real vector space,
(iii) $\mathbf{h} : \mathbf{V} \times \mathbf{V} \to \mathbf{B} \otimes \mathbf{B}$ is a non-degenerate symmetric bilinear map.

Remarks. (i) Non-degenerate means that if $\mathbf{h}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0}$ for all $\boldsymbol{x} \in \mathbf{V}$ then $\boldsymbol{y} = \mathbf{0}$.

(ii) $\mathbf{h}(\boldsymbol{x}, \boldsymbol{y})$ is often called the $\mathbf{h}$-*product* of $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{V}$. The elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of $\mathbf{V}$ are called $\mathbf{h}$-*orthogonal* if their $\mathbf{h}$-product is zero.

(iii) In mathematical literature one usually considers the case $\mathbf{B} = \mathbb{R}$, i.e. when — because of $\mathbb{R} \otimes \mathbb{R} \equiv \mathbb{R}$— the pseudo-Euclidean form $\mathbf{h}$ takes real values. Physical applications require the possibility $\mathbf{B} \neq \mathbb{R}$.

1.2. Definition. A basis $\{\boldsymbol{e}_i | \ i = 1, \dots, N\}$ of $\mathbf{V}$ is called $\mathbf{h}$-*orthogonal* if $\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_k) = \mathbf{0}$ for $i \neq k$.

An $\mathbf{h}$-orthogonal basis $\{\boldsymbol{e}_i | \ i = 1, \dots, N\}$ is *normed* to $\boldsymbol{a} \in \mathbf{B}$ if either $\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i) = \boldsymbol{a}^2$ or $\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i) = -\boldsymbol{a}^2$ for all i. If $\mathbf{B} = \mathbb{R}$, an $\mathbf{h}$-orthogonal basis normed to 1 is called $\mathbf{h}$-*orthonormal*. ■

Since $\mathbf{B} \otimes \mathbf{B}$ has a canonical orientation, it makes sense that $\mathbf{h}(\boldsymbol{x}, \boldsymbol{y})$ is negative or positive for $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{V}$.

We can argue like in the case of real-valued bilinear forms to have the following.

Proposition. $\mathbf{h}$-orthogonal bases in $\mathbf{V}$ exist and there is a non-negative integer $\neg(\mathbf{h})$ such that for every $\mathbf{h}$-orthogonal basis $\{\boldsymbol{e}_i | \ i = 1, \dots, N\}$

$$\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i) < \mathbf{0} \quad \text{for} \quad \neg(\mathbf{h}) \quad \text{indices} \quad i,$$
$$\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i) > \mathbf{0} \quad \text{for} \quad N - \neg(\mathbf{h}) \quad \text{indices} \quad i. \qquad ■$$

An $\mathbf{h}$-orthogonal basis can always be normed to an arbitrary $\mathbf{0} \neq \boldsymbol{a} \in \mathbf{B}$. Further on we deal with $\mathbf{h}$-orthogonal bases normed to an element of $\mathbf{B}$ and such a basis will be numbered so that $\mathbf{h}$ takes negative values on the first $\neg(\mathbf{h})$ elements, i.e.

$$\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i) = \alpha(i)\boldsymbol{a}^2,$$

$$\alpha(i) = \begin{cases} -1 & \text{if } i = 1, \dots, \neg(\mathbf{h}) \\ 1 & \text{if } i = \neg(\mathbf{h}) + 1, \dots, N. \end{cases}$$

We say that $\mathbf{h}$ is *positive definite* if $\mathbf{h}(\boldsymbol{x}, \boldsymbol{x}) > \mathbf{0}$ for all non-zero $\boldsymbol{x}$. $\mathbf{h}$ is positive definite if and only if $\neg(\mathbf{h}) = \mathbf{0}$.

1.3. An important property of pseudo-Euclidean vector spaces is that a natural correspondence exists between $\mathbf{V}^*$ and $\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}}$. Note that every element of $\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}}$ is of the form $\frac{\boldsymbol{y}}{\boldsymbol{ab}}$ where $\boldsymbol{y} \in \mathbf{V}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B} \setminus \{\mathbf{0}\}$. Take such an element of $\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}}$. Then

$$\mathbf{V} \to \mathbb{R}, \qquad \boldsymbol{x} \mapsto \frac{\mathbf{h}(\boldsymbol{y}, \boldsymbol{x})}{\boldsymbol{ab}}$$

is a linear map, i.e. an element of $\mathbf{V}^*$, which we write in the form $\frac{\mathbf{h}(\boldsymbol{y},\cdot)}{\boldsymbol{ab}}$.

Proposition. $\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}} \to \mathbf{V}^*$, $\frac{\boldsymbol{y}}{\boldsymbol{ab}} \mapsto \frac{\mathbf{h}(\boldsymbol{y},\cdot)}{\boldsymbol{ab}}$ is a linear bijection.

Proof. It is linear and injective because $\mathbf{h}$ is bilinear and non-degenerate, and surjective because the two vector spaces in question have the same dimension. ■

We find this linear bijection so natural that we use it for identifying the vector spaces:

$$\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}} \equiv \mathbf{V}^*, \qquad \frac{\boldsymbol{y}}{\boldsymbol{ab}} \equiv \frac{\mathbf{h}(\boldsymbol{y},\cdot)}{\boldsymbol{ab}}.$$

1.4. (i) In the above identification the dual of an $\mathbf{h}$-orthogonal basis $\{\boldsymbol{e}_i |\, i = 1, \dots, N\}$ becomes

$$\left\{ \frac{\boldsymbol{e}_i}{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i)} \,\Big|\, i = 1, \dots, N \right\}$$

which equals

$$\left\{ \frac{\alpha(i)\boldsymbol{e}_i}{\boldsymbol{a}^2} \,\Big|\, i = 1, \dots, N \right\}$$

if the $\mathbf{h}$-orthogonal basis is normed to $\boldsymbol{a}$.

As a consequence, for all $\boldsymbol{x} \in \mathbf{V}$ (see IV.1.1),

$$\boldsymbol{x} = \sum_{i=1}^{N} \frac{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{x})}{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i)} \boldsymbol{e}_i,$$

and $\boldsymbol{x} = \mathbf{0}$ if and only if $\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{x}) = \mathbf{0}$ for all $i = 1, \dots, N$.

(ii) If $\mathbf{V}$ is oriented then both $\mathbf{V}^*$ and $\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}}$ are oriented. The above identification is orientation-preserving if $\neg(\mathbf{h})$ is even and is orientation-reversing if $\neg(\mathbf{h})$ is odd.

1.5. Let us take a linear map $\boldsymbol{F} : \mathbf{V} \to \mathbf{V}$. As we know, its transpose is a linear map $\boldsymbol{F}^* : \mathbf{V}^* \to \mathbf{V}^*$; according to the previous identification we can consider it to be a linear map $\boldsymbol{F}^* : \frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}} \to \frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}}$. Consequently, we can define the $\mathbf{h}$-adjoint of $\boldsymbol{F}$,

$$\boldsymbol{F}^{\circledast} : \mathbf{V} \to \mathbf{V}, \qquad \boldsymbol{y} \mapsto (\boldsymbol{ab})\left(\boldsymbol{F}^* \cdot \frac{\boldsymbol{y}}{\boldsymbol{ab}}\right).$$

Observe that this is equivalent to

$$\frac{\boldsymbol{F}^{\circledast}\cdot\boldsymbol{y}}{\boldsymbol{ab}} = \boldsymbol{F}^*\cdot\frac{\boldsymbol{y}}{\boldsymbol{ab}} \qquad (\boldsymbol{y}\in\mathbf{V},\ \boldsymbol{a},\boldsymbol{b}\in\mathbf{B}\setminus\{\mathbf{0}\}).$$

According to the definition of the transpose we have

$$\frac{\boldsymbol{y}}{\boldsymbol{ab}}\cdot\boldsymbol{F}\cdot\boldsymbol{x} = \left(\boldsymbol{F}^*\cdot\frac{\boldsymbol{y}}{\boldsymbol{ab}}\right)\cdot\boldsymbol{x} = \frac{\boldsymbol{F}^{\circledast}\cdot\boldsymbol{y}}{\boldsymbol{ab}}\cdot\boldsymbol{x},$$

which means

$$\frac{\mathbf{h}(\boldsymbol{y},\boldsymbol{F}\cdot\boldsymbol{x})}{\boldsymbol{ab}} = \frac{\mathbf{h}(\boldsymbol{F}^{\circledast}\cdot\boldsymbol{y},\boldsymbol{x})}{\boldsymbol{ab}},$$

i.e.

$$\mathbf{h}(\boldsymbol{y},\boldsymbol{F}\cdot\boldsymbol{x}) = \mathbf{h}(\boldsymbol{F}^{\circledast}\cdot\boldsymbol{y},\boldsymbol{x}) = \mathbf{h}(\boldsymbol{x},\boldsymbol{F}^{\circledast}\cdot\boldsymbol{y}) \qquad (\boldsymbol{x},\boldsymbol{y}\in\mathbf{V}).$$

The definition of $\mathbf{h}$-adjoints involves that the formulae in IV. 1.4 remain valid for $\mathbf{h}$-adjoints as well: if $\boldsymbol{F},\boldsymbol{G}\in\mathrm{Lin}(\mathbf{V})$, $\alpha\in\mathbb{R}$, then

$$\begin{aligned}(\boldsymbol{F}+\boldsymbol{G})^{\circledast} =& \boldsymbol{F}^{\circledast}+\boldsymbol{G}^{\circledast},\\ (\alpha\boldsymbol{F})^{\circledast} =& \alpha\boldsymbol{F}^{\circledast},\\ (\boldsymbol{F}\cdot\boldsymbol{G})^{\circledast} =& \boldsymbol{G}^{\circledast}\cdot\boldsymbol{F}^{\circledast}.\end{aligned}$$

Moreover,

$$\det\boldsymbol{F}^{\circledast} = \det\boldsymbol{F}.$$

1.6. Let $(\mathbf{V},\mathbf{B},\mathbf{h})$ and $(\mathbf{V}',\mathbf{B},'\mathbf{h}')$ be pseudo-Euclidean vector spaces. A linear map $\boldsymbol{L} : \mathbf{V}\to\mathbf{V}'$ is called $\mathbf{h}$-$\mathbf{h}'$-orthogonal if there is a linear bijection $\boldsymbol{Z} : \mathbf{B}\to\mathbf{B}'$ such that $\mathbf{h}'\circ(\boldsymbol{L}\times\boldsymbol{L}) = (\boldsymbol{Z}\otimes\boldsymbol{Z})\circ\mathbf{h}$ i.e.

$$\mathbf{h}'(\boldsymbol{L}\cdot\boldsymbol{x},\boldsymbol{L}\cdot\boldsymbol{y}) = (\boldsymbol{Z}\otimes\boldsymbol{Z})\mathbf{h}(\boldsymbol{x},\boldsymbol{y}) \qquad (\boldsymbol{x},\boldsymbol{y}\in\mathbf{V}).$$

Note that according to our identification, $\boldsymbol{Z}$ is an element of $\frac{\mathbf{B}'}{\mathbf{B}}$.

It is quite trivial that there is a $\mathbf{h}$-$\mathbf{h}'$-orthogonal linear map between the pseudo-Euclidean vector spaces if and only if $\dim \mathbf{V} = \dim \mathbf{V}'$ and $\neg(\mathbf{h}) = \neg(\mathbf{h}')$.

In particular, if $\{\boldsymbol{e}_i |\ i = 1, \dots, N\}$ is an $\mathbf{h}$-orthogonal basis, normed to $\boldsymbol{a}$, of $\mathbf{V}$, and $\{\boldsymbol{e}'_i |\ i = 1, \dots, N\}$ is an $\mathbf{h}'$-orthogonal basis, normed to $\boldsymbol{a}'$, of $\mathbf{V}'$ then

$$\boldsymbol{L} \cdot \boldsymbol{e}_i := \boldsymbol{e}'_i \qquad (i = 1, \dots, N)$$

determine an $\mathbf{h}$-$\mathbf{h}'$-orthogonal map for which $\boldsymbol{Z} = \frac{\boldsymbol{a}'}{\boldsymbol{a}}$.

1.7. Let n and N be natural numbers, $N \geq 1$, $n \leq N$. The map

$$\mathrm{H}_n : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \qquad (\boldsymbol{x}, \boldsymbol{y}) \mapsto -\sum_{i=1}^{n} x^i y^i + \sum_{i=n+1}^{N} x^i y^i = \sum_{i=1}^{N} \alpha(i) x^i y^i$$

(where $\alpha(i) := -1$ for $i = 1, \dots, n$ and $\alpha(i) := 1$ for $i = n+1, \dots, N$) is a non-degenerate symmetric bilinear map, i.e. $(\mathbb{R}^N, \mathbb{R}, \mathrm{H}_n)$ is a pseudo-Euclidean vector space and $\neg(\mathrm{H}_n) = n$.

The standard basis of $\mathbb{R}^N$ is H_n-orthonormal.

According to 1.3, we have the identification $\left(\mathbb{R}^N\right)^* \equiv \mathbb{R}^N$, but we must pay attention to the fact that if $n \neq 0$ it differs from the standard one mentioned in IV.1.6.

The standard identification is a linear bijection $\boldsymbol{S} : \mathbb{R}^N \to (\mathbb{R}^N)^*$, and the present identification is another one: $\boldsymbol{J}_n : \mathbb{R}^N \to (\mathbb{R}^N)^*$, $\boldsymbol{x} \mapsto \mathrm{H}_n(x, \cdot)$. We easily see that

$$(x_i |\ i = 1, \dots, N) := \boldsymbol{J}_n \cdot \left(x^i |\ i = 1, \dots, N\right) = \left(\alpha(i) x^i |\ i = 1, \dots, N\right).$$

The standard identification coincides with $\boldsymbol{J}_0$, the one corresponding to $\mathrm{H}_{\mathbf{0}}$.

According to the identification induced by H_n, the dual of the standard basis $\{\boldsymbol{\chi}_i |\ i = 1, \dots, N\}$ is $\{\alpha(i)\boldsymbol{\chi}_i |\ i = 1, \dots, N\}$.

It is useful to regard H_n as the diagonal matrix in which the first n elements in the diagonal are -1 and the others equal 1.

For the H_n-adjoint of the linear map (matrix) $\boldsymbol{F}$ we have $\boldsymbol{x} \cdot \mathrm{H}_n \cdot \boldsymbol{F}^{\circledast} \cdot \boldsymbol{y} = (\boldsymbol{F} \cdot \boldsymbol{x}) \cdot \mathrm{H}_n \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{F}^* \cdot \mathrm{H}_n \cdot \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$, where $\boldsymbol{F}^*$ denotes the usual transpose of the matrix $\boldsymbol{F}$; thus $\boldsymbol{F}^* \cdot \mathrm{H}_n = \mathrm{H}_n \cdot \boldsymbol{F}^{\circledast}$ or

$$\boldsymbol{F}^{\circledast} = \mathrm{H}_n \cdot \boldsymbol{F}^* \cdot \mathrm{H}_n.$$

1.8. Exercises

1. Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be pairwise $\mathbf{h}$-orthogonal vectors in the pseudo-Euclidean vector space $(\mathbf{V}, \mathbf{B}, \mathbf{h})$ such that $\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i) \neq \mathbf{0}$ for all $i = 1, \dots, n$. Prove that the following statements are equivalent:

(i) $\quad n = \dim \mathbf{V}$ (i.e. the vectors form a basis),

(ii) $\quad$ if $\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{x}) = \mathbf{0}$ for all $i = 1, \dots, n$ then $\boldsymbol{x} = \mathbf{0}$,

(iii)
$$\boldsymbol{x} = \sum_{i=1}^{n} \frac{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{x})}{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i)} \boldsymbol{e}_i \quad \text{for all} \quad \boldsymbol{x} \in \mathbf{V},$$

(iv)
$$\mathbf{h}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} \frac{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{x}) \otimes \mathbf{h}(\boldsymbol{e}_i, \boldsymbol{y})}{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i)} \quad \text{for all} \quad \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V},$$

(v)
$$\mathbf{h}(\boldsymbol{x}, \boldsymbol{x}) = \sum_{i=1}^{n} \frac{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{x}) \otimes \mathbf{h}(\boldsymbol{e}_i, \boldsymbol{x})}{\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i)} \quad \text{for all} \quad \boldsymbol{x} \in \mathbf{V}.$$

2. Demonstrate that the set $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of pairwise $\mathbf{h}$-orthogonal vectors can be completed to an $\mathbf{h}$-orthogonal basis if and only if $\mathbf{h}(\boldsymbol{e}_i, \boldsymbol{e}_i) \neq \mathbf{0}$ for all $i = 1, \dots, n$.

2. Tensors of pseudo-Euclidean vector spaces

2.1. Let $\mathbf{V}$ and $\mathbf{A}$ be finite dimensional vector spaces, $\dim \mathbf{A} = 1$. Suppose $\boldsymbol{F} : \mathbf{V} \to \mathbf{V}$ is a linear map. Then we can define the linear maps

$$\boldsymbol{F}^{\mathbf{A}} : \mathbf{A} \otimes \mathbf{V} \to \mathbf{A} \otimes \mathbf{V}, \qquad \boldsymbol{a}\boldsymbol{v} \mapsto \boldsymbol{a} \otimes (\boldsymbol{F} \cdot \boldsymbol{v}),$$
$$\boldsymbol{F}_{\mathbf{A}} : \frac{\mathbf{V}}{\mathbf{A}} \to \frac{\mathbf{V}}{\mathbf{A}}, \qquad \frac{\boldsymbol{v}}{\boldsymbol{a}} \mapsto \frac{\boldsymbol{F} \cdot \boldsymbol{v}}{\boldsymbol{a}}$$

($\boldsymbol{F}^{\mathbf{A}} = \mathrm{id}_{\mathbf{A}} \otimes \boldsymbol{F}$, $\boldsymbol{F}_{\mathbf{A}} = \frac{\boldsymbol{F}}{\mathrm{id}_{\mathbf{A}}}$, see IV.3.12 and IV.4.6).

According to the usual identifications

$$\begin{aligned}\mathrm{Lin}(\mathbf{A} \otimes \mathbf{V}) \equiv& (\mathbf{A} \otimes \mathbf{V}) \otimes (\mathbf{A} \otimes \mathbf{V})^* \equiv \mathbf{A} \otimes \mathbf{V} \otimes \mathbf{A}^* \otimes \mathbf{V}^* \equiv \\ \equiv& \mathbf{A} \otimes \mathbf{A}^* \otimes \mathbf{V} \otimes \mathbf{V}^* \equiv \mathbb{R} \otimes \mathbf{V} \otimes \mathbf{V}^* \equiv \mathbf{V} \otimes \mathbf{V}^* \equiv \\ \equiv& \mathrm{Lin}(\mathbf{V}),\end{aligned}$$

we have $\boldsymbol{F}^{\mathbf{A}} \equiv \boldsymbol{F}$ and similarly $\boldsymbol{F}_{\mathbf{A}} \equiv \boldsymbol{F}$. Therefore we shall write $\boldsymbol{F}$ instead of $\boldsymbol{F}^{\mathbf{A}}$ and $\boldsymbol{F}_{\mathbf{A}}$:

$$\text{for} \quad \boldsymbol{s} \in \mathbf{A} \otimes \mathbf{V} \qquad \text{we have} \qquad \boldsymbol{F} \cdot \boldsymbol{s} \in \mathbf{A} \otimes \mathbf{V},$$
$$\text{for} \quad \boldsymbol{n} \in \frac{\mathbf{V}}{\mathbf{A}} \qquad \text{we have} \qquad \boldsymbol{F} \cdot \boldsymbol{n} \in \frac{\mathbf{V}}{\mathbf{A}}.$$

2.2. Let us formulate the previous convention in another way. $\mathbf{V}\otimes\mathbf{A}\equiv \mathrm{Lin}(\mathbf{A}^*,\mathbf{V})$, hence we have the composition $\boldsymbol{F}\cdot\boldsymbol{s}$ of $\boldsymbol{F}\in\mathrm{Lin}(\mathbf{V})\equiv\mathbf{V}\otimes\mathbf{V}^*$ and $\boldsymbol{s}\in\mathrm{Lin}(\mathbf{A}^*,\mathbf{V})\equiv\mathbf{V}\otimes\mathbf{A}$.

More generally, if $\mathbf{U}$ and $\mathbf{W}$ are finite dimensional vector spaces, the *dot product* of an element from $\mathbf{U}\otimes\mathbf{V}^*$ and an element from $\mathbf{V}\otimes\mathbf{W}$ is defined to be an element in $\mathbf{U}\otimes\mathbf{W}$; this dot product can be regarded as the composition of the corresponding linear maps and is characterized by

$$(\boldsymbol{u}\otimes\boldsymbol{p})\cdot(\boldsymbol{v}\otimes\boldsymbol{w})=(\boldsymbol{p}\cdot\boldsymbol{v})\boldsymbol{u}\otimes\boldsymbol{w}.$$

The scheme is worth repeating:

$$\mathbf{U}\otimes\mathbf{V}^*\qquad\text{dot}\qquad\mathbf{V}\otimes\mathbf{W}\qquad\text{results in}\qquad\mathbf{U}\otimes\mathbf{W}.$$

Evidently, we can have $\mathbf{U}=\frac{\mathbb{K}}{\mathbf{A}}$ or $\mathbf{W}=\frac{\mathbb{K}}{\mathbf{A}}$, thus similar formulae are valid for tensor quotients as well.

2.3. What we have said in the previous paragraph concerns any vector spaces.

In the following $(\mathbf{V},\mathbf{B},\mathbf{h})$ denotes a pseudo-Euclidean vector space.

The identification described in 1.3 and the corresponding formula suggests us a new notation: "removing" the denominator from both sides we arrive at the definition

$$\boldsymbol{x}\cdot\boldsymbol{y}:=\mathbf{h}(\boldsymbol{x},\boldsymbol{y}),$$

i.e. in the sequel we omit $\mathbf{h}$, denoting the $\mathbf{h}$-product of vectors by a simple dot.

The dot product of two elements of $\mathbf{V}$ is an element of $\mathbf{B}\otimes\mathbf{B}$. Then we can extend the previous dot product formalism as follows:

$$\mathbf{U}\otimes\mathbf{V}\qquad\text{dot}\qquad\mathbf{V}\otimes\mathbf{W}\qquad\text{results in}\qquad(\mathbf{B}\otimes\mathbf{B})\otimes\mathbf{U}\otimes\mathbf{W},$$

$$(\boldsymbol{u}\otimes\boldsymbol{v}')\cdot(\boldsymbol{v}\otimes\boldsymbol{w}):=(\boldsymbol{v}'\cdot\boldsymbol{v})\boldsymbol{u}\otimes\boldsymbol{w}.$$

2.4. According to the convention introduced in 2.1, the $\mathbf{h}$-adjoint and the transpose of a linear map $\boldsymbol{F}:\mathbf{V}\to\mathbf{V}$ can be identified, for $\boldsymbol{F}^*=\frac{\boldsymbol{F}^{\circledast}}{\mathrm{id}_{\mathbf{B}\otimes\mathbf{B}}}$. However, we continue to distinguish between the transpose and the $\mathbf{h}$-adjoint because of the following reason.

In the pseudo-Euclidean vector space $(\mathbb{R}^N,\mathbb{R},\mathrm{H}_n)$ for $n\neq 0$, $n\neq N$ a linear map $\mathbb{R}^N\to\mathbb{R}^N$ is represented by a matrix, and the transpose of a matrix has a generally accepted meaning, and the H_n-adjoint of a matrix differs from its transpose.

The $\mathbf{h}$-adjoint of $\boldsymbol{F}\in\mathrm{Lin}(\mathbf{V})$ is characterized in the new notation of dot products as follows:

$$\boldsymbol{y}\cdot\boldsymbol{F}\cdot\boldsymbol{x}=(\boldsymbol{F}^{\circledast}\cdot\boldsymbol{y})\cdot\boldsymbol{x}=\boldsymbol{x}\cdot\boldsymbol{F}^{\circledast}\boldsymbol{y}\qquad(\boldsymbol{y},\boldsymbol{x}\in\mathbf{V}).$$

2.5. According to our convention we have

$$\begin{aligned}
&\boldsymbol{n}\cdot\boldsymbol{x} \qquad \text{is in} \quad \mathbf{B} \quad \text{for} \qquad \boldsymbol{n}\in\frac{\mathbf{V}}{\mathbf{B}} \quad \text{and} \quad \boldsymbol{x}\in\mathbf{V},\\
&\boldsymbol{n}\cdot\boldsymbol{m} \qquad \text{is in} \quad \mathbb{R} \quad \text{for} \qquad \boldsymbol{n},\boldsymbol{m}\in\frac{\mathbf{V}}{\mathbf{B}}.
\end{aligned}$$

If $\{\boldsymbol{e}_i|\ i=1,...,N\}$ is an $\mathbf{h}$-orthogonal basis, normed to $\boldsymbol{a}\in\mathbf{B}$, in $\mathbf{V}$, then $\left\{\boldsymbol{n}_i := \frac{\boldsymbol{e}_i}{\boldsymbol{a}}\middle|\ i=1,...,N\right\}$ is an $\mathbf{h}$-orthonormal basis of $\frac{\mathbf{V}}{\mathbf{B}}$:

$$\boldsymbol{n}_i\cdot\boldsymbol{n}_k=\alpha(i)\delta_{ik} \qquad (i,k=1,...,N).$$

It is more convenient to use this basis instead of the original one; for all $\boldsymbol{x}\in\mathbf{V}$ we have

$$\boldsymbol{x}=\sum_{i=1}^{N}\alpha(i)(\boldsymbol{n}_i\cdot\boldsymbol{x})\boldsymbol{n}_i.$$

2.6. (i) We have the identifications $\left(\frac{\mathbf{V}}{\mathbf{B}}\right)^*\equiv\frac{\mathbf{V}^*}{\mathbf{B}^*}\equiv\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}\otimes\mathbf{B}^*}\equiv\frac{\mathbf{V}}{\mathbf{B}}$; the element $\boldsymbol{n}$ of $\frac{\mathbf{V}}{\mathbf{B}}$ is identified with the linear functional $\frac{\mathbf{V}}{\mathbf{B}}\to\mathbb{R}$, $\boldsymbol{m}\mapsto\boldsymbol{n}\cdot\boldsymbol{m}$.

(ii) In view of the identifications $\mathrm{Lin}(\mathbf{V})\equiv\mathbf{V}\otimes\mathbf{V}^*\equiv\mathbf{V}\otimes\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}}\equiv\frac{\mathbf{V}}{\mathbf{B}}\otimes\frac{\mathbf{V}}{\mathbf{B}}$, or in view of our dot product convention, for $\boldsymbol{n},\boldsymbol{m}\in\frac{\mathbf{V}}{\mathbf{B}}$,

$$\boldsymbol{m}\otimes\boldsymbol{n}:\mathbf{V}\to\mathbf{V}, \qquad \boldsymbol{x}\mapsto(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{m}$$

is a linear map, and every element of $\mathrm{Lin}(\mathbf{V})$ is the sum of such linear maps.

Evidently,

$$(\boldsymbol{m}\otimes\boldsymbol{n})\cdot(\boldsymbol{m}'\otimes\boldsymbol{n}')=(\boldsymbol{n}\cdot\boldsymbol{m}')\boldsymbol{m}\otimes\boldsymbol{n}'$$

and

$$(\boldsymbol{m}\otimes\boldsymbol{n})^{\circledast}=\boldsymbol{n}\otimes\boldsymbol{m}.$$

2.7. Definition. For the pseudo-Euclidean vector space $(\mathbf{V},\mathbf{B},\mathbf{h})$ we put

$$\begin{aligned}
\mathcal{O}(\mathbf{h}) :=& \{\boldsymbol{L}\in\mathrm{Lin}(\mathbf{V})|\ \boldsymbol{L}^{\circledast}=\boldsymbol{L}^{-1}\},\\
\mathrm{A}(\mathbf{h}) :=& \{\boldsymbol{A}\in\mathrm{Lin}(\mathbf{V})|\ \boldsymbol{A}^{\circledast}=-\boldsymbol{A}\},
\end{aligned}$$

and the elements of $\mathcal{O}(\mathbf{h})$ and $\mathrm{A}(\mathbf{h})$ are called $\mathbf{h}$-*orthogonal* and $\mathbf{h}$-*antisymmetric,* respectively.

Proposition. (i) For $\boldsymbol{L}\in\mathrm{Lin}(\mathbf{V})$ the following three statements are equivalent:

— $\boldsymbol{L}$ is in $\mathcal{O}(\mathbf{h})$,

— $(\boldsymbol{L}\cdot\boldsymbol{y})\cdot(\boldsymbol{L}\cdot\boldsymbol{x})=\boldsymbol{y}\cdot\boldsymbol{x}$ for all $\boldsymbol{y},\boldsymbol{x}\in\mathbf{V}$,

— $(\boldsymbol{L}\cdot\boldsymbol{x})\cdot(\boldsymbol{L}\cdot\boldsymbol{x})=\boldsymbol{x}\cdot\boldsymbol{x}$ for all $\boldsymbol{x}\in\mathbf{V}$.

(ii) For $\boldsymbol{A}\in\mathrm{Lin}(\mathbf{V})$ the following three statements are equivalent:

— $\boldsymbol{A}$ is in $\mathrm{A}(\mathbf{h})$,

— $\boldsymbol{y}\cdot\boldsymbol{A}\cdot\boldsymbol{x}=-(\boldsymbol{A}\cdot\boldsymbol{y})\cdot\boldsymbol{x}=-\boldsymbol{x}\cdot\boldsymbol{A}\cdot\boldsymbol{y}$ for all $\boldsymbol{y},\boldsymbol{x}\in\mathbf{V}$,

— $\boldsymbol{x}\cdot\boldsymbol{A}\cdot\boldsymbol{x}=\mathbf{0}$ for all $\boldsymbol{x}\in\mathbf{V}$.

2.8. Proposition.

(i) $|\mathrm{det}\boldsymbol{L}|=1$ for $\boldsymbol{L}\in\mathcal{O}(\mathbf{h})$;

(ii) $\mathrm{Tr}\boldsymbol{A}=0$ for $\boldsymbol{A}\in\mathrm{A}(\mathbf{h})$.

Proof. It is convenient to regard now the linear maps in question as linear maps $\frac{\mathbf{V}}{\mathbf{B}}\to\frac{\mathbf{V}}{\mathbf{B}}$, according to our identifications described in 2.1. It is not hard to see that this does not influence determinants and traces.

(i) Let $\{\boldsymbol{n}_1,\dots,\boldsymbol{n}_N\}$ be an $\mathbf{h}$-orthonormal basis in $\frac{\mathbf{V}}{\mathbf{B}}$. According to IV.3.15 and to the identification $\left(\frac{\mathbf{V}}{\mathbf{B}}\right)^*\equiv\frac{\mathbf{V}}{\mathbf{B}}$ we have

$$\begin{aligned}0\neq\left(\bigwedge_{k=1}^{N}\boldsymbol{n}_k\right)(\boldsymbol{n}_1,...,\boldsymbol{n}_N)=\left(\bigwedge_{k=1}^{N}\boldsymbol{L}\cdot\boldsymbol{n}_k\right)(\boldsymbol{L}\cdot\boldsymbol{n}_1,\dots,\boldsymbol{L}\cdot\boldsymbol{n}_N)=\\ =(\mathrm{det}\boldsymbol{L})\left(\bigwedge_{k=1}^{N}\boldsymbol{L}\cdot\boldsymbol{n}_k\right)(\boldsymbol{n}_1,\dots,\boldsymbol{n}_N)=\\ =(\mathrm{det}\boldsymbol{L})^2\left(\bigwedge_{k=1}^{N}\boldsymbol{n}_k\right)(\boldsymbol{n}_1,\dots,\boldsymbol{n}_N).\end{aligned}$$

(ii) We know that the dual of the preceding basis is $\{\alpha(i)\boldsymbol{n}_i|\,i=1,\dots,N\}$, thus in view of IV.3.9,

$$\mathrm{Tr}\boldsymbol{A}=\sum_{i=1}^{N}\alpha(i)\boldsymbol{n}_i\cdot\boldsymbol{A}\cdot\boldsymbol{n}_i=0.$$

2.9. A linear map $\boldsymbol{S}:\mathbf{V}\to\mathbf{V}$ is called $\mathbf{h}$-*symmetric* if $\boldsymbol{S}^*=\boldsymbol{S}$ or, equivalently, $\boldsymbol{x}\cdot\boldsymbol{S}\cdot\boldsymbol{y}=\boldsymbol{y}\cdot\boldsymbol{S}\cdot\boldsymbol{x}$ for all $\boldsymbol{x},\boldsymbol{y}\in\mathbf{V}$. The set of $\mathbf{h}$-symmetric linear maps is denoted by $\mathrm{S}(\mathbf{h})$.

$\mathrm{A}(\mathbf{h})$ and $\mathrm{S}(\mathbf{h})$ are complementary subspaces of $\mathrm{Lin}(\mathbf{V})\equiv\mathbf{V}\otimes\mathbf{V}^*$, i.e. their intersection is the zero subspace and they span the whole space $\mathbf{V}\otimes\mathbf{V}^*$. Indeed, only the zero linear map is both symmetric and antisymmetric, and for any linear map $\boldsymbol{F}:\mathbf{V}\to\mathbf{V}$ we have that

$$\boldsymbol{S}:=\frac{\boldsymbol{F}+\boldsymbol{F}^*}{2},\qquad\boldsymbol{A}:=\frac{\boldsymbol{F}-\boldsymbol{F}^*}{2}$$

are $\mathbf{h}$-symmetric and $\mathbf{h}$-antisymmetric, respectively, and $\boldsymbol{F}=\boldsymbol{S}+\boldsymbol{A}$.

Taking the identification $\mathbf{V}\otimes\mathbf{V}^* \equiv \frac{\mathbf{V}}{\mathbf{B}}\otimes\frac{\mathbf{V}}{\mathbf{B}}$ we can easily see that $\frac{\mathbf{V}}{\mathbf{B}}\vee\frac{\mathbf{V}}{\mathbf{B}} \subset \mathrm{S}(\mathbf{h})$ and $\frac{\mathbf{V}}{\mathbf{B}}\wedge\frac{\mathbf{V}}{\mathbf{B}} \subset \mathrm{A}(\mathbf{h})$; since these subspaces are complementary, too, equalities hold necessarily:

$$\mathbf{V}\vee\mathbf{V}^* := \frac{\mathbf{V}}{\mathbf{B}}\vee\frac{\mathbf{V}}{\mathbf{B}} = \mathrm{S}(\mathbf{h}), \qquad \mathbf{V}\wedge\mathbf{V}^* := \frac{\mathbf{V}}{\mathbf{B}}\wedge\frac{\mathbf{V}}{\mathbf{B}} = \mathrm{A}(\mathbf{h}).$$

As a consequence,

$$\dim \mathrm{S}(\mathbf{h}) = \frac{N(N+1)}{2}, \qquad \dim \mathrm{A}(\mathbf{h}) = \frac{N(N-1)}{2}.$$

Recall that for $\boldsymbol{m}, \boldsymbol{n} \in \frac{\mathbf{V}}{\mathbf{B}}$ we have

$$\boldsymbol{m}\vee\boldsymbol{n} = \boldsymbol{m}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{m}, \qquad \boldsymbol{m}\wedge\boldsymbol{n} = \boldsymbol{m}\otimes\boldsymbol{n} - \boldsymbol{n}\otimes\boldsymbol{m}.$$

2.10. Proposition.

$$(\mathbf{V}\otimes\mathbf{V}^*)\times(\mathbf{V}\otimes\mathbf{V}^*) \to \mathbb{R}, \qquad (\boldsymbol{F},\boldsymbol{G}) \mapsto \boldsymbol{F}:\boldsymbol{G} := \mathrm{Tr}(\boldsymbol{F}^{\circledast}\cdot\boldsymbol{G})$$

is a non-degenerate symmetric bilinear form, which is positive definite if and only if $\mathbf{h}$ is positive definite.

Proof. It is trivially bilinear and symmetric because of the properties of Tr and $\mathbf{h}$-adjoints.

Suppose that $\mathrm{Tr}(\boldsymbol{F}^{\circledast}\cdot\boldsymbol{G}) = 0$ for all $\boldsymbol{F}\in\mathbf{V}\otimes\mathbf{V}^*$, i.e.

$$0 = \sum_{i=1}^{N} \alpha(i)\boldsymbol{n}_i\cdot\boldsymbol{F}^{\circledast}\cdot\boldsymbol{G}\cdot\boldsymbol{n}_i$$

for all $\mathbf{h}$-orthonormal bases $\{\boldsymbol{n}_1,\dots,\boldsymbol{n}_N\}$ of $\frac{\mathbf{V}}{\mathbf{B}}$. Then taking $\boldsymbol{F} := \boldsymbol{n}_j\otimes\boldsymbol{n}_k$ for all $j,k = 1,\dots,N$, and using $(\boldsymbol{n}_j\otimes\boldsymbol{n}_k)^{\circledast} = \boldsymbol{n}_k\otimes\boldsymbol{n}_j$ we conclude that $\boldsymbol{n}_j\cdot\boldsymbol{G}\cdot\boldsymbol{n}_k = 0$ for all j,k which results in $\boldsymbol{G} = \mathbf{0}$.

Since

$$\mathrm{Tr}(\boldsymbol{F}^{\circledast}\cdot\boldsymbol{F}) = \sum_{i=1}^{N}\alpha(i)(\boldsymbol{F}\cdot\boldsymbol{n}_i)\cdot(\boldsymbol{F}\cdot\boldsymbol{n}_i),$$

we see that if $\mathbf{h}$ is positive definite then $\mathrm{Tr}(\boldsymbol{F}^{\circledast}\cdot\boldsymbol{F}) > 0$; if $\mathbf{h}$ is not positive definite then we can easily construct an $\boldsymbol{F}$ such that $\boldsymbol{F}:\boldsymbol{F} < 0$.

Remark. (i) Compare the present bilinear form with that of the duality treated in IV.3.10; take into account the identification $\mathbf{V}\otimes\mathbf{V}^* \equiv \mathbf{V}\otimes\frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}} \equiv \frac{\mathbf{V}}{\mathbf{B}\otimes\mathbf{B}}\otimes\mathbf{V} \equiv \mathbf{V}^*\otimes\mathbf{V}$.

(*ii*) The bilinear form is not positive definite, in general, either on the linear subspace of $\mathbf{h}$-symmetric linear maps or on the linear subspace of $\mathbf{h}$-antisymmetric linear maps.

(*iii*) For $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{n}_1, \boldsymbol{n}_2 \in \frac{\mathbf{V}}{\mathbf{B}}$ we have

$$\begin{aligned}
(\boldsymbol{k}_1 \otimes \boldsymbol{n}_1) : (\boldsymbol{k}_2 \otimes \boldsymbol{n}_2) =& (\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2),\\
(\boldsymbol{k}_1 \vee \boldsymbol{n}_1) : (\boldsymbol{k}_2 \vee \boldsymbol{n}_2) =& 2\left((\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) + (\boldsymbol{k}_1 \cdot \boldsymbol{n}_2)(\boldsymbol{k}_2 \cdot \boldsymbol{n}_1)\right),\\
(\boldsymbol{k}_1 \wedge \boldsymbol{n}_1) : (\boldsymbol{k}_2 \wedge \boldsymbol{n}_2) =& 2\left((\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) - (\boldsymbol{k}_1 \cdot \boldsymbol{n}_2)(\boldsymbol{k}_2 \cdot \boldsymbol{n}_1)\right),
\end{aligned}$$

which shows that sometimes it is convenient to use the half of this bilinear form for $\mathbf{h}$-symmetric and $\mathbf{h}$-antisymmetric linear maps:

$$\begin{aligned}
\boldsymbol{F} \bullet \boldsymbol{G} :=& \frac{1}{2}\boldsymbol{F} : \boldsymbol{G} = \frac{1}{2}\mathrm{Tr}(\boldsymbol{F}^* \cdot \boldsymbol{G}) = \frac{1}{2}\mathrm{Tr}(\boldsymbol{F} \cdot \boldsymbol{G} \qquad (\boldsymbol{F}, \boldsymbol{G} \in \mathrm{S}(\mathbf{h})),\\
\boldsymbol{F} \bullet \boldsymbol{G} :=& \frac{1}{2}\boldsymbol{F} : \boldsymbol{G} = \frac{1}{2}\mathrm{Tr}(\boldsymbol{F}^* \cdot \boldsymbol{G}) = -\frac{1}{2}\mathrm{Tr}(\boldsymbol{F} \cdot \boldsymbol{G}) \qquad (\boldsymbol{F}, \boldsymbol{G} \in \mathrm{A}(\mathbf{h})).
\end{aligned}$$

2.11. Proposition. Let $\boldsymbol{L}$ be an $\mathbf{h}$-orthogonal map. Then for all $\boldsymbol{F}, \boldsymbol{G} \in \mathrm{Lin}(\mathbf{V})$

(*i*) $(\boldsymbol{L} \cdot \boldsymbol{F} \cdot \boldsymbol{L}^{-1}) : (\boldsymbol{L} \cdot \boldsymbol{G} \cdot \boldsymbol{L}^{-1}) = \boldsymbol{F} : \boldsymbol{G}$;

(*ii*) if $\boldsymbol{F}$ is $\mathbf{h}$-symmetric or $\mathbf{h}$-antisymmetric then $\boldsymbol{L} \cdot \boldsymbol{F} \cdot \boldsymbol{L}^{-1}$ is $\mathbf{h}$-symmetric or $\mathbf{h}$-antisymmetric, respectively.

2.12. Proposition. If $(\boldsymbol{n}_1, \dots, \boldsymbol{n}_N)$ and $(\boldsymbol{n}'_1, \dots, \boldsymbol{n}'_N)$ are equally oriented $\mathbf{h}$-orthonormal bases in $\frac{\mathbf{V}}{\mathbf{B}}$ then

$$\bigwedge_{i=1}^{N} \boldsymbol{n}_i = \bigwedge_{i=1}^{N} \boldsymbol{n}'_i.$$

Proof. Evidently, $\boldsymbol{L} \cdot \boldsymbol{n}_i := \boldsymbol{n}'_i$ $(i = 1, \dots, N)$ determines an $\mathbf{h}$-orthogonal map $\boldsymbol{L}$ whose determinant is positive since the bases are equally oriented. Then proposition in IV.3.18 gives the desired result. ■

Suppose $\mathbf{V}$ and $\mathbf{B}$ are oriented; then $\frac{\mathbf{V}}{\mathbf{B}}$ is oriented as well and the *Levi–Civita tensor* of $(\mathbf{V}, \mathbf{B}, \mathbf{h})$,

$$\varepsilon := \bigwedge_{i=1}^{N} \boldsymbol{n}_i = \bigwedge_{i=1}^{N} \frac{\boldsymbol{e}_i}{\boldsymbol{a}} \in \bigwedge^{N} \left(\frac{\mathbf{V}}{\mathbf{B}}\right) \equiv \frac{\bigwedge^{N} \mathbf{V}}{\bigotimes^{N} \mathbf{B}},$$

is well defined, where $(\boldsymbol{n}_1, \dots, \boldsymbol{n}_N)$ is a positively oriented orthonormal basis in $\frac{\mathbf{V}}{\mathbf{B}}$, and $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_N)$ is a positively oriented orthogonal basis in $\mathbf{V}$, normed to $\boldsymbol{a} \in \mathbf{B}^+$.

2.13. Exercises

1. According to the theory of tensor quotients, the Levi–Civita tensor can be considered to be a linear map

$$\varepsilon : \overset{N}{\otimes} \mathbf{B} \to \overset{N}{\wedge} \mathbf{V}, \qquad \underset{i=1}{\overset{N}{\otimes}} \boldsymbol{a}_i \mapsto \left(\underset{i=1}{\overset{N}{\otimes}} \boldsymbol{a}_i \right) \otimes \varepsilon.$$

Prove that $\left(\underset{i=1}{\overset{N}{\otimes}} \boldsymbol{a}_i \right) \otimes \varepsilon = \underset{i=1}{\overset{N}{\wedge}} \boldsymbol{e}_i$ where $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_N)$ is a positively oriented $\mathbf{h}$-orthogonal basis such that $|\boldsymbol{e}_i \cdot \boldsymbol{e}_i| = |\boldsymbol{a}_i^2|$ $(i = 1, .., N)$.

2. The previous linear map is a bijection whose inverse is $\frac{1}{\varepsilon} \in \frac{\overset{N}{\otimes} \mathbf{B}}{\overset{N}{\wedge} \mathbf{V}}$, regarded as a linear map $\overset{N}{\wedge} \mathbf{V} \to \overset{N}{\otimes} \mathbf{B}$, $\underset{i=1}{\overset{N}{\wedge}} \boldsymbol{x}_i \mapsto \frac{\overset{N}{\underset{i=1}{\wedge}} \boldsymbol{x}_i}{\varepsilon}$.

Prove that $\frac{\overset{N}{\underset{i=1}{\wedge}} \boldsymbol{x}_i}{\varepsilon} = \sum_{\pi \in \mathrm{Perm}_N} \mathrm{sign}\pi \prod_{i=1}^{N} (\boldsymbol{n}_{\pi(i)} \cdot \boldsymbol{x}_i) =: \varepsilon(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$.

3. Euclidean vector spaces

3.1. A pseudo-Euclidean vector space $(\mathbf{V}, \mathbf{B}, \mathbf{h})$ is called *Euclidean* if $\mathbf{h}$ is positive definite or, equivalently, $\neg(\mathbf{h}) = 0$.

For a clear distinction, in the following $(\mathbf{E}, \mathbf{D}, \mathbf{b})$ denotes a Euclidean vector space.

The notations introduced for pseudo-Euclidean vector spaces will be used, e.g.

$$\boldsymbol{x} \cdot \boldsymbol{y} := \mathbf{b}(\boldsymbol{x}, \boldsymbol{y}) \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathbf{E});$$

note that if $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{E}$ and $\boldsymbol{k}, \boldsymbol{n} \in \frac{\mathbf{E}}{\mathbf{D}}$ then

$$\boldsymbol{x} \cdot \boldsymbol{y} \in \mathbf{D} \otimes \mathbf{D}, \qquad \boldsymbol{n} \cdot \boldsymbol{x} \in \mathbf{D}, \qquad \boldsymbol{k} \cdot \boldsymbol{n} \in \mathbb{R}.$$

Moreover, we put

$$|\boldsymbol{x}|^2 := \mathbf{b}(\boldsymbol{x}, \boldsymbol{x}) \qquad (\boldsymbol{x} \in \mathbf{E}).$$

Lastly, we say orthogonal, adjoint etc. instead of $\mathbf{b}$-orthogonal, $\mathbf{b}$-adjoint etc.

3.2. Recall that a canonical order is given in $\mathbf{D} \otimes \mathbf{D}$ (IV.5.4) and so in $(\mathbf{D} \otimes \mathbf{D}) \otimes (\mathbf{D} \otimes \mathbf{D})$ as well. Thus the absolute value of elements in $\mathbf{D} \otimes \mathbf{D}$ and the square root of elements in $(\mathbf{D} \otimes \mathbf{D}) \otimes (\mathbf{D} \otimes \mathbf{D})$ make sense.

Proposition (Cauchy–Schwartz inequality). For all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{E}$ we have

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \sqrt{|\boldsymbol{x}|^2 |\boldsymbol{y}|^2}$$

and equality holds if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are parallel.

Proof. Exclude the trivial cases $\boldsymbol{x} = \mathbf{0}$ or $\boldsymbol{y} = \mathbf{0}$. Then the positive definiteness of $\mathbf{b}$ yields

$$\begin{aligned}\mathbf{0} \leq \left|\boldsymbol{x} - \frac{\boldsymbol{y}\cdot\boldsymbol{x}}{\boldsymbol{y}\cdot\boldsymbol{y}}\boldsymbol{y}\right|^2 &= |\boldsymbol{x}|^2 - 2\frac{\boldsymbol{y}\cdot\boldsymbol{x}}{\boldsymbol{y}\cdot\boldsymbol{y}}(\boldsymbol{x}\cdot\boldsymbol{y}) + \left(\frac{\boldsymbol{y}\cdot\boldsymbol{x}}{\boldsymbol{y}\cdot\boldsymbol{y}}\right)^2|\boldsymbol{y}|^2 = \\ &= |\boldsymbol{x}|^2 - \frac{(\boldsymbol{y}\cdot\boldsymbol{x})(\boldsymbol{x}\cdot\boldsymbol{y})}{|\boldsymbol{y}|^2}\end{aligned}$$

where equality holds if and only if $\boldsymbol{x} = \frac{\boldsymbol{y}\cdot\boldsymbol{x}}{\boldsymbol{y}\cdot\boldsymbol{y}}\boldsymbol{y}$. ■

In general, the right-hand side of the Cauchy inequality cannot be written in a simpler form because $|\boldsymbol{x}|$ and $|\boldsymbol{y}|$ make no sense, unless $\mathbf{D}$ is oriented.

3.3. Suppose now that $\mathbf{D}$ is oriented as well. Then we can define the *magnitude* or *length* of $\boldsymbol{x} \in \mathbf{E}$ as a non-negative element of $\mathbf{D}$:

$$|\boldsymbol{x}| := \sqrt{|\boldsymbol{x}|^2}.$$

The following fundamental relations hold:
(i) $|\boldsymbol{x}| = \mathbf{0}$ if and only if $\boldsymbol{x} = \mathbf{0}$,
(ii) $|\alpha\boldsymbol{x}| = |\alpha||\boldsymbol{x}|$,
(iii) $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$
for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{E}$ and $\alpha \in \mathbb{R}$. The third relation is called the *triangle inequality* and is proved by the Cauchy–Schwartz inequality.

Moreover, the Cauchy–Schwartz inequality allows us to define the *angle* formed by $\boldsymbol{x} \neq 0$ and $\boldsymbol{y} \neq 0$:

$$\arg(\boldsymbol{x}, \boldsymbol{y}) := \arccos\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}||\boldsymbol{y}|}.$$

3.4. The identification

$$\frac{\mathbf{E}}{\mathbf{D}\otimes\mathbf{D}} \equiv \mathbf{E}^*$$

(see 1.3) is a fundamental property of the Euclidean vector space $(\mathbf{E}, \mathbf{D}, \mathbf{b})$.

The dual of an orthogonal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_N$, normed to $\boldsymbol{m} \in \mathbf{D}$, in this identification becomes $\left\{\frac{\boldsymbol{e}_1}{\boldsymbol{m}^2}, \dots, \frac{\boldsymbol{e}_N}{\boldsymbol{m}^2}\right\}$.

Accordingly, $\boldsymbol{n}_i := \frac{\boldsymbol{e}_i}{\boldsymbol{m}}$ $(i = 1, \dots, N)$ form an orthonormal basis in $\frac{\mathbf{E}}{\mathbf{D}}$ which coincides with its dual basis in the identification $\frac{\mathbf{E}}{\mathbf{D}} \equiv \left(\frac{\mathbf{E}}{\mathbf{D}}\right)^*$:

$$\boldsymbol{n}_i \cdot \boldsymbol{n}_k = \delta_{ik} \qquad (i, k = 1, \dots, N).$$

For all $\boldsymbol{x} \in \mathbf{E}$ we have

$$\boldsymbol{x} = \sum_{i=1}^{N}(\boldsymbol{n}_i\cdot\boldsymbol{x})\boldsymbol{n}_i.$$

In the following $\frac{\mathbf{E}}{\mathbf{D}}$ will be used frequently, so we find it convenient to introduce a shorter notation:

$$\mathbf{N} := \frac{\mathbf{E}}{\mathbf{D}}.$$

3.5. If $\mathbf{S}$ is a linear subspace of $\mathbf{E}$ then

$$\mathbf{S}^\perp := \{\boldsymbol{x} \in \mathbf{E} |\ \boldsymbol{x} \cdot \boldsymbol{y} = \mathbf{0} \quad \text{for all} \quad \boldsymbol{y} \in \mathbf{S}\}$$

is called the *orthocomplement* of $\mathbf{S}$. It can be shown that $\mathbf{S}^\perp$ is a linear subspace, complementary to $\mathbf{S}$, i.e. their intersection is the zero subspace and they span the whole $\mathbf{E}$.

Every vector $\boldsymbol{x} \in \mathbf{E}$ can be uniquely decomposed into a sum of two vectors, one in $\mathbf{S}$ and the other in $\mathbf{S}^\perp$, called the *orthogonal projections* of $\boldsymbol{x}$ in $\mathbf{S}$ and in $\mathbf{S}^\perp$, respectively.

Let $\boldsymbol{n}$ be a unit vector in $\mathbf{N}$, i.e. $|\boldsymbol{n}|^2 := \boldsymbol{n} \cdot \boldsymbol{n} = 1$. Then

$$\boldsymbol{n} \otimes \mathbf{D} := \{\boldsymbol{n}\boldsymbol{d} |\ \boldsymbol{d} \in \mathbf{D}\}$$

is a one-dimensional subspace of $\mathbf{E}$. Furthermore, its orthocomplement,

$$\{\boldsymbol{x} \in \mathbf{E} |\ \boldsymbol{n} \cdot \boldsymbol{x} = \mathbf{0}\} = (\boldsymbol{n} \otimes \mathbf{D})^\perp$$

is an $(N-1)$-dimensional subspace. The corresponding projections of $\boldsymbol{x} \in \mathbf{E}$ in $\boldsymbol{n} \otimes \mathbf{D}$ and in $(\boldsymbol{n} \otimes \mathbf{D})^\perp$ are

$$(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} \qquad \text{and} \qquad \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n},$$

respectively.

3.6. Proposition. For $\boldsymbol{F} \in \mathrm{Lin}(\mathbf{E})$ we have $\mathrm{Ker}\ \boldsymbol{F}^* = (\mathrm{Ran}\ \boldsymbol{F})^\perp$.

Proof. $\boldsymbol{x}$ is in Ker $\boldsymbol{F}^*$, i.e. $\boldsymbol{F}^* \cdot \boldsymbol{x} = \mathbf{0}$ if and only if $\boldsymbol{y} \cdot \boldsymbol{F}^* \cdot \boldsymbol{x} = \mathbf{0}$ for all $\boldsymbol{y} \in \mathbf{E}$, which is equivalent to $\boldsymbol{x} \cdot \boldsymbol{F} \cdot \boldsymbol{y} = \mathbf{0}$ for all $\boldsymbol{y} \in \mathbf{E}$, thus $\boldsymbol{x}$ is in Ker $\boldsymbol{F}^*$ if and only if it is orthogonal to the range of $\boldsymbol{F}$.

3.7. We know that a linear map $\boldsymbol{L} : \mathbf{E} \to \mathbf{E}$ is orthogonal, i.e. $\boldsymbol{y} \cdot \boldsymbol{x} = (\boldsymbol{L} \cdot \boldsymbol{y}) \cdot (\boldsymbol{L} \cdot \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{E}$ if and only if $|\boldsymbol{L} \cdot \boldsymbol{x}|^2 = |\boldsymbol{x}|^2$ for all $\boldsymbol{x} \in \mathbf{E}$ (see 2.7). Because of the Euclidean structure we need not assume the linearity of $\boldsymbol{L}$, according to the following result.

Proposition. Let $\boldsymbol{L} : \mathbf{E} \to \mathbf{E}$ be a map such that $\boldsymbol{L}(\boldsymbol{y}) \cdot \boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{y} \cdot \boldsymbol{x}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{E}$. Then $\boldsymbol{L}$ is necessarily linear.

Proof. First of all note that if $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_N\}$ is an orthogonal basis in $\mathbf{E}$ then $\{\boldsymbol{L}\cdot\boldsymbol{e}_1, \dots, \boldsymbol{L}\cdot\boldsymbol{e}_N\}$ is an orthogonal basis as well. As a consequence, Ran $\boldsymbol{L}$ spans $\mathbf{E}$.

If $\boldsymbol{L}(\boldsymbol{y}) = \boldsymbol{L}(\boldsymbol{x})$ then $|\boldsymbol{y}|^2 = |\boldsymbol{x}|^2 = \boldsymbol{y}\cdot\boldsymbol{x}$ and so $|\boldsymbol{y}-\boldsymbol{x}|^2 = \mathbf{0}$, hence $\boldsymbol{L}$ is injective.

Writing $\boldsymbol{x}' := \boldsymbol{L}(\boldsymbol{x})$, $\boldsymbol{y}' := \boldsymbol{L}(\boldsymbol{y})$ and then omitting the prime, we find that $\boldsymbol{L}^{-1}(\boldsymbol{y})\cdot\boldsymbol{L}^{-1}(\boldsymbol{x}) = \boldsymbol{y}\cdot\boldsymbol{x}$ for all $\boldsymbol{x},\boldsymbol{y}\in$ Ran $\boldsymbol{L}$ and $\boldsymbol{L}^{-1}(\boldsymbol{y})\cdot\boldsymbol{x} = \boldsymbol{y}\cdot\boldsymbol{L}(\boldsymbol{x})$ for all $\boldsymbol{x}\in\mathbf{E}$, $\boldsymbol{y}\in$ Ran $\boldsymbol{L}$.

Consequently, for all $\boldsymbol{y}\in$ Ran $\boldsymbol{L}$ and $\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathbf{E}$ we have

$$\begin{aligned}\boldsymbol{y}\cdot\boldsymbol{L}(\boldsymbol{x}_1+\boldsymbol{x}_2) =& \boldsymbol{L}^{-1}(\boldsymbol{y})\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2) = \boldsymbol{L}^{-1}(\boldsymbol{y})\cdot\boldsymbol{x}_1 + \boldsymbol{L}^{-1}(\boldsymbol{y})\cdot\boldsymbol{x}_2 = \\ =& \boldsymbol{y}\cdot\boldsymbol{L}(\boldsymbol{x}_1) + \boldsymbol{y}\cdot\boldsymbol{L}(\boldsymbol{x}_2) = \boldsymbol{y}\cdot(\boldsymbol{L}(\boldsymbol{x}_1)+\boldsymbol{L}(\boldsymbol{x}_2))\,;\end{aligned}$$

since $\boldsymbol{y}$ is arbitrary in Ran $\boldsymbol{L}$ which spans $\mathbf{E}$, this means that

$$\boldsymbol{L}(\boldsymbol{x}_1+\boldsymbol{x}_2) = \boldsymbol{L}(\boldsymbol{x}_1) + \boldsymbol{L}(\boldsymbol{x}_2) \qquad (\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathbf{E}).$$

A similar argument shows that

$$\boldsymbol{L}(\alpha\boldsymbol{x}) = \alpha\boldsymbol{L}(\boldsymbol{x}) \qquad (\alpha\in\mathbb{R}, \boldsymbol{x}\in\mathbf{E}).\blacksquare$$

This has the simple but important consequence that if $\boldsymbol{L}:\mathbf{E}\to\mathbf{E}$ is a map such that $|\boldsymbol{L}(\boldsymbol{y})-\boldsymbol{L}(\boldsymbol{x})|^2 = |\boldsymbol{y}-\boldsymbol{x}|^2$ for all $\boldsymbol{x},\boldsymbol{y}\in\mathbf{E}$ and $\boldsymbol{L}(\mathbf{0}) = \mathbf{0}$ then $\boldsymbol{L}$ is necessarily linear. The proof is left to the reader as an exercise.

3.8. In the following, assuming that

$$\dim\mathbf{E} = 3,$$

we shall examine the structure of the antisymmetric linear maps of $\mathbf{E}$. As we know, (see 2.9) $\mathrm{A}(\mathbf{b}) \equiv \frac{\mathbf{E}}{\mathbf{D}}\wedge\frac{\mathbf{E}}{\mathbf{D}} = \mathbf{N}\wedge\mathbf{N}$ is a three-dimensional vector space endowed (see 2.10) with a real-valued positive definite symmetric bilinear form — an inner product — :

$$\boldsymbol{A}\bullet\boldsymbol{B} = \frac{1}{2}\mathrm{Tr}(\boldsymbol{A}^*\cdot\boldsymbol{B}) = -\frac{1}{2}\mathrm{Tr}(\boldsymbol{A}\cdot\boldsymbol{B}).$$

The magnitude of the antisymmetric linear map $\boldsymbol{A}$ is the real number

$$|\boldsymbol{A}| := \sqrt{\boldsymbol{A}\bullet\boldsymbol{A}}.$$

Recall that for $\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{n}_1,\boldsymbol{n}_2\in\mathbf{N}$ we have

$$(\boldsymbol{k}_1\wedge\boldsymbol{k}_2)\bullet(\boldsymbol{n}_1\wedge\boldsymbol{n}_2) = (\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(\boldsymbol{n}_1\cdot\boldsymbol{n}_2) - (\boldsymbol{k}_1\cdot\boldsymbol{n}_2)(\boldsymbol{k}_2\cdot\boldsymbol{n}_1),$$

in particular, if $\boldsymbol{k}_1 = \boldsymbol{k}_2 =: \boldsymbol{k}$, $\boldsymbol{n}_1 = \boldsymbol{n}_2 =: \boldsymbol{n}$,

$$|\boldsymbol{k} \wedge \boldsymbol{n}|^2 = |\boldsymbol{k}|^2 |\boldsymbol{n}|^2 - (\boldsymbol{k} \cdot \boldsymbol{n})^2.$$

3.9. If $\boldsymbol{A} \in \mathbf{N} \wedge \mathbf{N}$ then $\boldsymbol{A}^* = -\boldsymbol{A}$, thus proposition 3.6 yields that Ker $\boldsymbol{A}$ is the orthogonal complement of Ran $\boldsymbol{A}$.

Proposition. If $\mathbf{0} \neq \boldsymbol{A} \in \mathbf{N} \wedge \mathbf{N}$ then Ran $\boldsymbol{A}$ is two-dimensional, consequently, Ker $\boldsymbol{A}$ is one-dimensional.

Proof. Since $\boldsymbol{A} \neq \mathbf{0}$, there is a $\mathbf{0} \neq \boldsymbol{x} \in$ Ran $\boldsymbol{A}$. Then $\boldsymbol{x} \notin$ Ker $\boldsymbol{A}$, thus $\mathbf{0} \neq \boldsymbol{A} \cdot \boldsymbol{x} \in$ Ran $\boldsymbol{A}$. $\boldsymbol{x}$ and $\boldsymbol{A} \cdot \boldsymbol{x}$ are orthogonal to each other because $\boldsymbol{A}$ is antisymmetric. Consequently, the subspace spanned by $\boldsymbol{x}$ and $\boldsymbol{A} \cdot \boldsymbol{x}$ is two-dimensional in the range of $\boldsymbol{A}$: Ran $\boldsymbol{A}$ is at least two-dimensional, Ker $\boldsymbol{A}$ is at most one-dimensional. Suppose Ker $\boldsymbol{A} = \{\mathbf{0}\}$. Take a $\mathbf{0} \neq \boldsymbol{y}$, orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{A} \cdot \boldsymbol{x}$. $\mathbf{E}$ is three-dimensional, $\boldsymbol{A} \cdot \boldsymbol{y}$ is orthogonal to $\boldsymbol{y}$, so it lies in the subspace generated by $\boldsymbol{x}$ and $\boldsymbol{A} \cdot \boldsymbol{x}$, i.e. $\boldsymbol{A} \cdot \boldsymbol{y} = \alpha \boldsymbol{x} + \beta \boldsymbol{A} \cdot \boldsymbol{x}$. Multiplying by $\boldsymbol{x}$ and using $\boldsymbol{x} \cdot \boldsymbol{A} \cdot \boldsymbol{x} = \mathbf{0}$, $\boldsymbol{x} \cdot \boldsymbol{A} \cdot \boldsymbol{y} = -\boldsymbol{y} \cdot \boldsymbol{A} \cdot \boldsymbol{x} = \mathbf{0}$, we get $\alpha = 0$. As a consequence, $\boldsymbol{A} \cdot (\boldsymbol{y} - \beta \boldsymbol{x}) = \mathbf{0}$, the vector $\boldsymbol{y} - \beta \boldsymbol{x}$ is in Ker $\boldsymbol{A}$, thus $\boldsymbol{y} - \beta \boldsymbol{x} = \mathbf{0}$, $\boldsymbol{y} = \beta \boldsymbol{x}$, a contradiction.

3.10. Let us take a non-zero $\boldsymbol{A} \in \mathbf{N} \wedge \mathbf{N}$. There is an orthonormal basis $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$ in $\mathbf{N}$ such that $\boldsymbol{n}_3 \otimes \mathbf{D} =$ Ker $\boldsymbol{A}$. $\{\boldsymbol{n}_1 \wedge \boldsymbol{n}_2, \boldsymbol{n}_3 \wedge \boldsymbol{n}_1, \boldsymbol{n}_2 \wedge \boldsymbol{n}_3\}$ is a basis in $\mathbf{N} \wedge \mathbf{N}$, thus there are real numbers $\alpha_1, \alpha_2, \alpha_3$ such that

$$\boldsymbol{A} = \alpha_3 (\boldsymbol{n}_1 \wedge \boldsymbol{n}_2) + \alpha_2 (\boldsymbol{n}_3 \wedge \boldsymbol{n}_1) + \alpha_1 (\boldsymbol{n}_2 \wedge \boldsymbol{n}_3).$$

Since $\boldsymbol{A} \cdot \boldsymbol{n}_3 = \mathbf{0}$, we easily find that $\alpha_2 = \alpha_1 = 0$ and, consequently, $|\alpha_3| = |\boldsymbol{A}|$. Renaming $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ and taking their antisymmetric tensor product in a convenient order we arrive at the following result.

Proposition. If $\mathbf{0} \neq \boldsymbol{A} \in \mathbf{N} \wedge \mathbf{N}$ and $\boldsymbol{n}$ is an arbitrary unit vector in $\mathbf{N}$, orthogonal to the kernel of $\boldsymbol{A}$, then there is a unit vector $\boldsymbol{k}$, orthogonal to the kernel of $\boldsymbol{A}$ and to $\boldsymbol{n}$ such that

$$\boldsymbol{A} = |\boldsymbol{A}| \boldsymbol{k} \wedge \boldsymbol{n}. \quad \blacksquare$$

As a consequence, we have

$$\boldsymbol{A}^3 = -|\boldsymbol{A}|^2 \boldsymbol{A}.$$

Moreover, for non-zero $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathbf{N} \wedge \mathbf{N}$ the following statements are equivalent:

— $\boldsymbol{A}$ is a multiple of $\boldsymbol{B}$,

— Ker $\boldsymbol{A} = $ Ker $\boldsymbol{B}$,
— Ran $\boldsymbol{A} = $ Ran $\boldsymbol{B}$.

3.11. If $\boldsymbol{A}, \boldsymbol{B} \in \mathbf{N} \wedge \mathbf{N}$, their commutator

$$[\boldsymbol{A}, \boldsymbol{B}] := \boldsymbol{A} \cdot \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{A}$$

is in $\mathbf{N} \wedge \mathbf{N}$, too. Moreover, the properties of the trace imply that for all $\boldsymbol{C} \in \mathbf{N} \wedge \mathbf{N}$

$$[\boldsymbol{A}, \boldsymbol{B}] \bullet \boldsymbol{C} = [\boldsymbol{C}, \boldsymbol{A}] \bullet \boldsymbol{B} = [\boldsymbol{B}, \boldsymbol{C}] \bullet \boldsymbol{A}.$$

Proposition.

$$|[\boldsymbol{A}, \boldsymbol{B}]|^2 = |\boldsymbol{A}|^2 |\boldsymbol{B}|^2 - (\boldsymbol{A} \bullet \boldsymbol{B})^2.$$

Proof. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel (in particular, if one of them is zero) then the equality holds trivially. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are not parallel, dividing the equality by $|\boldsymbol{A}|^2 |\boldsymbol{B}|^2$ we reduce the problem to the case $|\boldsymbol{A}| = |\boldsymbol{B}| = 1$. The ranges of $\boldsymbol{A}$ and $\boldsymbol{B}$ are different two-dimensional subspaces, hence their intersection is a one-dimensional subspace (because $\mathbf{E}$ is three-dimensional). Let $\boldsymbol{n}$ be a unit vector in $\mathbf{N}$ such that $\boldsymbol{n} \otimes \mathbf{D} = $ Ran $\boldsymbol{A} \cap$Ran $\boldsymbol{B}$. Then there are unit vectors $\boldsymbol{k}$ and $\boldsymbol{r}$ in $\mathbf{N}$, orthogonal to $\boldsymbol{n}$, such that $\boldsymbol{A} = \boldsymbol{k} \wedge \boldsymbol{n}$, $\boldsymbol{B} = \boldsymbol{r} \wedge \boldsymbol{n}$. Simple calculations yield

$$[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{k} \wedge \boldsymbol{r}, \qquad |[\boldsymbol{A}, \boldsymbol{B}]|^2 = 1 - (\boldsymbol{k} \cdot \boldsymbol{r})^2$$

which gives the desired result in view of 3.8.

3.12. According to the formula cited at the beginning of the previous paragraph, $[\boldsymbol{A}, \boldsymbol{B}]$ is orthogonal to both $\boldsymbol{A}$ and $\boldsymbol{B}$. Consequently, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal, $|\boldsymbol{A}| = |\boldsymbol{B}| = 1$ then $\boldsymbol{A}, \mathbf{B}$ and $[\boldsymbol{A}, \boldsymbol{B}]$ form an orthonormal basis in $\mathbf{N} \wedge \mathbf{N}$.

Proposition. For all $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbf{N} \wedge \mathbf{N}$ we have

$$[[\boldsymbol{A}, \boldsymbol{B}], \boldsymbol{C}] = (\boldsymbol{A} \bullet \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B} \bullet \boldsymbol{C})\boldsymbol{A}.$$

Proof. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel, both sides are zero. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are not parallel (in particular, neither of them is zero) then $\boldsymbol{B} = \alpha \boldsymbol{A} + \mathbf{B}'$ where α is a number and $\mathbf{B}' \neq \mathbf{0}$ is orthogonal to $\mathbf{A}$. $\alpha \boldsymbol{A}$ results in zero on both sides, hence it is sufficient to consider an arbitrary $\boldsymbol{A}$, a $\boldsymbol{B}$ orthogonal to $\boldsymbol{A}$, and three linearly independent elements in the role of $\boldsymbol{C}$; they will be $\boldsymbol{A}$, $\boldsymbol{B}$ and $[\boldsymbol{A}, \boldsymbol{B}]$.

For $\boldsymbol{C} = [\boldsymbol{A}, \boldsymbol{B}]$ the equality is trivial, both sides are zero.

For $\boldsymbol{C} = \boldsymbol{A}$, the right-hand side equals $|\boldsymbol{A}|^2 \boldsymbol{B}$; $[[\boldsymbol{A}, \boldsymbol{B}], \boldsymbol{A}]$ on the left-hand side is orthogonal to both $[\boldsymbol{A}, \boldsymbol{B}]$ and $\boldsymbol{A}$, hence it is parallel to $\boldsymbol{B}$: there is a number α such that $[[\boldsymbol{A},\boldsymbol{B}],\boldsymbol{A}]=\alpha\boldsymbol{B}$. Take the inner product of both sides by $\boldsymbol{B}$, apply the formula at the beginning of 3.11 to have $|[\boldsymbol{A},\boldsymbol{B}]|^2=\alpha|\boldsymbol{B}|^2$ which implies $\alpha=|\boldsymbol{A}|^2$ according to the previous result.

A similar argument is applied to $\boldsymbol{C}=\boldsymbol{B}$.

3.13. Let us continue to consider the Euclidean vector space $(\mathbf{E},\mathbf{D},\mathbf{b})$, $\dim\mathbf{E}=3$, and suppose that $\mathbf{E}$ and $\mathbf{D}$ are oriented. Then $\mathbf{N}=\frac{\mathbf{E}}{\mathbf{D}}$ is oriented as well (see IV.5.2). According to 2.12, there is a well-defined ε in $\overset{3}{\wedge}\mathbf{N}$ such that

$$\varepsilon=\bigwedge_{i=1}^{3}\boldsymbol{n}_i$$

for an arbitrary positively oriented orthonormal basis $(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)$ of $\mathbf{N}$. ε is called the *Levi–Civita tensor* of $(\mathbf{E},\mathbf{D},\mathbf{b})$.

The Levi–Civita tensor establishes a linear bijection

$$\mathbf{j}:\mathbf{N}\wedge\mathbf{N}\to\mathbf{N},\qquad \boldsymbol{k}\wedge\boldsymbol{n}\mapsto\varepsilon(\cdot,\boldsymbol{k},\boldsymbol{n}).$$

Let us examine more closely what this is. The dual of $\mathbf{N}$ is identified with $\mathbf{N}$, thus ε can be considered to be a trilinear map

$$\mathbf{N}^3\to\mathbb{R},\qquad (\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\mapsto\sum_{\pi\in Perm_3}\operatorname{sign}\pi\prod_{i=1}^{3}\boldsymbol{n}_{\pi(i)}\cdot\boldsymbol{k}_i$$

(see IV.3.15). Thus, for given $\boldsymbol{k}$ and $\boldsymbol{n}$, $\varepsilon(\cdot,\boldsymbol{k},\boldsymbol{n})$ is the linear map $\mathbf{N}\to\mathbb{R}$, $\boldsymbol{r}\mapsto\varepsilon(\boldsymbol{r},\boldsymbol{k},\boldsymbol{n})$, i.e. it is an element of $\mathbf{N}^*\equiv\mathbf{N}$.

In other words, $\mathbf{j}(\boldsymbol{k}\wedge\boldsymbol{n})$ is the element of $\mathbf{N}$ determined by

$$\boldsymbol{r}\cdot\mathbf{j}(\boldsymbol{k}\wedge\boldsymbol{n})=\varepsilon(\boldsymbol{r},\boldsymbol{k},\boldsymbol{n})$$

for all $\boldsymbol{r}\in\mathbf{N}$.

The Levi–Civita tensor is antisymmetric, hence $\mathbf{j}(\boldsymbol{k}\wedge\boldsymbol{n})$ is orthogonal to both $\boldsymbol{k}$ and $\boldsymbol{n}$.

If $(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)$ is a positively oriented orthonormal basis in $\mathbf{N}$, then

$$\mathbf{j}(\boldsymbol{n}_1\wedge\boldsymbol{n}_2)=-\boldsymbol{n}_3,\qquad \mathbf{j}(\boldsymbol{n}_2\wedge\boldsymbol{n}_3)=-\boldsymbol{n}_1,\qquad \mathbf{j}(\boldsymbol{n}_3\wedge\boldsymbol{n}_1)=-\boldsymbol{n}_2.$$

Proposition. For all $\boldsymbol{A}\in\mathbf{N}\wedge\mathbf{N}$ we have

(*i*) $\boldsymbol{A}\cdot\mathbf{j}(\boldsymbol{A})=\mathbf{0}$,

(*ii*) $|\mathbf{j}(\boldsymbol{A})|=|\boldsymbol{A}|$,

(*iii*) if $\boldsymbol{A}\neq 0$ then $(\mathbf{j}(\boldsymbol{A}),\boldsymbol{n},\boldsymbol{A}\cdot\boldsymbol{n})$ is a positively oriented orthogonal basis in $\mathbf{N}$ for arbitrary non zero $\boldsymbol{n}$, orthogonal to Ker $\boldsymbol{A}$.

Proof. There is a positively oriented orthonormal basis $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ such that $\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{n}_1 \wedge \boldsymbol{n}_2$ (and so $\boldsymbol{A} \cdot \boldsymbol{n}_3 = \boldsymbol{0}$); then $\mathbf{j}(\boldsymbol{A}) = -|\boldsymbol{A}|\boldsymbol{n}_3$ from which (i) and (ii) follow immediately. Moreover, we can choose $\boldsymbol{n}_1 := \frac{\boldsymbol{n}}{|\boldsymbol{n}|}$ where $\boldsymbol{n}$ is an arbitrary non-zero vector orthogonal to Ker $\boldsymbol{A}$. ■

The kernel of a non-zero $\boldsymbol{A}$ is one-dimensional; according to (i), $\mathbf{j}(\boldsymbol{A})$ spans the kernel of $\boldsymbol{A}$. The one-dimensional vector space Ker $\boldsymbol{A}$ will be *oriented* by $\mathbf{j}(\boldsymbol{A})$.

Since $\mathbf{j}$ is linear, (ii) is equivalent to $\mathbf{j}(\boldsymbol{A}) \cdot \mathbf{j}(\boldsymbol{B}) = \boldsymbol{A} \bullet \boldsymbol{B}$ for all $\boldsymbol{A}$ and $\boldsymbol{B}$ which can also be proved directly using that $\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{k} \wedge \boldsymbol{n}$, $\boldsymbol{B} = |\boldsymbol{B}|\boldsymbol{r} \wedge \boldsymbol{n}$.

3.14. Definition. The map

$$\mathbf{N} \times \mathbf{N} \to \mathbf{N}, \qquad (\boldsymbol{k}, \boldsymbol{n}) \mapsto \boldsymbol{k} \times \boldsymbol{n} := -\mathbf{j}(\boldsymbol{k} \wedge \boldsymbol{n})$$

is called the *vectorial product.* ■

It is evident from the properties of $\mathbf{j}$ that the vectorial product is an antisymmetric bilinear mapping, $\boldsymbol{k} \times \boldsymbol{n} = \boldsymbol{0}$ if and only if $\boldsymbol{k}$ and $\boldsymbol{n}$ are parallel, $\boldsymbol{k} \times \boldsymbol{n}$ is orthogonal to both $\boldsymbol{k}$ and $\boldsymbol{n}$,

$$|\boldsymbol{k} \times \boldsymbol{n}|^2 = |\boldsymbol{k}|^2 |\boldsymbol{n}|^2 - (\boldsymbol{k} \cdot \boldsymbol{n})^2.$$

If $\boldsymbol{k}$ and $\boldsymbol{n}$ are not parallel then $\boldsymbol{k}$, $\boldsymbol{n}$ and $\boldsymbol{k} \times \boldsymbol{n}$ form a positively oriented basis in $\mathbf{N}$; moreover, if $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ is a positively oriented orthonormal basis in $\mathbf{N}$ then

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \boldsymbol{n}_3, \qquad \boldsymbol{n}_2 \times \boldsymbol{n}_3 = \boldsymbol{n}_1, \qquad \boldsymbol{n}_3 \times \boldsymbol{n}_1 = \boldsymbol{n}_2.$$

Proposition.

(i)
$$\mathbf{j}([\boldsymbol{A}, \boldsymbol{B}]) = \mathbf{j}(\boldsymbol{A}) \times \mathbf{j}(\boldsymbol{B}) \qquad (\boldsymbol{A}, \boldsymbol{B} \in \mathbf{N} \wedge \mathbf{N})$$

or, equivalently,

$$\mathbf{j}(\boldsymbol{A}) \wedge \mathbf{j}(\boldsymbol{B}) = -[\boldsymbol{A}, \boldsymbol{B}];$$

(ii)
$$\boldsymbol{A} \cdot \boldsymbol{n} = \mathbf{j}(\boldsymbol{A}) \times \boldsymbol{n} \qquad (\boldsymbol{A} \in \mathbf{N} \wedge \mathbf{N},\ \boldsymbol{n} \in \mathbf{N})$$

which implies

$$\boldsymbol{A} \cdot \mathbf{j}(\boldsymbol{B}) = \mathbf{j}([\boldsymbol{A}, \boldsymbol{B}]) \qquad (\boldsymbol{A}, \boldsymbol{B} \in \mathbf{N} \wedge \mathbf{N}).$$

Proof. There is an orthonormal basis $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$ in $\mathbf{N}$ such that $\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{n}_1 \wedge \boldsymbol{n}_2$.

(i) $\boldsymbol{n}_1 \wedge \boldsymbol{n}_2$, $\boldsymbol{n}_2 \wedge \boldsymbol{n}_3$ and $\boldsymbol{n}_3 \wedge \boldsymbol{n}_1$ form a basis in $\mathbf{N} \wedge \mathbf{N}$, thus it is sufficient to consider them in the role of $\boldsymbol{B}$; then a simple calculation yields the desired result.

(ii) Take $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ in the role of $\boldsymbol{n}$. ■

As a consequence of our results we have

$$(\boldsymbol{k} \times \boldsymbol{n}) \cdot \boldsymbol{r} = (\boldsymbol{n} \times \boldsymbol{r}) \cdot \boldsymbol{k} = (\boldsymbol{r} \times \boldsymbol{k}) \cdot \boldsymbol{n} = \varepsilon(\boldsymbol{r}, \boldsymbol{k}, \boldsymbol{n})$$

and

$$(\boldsymbol{k} \times \boldsymbol{n}) \times \boldsymbol{r} = (\boldsymbol{k} \cdot \boldsymbol{r})\boldsymbol{n} - (\boldsymbol{n} \cdot \boldsymbol{r})\boldsymbol{k}$$

for all $\boldsymbol{k}, \boldsymbol{n}, \boldsymbol{r} \in \mathbf{N}$.

3.15. The Levi–Civita tensor establishes another linear bijection as well:

$$\mathrm{j_o} : \overset{3}{\wedge} \mathbf{N} \to \mathbb{R}, \qquad \overset{3}{\underset{i=1}{\wedge}} \boldsymbol{k}_i \mapsto \varepsilon(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3).$$

It is quite trivial that ${\mathrm{j_o}}^{-1}(\alpha) = \alpha\varepsilon$ for all $\alpha \in \mathbb{R}$.

3.16. An orthogonal map $\boldsymbol{R} : \mathbf{E} \to \mathbf{E}$ (also regarded as an orthogonal map $\mathbf{N} \to \mathbf{N}$, see 2.1) sends orthogonal bases into orthogonal ones, preserves and changes orientation according to whether $\det\boldsymbol{R} = 1$ or $\det\boldsymbol{R} = -1$. In view of IV.3.18,

$$\varepsilon \circ \left(\overset{3}{\times} \boldsymbol{R} \right) = (\det\boldsymbol{R})\varepsilon.$$

Then one proves without difficulty that

$$\mathbf{j}(\boldsymbol{R} \cdot \boldsymbol{k} \wedge \boldsymbol{R} \cdot \boldsymbol{n}) = (\det\boldsymbol{R})\boldsymbol{R} \cdot \mathbf{j}(\boldsymbol{k} \wedge \boldsymbol{n}) \qquad (\boldsymbol{k}, \boldsymbol{n} \in \mathbf{N}).$$

Since $\boldsymbol{R} \cdot \boldsymbol{k} \wedge \boldsymbol{R} \cdot \boldsymbol{n} = \boldsymbol{R} \cdot (\boldsymbol{k} \wedge \boldsymbol{n}) \cdot \boldsymbol{R}^{-1}$, the previous result can be written in the form

$$\mathbf{j}(\boldsymbol{R} \cdot \boldsymbol{A} \cdot \boldsymbol{R}^{-1}) = (\det\boldsymbol{R})\boldsymbol{R} \cdot \mathbf{j}(\boldsymbol{A}) \qquad (\boldsymbol{A} \in \mathbf{N} \wedge \mathbf{N}).$$

Moreover,

$$\mathrm{j_o}\left(\overset{3}{\underset{i=1}{\wedge}} \boldsymbol{R} \cdot \boldsymbol{k}_i \right) = (\det\boldsymbol{R}) \ \mathrm{j_o}\left(\overset{3}{\underset{i=1}{\wedge}} \boldsymbol{k}_i \right).$$

3.17. In the usual way, the linear bijection $\mathbf{j}$ can be lifted to a linear bijection

$$\mathbf{E} \wedge \mathbf{E} \to \mathbf{E} \otimes \mathbf{D}, \qquad \text{defined by} \qquad \boldsymbol{x} \wedge \boldsymbol{y} \mapsto \mathbf{j}\left(\frac{\boldsymbol{x}}{\boldsymbol{m}} \wedge \frac{\boldsymbol{y}}{\boldsymbol{m}}\right) \boldsymbol{m}^2$$

where $\boldsymbol{m}$ is an arbitrary non-zero element of $\mathbf{D}$.

Similarly, the linear bijection $\mathrm{j_o}$ can be lifted to a linear bijection

$$\overset{3}{\wedge}\, \boldsymbol{E} \to \overset{3}{\otimes}\, \mathbf{D}, \qquad \overset{3}{\underset{i=1}{\wedge}} \boldsymbol{x}_i \mapsto \mathrm{j_o}\left(\overset{3}{\underset{i=1}{\wedge}} \frac{\boldsymbol{x}_i}{\boldsymbol{m}}\right) \boldsymbol{m}^3.$$

We have utilized here that $\mathbf{E} = \mathbf{N} \otimes \mathbf{D}$. Evidently, similar formulae are valid for $\mathbf{N} \otimes \mathbf{A}$ where $\mathbf{A}$ is an arbitrary one-dimensional vector space.

3.18. Let us consider the Euclidean vector space $(\mathbb{R}^3, \mathbb{R}, \mathrm{B})$ where

$$\mathrm{B}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{3} x^i y^i =: \boldsymbol{x} \cdot \boldsymbol{y}$$

(i.e. $\mathrm{B} = \mathrm{H}_0$ in the notation of 1.7).

The identification $\mathbb{R}^3 \equiv \left(\mathbb{R}^3\right)^*$ is the usual one: the functional corresponding to $\boldsymbol{x}$ and represented by the usual matrix multiplication rule coincides with $\boldsymbol{x}$. In customary notations $\boldsymbol{x}$ considered to be a vector has the components (x^1, x^2, x^3) and $\boldsymbol{x}$ considered to be a covector has the components (x_1, x_2, x_3); the previous statement says that $x_i = x^i$, $i = 1, 2, 3$.

That is why in this case one usually writes only subscripts.

The adjoint of a 3×3 matrix (as a linear map $\mathbb{R}^3 \to \mathbb{R}^3$) coincides with the transpose of the matrix.

$\mathbb{R}^3$ and $\mathbb{R}$ are endowed with the usual orientations: the naturally ordered standard bases are taken to be positively oriented.

The Levi–Civita tensor is given by a matrix of three indices:

$$\varepsilon = (\varepsilon_{ijk} \mid i, j, k = 1, 2, 3),$$

$$\varepsilon(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} x_i y_j z_k,$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases}$$

Then it is an easy task to show that

$$\mathbf{j}\left(L_{jk} \mid j, k = 1, 2, 3\right) = \left(-\frac{1}{2} \sum_{j,k=1}^{3} \varepsilon_{ijk} L_{jk} \,\middle|\, i = 1, 2, 3\right),$$

$$\mathbf{j}^{-1}\left(x_k \mid k = 1, 2, 3\right) = \left(-\sum_{k=1}^{3} \varepsilon_{ijk} x_k \,\middle|\, i, j = 1, 2, 3\right),$$

in other notation,

$$\mathbf{j}^{-1}(x_1, x_2, x_3) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

Moreover,

$$\boldsymbol{x} \times \boldsymbol{y} = \left(\sum_{j,k=1}^{3} \varepsilon_{ijk} x_j y_k | \; i = 1, 2, 3 \right).$$

3.19. Consider the Euclidean vector space $(\mathbf{E}, \mathbf{D}, \mathbf{b})$, $\dim \mathbf{E} = 3$.

A linear coordinatization $\boldsymbol{K}$ of $\mathbf{E}$ is called *orthogonal* if it corresponds to an ordered orthogonal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ normed to an $\boldsymbol{m} \in \mathbf{D}$.

Since the dual of the basis is $\left(\frac{\boldsymbol{e}_i}{\boldsymbol{m}^2} | \; i = 1, 2, 3\right)$ (see 3.4), we have

$$\boldsymbol{K} \cdot \boldsymbol{x} = \left(\frac{\boldsymbol{e}_i \cdot \boldsymbol{x}}{\boldsymbol{m}^2} | \; i = 1, 2, 3 \right) =: \left(x^1, x^2, x^3\right).$$

Consider the identification $\mathbf{E} \equiv \mathbf{D} \otimes \mathbf{D} \otimes \mathbf{E}^* \equiv \left(\frac{\mathbf{E}}{\mathbf{D} \otimes \mathbf{D}}\right)^*$; then $\boldsymbol{x}$, as an element of the dual of $\frac{\mathbf{E}}{\mathbf{D} \otimes \mathbf{D}}$, has the coordinates

$$\left(\boldsymbol{x} \cdot \frac{\boldsymbol{e}_i}{\boldsymbol{m}^2} | \; i = 1, 2, 3 \right) =: (x_1, x_2, x_3).$$

We see, in accordance with the previous paragraph, that $x^i = x_i$ $(i = 1, 2, 3)$ and we can use only subscripts.

Then all the operations regarding the Euclidean structure can be represented by the corresponding operations in $(\mathbb{R}^3, \mathbb{R}, \mathrm{B})$, e.g.

— the $\mathbf{b}$-product of elements $\boldsymbol{x}, \boldsymbol{y}$ of $\mathbf{E}$ is computed by the inner product of their coordinates in $\mathbb{R}^3$:

$$\text{if} \quad \boldsymbol{K} \cdot \boldsymbol{x} = (x_1, x_2, x_3) \quad \text{and} \quad \boldsymbol{K} \cdot \boldsymbol{y} = (y_1, y_2, y_3)$$

i.e.

$$\boldsymbol{x} = \sum_{i=1}^{3} x_i \boldsymbol{e}_i, \qquad \boldsymbol{y} = \sum_{i=1}^{3} y_i \boldsymbol{e}_i$$

$$\text{then} \quad \boldsymbol{x} \cdot \boldsymbol{y} = \left(\sum_{i=1}^{3} x_i y_i \right) \boldsymbol{m}^2;$$

— the matrix in the coordinatization of an adjoint map will be the transpose of the matrix representing the linear map in question:

$$\begin{aligned} \text{if} \qquad & \boldsymbol{K}\cdot\boldsymbol{L}\cdot\boldsymbol{K}^{-1}=(L_{ik}|\ i,k=1,2,3) \\ \text{then} \qquad & \boldsymbol{K}\cdot\boldsymbol{L}^{*}\cdot\boldsymbol{K}^{-1}=(L_{ki}|\ i,k=1,2,3)\,; \end{aligned}$$

— if both $\mathbf{E}$ and $\mathbf{D}$ are oriented and the basis establishing the coordinatization is positively oriented then the vectorial product can be computed by the vectorial product of coordinates:

$$\boldsymbol{K}\cdot(\boldsymbol{x}\times\boldsymbol{y})=\left(\sum_{j,k=1}^{3}\varepsilon_{ijk}x_j y_k|\ i=1,2,3\right).$$

These statements fail, in general, for a non-orthogonal coordinatization.

3.20. Let $(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)$ be an arbitrary ordered basis in $\mathbf{E}$, chose a positive element $\boldsymbol{m}$ of $\mathbf{D}$ and put

$$\mathrm{b}_{ik}:=\frac{\boldsymbol{v}_i\cdot\boldsymbol{v}_k}{\boldsymbol{m}^2}\ \left(:=\frac{\mathbf{b}(\boldsymbol{v}_i,\boldsymbol{v}_k)}{\boldsymbol{m}^2}\right)\in\mathbb{R}\qquad(i,k=1,2,3).$$

The dual of the basis can be represented by vectors $\boldsymbol{r}^1,\boldsymbol{r}^2,\boldsymbol{r}^3$ in $\frac{\mathbf{E}}{\mathbf{D}\otimes\mathbf{D}}$— usually called the *reciprocal system* of the given basis — in such a way that

$$\boldsymbol{r}^i\cdot\boldsymbol{v}_k=\delta_{ik}\qquad(i,k=1,2,3).$$

It is not hard to see that

$$\boldsymbol{r}^1:=\frac{\varepsilon(\cdot,\boldsymbol{v}_2,\boldsymbol{v}_3)}{\boldsymbol{\Delta}},\quad\text{etc.}$$

where $\boldsymbol{\Delta}:=\varepsilon(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)$.

Put

$$\mathrm{b}^{ik}:=(\boldsymbol{r}^i\cdot\boldsymbol{r}^k)\boldsymbol{m}^2\in\mathbb{R}\qquad(i,k=1,2,3).$$

Let us take the coordinatization $\boldsymbol{K}$ of $\mathbf{E}$ defined by the basis $(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)$. Now we must distinguish between subscripts and superscripts. We agree to write the elements of $\mathbb{R}^3$ in the form (x^i) and the elements of $(\mathbb{R}^3)^*$ in the form (x_i). Then

$$\boldsymbol{K}\cdot\boldsymbol{x}=\left(\boldsymbol{r}^i\cdot\boldsymbol{x}\right)=:\left(x^i\right).$$

Consider the identification $\mathbf{E}\equiv\mathbf{D}\otimes\mathbf{D}\otimes\mathbf{E}^*\equiv\left(\frac{\mathbf{E}}{\mathbf{D}\otimes\mathbf{D}}\right)^*$; then $\left(\frac{\boldsymbol{v}_i}{\boldsymbol{m}^2}|\ i=1,2,3\right)$ is an ordered basis in $\frac{\mathbf{E}}{\mathbf{D}\otimes\mathbf{D}}$ and $\boldsymbol{x}$, as an element of the dual of $\frac{\mathbf{E}}{\mathbf{D}\otimes\mathbf{D}}$, has the coordinates

$$\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_i}{\boldsymbol{m}^2}\right)=:(x_i)\,.$$

Writing $\boldsymbol{x} = \sum_{i=1}^{3} x^k \boldsymbol{v}_k = \sum_{k=1}^{3} x_k \boldsymbol{r}^k \boldsymbol{m}^2$ we find that

$$x_i = \sum_{k=1}^{3} \mathrm{b}_{ik} x^k, \qquad x^k = \sum_{i=1}^{3} \mathrm{b}^{ki} x_i,$$

i.e., in general, $x^i \neq x_i$.

Now if

$$\boldsymbol{K} \cdot \boldsymbol{x} = \left(x^i\right) \qquad \text{and} \qquad \boldsymbol{K} \cdot \boldsymbol{y} = \left(y^i\right)$$

i.e.

$$\boldsymbol{x} = \sum_{i=1}^{3} x^i \boldsymbol{v}_i, \qquad \boldsymbol{y} = \sum_{i=1}^{3} y^i \boldsymbol{v}_i$$

then

$$\boldsymbol{x} \cdot \boldsymbol{y} = \left(\sum_{i,k=1}^{3} \mathrm{b}_{ik} x^i y^k \right) \boldsymbol{m}^2 = \left(\sum_{k=1}^{3} x_k y^k \right) \boldsymbol{m}^2 = \left(\sum_{i=1}^{3} x^i y_i \right) \boldsymbol{m}^2.$$

3.21. Exercises

In the following we keep assuming that $\dim \mathbf{E} = 3$.

1. Let $\boldsymbol{A}$ be a non-zero element of $\mathrm{A}(\mathbf{b})$. If $\boldsymbol{x}$ is a non-zero vector in $\mathbf{E}$, orthogonal to Ker $\boldsymbol{A}$, then

(i) $\boldsymbol{A}^2 \cdot \boldsymbol{x} = -|\boldsymbol{A}|^2 \boldsymbol{x}$,

(ii) $|\boldsymbol{A} \cdot \boldsymbol{x}| = |\boldsymbol{A}||\boldsymbol{x}|$,

(iii) $\boldsymbol{A} = \frac{(\boldsymbol{A}\cdot\boldsymbol{x})\wedge\boldsymbol{x}}{|\boldsymbol{x}|^2}$.

2. Show that Ker $(\boldsymbol{A}^2) =$ Ker $\boldsymbol{A}$ for $\boldsymbol{A} \in \mathbf{N} \wedge \mathbf{N} \equiv \mathrm{A}(\mathbf{b})$.

3. Prove that

$$(i) \qquad \begin{aligned} \varepsilon_{ijk}\varepsilon_{rst} =& \delta_{ir}\delta_{js}\delta_{kt} + \delta_{is}\delta_{jt}\delta_{kr} + \delta_{it}\delta_{jr}\delta_{ks} \\ & -\delta_{it}\delta_{js}\delta_{kr} - \delta_{is}\delta_{jr}\delta_{kt} - \delta_{ir}\delta_{jt}\delta_{ks}, \end{aligned}$$

$$(ii) \qquad \sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr},$$

$$(iii) \qquad \sum_{j,k=1}^{3} \varepsilon_{ijk}\varepsilon_{rjk} = 2\delta_{ir}.$$

4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be one-dimensional vector spaces. Define the vectorial product
$$(\mathbf{N} \otimes \mathbf{A}) \times (\mathbf{N} \otimes \mathbf{B}) \to \mathbf{N} \otimes \mathbf{A} \otimes \mathbf{B}.$$

4. Minkowskian vector spaces

4.1. A pseudo-Euclidean vector space $(\mathbf{V}, \mathbf{B}, \mathbf{h})$ is called *Minkowskian* if $\dim \mathbf{V} > 1$ and $\neg(\mathbf{h}) = 1$.

For a clear distinction, in the following $(\mathbf{M}, \mathbf{I}, \mathbf{g})$ denotes a Minkowskian vector space, and
$$\dim \mathbf{M} = 1 + N$$
where $N \geq 1$.

We call attention to the fact that $\mathbf{g}$ is usually called a Lorentz metric; since $\mathbf{g}$ does not define a metric (distances and angles, see later) we prefer to call it a *Lorentz form.*

The notations introduced for pseudo-Euclidean vector spaces will be used, e.g.
$$\boldsymbol{x} \cdot \boldsymbol{y} := \mathbf{g}(\boldsymbol{x}, \boldsymbol{y}) \qquad\qquad (\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{M});$$
note that if $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{M}$ and $\boldsymbol{u}, \boldsymbol{v} \in \frac{\mathbf{M}}{\mathbf{I}}$ then
$$\boldsymbol{x} \cdot \boldsymbol{y} \in \mathbf{I} \otimes \mathbf{I}, \qquad \boldsymbol{u} \cdot \boldsymbol{x} \in \mathbf{I}, \qquad \boldsymbol{u} \cdot \boldsymbol{v} \in \mathbb{R}.$$

Moreover, we put
$$\boldsymbol{x}^2 := \boldsymbol{x} \cdot \boldsymbol{x} \qquad\qquad (\boldsymbol{x} \in \mathbf{M}).$$

In contradistinction to Euclidean spaces, here we keep saying $\mathbf{g}$-orthogonal, $\mathbf{g}$-adjoint etc.

The elements of a $\mathbf{g}$-orthogonal basis will be numbered from 0 to N, in such a way that for $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\}$ we have $\boldsymbol{e}_0{}^2 < \mathbf{0}$, $\boldsymbol{e}_i{}^2 > \mathbf{0}$ if $i = 1, \ldots, N$.

4.2. Recall that there is a canonical orientation on $\mathbf{I} \otimes \mathbf{I}$, hence it makes sense that an element of $\mathbf{I} \otimes \mathbf{I}$ is positive or negative. Let us introduce the notations
$$\begin{aligned}
\mathrm{S} &:= \{\boldsymbol{x} \in \mathbf{M} \mid \boldsymbol{x}^2 > \mathbf{0}\}, & \mathrm{S}_\mathbf{0} &:= \mathrm{S} \cup \{\mathbf{0}\}, \\
\mathrm{T} &:= \{\boldsymbol{x} \in \mathbf{M} \mid \boldsymbol{x}^2 < \mathbf{0}\}, & \mathrm{T}_\mathbf{0} &:= \mathrm{T} \cup \{\mathbf{0}\}, \\
\mathrm{L} &:= \{\boldsymbol{x} \in \mathbf{M} \mid \boldsymbol{x}^2 = \mathbf{0}, \boldsymbol{x} \neq \mathbf{0}\} & \mathrm{L}_\mathbf{0} &:= \mathrm{L} \cup \{\mathbf{0}\}.
\end{aligned}$$
The elements of $\mathrm{S}_\mathbf{0}$, T and L are called *spacelike, timelike* and *lightlike* vectors, respectively.

Neither of $\mathrm{S}_\mathbf{0}$, $\mathrm{T}_\mathbf{0}$ and $\mathrm{L}_\mathbf{0}$ is a linear subspace.

The bilinear map $\mathbf{g}$ is continuous (see VI.3.1). Thus S and T are open subsets, $\mathrm{L}_{\mathbf{0}}$ is a closed subset.

4.3. Take a non-zero element $\boldsymbol{x}$ of $\mathbf{M}$. The Lorentz form $\mathbf{g}$ is non-degenerate, hence the linear map $\mathbf{M} \to \mathbf{I} \otimes \mathbf{I}$, $\boldsymbol{y} \mapsto \boldsymbol{x} \cdot \boldsymbol{y}$ is a surjection, i.e. it has an N-dimensional kernel. In other words,

$$\mathbf{H}_{\boldsymbol{x}} := \{\boldsymbol{y} \in \mathbf{M} |\ \boldsymbol{x} \cdot \boldsymbol{y} = \mathbf{0}\}$$

is an N-dimensional linear subspace of $\mathbf{M}$. Let $\mathbf{g}_{\boldsymbol{x}}$ be the restriction of $\mathbf{g}$ onto $\mathbf{H}_{\boldsymbol{x}} \times \mathbf{H}_{\boldsymbol{x}}$; it is an $\mathbf{I} \otimes \mathbf{I}$-valued symmetric bilinear map.

(i) Suppose $\boldsymbol{x} \in \mathrm{S}$ or $\boldsymbol{x} \in \mathrm{T}$. Then $\boldsymbol{x}$ is not in $\mathbf{H}_{\boldsymbol{x}}$. $\mathbb{R}\boldsymbol{x}$ and $\mathbf{H}_{\boldsymbol{x}}$ are complementary subspaces. As a consequence, $\mathbf{g}_{\boldsymbol{x}}$ is non-degenerate, i.e. $(\mathbf{H}_{\boldsymbol{x}}, \mathbf{I}, \mathbf{g}_{\boldsymbol{x}})$ is an N-dimensional pseudo-Euclidean vector space. Thus there is a $\mathbf{g}_{\boldsymbol{x}}$-orthogonal basis in $\mathbf{H}_{\boldsymbol{x}}$; such a basis, supplemented by $\boldsymbol{x}$, will be a $\mathbf{g}$-orthogonal basis in $\mathbf{M}$.

— if $\boldsymbol{x}$ is in S then $\boldsymbol{x}^2 > \mathbf{0}$, so one and only one element of a $\mathbf{g}_{\boldsymbol{x}}$-orthogonal basis belongs to T, the other ones belong to S. Consequently, $(\mathbf{H}_{\boldsymbol{x}}, \mathbf{I}, \mathbf{g}_{\boldsymbol{x}})$ is an N-dimensional Minkowskian vector space.

— if $\boldsymbol{x}$ is in T then $\boldsymbol{x}^2 < \mathbf{0}$, so all the elements of a $\mathbf{g}_{\boldsymbol{x}}$-orthogonal basis belong to S. Consequently, $\mathbf{H}_{\boldsymbol{x}} \subset \mathrm{S}_{\mathbf{0}}$ and $(\mathbf{H}_{\boldsymbol{x}}, \mathbf{I}, \mathbf{g}_{\boldsymbol{x}})$ is an N-dimensional Euclidean vector space.

(ii) Suppose $\boldsymbol{x} \in \mathrm{L}$. Then $\boldsymbol{x}$ itself is in $\mathbf{H}_{\boldsymbol{x}}$, in other words, $\mathbb{R}\boldsymbol{x}$ is contained in $\mathbf{H}_{\boldsymbol{x}}$. One cannot give naturally a subspace complementary to $\mathbf{H}_{\boldsymbol{x}}$. Moreover, $\mathbf{g}_{\boldsymbol{x}}$ is degenerate.

Let $\boldsymbol{e}_0$ be an element of T; let $\boldsymbol{s}$ be an element of $\mathbf{I}$ such that $\boldsymbol{e}_0^2 = -\boldsymbol{s}^2$. As we have seen, $\mathbf{H}_{\boldsymbol{e}_0}$ is contained in $\mathrm{S}_{\mathbf{0}}$, so $\boldsymbol{e}_0 \cdot \boldsymbol{x} \neq \mathbf{0}$, and $\boldsymbol{e}_1 := \frac{\boldsymbol{e}_0 \cdot \boldsymbol{e}_0}{\boldsymbol{e}_0 \cdot \boldsymbol{x}} \boldsymbol{x} - \boldsymbol{e}_0$ belongs to S, $\boldsymbol{e}_0 \cdot \boldsymbol{e}_1 = \mathbf{0}$ and $\boldsymbol{e}_1{}^2 = \boldsymbol{s}^2$. $\{\boldsymbol{e}_0, \boldsymbol{e}_1\}$ can be completed to a $\mathbf{g}$-orthogonal basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\}$, normed to $\boldsymbol{s}$, of $\mathbf{M}$. The vector $\boldsymbol{x}$ is a linear combination of $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$, thus $\{\boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\}$ is contained in $\mathbf{H}_{\boldsymbol{x}}$ and $\{\boldsymbol{x}, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N\}$ is a basis of $\mathbf{H}_{\boldsymbol{x}}$.

This has the immediate consequence that every element of $\mathbf{H}_{\boldsymbol{x}}$ which is not parallel to $\boldsymbol{x}$ belongs to $\mathrm{S}_{\mathbf{0}}$.

4.4. It follows from 4.3(i) that if $\boldsymbol{x} \in \mathrm{T}$ then $\boldsymbol{x} \cdot \boldsymbol{y} \neq \mathbf{0}$ for all $\boldsymbol{y} \in \mathrm{T}$ and for all $\boldsymbol{y} \in \mathrm{L}$.

Moreover, the results in the preceding paragraph imply that

— there are N-dimensional linear subspaces in $\mathrm{S}_{\mathbf{0}}$,

— there are at most one-dimensional linear subspaces in $\mathrm{T}_{\mathbf{0}}$ and $\mathrm{L}_{\mathbf{0}}$,

— there is a one-to-one correspondence between N-dimensional linear subspaces in $\mathrm{S}_{\mathbf{0}}$ and one-dimensional linear subspaces in $\mathrm{T}_{\mathbf{0}}$ in such a way that the subspaces in correspondence are $\mathbf{g}$-orthogonal to each other.

4.5. The identification

$$\frac{\mathbf{M}}{\mathbf{I} \otimes \mathbf{I}} \equiv \mathbf{M}^*$$

(see 1.3) is a fundamental property of the Minkowskian vector space $(\mathbf{M}, \mathbf{I}, \mathbf{g})$.

The dual of a $\mathbf{g}$-orthogonal basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_N\}$, normed to $\boldsymbol{s} \in \mathbf{I}$, in this identification becomes $\left\{-\frac{\boldsymbol{e}_0}{\boldsymbol{s}^2}, \frac{\boldsymbol{e}_1}{\boldsymbol{s}^2}, \dots, \frac{\boldsymbol{e}_N}{\boldsymbol{s}^2}\right\}$.

Accordingly, $\boldsymbol{n}_i := \frac{\boldsymbol{e}_i}{\boldsymbol{s}}$ $(i = 0, 1, \dots, N)$ form a $\mathbf{g}$-orthonormal basis in $\frac{\mathbf{M}}{\mathbf{I}}$:

$$\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 = -1, \qquad \boldsymbol{n}_i \cdot \boldsymbol{n}_k = \delta_{ik} \qquad (i, k = 1, \dots, N).$$

The corresponding dual basis in the identification $\frac{\mathbf{M}}{\mathbf{I}} \equiv \left(\frac{\mathbf{M}}{\mathbf{I}}\right)^*$ is $\{-\boldsymbol{n}_0, \boldsymbol{n}_1, \dots, \boldsymbol{n}_N\}$.

For all $\boldsymbol{x} \in \mathbf{M}$ we have

$$\boldsymbol{x} = -(\boldsymbol{n}_0 \cdot \boldsymbol{x})\boldsymbol{n}_0 + \sum_{i=1}^{N} (\boldsymbol{n}_i \cdot \boldsymbol{x})\boldsymbol{n}_i.$$

4.6. The following relation will be a starting point of important results. If $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{T} \cup \mathrm{L}$, $\boldsymbol{x}$ is not parallel to $\boldsymbol{y}$, $\boldsymbol{z} \in \mathrm{T}$, then

$$2(\boldsymbol{x} \cdot \boldsymbol{y})(\boldsymbol{y} \cdot \boldsymbol{z})(\boldsymbol{z} \cdot \boldsymbol{x}) < (\boldsymbol{x} \cdot \boldsymbol{z})^2 \boldsymbol{y}^2 + (\boldsymbol{y} \cdot \boldsymbol{z})^2 \boldsymbol{x}^2 \leq \mathbf{0}.$$

This is implied by the simple fact that $\boldsymbol{a} := \frac{\boldsymbol{y} \cdot \boldsymbol{z}}{\boldsymbol{x} \cdot \boldsymbol{z}} \boldsymbol{x} - \boldsymbol{y}$ is $\mathbf{g}$-orthogonal to $\boldsymbol{z}$, thus $\boldsymbol{a}$ is in $\mathrm{S} : \boldsymbol{a}^2 > \mathbf{0}$.

4.7. Since $\mathbf{I} \otimes \mathbf{I}$ is canonically oriented (see IV.5.4), the absolute value of its elements makes sense.

Proposition (reversed Cauchy inequality). If $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{T}$ then

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \geq \sqrt{|\boldsymbol{x}^2||\boldsymbol{y}^2|} > \mathbf{0}$$

and equality holds if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are parallel.

Proof. Put $\boldsymbol{z} := \boldsymbol{x}$ in the previous formula, and recall that we have the square root mapping from $(\mathbf{I} \otimes \mathbf{I}) \otimes (\mathbf{I} \otimes \mathbf{I})$ into $\mathbf{I} \otimes \mathbf{I}$. ■

In general, the right-hand side of this equality cannot be written in a simpler form because $|\boldsymbol{x}|$ and $|\boldsymbol{y}|$ make no sense, unless $\mathbf{I}$ is oriented.

4.8. Definition. The elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of T *have the same arrow* if $\boldsymbol{x} \cdot \boldsymbol{y} < \mathbf{0}$.

Proposition. Having the same arrow is an equivalence relation on T and there are two equivalence classes (called *arrow classes).*

Proof. The relation having the same arrow is evidently reflexive and symmetric. Suppose now that $\boldsymbol{x}$ and $\boldsymbol{y}$ as well as $\boldsymbol{y}$ and $\boldsymbol{z}$ have the same arrow.

Then 4.6 implies that $\boldsymbol{x}$ and $\boldsymbol{z}$ have the same arrow as well, hence the relation is transitive.

Let $\boldsymbol{x}$ be an element of T. It is obvious that $\boldsymbol{x}$ and $-\boldsymbol{x}$ have not the same arrow: there are at least two arrow classes. On the other hand, since $\boldsymbol{x}\cdot\boldsymbol{y}\neq\mathbf{0}$ for all $\boldsymbol{y}\in\mathrm{T}$, $\boldsymbol{y}$ and $\boldsymbol{x}$ or $-\boldsymbol{y}$ and $\boldsymbol{x}$ have the same arrow: there are at most two arrow classes.

4.9. Proposition. The arrow classes of T are convex cones, i.e. if $\boldsymbol{x}$ and $\boldsymbol{y}$ have the same arrow then $\alpha\boldsymbol{x}+\beta\boldsymbol{y}$ is in their arrow class for all $\alpha,\beta\in\mathbb{R}^+$.

Proof. It is quite evident that $(\alpha\boldsymbol{x}+\beta\boldsymbol{y})^2<\mathbf{0}$ and $(\alpha\boldsymbol{x}+\beta\boldsymbol{y})\cdot\boldsymbol{x}<\mathbf{0}$, thus $\alpha\boldsymbol{x}+\beta\boldsymbol{y}$ is in T, moreover, $\alpha\boldsymbol{x}+\beta\boldsymbol{y}$ and $\boldsymbol{x}$ have the same arrow. ■

The arrow classes are open subsets of $\mathbf{M}$ because the arrow class of $\boldsymbol{x}\in\mathrm{T}$ is $\{\boldsymbol{y}\in\mathrm{T}\,|\,\boldsymbol{x}\cdot\boldsymbol{y}<\mathbf{0}\}$.

4.10. Suppose now that $\mathbf{I}$ is oriented. Then we can take the square root of non-negative elements of $\mathbf{I}\otimes\mathbf{I}$, so we define the *pseudo-length* of vectors:

$$|\boldsymbol{x}|:=\sqrt{|\boldsymbol{x}^2|}\qquad(\boldsymbol{x}\in\mathbf{M}).$$

The length of vectors in Euclidean vector spaces has the fundamental properties listed in 3.3. Now we find that

(i) $|\boldsymbol{x}|=\mathbf{0}$ if $\boldsymbol{x}=\mathbf{0}$ but $|\boldsymbol{x}|=\mathbf{0}$ does not imply $\boldsymbol{x}=\mathbf{0}$;

(ii) $|\alpha\boldsymbol{x}|=|\alpha||\boldsymbol{x}|$ for all $\alpha\in\mathbb{R}$;

(iii) there is no definite relation between $|\boldsymbol{x}+\boldsymbol{y}|$ and $|\boldsymbol{x}|+|\boldsymbol{y}|$:

— if $\mathbf{H}\subset\mathrm{S}_\mathbf{0}$ is a linear subspace then $\left(\mathbf{H},\mathbf{I},\mathbf{g}|_{\mathbf{H}\times\mathbf{H}}\right)$ is a Euclidean vector space, consequently, for $\boldsymbol{x},\boldsymbol{y}\in\mathbf{H}$ the triangle inequality $|\boldsymbol{x}+\boldsymbol{y}|\leq|\boldsymbol{x}|+|\boldsymbol{y}|$ holds,

— for vectors in T a reverse relation can hold, as follows.

Proposition (reversed triangle inequality). If the elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of T have the same arrow then

$$|\boldsymbol{x}+\boldsymbol{y}|\geq|\boldsymbol{x}|+|\boldsymbol{y}|$$

and equality holds if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are parallel.

Proof. According to the previous statement $\boldsymbol{x}+\boldsymbol{y}$ belongs to T, thus we can apply the reversed Cauchy inequality:

$$\begin{aligned}|\boldsymbol{x}+\boldsymbol{y}|^2=-(\boldsymbol{x}+\boldsymbol{y})^2=&|\boldsymbol{x}|^2-2(\boldsymbol{x}\cdot\boldsymbol{y})+|\boldsymbol{y}|^2\geq\\ \geq&|\boldsymbol{x}|^2+2\sqrt{|\boldsymbol{x}|^2|\boldsymbol{y}|^2}+|\boldsymbol{y}|^2=(|\boldsymbol{x}|+|\boldsymbol{y}|)^2.\end{aligned}\qquad■$$

The triangle inequality and “non-zero vector has non-zero length” are indispensable properties of a length; that is why we use the name pseudo-length.

4.11. Definition. The elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of L *have the same arrow* if $\boldsymbol{x}\cdot\boldsymbol{y} < 0$.

Proposition. Having the same arrow is an equivalence relation on L and there are two equivalence classes (called *arrow classes).*

Proof. Argue as in 4.8.

4.12. Now we relate the arrow classes of L to those of T. It is evident that the elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of L have the same arrow if and only if $\alpha\boldsymbol{x} + \beta\boldsymbol{y}$ are in T and have the same arrow for all $\alpha, \beta \in \mathbb{R}^+$.

Proposition. (*i*) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{L}$, $\boldsymbol{z} \in \mathrm{T}$. Then $\boldsymbol{x}$ and $\boldsymbol{y}$ have the same arrow if and only if $\boldsymbol{x}\cdot\boldsymbol{z}$ and $\boldsymbol{y}\cdot\boldsymbol{z}$ have the same sign (in the ordered one-dimensional vector space $\mathbf{I}\otimes\mathbf{I}$).

(*ii*) Let $\boldsymbol{x} \in \mathrm{L}$, $\boldsymbol{y}, \boldsymbol{z} \in \mathrm{T}$. Then $\boldsymbol{y}$ and $\boldsymbol{z}$ have the same arrow if and only if $\boldsymbol{x}\cdot\boldsymbol{y}$ and $\boldsymbol{x}\cdot\boldsymbol{z}$ have the same sign.

Proof. Apply the inequality in 4.6. ∎

As a consequence, the arrow classes of T and those of L determine each other uniquely. We say that the elements $\boldsymbol{x}$ of L and $\boldsymbol{y}$ of T have the same arrow if $\boldsymbol{x}\cdot\boldsymbol{y} < 0$. According to the previous proposition, if we select an arrow class $\mathrm{T}^{\rightarrow}$ from T then there is an arrow class $\mathrm{L}^{\rightarrow}$ in L such that all the elements of $\mathrm{T}^{\rightarrow}$ and $\mathrm{L}^{\rightarrow}$ have the same arrow:

$$\mathrm{L}^{\rightarrow} = \{\boldsymbol{y} \in \mathrm{L} |\ \boldsymbol{x}\cdot\boldsymbol{y} < \mathbf{0},\ \boldsymbol{x} \in \mathrm{T}^{\rightarrow}\}.$$

It can be shown that $\mathrm{L}^{\rightarrow} \cup \{\mathbf{0}\}$ is the boundary of $\mathrm{T}^{\rightarrow}$.

4.13. We say that $(\mathbf{M}, \mathbf{I}, \mathbf{g})$ is *arrow-oriented* or an *arrow orientation is associated* with $\mathbf{g}$ if we select one of the arrow classes of T. More precisely, an arrow-oriented Minkowskian vector space is $(\mathbf{M}, \mathbf{I}, \mathbf{g}, \mathrm{T}^{\rightarrow})$ where $(\mathbf{M}, \mathbf{I}, \mathbf{g})$ is a Minkowskian vector space and $\mathrm{T}^{\rightarrow}$ is one of the arrow classes of T.

A linear isomorphism between arrow-oriented Minkowskian vector spaces is called *arrow-preserving* or *arrow-reversing* if it maps the chosen arrow classes into each other or into the opposite ones, respectively.

4.14. In the following we assume that $\mathbf{M}$ and $\mathbf{I}$ are oriented and $\mathbf{g}$ is arrow-oriented; moreover

$$\dim \mathbf{M} = 4.$$

We introduce the notation

$$\mathrm{V}(1) := \left\{\boldsymbol{u} \in \frac{\mathbf{M}}{\mathbf{I}} |\ \boldsymbol{u}^2 = -1,\ \boldsymbol{u}\otimes\mathbf{I}^+ \subset \mathrm{T}^{\rightarrow}\right\}.$$

If $\boldsymbol{u} \in \mathrm{V}(1)$ then

$$\boldsymbol{u} \otimes \mathbf{I} := \{\boldsymbol{u}t \mid t \in \mathbf{I}\} \subset \mathrm{T}_0,$$
$$\mathbf{E}_{\boldsymbol{u}} := \{\boldsymbol{x} \in \mathbf{M} \mid \boldsymbol{u} \cdot \boldsymbol{x} = \mathbf{0}\} \subset \mathrm{S}_0$$

are complementary subspaces. The corresponding projections of $\boldsymbol{x} \in \mathbf{M}$ in $\boldsymbol{u} \otimes \mathbf{I}$ and in $\mathbf{E}_{\boldsymbol{u}}$ are

$$-(\boldsymbol{u} \cdot \boldsymbol{x})\boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{x} + (\boldsymbol{u} \cdot \boldsymbol{x})\boldsymbol{u}.$$

Let $\mathbf{b}_{\boldsymbol{u}}$ denote the restriction of $\mathbf{g}$ onto $\mathbf{E}_{\boldsymbol{u}} \times \mathbf{E}_{\boldsymbol{u}}$. According to 4.3.$(i)$ — $\boldsymbol{us} \in \mathrm{T}$ for $\boldsymbol{s} \in \mathbf{I}$ and $\mathbf{E}_{\boldsymbol{u}} = \mathbf{H}_{\boldsymbol{su}}$— we have that $(\mathbf{E}_{\boldsymbol{u}}, \mathbf{I}, \mathbf{b}_{\boldsymbol{u}})$ is a three-dimensional Euclidean vector space.

4.15. We shall examine the structure of $\mathbf{g}$-antisymmetric linear maps of $\mathbf{M}$.

As we know (see 2.9) $\mathrm{A}(\mathbf{g}) \equiv \frac{\mathbf{M}}{\mathbf{I}} \wedge \frac{\mathbf{M}}{\mathbf{I}}$ is a six-dimensional vector space endowed (see 2.10) with a real-valued non-degenerate symmetric bilinear form:

$$\boldsymbol{H} \bullet \boldsymbol{G} := \frac{1}{2}\mathrm{Tr}(\boldsymbol{H}^* \cdot \boldsymbol{G}) = -\frac{1}{2}\mathrm{Tr}(\boldsymbol{H} \cdot \boldsymbol{G}).$$

In particular, for $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{n}_1, \boldsymbol{n}_2 \in \frac{\mathbf{M}}{\mathbf{I}}$ we have

$$(\boldsymbol{k}_1 \wedge \boldsymbol{k}_2) \bullet (\boldsymbol{n}_1 \wedge \boldsymbol{n}_2) = (\boldsymbol{k}_1 \cdot \boldsymbol{n}_1)(\boldsymbol{k}_2 \cdot \boldsymbol{n}_2) - (\boldsymbol{k}_1 \cdot \boldsymbol{n}_2)(\boldsymbol{k}_2 \cdot \boldsymbol{n}_1).$$

If $\{\boldsymbol{n}_0, \boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$ is a $\mathbf{g}$-orthonormal basis of $\frac{\mathbf{M}}{\mathbf{I}}$ then

$$\begin{array}{lll} \boldsymbol{n}_0 \wedge \boldsymbol{n}_1, & \boldsymbol{n}_0 \wedge \boldsymbol{n}_2, & \boldsymbol{n}_0 \wedge \boldsymbol{n}_3, \\ \boldsymbol{n}_1 \wedge \boldsymbol{n}_2, & \boldsymbol{n}_2 \wedge \boldsymbol{n}_3, & \boldsymbol{n}_3 \wedge \boldsymbol{n}_1 \end{array}$$

constitute a basis in $\frac{\mathbf{M}}{\mathbf{I}} \wedge \frac{\mathbf{M}}{\mathbf{I}}$ (see IV.3.15). We can take $\boldsymbol{u} := \boldsymbol{n}_0 \in \mathrm{V}(1)$; then every $\mathbf{g}$-antisymmetric map can be written in the form $\sum_{i=1}^{3} \alpha_i \boldsymbol{u} \wedge \boldsymbol{n}_i + \sum_{i=1}^{3} \sum_{k<i} \alpha_{ki} \boldsymbol{n}_k \wedge \boldsymbol{n}_i$. The vectors $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$ span the three-dimensional Euclidean vector space $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ (more precisely, $\left(\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}, \mathbb{R}, \cdot\right)$), hence according to the results of the previous chapter there are a real number β and unit vectors $\boldsymbol{k}$ and $\boldsymbol{n}$, $\mathbf{g}$-orthogonal to each other and to $\boldsymbol{u}$ such that $\sum_{i=1}^{3} \sum_{k<i} \alpha_{ki} \boldsymbol{n}_k \wedge \boldsymbol{n}_i = \beta \boldsymbol{k} \wedge \boldsymbol{n}$. Furthermore, $\sum_{i=1}^{3} \alpha_i \boldsymbol{u} \wedge \boldsymbol{n}_i = \boldsymbol{u} \wedge \sum_{i=1}^{3} \alpha_i \boldsymbol{n}_i$; thus we arrive at the following result.

Proposition. Let $\boldsymbol{H}$ be an element of $\frac{\mathbf{M}}{\mathbf{I}} \wedge \frac{\mathbf{M}}{\mathbf{I}}$. Then for all $\boldsymbol{u} \in \mathrm{V}(1)$ there are $\alpha, \beta \in \mathbb{R}$, $\boldsymbol{r}, \boldsymbol{k}, \boldsymbol{n} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $\boldsymbol{r}^2 = \boldsymbol{k}^2 = \boldsymbol{n}^2 = 1$, $\boldsymbol{k} \cdot \boldsymbol{n} = 0$ such that

$$\boldsymbol{H} = \alpha \boldsymbol{u} \wedge \boldsymbol{r} + \beta \boldsymbol{k} \wedge \boldsymbol{n}. \quad \blacksquare$$

Observe that then

$$\boldsymbol{H} \bullet \boldsymbol{H} = -\alpha^2 + \beta^2.$$

4.16. Proposition. Take an $\boldsymbol{H} \in \frac{\mathrm{M}}{\mathrm{I}} \wedge \frac{\mathrm{M}}{\mathrm{I}}$ in the form given by the previous proposition. Then Ker $\boldsymbol{H} = \{\mathbf{0}\}$ if and only if $\alpha \neq 0$, $\beta \neq 0$ and $\boldsymbol{r}$ is linearly independent from $\boldsymbol{k}$ and $\boldsymbol{n}$.

Proof. If $\boldsymbol{r}$ is linearly independent from $\boldsymbol{k}$ and $\boldsymbol{n}$ then $\boldsymbol{u}$, $\boldsymbol{r}$, $\boldsymbol{k}$, $\boldsymbol{n}$ are linearly independent vectors. Furthermore, if neither of α and β is zero then for all $\boldsymbol{x} \in \mathrm{M}$

$$\boldsymbol{H} \cdot \boldsymbol{x} = \alpha(\boldsymbol{r} \cdot \boldsymbol{x})\boldsymbol{u} - \alpha(\boldsymbol{u} \cdot \boldsymbol{x})\boldsymbol{r} + \beta(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{k} - \beta(\boldsymbol{k} \cdot \boldsymbol{x})\boldsymbol{n} = \mathbf{0}$$

implies $\boldsymbol{r} \cdot \boldsymbol{x} = \boldsymbol{u} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{k} \cdot \boldsymbol{x} = \mathbf{0}$; as a consequence, $\boldsymbol{x} = \mathbf{0}$. This means that Ker $\boldsymbol{H} = \{\mathbf{0}\}$.

If $\alpha = 0$ then $\boldsymbol{H} \cdot \boldsymbol{u} = \mathbf{0}$; if $\beta = 0$ then $\boldsymbol{H} \cdot \boldsymbol{m} = \mathbf{0}$ for $\boldsymbol{m} \in \frac{\mathrm{M}}{\mathrm{I}}$, $\boldsymbol{u} \cdot \boldsymbol{m} = 0$, $\boldsymbol{r} \cdot \boldsymbol{m} = 0$. If $\alpha \neq 0$ and $\beta \neq 0$ but $\boldsymbol{r}$ is a linear combination of $\boldsymbol{k}$ and $\boldsymbol{n}$ then $\boldsymbol{H} \cdot \boldsymbol{m} = \mathbf{0}$ for $\boldsymbol{m} \in \frac{\mathrm{M}}{\mathrm{I}}$, $\boldsymbol{u} \cdot \boldsymbol{m} = \boldsymbol{k} \cdot \boldsymbol{m} = \boldsymbol{n} \cdot \boldsymbol{m} = \mathbf{0}$. This means that Ker $\boldsymbol{H} \neq \{\mathbf{0}\}$. ■

Since if $\boldsymbol{k}'$ and $\boldsymbol{n}'$ are $\mathbf{g}$-orthogonal unit vectors in the plane spanned by $\boldsymbol{k}$ and $\boldsymbol{n}$ (do not forget that $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ is a Euclidean vector space) then $\boldsymbol{k}' \wedge \boldsymbol{n}' = \pm \boldsymbol{k} \wedge \boldsymbol{n}$, we can choose $\boldsymbol{n} = \boldsymbol{r}$ if Ker $\boldsymbol{H} \neq \{\mathbf{0}\}$; then for all $\boldsymbol{u} \in \mathrm{V}(1)$ there are $\alpha, \beta \in \mathbb{R}$ and $\boldsymbol{k}, \boldsymbol{n} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $\boldsymbol{k}^2 = \boldsymbol{n}^2 = 1$, $\boldsymbol{k} \cdot \boldsymbol{n} = 0$ such that

$$\boldsymbol{H} = (\alpha \boldsymbol{u} + \beta \boldsymbol{k}) \wedge \boldsymbol{n}.$$

4.17. Proposition. Suppose $\boldsymbol{H} \in \frac{\mathrm{M}}{\mathrm{I}} \wedge \frac{\mathrm{M}}{\mathrm{I}}$, Ker $\boldsymbol{H} \neq \{\mathbf{0}\}$ and put $|\boldsymbol{H}| := \sqrt{|\boldsymbol{H} \bullet \boldsymbol{H}|}$. Then

(i) $\boldsymbol{H} \bullet \boldsymbol{H} > 0$ if and only if there are a $\boldsymbol{u} \in \mathrm{V}(1)$, $\boldsymbol{k}, \boldsymbol{n} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $\boldsymbol{k}^2 = \boldsymbol{n}^2 = 1$, $\boldsymbol{k} \cdot \boldsymbol{n} = 0$ such that

$$\boldsymbol{H} = |\boldsymbol{H}| \boldsymbol{k} \wedge \boldsymbol{n}.$$

($\boldsymbol{H}$ is the antisymmetric tensor product of two $\mathbf{g}$-orthogonal spacelike vectors.)

(ii) $\boldsymbol{H} \bullet \boldsymbol{H} < 0$ if and only if there are a $\boldsymbol{u} \in \mathrm{V}(1)$, an $\boldsymbol{n} \in \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$, $\boldsymbol{n}^2 = 1$ such that

$$\boldsymbol{H} = |\boldsymbol{H}| \boldsymbol{u} \wedge \boldsymbol{n}.$$

($\boldsymbol{H}$ is the antisymmetric tensor product of a timelike vector and a spacelike vector, $\mathbf{g}$-orthogonal to each other.)

(iii) $\boldsymbol{H} \bullet \boldsymbol{H} = 0$, $\boldsymbol{H} \neq \mathbf{0}$ if and only if there are $\boldsymbol{w}, \boldsymbol{n} \in \frac{\mathrm{M}}{\mathrm{I}}$, $\boldsymbol{w} \neq \mathbf{0}$, $\boldsymbol{w}^2 = 0$, $\boldsymbol{n}^2 = 1$, $\boldsymbol{w} \cdot \boldsymbol{n} = 0$ such that

$$\boldsymbol{H} = \boldsymbol{w} \wedge \boldsymbol{n}.$$

($\boldsymbol{H}$ is the antisymmetric tensor product of a lightlike vector and a spacelike vector, $\mathbf{g}$-orthogonal to each other.)

Proof. Let us write the formula of the preceding paragraph in the form $\boldsymbol{H} = (\alpha\boldsymbol{u}' + \beta\boldsymbol{k}') \wedge \boldsymbol{n}'$; then $\boldsymbol{H} \bullet \boldsymbol{H} = -\alpha^2 + \beta^2$.

(i) Put $\quad \boldsymbol{u} := \dfrac{\beta\boldsymbol{u}' + \alpha\boldsymbol{k}'}{-\alpha^2 + \beta^2}, \quad \boldsymbol{k} := \dfrac{\alpha\boldsymbol{u}' + \beta\boldsymbol{k}'}{-\alpha^2 + \beta^2}, \quad \boldsymbol{n} := \boldsymbol{n}'.$

(ii) Put $\quad \boldsymbol{u} := \dfrac{\alpha\boldsymbol{u}' + \beta\boldsymbol{k}'}{\alpha^2 - \beta^2}, \quad \boldsymbol{n} := \boldsymbol{n}'.$

(iii) Put $\quad \boldsymbol{w} := \alpha\boldsymbol{u}' + \beta\boldsymbol{k}', \quad \boldsymbol{n} := \boldsymbol{n}'.$

4.18. (i) We see that Ker $\boldsymbol{H} \neq \{\mathbf{0}\}$ and $\boldsymbol{H} \bullet \boldsymbol{H} > 0$ is equivalent to the statement that there is a $\boldsymbol{u} \in \mathrm{V}(1)$ such that $\boldsymbol{H} \cdot \boldsymbol{u} = \mathbf{0}$.

Note that then $\boldsymbol{H}^3 = -|\boldsymbol{H}|^2\boldsymbol{H}$.

(ii) On the contrary, if Ker $\boldsymbol{H} \neq \{\mathbf{0}\}$ and $\boldsymbol{H} \bullet \boldsymbol{H} < 0$ then $\boldsymbol{H}^3 = |\boldsymbol{H}|^2\boldsymbol{H}$.

4.19. According to our convention introduced in 4.1, let us number the coordinates of elements of $\mathbb{R}^{1+3}$ from 0 to 3 and let us consider the Minkowskian vector space $(\mathbb{R}^{1+3}, \mathbb{R}, \mathrm{G})$ where

$$\mathrm{G}(\boldsymbol{x}, \boldsymbol{y}) = -x^0y^0 + \sum_{i=1}^{3} x^iy^i =: \boldsymbol{x} \cdot \boldsymbol{y}$$

(i.e. $\mathrm{G} = \mathrm{H}_1$ in the notation of 1.7).

Now the identification $\mathbb{R}^{1+3} \equiv \left(\mathbb{R}^{1+3}\right)^*$ induced by G is described in usual notations as follows. $\boldsymbol{x} \in \mathbb{R}^{1+3}$ regarded as a vector has the components (x^0, x^1, x^2, x^3); $\boldsymbol{x}$ regarded as a covector has the components (x_0, x_1, x_2, x_3), and the values of the linear functional $\boldsymbol{x}$ are computed by the usual matrix multiplication:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=0}^{3} x_iy^i.$$

Then we have

$$x_0 = -x^0, \qquad x_i = x^i \qquad (i = 1, 2, 3).$$

As usual, we apply the symbols (x^i) and (x_i) for the vectors and covectors and we accept the *Einstein summation rule:* a summation from 0 to 3 is to be carried out for equal subscripts and superscripts.

Introducing

$$\mathrm{g}_{ik} := \mathrm{g}^{ik} := \begin{cases} -1 & \text{if } i = k = 0 \\ \ \ 1 & \text{if } i = k \in \{1, 2, 3\} \\ \ \ 0 & \text{if } i \neq k \end{cases}$$

we can write that

$$x^i = \mathrm{g}^{ik} x_k, \qquad x_i = \mathrm{g}_{ik} x^k \qquad\qquad \text{(summation!)}.$$

Observe that $\mathrm{g}_{ik} = \mathrm{G}(\boldsymbol{\chi}_i, \boldsymbol{\chi}_k)$ where $\{\boldsymbol{\chi}_0, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3\}$ is the standard basis of $\mathbb{R}^{1+3}$.

According to the identification induced by G, the dual of the standard basis $\{\boldsymbol{\chi}_0, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3\}$ is $\{-\boldsymbol{\chi}_0, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3\}$.

It is useful to regard G as the diagonal matrix in which the first ("zeroth") element in the diagonal is -1 and the other ones equal 1.

For the G-adjoint $\boldsymbol{L}^{\circledast}$ of the linear map (matrix) $\boldsymbol{L}$ we have

$$\boldsymbol{L}^{\circledast} = \mathrm{G} \cdot \boldsymbol{L}^* \cdot \mathrm{G}$$

where $\boldsymbol{L}^*$ is the transpose of $\boldsymbol{L}$ (see 1.7).

A linear map $\boldsymbol{L} : \mathbb{R}^{1+3} \to \mathbb{R}^{1+3}$ is given by its matrix $\left(L^i{}_k\right)$,

a linear map $\boldsymbol{P} : (\mathbb{R}^{1+3})^* \to (\mathbb{R}^{1+3})^*$ is given by its matrix $\left(P_i{}^k\right)$ etc. see IV.1.6.

For the transpose of $L : \mathbb{R}^{1+3} \to \mathbb{R}^{1+3}$

$$(L^*)_i{}^k = L^k{}_i,$$

holds, thus for the G-adjoint we have

$$(L^{\circledast})^i{}_k = \mathrm{g}^{im} L^n{}_m \mathrm{g}_{nk} \qquad\qquad \text{(summation!)}.$$

Consequently, a G-antisymmetric linear map has the form

$$\begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & 0 & -\beta_3 & \beta_2 \\ \alpha_2 & \beta_3 & 0 & -\beta_1 \\ \alpha_3 & -\beta_2 & \beta_1 & 0 \end{pmatrix}$$

If (x^i) is in T, i.e. $x^i \cdot x_i < 0$ then $x^0 \neq 0$. It is not hard to see that (x^i) and (y^i) have the same arrow if and only if x^0 and y^0 have the same sign. As a consequence, an arrow class is characterized by the sign of the zeroth component of its element. One usually takes the arrow orientation in such a way that

$$\mathrm{T}^{\rightarrow} := \{(x^i) \in \mathbb{R}^{1+3} \,|\, x^i x_i < 0\,,\ x^0 > 0\}.$$

4.20. Consider the four-dimensional Minkowskian vector space $(\mathbf{M}, \mathbf{I}, \mathbf{g})$. A linear coordinatization $\boldsymbol{K}$ of $\mathbf{M}$ is called $\mathbf{g}$-orthogonal if it corresponds to an ordered $\mathbf{g}$-orthogonal basis $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ normed to an $\boldsymbol{s} \in \mathbf{I}$. According to the identification $\mathbf{M}^* \equiv \frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$, the basis in question has the dual $\left(-\frac{\boldsymbol{e}_0}{\boldsymbol{s}^2}, \frac{\boldsymbol{e}_1}{\boldsymbol{s}^2}, \frac{\boldsymbol{e}_2}{\boldsymbol{s}^2}, \frac{\boldsymbol{e}_3}{\boldsymbol{s}^2}\right)$; thus we have

$$\boldsymbol{K}\cdot\boldsymbol{x} = \left(-\frac{\boldsymbol{e}_0\cdot\boldsymbol{x}}{\boldsymbol{s}^2}, \frac{\boldsymbol{e}_1\cdot\boldsymbol{x}}{\boldsymbol{s}^2}, \frac{\boldsymbol{e}_2\cdot\boldsymbol{x}}{\boldsymbol{s}^2}, \frac{\boldsymbol{e}_3\cdot\boldsymbol{x}}{\boldsymbol{s}^2}\right) =: \left(x^i\right) \qquad (\boldsymbol{x}\in\mathbf{M}).$$

Consider the identification $\mathbf{M} \equiv \mathbf{I}\otimes\mathbf{I}\otimes\mathbf{M}^* \equiv \left(\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}\right)^*$; then $\boldsymbol{x}$, as an element of the dual of $\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$, has the coordinates

$$\left(\boldsymbol{x}\cdot\frac{\boldsymbol{e}_i}{\boldsymbol{s}^2} \,\middle|\, i=0,1,2,3\right) =: (x_i)\,.$$

We see, in accordance with the previous paragraph, that $x^0 = -x_0$ and $x^i = x_i$ $(i=1,2,3)$.

Then all the operations regarding the Minkowskian structure can be represented by the corresponding operations in $(\mathbb{R}^{1+3}, \mathbb{R}, \mathrm{G})$, e.g.

— the $\mathbf{g}$-product of elements $\boldsymbol{x}, \boldsymbol{y}$ of $\mathbf{M}$ is computed by the G-product of their coordinates in $\mathbb{R}^{1+3}$:

$$\text{if} \qquad \boldsymbol{K}\cdot\boldsymbol{x} = \left(x^i\right) \qquad \text{and} \qquad \boldsymbol{K}\cdot\boldsymbol{y} = \left(y^i\right)$$

i.e.
$$\boldsymbol{x} = \sum_{0=1}^{3} x^i\boldsymbol{e}_i, \qquad \boldsymbol{y} = \sum_{0=1}^{3} y^i\boldsymbol{e}_i$$

$$\text{then} \qquad \boldsymbol{x}\cdot\boldsymbol{y} = \left(-x^0y^0 + \sum_{i=1}^{3} x^iy^i\right)\boldsymbol{s}^2;$$

— the matrix in the coordinatization of a $\mathbf{g}$-adjoint map will be the G-adjoint of the matrix representing the linear map in question:

$$\begin{aligned} \text{if} \quad & \boldsymbol{K}\cdot\boldsymbol{L}\cdot\boldsymbol{K}^{-1} = \left({L^i}_k \,\middle|\, i,k=1,2,3\right) \\ \text{then} \quad & \boldsymbol{K}\cdot\boldsymbol{L}^{*}\cdot\boldsymbol{K}^{-1} = \left({L^k}_i \,\middle|\, i,k=1,2,3\right). \end{aligned}$$

4.21. Let $(\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ be an arbitrary ordered basis in $\mathbf{M}$, choose a positive element $\boldsymbol{s}$ of $\mathbf{I}$ and put

$$\mathrm{g}_{ik} := \frac{\boldsymbol{v}_i\cdot\boldsymbol{v}_k}{\boldsymbol{s}^2}\left(:= \frac{\mathbf{g}(\boldsymbol{v}_i, \boldsymbol{v}_k)}{\boldsymbol{s}^2}\right) \in \mathbb{R} \qquad (i,k=0,1,2,3).$$

The dual of the basis can be represented by vectors $\boldsymbol{r}^0, \boldsymbol{r}^1, \boldsymbol{r}^2, \boldsymbol{r}^3$ in $\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$ — usually called the *reciprocal system* of the given basis — in such a way that

$$\boldsymbol{r}^i\cdot\boldsymbol{v}_k = \delta_{ik} \qquad (i,k=0,1,2,3).$$

It is not hard to see that

$$\boldsymbol{r}^0 := \frac{\varepsilon(\cdot, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)}{\boldsymbol{\Delta}} \quad \text{etc.}$$

where ε is the Levi–Civita tensor (see V.2.12) and $\boldsymbol{\Delta} := \varepsilon(\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$.

Put

$$\mathrm{g}^{ik} := (\boldsymbol{r}^i \cdot \boldsymbol{r}^k)\boldsymbol{s}^2 \in \mathbb{R} \qquad (i, k = 0, 1, 2, 3).$$

Let us take the coordinatization $\boldsymbol{K}$ of $\mathbf{M}$ defined by the basis $(\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$. Then

$$\boldsymbol{K} \cdot \boldsymbol{x} = \left(\boldsymbol{r}^i \cdot \boldsymbol{x}\right) =: \left(x^i\right).$$

Consider the identification $\mathbf{M} \equiv \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{M}^* \equiv \left(\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}\right)^*$; then $\left(\frac{\boldsymbol{v}_i}{\boldsymbol{s}^2} \middle| i = 0, 1, 2, 3\right)$ is an ordered basis in $\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$, and $\boldsymbol{x}$, as an element of the dual of $\frac{\mathbf{M}}{\mathbf{I}\otimes\mathbf{I}}$, has the coordinates

$$\left(\boldsymbol{x} \cdot \frac{\boldsymbol{v}_i}{\boldsymbol{s}^2}\right) =: (x_i).$$

Writing $\boldsymbol{x} = \sum_{i=0}^{3} x^k \boldsymbol{v}_k = \sum_{k=0}^{3} x_k \boldsymbol{r}^k \boldsymbol{s}^2$ we find that

$$x_i = \mathrm{g}_{ik} x^k, \qquad x^k = \mathrm{g}^{ki} x_i \qquad\qquad \text{(summation!)}.$$

Now if

$$\boldsymbol{K} \cdot \boldsymbol{x} = \left(x^i\right) \qquad \text{and} \qquad \boldsymbol{K} \cdot \boldsymbol{y} = \left(y^i\right)$$

i.e.

$$\boldsymbol{x} = \sum_{i=0}^{3} x^i \boldsymbol{v}_i, \qquad \boldsymbol{y} = \sum_{i=0}^{3} y^i \boldsymbol{v}_i$$

then

$$\boldsymbol{x} \cdot \boldsymbol{y} = \left(\mathrm{g}_{ik} x^i y^k\right) \boldsymbol{s}^2 = \left(x_k y^k\right) \boldsymbol{s}^2 = \left(x^i y_i\right) \boldsymbol{s}^2.$$

4.22. Exercises

1. Let $\mathrm{T}^{\rightarrow}$ and $\mathrm{L}^{\rightarrow}$ be the arrow classes corresponding to each other according to 4.12. Prove that

$$\mathrm{T}^{\rightarrow} + \mathrm{L}^{\rightarrow} = \mathrm{T}^{\rightarrow},$$
$$\mathrm{L}^{\rightarrow} + \mathrm{L}^{\rightarrow} = \mathrm{T}^{\rightarrow} \cup \mathrm{L}^{\rightarrow}.$$

2. If $\boldsymbol{x}$ is in $\mathrm{T}^{\rightarrow}$ and $\boldsymbol{y}$ is in S then $\boldsymbol{x} + \boldsymbol{y}$ is not in $\mathrm{T}^{\leftarrow}$. (Hint: suppose $-(\boldsymbol{x} + \boldsymbol{y}) \in \mathrm{T}^{\rightarrow}$ and use $\mathrm{T}^{\rightarrow} + \mathrm{T}^{\rightarrow} = \mathrm{T}^{\rightarrow}$.)

3. If $\mathbf{H}$ is a linear subspace in $\mathrm{S_0}$ then $\left(\mathbf{H}, \mathbf{I}, \mathbf{g}|_{\mathbf{H}\times\mathbf{H}}\right)$ is a Euclidean vector space. Consequently, the length of vectors and the angle between vectors in $\mathbf{H}$ makes sense. Since every $\boldsymbol{x} \in \mathrm{S_0}$ belongs to some linear subspace in $\mathrm{S_0}$ (e.g. to the linear subspace generated by $\boldsymbol{x}$), the length of every element in $\mathrm{S_0}$ makes sense. On the other hand, if $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{S_0}$, the linear subspace generated by $\boldsymbol{x}$ and $\boldsymbol{y}$ need not be contained in $\mathrm{S_0}$; as a consequence, the angle between two elements of $\mathrm{S_0}$ may not be meaningful.

Take a $\mathbf{g}$-orthogonal basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_N\}$, normed to $\boldsymbol{s} \in \mathbf{I}$. Then $\boldsymbol{e}_1$ and $\boldsymbol{x} := 2\boldsymbol{e}_1 + \boldsymbol{e}_0$ are vectors in S that do not satisfy the Cauchy inequality and the triangle inequality.

4. Suppose $\dim \mathbf{M} = 4$, $\mathbf{M}$ and $\mathbf{I}$ are oriented. Then the Levi–Civita tensor of $(\mathbf{M}, \mathbf{I}, \mathbf{g})$ can be defined by $\varepsilon := \bigwedge_{i=0}^{3} \frac{\boldsymbol{e}_i}{\boldsymbol{s}}$ where $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a positively oriented ordered basis, normed to $\boldsymbol{s} \in \mathbf{I}$.

Prove that

$$\frac{\mathbf{M}}{\mathbf{I}} \to \overset{3}{\wedge} \frac{\mathbf{M}}{\mathbf{I}}, \qquad \boldsymbol{n} \mapsto \varepsilon(\cdot, \cdot, \cdot, \boldsymbol{n})$$

and

$$\mathbf{J} : \frac{\mathbf{M}}{\mathbf{I}} \wedge \frac{\mathbf{M}}{\mathbf{I}} \to \frac{\mathbf{M}}{\mathbf{I}} \wedge \frac{\mathbf{M}}{\mathbf{I}}, \qquad \boldsymbol{k} \wedge \boldsymbol{n} \mapsto \varepsilon(\cdot, \cdot, \boldsymbol{k}, \boldsymbol{n})$$

are linear bijections. Moreover,

$$\mathbf{J}(\mathbf{J}(\boldsymbol{H})) = -\boldsymbol{H}, \qquad \boldsymbol{H} \bullet \mathbf{J}(\boldsymbol{G}) = \mathbf{J}(\boldsymbol{G}) \bullet \boldsymbol{G},$$

thus $\mathbf{J}(\boldsymbol{H}) \bullet \mathbf{J}(\boldsymbol{G}) = -\boldsymbol{H} \bullet \boldsymbol{G} \quad$ for all $\quad \boldsymbol{H}, \boldsymbol{G} \in \frac{\mathbf{M}}{\mathbf{I}} \wedge \frac{\mathbf{M}}{\mathbf{I}}$.

5. Give the actual form of the previous bijections in the case $(\mathbb{R}^{1+3}, \mathbb{R}, \mathrm{G})$.

6. Let ε be the Levi–Civita tensor of the Minkowskian vector space $(\mathbf{M}, \mathbf{I}, \mathbf{g})$, $\dim \mathbf{M} = 4$. If $\boldsymbol{u} \in \mathrm{V}(1)$ then $\varepsilon(\boldsymbol{u}, \cdot, \cdot, \cdot)$ is the Levi–Civita tensor of the three-dimensional Euclidean vector space $(\mathbf{E}_{\boldsymbol{u}}, \mathbf{I}, \mathbf{b}_{\boldsymbol{u}})$ where $\mathbf{b}_{\boldsymbol{u}}$ is the restriction of $\mathbf{g}$ onto $\mathbf{E}_{\boldsymbol{u}} \times \mathbf{E}_{\boldsymbol{u}}$.

VI. AFFINE SPACES

1. Fundamentals

1.1. Definition. An *affine space* is a triplet $(\mathrm{V}, \mathbf{V}, -)$ where

(i) V is a non-void set,
(ii) $\mathbf{V}$ is a vector space,
(iii) $-$ is a map from $\mathrm{V} \times \mathrm{V}$ into $\mathbf{V}$, denoted by

$$(x, y) \mapsto x - y,$$

having the properties
1) for every $o \in \mathrm{V}$ the map $O_o : \mathrm{V} \to \mathbf{V}$, $x \mapsto x - o$ is bijective,
2) $(x - y) + (y - z) + (z - x) = \mathbf{0}$ for all $x, y, z \in \mathrm{V}$. ■

O_o is often called the *vectorization* of V *with origin* o.

As usual, we shall denote an affine space by a single letter; we say that V is an affine space over the vector space $\mathbf{V}$ and we call the map $-$*subtraction.*

The *dimension* of an affine space V is, by definition, the dimension of the underlying vector space $\mathbf{V}$. V is *oriented* if $\mathbf{V}$ is oriented (in this case $\mathbf{V}$ is necessarily a finite-dimensional real vector space).

Proposition. Let V be an affine space. Then

(i) $x - y = \mathbf{0}$ if and only if $x = y$ $(x, y \in \mathrm{V})$,
(ii) $x - y = -(y - x)$ $(x, y \in \mathrm{V})$,
(iii) for a natural number $n \geq 3$ and $x_1, x_2, \dots, x_n \in \mathrm{V}$,

$$(x_1 - x_2) + (x_2 - x_3) + \dots + (x_n - x_1) = \mathbf{0}.$$

Proof. (i) Put $z := x$, $y := x$ in 2) of the above definition to have $x - x = \mathbf{0}$. Property 1) says then that $x - y \neq \mathbf{0}$ for $x \neq y$.
(ii) Put $z := x$ in 2) and use the previous result.
(iii) Starting with 2) we can prove by induction. ■

As a consequence, we can rearrange the parentheses as follows:

$$(x - y) + (u - v) = (x - v) + (u - y) \qquad (x, y, u, v \in \mathrm{V}). \tag{1}$$

1.2. Observe that the sign $-$ in (ii) of the previous proposition denotes two different objects. Inside the parantheses it means the subtraction in the affine space, outside it means the subtraction in the underlying vector space. This ambiguity does not cause confusion if we are careful. We even find it convenient to increase a bit the ambiguity.

For given $y \in \mathrm{V}$, the inverse of the map O_y is denoted by

$$\mathbf{V} \to \mathrm{V}, \qquad \boldsymbol{x} \mapsto y + \boldsymbol{x}. \tag{2}$$

Hence, by definition,

$$y + (x - y) = x \qquad\qquad (x, y \in \mathrm{V}), \tag{3}$$

and a simple reasoning shows that

$$(x + \boldsymbol{x}) + \boldsymbol{y} = x + (\boldsymbol{x} + \boldsymbol{y}) \qquad\qquad (x \in \mathrm{V}, \boldsymbol{x}, \boldsymbol{y} \in \mathbf{V}). \tag{4}$$

Here the symbol $+$ on the left-hand side stands twice for the operation introduced by (2), on the right-hand side first it denotes this operation and then the addition of vectors.

Keep in mind the followings:

(i) the sum and the difference of two vectors, the multiple of a vector are meaningful, they are vectors;

(ii) the difference of two elements of the affine space is meaningful, it is a vector (sums and multiples make no sense);

(iii) the sum of an affine space element and of a vector is meaningful, it is an element of the affine space.

According to (1)–(4), we can apply the usual rules of addition and subtraction paying always attention to that the operations be meaningful; for instance, the rearrangement $(y + x) - y$ in (3) makes no sense.

1.3. Linear combinations of affine space elements cannot be defined in general, for multiples and sums make no sense. However, a good trick allows us to define convex combinations.

Proposition. Let $x_1, \dots, x_n$ be elements of the affine space V and let $\alpha_1, \dots, \alpha_n$ be non-negative real numbers such that $\sum\limits_{k=1}^{n} \alpha_k = 1$. Then there is a unique $x_{\mathrm{o}} \in \mathrm{V}$ for which

$$\sum_{k=1}^{n} \alpha_k (x_k - x_{\mathrm{o}}) = \mathbf{0}. \tag{$*$}$$

Proof. Let x be an arbitrary element of V; then a simple calculation based on $x_k - x_o = (x_k - x) + (x - x_o)$ shows that $x_o := x + \sum_{k=1}^{n} \alpha_k (x_k - x)$ satisfies equality $(*)$. Suppose y_o is another element with this property. Then

$$\mathbf{0} = \sum_{k=1}^{n} \alpha_k (x_k - x_o) - \sum_{k=1}^{n} \alpha_k (x_k - y_o) = \sum_{k=1}^{n} \alpha_k \left((x_k - x_o) - (x_k - y_o)\right) =$$
$$= \sum_{k=1}^{n} \alpha_k (y_o - x_o) = y_o - x_o. \qquad \blacksquare$$

Remove formally the parentheses in $(*)$ to arrive at the following definition.

Definition. The element x_o in the previous proposition is called the *convex combination* of the elements $x_1, \ldots, x_n$ with coefficients $\alpha_1, \ldots, \alpha_n$ and is denoted by

$$\sum_{k=1}^{n} \alpha_k x_k. \quad \blacksquare$$

Correspondingly we can define convex subsets and the convex hull of subsets in affine spaces as they are defined in vector spaces.

If $\beta_1, \ldots, \beta_n$ are non-negative real numbers, $\beta := \sum_{k=1}^{n} \beta_k > 0$, then we can take the convex combination of $x_1, \ldots, x_n$ with the coefficients $\alpha_k := \frac{\beta_k}{\beta}$ $(k = 1, \ldots, n)$ which will be denoted by

$$\frac{\sum_{k=1}^{n} \beta_k x_k}{\sum_{k=1}^{n} \beta_k}.$$

1.4. (i) A vector space $\mathbf{V}$, endowed with the vectorial subtraction, is an affine space over itself.

(ii) If $\mathbf{M}$ is a non-trivial linear subspace of the vector space $\mathbf{V}$ and $\boldsymbol{x} \in \mathbf{V}$, $\boldsymbol{x} \notin \mathbf{M}$ then $\boldsymbol{x} + \mathbf{M} := \{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{y} \in \mathrm{M}\}$ endowed with the vectorial subtraction is an affine space but is not a vector space regarding the vectorial operations in $\mathbf{V}$.

(iii) If I is an arbitrary non-void set and V_i is an affine space over $\mathbf{V}_i$ $(i \in I)$ then $\underset{i \in I}{\times} \mathrm{V}_i$, endowed with the subtraction

$$(x_i)_{i \in I} - (y_i)_{i \in I} := (x_i - y_i)_{i \in I}$$

is an affine space over $\underset{i \in I}{\times} \mathbf{V}_i$.

1.5. Definition. A non-void subset S of an affine space V is called an *affine subspace* if there is a linear subspace $\mathbf{S}$ of $\mathbf{V}$ such that $\{x - y |\ x, y \in \mathrm{S}\} = \mathbf{S}$.

S is called *directed* by $\mathbf{S}$ and the dimension of S is that of $\mathbf{S}$.

One-dimensional and two-dimensional affine subspaces of a real affine space are called *straight lines* and *planes,* respectively. *Hyperplanes* are affine subspaces having the dimension of V but one, in a finite dimensional affine space V.

Two affine subspaces are said to be *parallel* if they are directed by the same linear subspace. ■

An affine subspace S directed by $\mathbf{S}$, endowed with the subtraction inherited from V, is an affine space over $\mathbf{S}$.

If $\mathbf{S}$ is a linear subspace of $\mathbf{V}$ and $x \in \mathrm{V}$ then $x + \mathbf{S} := \{x + \boldsymbol{s} |\ \boldsymbol{s} \in \mathbf{S}\}$ is the unique affine subspace containing x and directed by $\mathbf{S}$.

Points of V are zero-dimensional affine subspaces.

1.6. A pseudo-Euclidean (Euclidean, Minkowskian) affine space is a triplet $(\mathrm{V}, \mathbf{B}, \mathbf{h})$ where V is an affine space over the vector space $\mathbf{V}$ and $(\mathbf{V}, \mathbf{B}, \mathbf{h})$ is a pseudo-Euclidean (Euclidean, Minkowskian) vector space.

1.7. Exercises

1. Prove that the following definition of affine spaces is equivalent to that given in 1.1.

A triplet $(\mathrm{V}, \mathbf{V}, +)$ is an *affine space* if

(i) V is a non-void set,

(ii) $\mathbf{V}$ is a vector space,

(iii) $+$ is a map from $\mathrm{V} \times \mathbf{V}$ into V, denoted by

$$(x, \boldsymbol{x}) \mapsto x + \boldsymbol{x}$$

having the properties

1) $(x + \boldsymbol{x}) + \boldsymbol{y} = x + (\boldsymbol{x} + \boldsymbol{y}) \qquad (x \in \mathrm{V}, \boldsymbol{x}, \boldsymbol{y} \in \mathbf{V})$,

2) for every $x \in \mathrm{V}$ the map $\mathbf{V} \to \mathrm{V}$, $\boldsymbol{x} \mapsto x + \boldsymbol{x}$ is bijective.

2. Let V be an affine space over $\mathbf{V}$. Let $\mathrm{V}/\mathbf{N}$ denote the set of affine subspaces in V, directed by a given linear subspace $\mathbf{N}$ of $\mathbf{V}$. If $\mathbf{M}$ is a linear subspace complementary to $\mathbf{N}$ then $\mathrm{V}/\mathbf{N}$ becomes an affine space over $\mathbf{M}$ if we define the subtraction by

$$\mathrm{S} - \mathrm{T} := x - y \qquad\qquad (x \in \mathrm{S},\ y \in \mathrm{T},\ x - y \in \mathbf{M}).$$

In other words, if $\boldsymbol{P}$ denotes the projection onto $\mathbf{M}$ along $\mathbf{N}$ then

$$(x + \mathbf{N}) - (y + \mathbf{N}) := \boldsymbol{P} \cdot (x - y).$$

Illustrate this fact by $\mathrm{V} := \mathbb{R}^2$, $\mathbf{N} := \{(\alpha, 0)|\ \alpha \in \mathbb{R}\}$, $\mathbf{M} := \{(\alpha, m\alpha)|\ \alpha \in \mathbb{R}\}$ where m is a given non-zero number.

3. Prove that the intersection of affine subspaces is an affine subspace, thus the affine subspace generated by a subset of an affine space is meaningful.

4. Let $\mathbf{V}$ be a vector space over the field $\mathbb{K}$. Then $\{1\} \times \mathbf{V}$ is an affine subspace of the vector space $\mathbb{K} \times \mathbf{V}$.

5. Let I be a one-dimensional oriented affine space over the vector space $\mathbf{I}$. Then an order can be defined on I by $a < b$ if and only if $a - b < \mathbf{0}$. Define the intervals of I.

2. Affine maps

2.1. Definition. Let V and U be affine spaces over $\mathbf{V}$ and $\mathbf{U}$, respectively. A map $L : \mathrm{V} \to \mathrm{U}$ is called *affine* if there is a linear map $\boldsymbol{L} : \mathbf{V} \to \mathbf{U}$ such that

$$L(y) - L(x) = \boldsymbol{L} \cdot (y - x) \qquad (x, y, \in \mathrm{V}).$$

We say that L is an affine map over $\mathbf{L}$. If L is a bijection, V and U are oriented, L is called *orientation-preserving* or *orientation-reversing* if $\boldsymbol{L}$ has that property. ■

The formula above is equivalent to

$$L(x + \boldsymbol{x}) = L(x) + \boldsymbol{L} \cdot \boldsymbol{x} \qquad (x \in V, \boldsymbol{x} \in \mathbf{V}).$$

It is easy to show that the linear map $\boldsymbol{L}$ in the definition is unique.

2.2. Proposition. Let $L : \mathrm{V} \to \mathrm{U}$ be an affine map. Then

(i) L is injective or surjective if and only if $\boldsymbol{L}$ is injective or surjective, respectively; if L is bijective then L^{-1} is an affine bijection over $\boldsymbol{L}^{-1}$;

(ii) $\boldsymbol{L} = \mathbf{0}$ if and only if L is a constant map;

(iii) Ran L is an affine subspace of U, directed by Ran $\boldsymbol{L}$;

(iv) if Z is an affine subspace of U, directed by $\mathbf{Z}$, and $(\mathrm{Ran}\ L) \cap \mathrm{Z} \neq \emptyset$ then $\overset{-1}{L}(\mathrm{Z})$ is an affine subspace of V, directed by $\overset{-1}{\boldsymbol{L}}(\mathbf{Z})$;

(v) L preserves convex combinations. ■

Observe that according to (iv), for all $u \in \mathrm{Ran}\ L$, $\overset{-1}{L}(\{u\})$ is an affine subspace of V, directed by Ker $\boldsymbol{L}$.

2.3. Proposition. (i) If L and K are affine maps such that $K \circ L$ exists then $K \circ L$ is affine map over $\boldsymbol{K} \cdot \boldsymbol{L}$;

(ii) Let I be a non-void set. If $L_i : \mathrm{V}_i \to \mathrm{U}_i$ ($i \in I$) are affine maps, then $\underset{i\in I}{\times} L_i$ is an affine map over $\underset{i\in I}{\times} \boldsymbol{L}_i$;

(iii) If $L_i : \mathrm{V} \to \mathrm{U}_i$ ($i \in I$) are affine maps then $(L_i)_{i\in I}$ is an affine map over $(\boldsymbol{L}_i)_{i\in I}$;

(iv) If L and K are affine maps from V into U then

$$K - L : V \to \boldsymbol{U}, \qquad x \mapsto K(x) - L(x)$$

is an affine map over $\boldsymbol{K} - \boldsymbol{L}$ (recall that the vector space $\mathbf{U}$ is an affine space over itself).

2.4. (i) Let $\mathbf{V}$ and $\mathbf{U}$ be vector spaces and consider them to be affine spaces. Take a linear map $\boldsymbol{L} : \mathbf{V} \to \mathbf{U}$ and an $\boldsymbol{a} \in \mathbf{U}$; then $\mathbf{V} \to \mathbf{U}$, $\boldsymbol{x} \mapsto \boldsymbol{a} + \boldsymbol{L} \cdot \boldsymbol{x}$ is an affine map over $\boldsymbol{L}$.

Conversely, suppose $L : \mathbf{V} \to \mathbf{U}$ is an affine map over the linear map $\boldsymbol{L}$. Put $\boldsymbol{a} := L(\mathbf{0})$. Then $L(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{L} \cdot \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbf{V}$.

Thus we have proved:

Proposition. We can identify the set of affine maps from $\mathbf{V}$ into $\mathbf{U}$ with $\{(\boldsymbol{a}, \boldsymbol{L}) |\ \boldsymbol{a} \in \mathbf{U},\ \boldsymbol{L} \in \mathrm{Lin}(\mathbf{V}, \mathbf{U})\} = \mathbf{U} \times \mathrm{Lin}(\mathbf{V}, \mathbf{U})$ in such a way that

$$(\boldsymbol{a}, \boldsymbol{L})(\boldsymbol{x}) := \boldsymbol{a} + \boldsymbol{L} \cdot \boldsymbol{x} \qquad\qquad (\boldsymbol{x} \in \boldsymbol{L}). \quad \blacksquare$$

Such an affine map $(\boldsymbol{a}, \boldsymbol{L}) : \mathbf{V} \to \mathbf{U}$ can be represented in more suitable ways, as follows.

(ii) $\mathbf{V} \to \mathbb{K}\times\mathbf{V}$, $\boldsymbol{x} \mapsto (1, \boldsymbol{x})$ (see Exercise 1.7.4) is an affine injection. We often find convenient to identify $\mathbf{V}$, considered to be an affine space, with $\{1\} \times \mathbf{V}$.

Take an affine map $(\boldsymbol{a}, \boldsymbol{L}) : \mathbf{V} \to \mathbf{U}$ and consider it to be an affine map from $\{1\}\times\mathbf{V}$ into $\{1\}\times\mathbf{U}$. It can be uniquely extended to a linear map $\mathbb{K}\times\mathbf{V} \to \mathbb{K}\times\mathbf{U}$, $(\alpha, \boldsymbol{x}) \mapsto (\alpha, \alpha\boldsymbol{a} + \boldsymbol{L} \cdot \boldsymbol{x})$.

Representing the linear maps from $\mathbb{K}\times\mathbf{V}$ into $\mathbb{K}\times\mathbf{U}$ by a matrix (see IV.3.7), we can write the extension of the affine map $(\boldsymbol{a}, \boldsymbol{L})$ in the form

$$\begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{L} \end{pmatrix}.$$

(iii) It often occurs that the vector space $\mathbf{V}$ is regarded as an affine space (i.e. we use only its affine structure, the subtraction of vectors) but the vector space $\mathbf{U}$ is continued to be regarded as a vector space (i.e. we use its vectorial structure, the sum of vectors and the multiple of vectors).

In this case we identify $\mathbf{V}$ with $\{1\} \times \mathbf{V}$ and $\mathbf{U}$ with $\{0\} \times \mathbf{U}$, and so we can conceive that $(\boldsymbol{a}, \boldsymbol{L})$ maps from $\{1\} \times \mathbf{V}$ into $\{0\} \times \mathbf{U}$. This map can be uniquely extended to a linear map $\mathbb{K} \times \mathbf{V} \to \mathbb{K} \times \mathbf{U}$, $(\alpha, \boldsymbol{x}) \mapsto (0, \alpha\boldsymbol{a} + \boldsymbol{L} \cdot \boldsymbol{x})$. Then the affine map $(\boldsymbol{a}, \boldsymbol{L})$ in a matrix representation has the form

$$\begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{L} \end{pmatrix}.$$

2.5. Exercises

1. Let o be an element of the affine space V. Then $O_o : \mathrm{V} \to \mathbf{V}$, $x \mapsto x - o$ is an affine map over $\mathrm{id}_{\mathbf{V}}$.

Consequently, if V is N-dimensional then there are affine bijections $\mathrm{V} \to \mathbb{K}^N$.

2. Let $L : \mathrm{V} \to \mathrm{V}$ be an affine map and o an element of V. Then $O_o \circ L \circ {O_o}^{-1}$ is an affine map $\mathbf{V} \to \mathbf{V}$. Using the matrix form given in the preceding paragraph show that

$$O_o \circ L \circ {O_o}^{-1} = \begin{pmatrix} 1 & 0 \\ L(o) - o & \boldsymbol{L} \end{pmatrix}.$$

3. Let $H : \mathrm{V} \to \mathbf{V}$ be an affine map and o an element of V. Then $H \circ {O_o}^{-1}$ is an affine map $\mathbf{V} \to \mathbf{V}$. Then the vector space $\mathbf{V}$ as the domain of this affine map is considered to be an affine space (representing the affine space V); and $\mathbf{V}$ as the range is considered to be a vector space. Using the matrix form given in the preceding paragraph show that

$$H \circ {O_o}^{-1} = \begin{pmatrix} 0 & \mathbf{0} \\ H(o) & \boldsymbol{H} \end{pmatrix}.$$

4. The matrix forms of affine maps $\mathbf{V} \to \mathbf{V}$ is extremely useful for obtaining the composition of such maps because we can apply the usual matrix multiplication rule. Find the composition of

(i) $(\boldsymbol{a}, \boldsymbol{L}) : \{1\} \times \mathbf{V} \to \{1\} \times \mathbf{V}$ and $(\boldsymbol{b}, \boldsymbol{K}) : \{1\} \times \mathbf{V} \to \{1\} \times \mathbf{V}$,

(ii) $(\boldsymbol{a}, \boldsymbol{L}) : \{1\} \times \mathbf{V} \to \{1\} \times \mathbf{V}$ and $(\boldsymbol{b}, \boldsymbol{K}) : \{1\} \times \mathbf{V} \to \{0\} \times \mathbf{V}$.

5. Let V be an affine space over $\mathbf{V}$.

(i) If $\boldsymbol{a} \in \mathbf{V}$ then $T_{\boldsymbol{a}} : \mathrm{V} \to \mathrm{V}$, $x \mapsto x + \boldsymbol{a}$ is an affine map over $\mathrm{id}_{\mathbf{V}}$.

(ii) If $L : \mathrm{V} \to \mathrm{V}$ is an affine map over $\mathrm{id}_{\mathbf{V}}$ then there is an $\boldsymbol{a} \in \mathbf{V}$ such that $L = T_{\boldsymbol{a}}$.

(iii) For all $x, y \in \mathrm{V}$ we have $O_y \circ {O_x}^{-1} = T_{x-y}$.

6. If $L : \mathrm{V} \to \mathrm{V}$ is an affine map over $-\mathrm{id}_{\mathbf{V}}$ then there is an $o \in \mathrm{V}$ such that $L(x) = o - (x - o)$ $(x \in \mathrm{V})$.

7. Let K and L be affine maps between the same affine spaces. Show that $\boldsymbol{K} = \boldsymbol{L}$ if and only if $K - L$ is a constant map.

8. Let $K, L, A : \mathrm{V} \to \mathrm{V}$ be affine maps. Show that $A \circ K - A \circ L = \boldsymbol{A} \circ (K - L)$.

3. Differentiation

3.1. Let $\mathbf{V}$ be a vector space. A *norm* on $\mathbf{V}$ is a map

$$\|\cdot\| : \mathbf{V} \to \mathbb{R}_0^+ \quad , \quad \boldsymbol{x} \mapsto \|\boldsymbol{x}\|$$

for which (i) $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$,

(ii) $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$ for all $\alpha \in \mathbb{K}$, $\boldsymbol{x} \in \mathbf{V}$,

(iii) $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{V}$.

The distance of $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{V}$ is defined to be $\|\boldsymbol{x} - \boldsymbol{y}\|$; the map

$$\mathbf{V} \times \mathbf{V} \to \mathbb{R}, \qquad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \|\boldsymbol{x} - \boldsymbol{y}\|$$

is called the *metrics* associated with the norm.

The reader is supposed to be familiar with the fundamental notions of analysis connected with metrics: open subsets, closed subsets, convergence, continuity, etc.

It is important that if $\mathbf{V}$ is finite-dimensional then all the norms on $\mathbf{V}$ are equivalent, i.e. they determine the same open subsets, closed subsets, convergent series, continuous functions etc.

As a consequence, in finite-dimensional vector spaces — e.g. in pseudo-Euclidean vector spaces — we can speak about open subsets, closed subsets, continuity etc. without giving an actual norm. Linear, bilinear, multilinear maps between finite-dimensional vector spaces are continuous.

3.2. If V is an affine space over $\mathbf{V}$ and there is a norm on $\mathbf{V}$ then

$$\mathrm{V} \times \mathrm{V} \to \mathbb{R}, \qquad (x, y) \mapsto \|x - y\|$$

is a metrics on V. Then the open subsets, closed subsets, convergence etc. are defined in V.

In the following we deal with finite dimensional real affine spaces; hence we speak about the fundamental notions of analysis without specifying norms on the underlying vector spaces.

As usual, if $\mathbf{V}$ and $\mathbf{U}$ are finite-dimensional vector spaces, $\mathrm{ordo} : \mathbf{V} \rightarrowtail \mathbf{U}$ denotes a function such that

(i) it is defined in a neighbourhood of $\mathbf{0} \in \mathbf{V}$,

(ii) $\lim\limits_{\boldsymbol{x}\to\mathbf{0}} \frac{\mathrm{ordo}(\boldsymbol{x})}{\|\boldsymbol{x}\|} = \mathbf{0}$ for some (hence for every) norm $\|\cdot\|$ on $\mathbf{V}$.

3.3. Definition. Let V and U be affine spaces. A map $F : \mathrm{V} \rightarrowtail \mathrm{U}$ is called *differentiable* at an interior point x of $\mathrm{Dom}\, F$ if there is a linear map $\mathrm{D}F(x) : \mathbf{V} \to \mathbf{U}$ and a neighbourhood $\mathcal{N}(x) \subset \mathrm{Dom}\, F$ of x such that

$$F(y) - F(x) = \mathrm{D}F(x) \cdot (y - x) + \mathrm{ordo}(y - x) \qquad\qquad (y \in \mathcal{N}(x)).$$

$\mathrm{D}F(x)$ is the derivative of F at x.

F is *differentiable on a subset* S of Dom F if it is differentiable at every point of S. F is *differentiable* if it is differentiable on its domain (which is necessarily open in this case). F is *continuously differentiable* if it is differentiable and Dom $F \to \mathrm{Lin}(\mathbf{V}, \mathbf{U})$, $x \mapsto \mathrm{D}F(x)$ is continuous.

If the real affine spaces V and U are oriented, a differentiable mapping $F : \mathrm{V} \rightarrowtail \mathrm{U}$ is called *orientation-preserving* if $\mathrm{D}F(x) : \mathbf{V} \to \mathbf{U}$ is an orientation-preserving linear bijection for all $x \in$ Dom F. ■

The differentiability of F at x is equivalent to the following: there is a neighbourhood $\mathcal{N}$ of $\mathbf{0} \in \mathbf{V}$ such that $x + \mathcal{N} \subset$ Dom F and

$$F(x + \boldsymbol{x}) - F(x) = \mathrm{D}F(x) \cdot \boldsymbol{x} + \mathrm{ordo}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \mathcal{N}).$$

This form shows immediately that $\mathrm{D}F(x)$ is uniquely determined.

3.4. If the affine spaces in question are actually vector spaces, i.e. F is a map between vector spaces then the above definition coincides with the one in standard analysis. Hence in the case of vector spaces we can apply the well-known results regarding differentiability. Moreover, for affine spaces one proves without difficulty that

(i) a differentiable map is continuous;

(ii) if $F : \mathrm{V} \rightarrowtail \mathrm{U}$ and $G : \mathrm{U} \rightarrowtail \mathrm{W}$ are differentiable then $G \circ F$ is differentiable, too, and

$$\mathrm{D}(G \circ F)(x) = \mathrm{D}G(F(x)) \cdot \mathrm{D}F(x) \qquad (x \in \mathrm{Dom}\ (G \circ F));$$

(iii) if $F, G : \mathrm{V} \rightarrowtail \mathbf{U}$ are differentiable then $F - G : \mathrm{V} \rightarrowtail \mathbf{U}$, $x \mapsto F(x) - G(x)$ is differentiable and

$$\mathrm{D}(F - G)(x) = \mathrm{D}F(x) - \mathrm{D}G(x) \qquad (x \in \mathrm{Dom}\ F \cap \mathrm{Dom}\ G).$$

(iv) An affine map $L : \mathrm{V} \to \mathrm{U}$ is differentiable, its derivative at every x equals the underlying linear map:

$$\mathrm{D}L(x) = \boldsymbol{L} \qquad (x \in \mathrm{V}).$$

3.5. Let V and U be affine spaces. If $F : \mathrm{V} \rightarrowtail \mathrm{U}$ is differentiable and its derivative map $\mathrm{V} \rightarrowtail \mathrm{Lin}(\mathbf{V}, \mathbf{U})$, $x \mapsto \mathrm{D}F(x)$ is differentiable then F is called *twice differentiable.*

Differentiability of higher order is defined similarly. An infinitely many times differentiable map is called *smooth.*

The second derivative of F at x is denoted by $\mathrm{D}^2F(x)$; by definition, it is an element of $\mathrm{Lin}(\mathbf{V}, \mathrm{Lin}(\mathbf{V}, \mathbf{U}))$.

The n-th derivative of F at x, $\mathrm{D}^nF(x)$ is an element of $\mathrm{Lin}(\mathbf{V}, \mathrm{Lin}(\mathbf{V}, \dots, \mathrm{Lin}(\mathbf{V}, \mathbf{U}) \dots))$.

This rather complicated object is significantly simplified with the aid of tensor products.

We know that $\mathrm{Lin}(\mathbf{V},\mathbf{U}) \equiv \mathbf{U} \otimes \mathbf{V}^*$. Thus $\mathrm{D}F(x) \in \mathbf{U} \otimes \mathbf{V}^*$.

Further,

$$\mathrm{Lin}(\mathbf{V}, \mathrm{Lin}(\mathbf{V},\mathbf{U})) \equiv \mathrm{Lin}(\mathbf{V}, \mathbf{U} \otimes \mathbf{V}^*) \equiv (\mathbf{U} \otimes \mathbf{V}^*) \otimes \mathbf{V}^* \equiv \mathbf{U} \otimes \mathbf{V}^* \otimes \mathbf{V}^*,$$

thus $\mathrm{D}^2F(x) \in \mathbf{U} \otimes \mathbf{V}^* \otimes \mathbf{V}^*$.

Similarly we have that $\mathrm{D}^nF(x) \in \mathbf{U} \otimes \left(\overset{n}{\otimes} \mathbf{V}^* \right)$.

Moreover, a well-known theorem states that the n-th derivative is symmetric, i.e. $\mathrm{D}^nF(x) \in \mathbf{U} \otimes \left(\overset{n}{\vee} \mathbf{V}^* \right)$.

3.6. We often need the following particular result.

Proposition. Let V, U and Z be affine spaces, $A : \mathrm{V} \to \mathrm{Z}$ an affine surjection. A mapping $f : \mathrm{Z} \rightarrowtail \mathrm{U}$ is $k$times (continuously) differentiable if and only if $f \circ A$ is k times (continuously) differentiable ($k \in \mathbb{N}$).

Proof. The first part of the statement is trivial.

Suppose that $F := f \circ A$ is k times (continuously) differentiable. We know that there is a linear injection $\boldsymbol{L} : \mathbf{Z} \to \mathbf{V}$ such that $\boldsymbol{A} \cdot \boldsymbol{L} = \mathrm{id}_{\mathbf{Z}}$. Then for $z \in \mathrm{Dom}\ f \subset \mathrm{Z}$, $\boldsymbol{h}$ in a neighbourhood of $\mathbf{0} \in \mathbf{Z}$ we have

$$f(z+\boldsymbol{h}) - f(z) = F(x + \boldsymbol{L}\cdot\boldsymbol{h}) - F(x) = \mathrm{D}F(x)\cdot\boldsymbol{L}\cdot\boldsymbol{h} + \mathrm{ordo}(\boldsymbol{L}\cdot\boldsymbol{h})$$

if $A(x) = z$. Since $\mathrm{ordo}(\boldsymbol{L}\cdot\boldsymbol{h}) = \mathrm{ordo}(\boldsymbol{h})$, we see that f is (continuously) differentiable and

$$\mathrm{D}f(z) = \mathrm{D}F(x)\cdot\boldsymbol{L} \qquad\qquad (z \in \mathrm{Dom}\ f,\ x \in \overset{-1}{A}\{z\}).$$

Moreover, $\mathrm{D}f : \mathrm{Z} \rightarrowtail \mathbf{U} \otimes \mathbf{Z}^*$ is a mapping such that $\mathrm{D}f \circ A = \mathrm{D}F \cdot \boldsymbol{L}$ and we can repeat the previous arguments to obtain that if F is twice (continuously) differentiable (i.e. $\mathrm{D}F$ is (continuously) differentiable) then f is twice (continuously) differentiable (i.e. $\mathrm{D}f$ is (continuously) differentiable).

Proceeding in this way we can demonstrate k times (continuously) differentiability.

3.7. (i) Let $\boldsymbol{C} : \mathrm{V} \rightarrowtail \mathbf{V}$ be a differentiable mapping (a *vector field* in V). Then $\mathrm{D}\boldsymbol{C}(x) \in \mathbf{V} \otimes \mathbf{V}^*$ for all $x \in \mathrm{Dom}\ \boldsymbol{C}$, thus we can take its trace:

$$\mathrm{D}\cdot\boldsymbol{C}(x) := \mathrm{Tr}\left(\mathrm{D}\boldsymbol{C}(x)\right).$$

The mapping $\mathrm{V} \rightarrowtail \mathbb{R}$, $x \mapsto \mathrm{D}\cdot\boldsymbol{C}(x)$ is called the *divergence* of $\boldsymbol{C}$.

If $\mathbf{Z}$ is a vector space, the divergence of differentiable mappings $\mathrm{V} \rightarrowtail \mathbf{Z} \otimes \mathbf{V}$ is defined similarly according to IV.3.9.

(*ii*) Let $\boldsymbol{S} : \mathrm{V} \rightarrowtail \mathbf{V}^*$ be a differentiable mapping (a *covector field* in V). Then $\mathrm{D}\boldsymbol{S}(x) \in \mathbf{V}^* \otimes \mathbf{V}^*$ for all $x \in \mathrm{Dom}\ \boldsymbol{S}$, and we can take

$$\mathrm{D} \wedge \boldsymbol{S}(x) := (\mathrm{D}\boldsymbol{S}(x))^* - \mathrm{D}\boldsymbol{S}(x).$$

The mapping $\mathrm{V} \rightarrowtail \mathbf{V}^* \wedge \mathbf{V}^*$, $x \mapsto \mathrm{D} \wedge \boldsymbol{S}(x)$ is called the *curl* of $\boldsymbol{S}$.

(*iii*) *Keep in mind that a vector field has no curl and a covector field has no divergence.*

3.8. (*i*) Let V_1, V_2 and U be affine spaces and consider a differentiable mapping $F : \mathrm{V}_1 \times \mathrm{V}_2 \rightarrowtail \mathrm{U}$. Take an $(x_1, x_2) \in \mathrm{Dom}\ F$ and fix x_2. Then $\mathrm{V}_1 \rightarrowtail \mathrm{U}$, $y_1 \mapsto F(y_1, x_2)$ is a differentiable mapping; its derivative at x_1 is called the *first partial derivative* of F at (x_1, x_2) and is denoted by $\mathrm{D}_1 F(x_1, x_2)$. By definition, $\mathrm{D}_1 F(x_1, x_2)$ is a linear map $\mathbf{V}_1 \to \mathbf{U}$.

The second partial derivative $\mathrm{D}_2 F(x_1, x_2)$ of F is defined similarly, and an evident generalization can be made for the k-th partial derivative ($k = 1, \dots, n$) of a mapping $\underset{k=1}{\overset{n}{\times}} \mathrm{V}_k \rightarrowtail \mathrm{U}$.

For a vector field $\boldsymbol{C} : \mathrm{V}_1 \times \mathrm{V}_2 \rightarrowtail \mathbf{V}_1 \times \mathbf{V}_2$ we define the components $\boldsymbol{C}^i : \mathrm{V}_1 \times \mathrm{V}_2 \rightarrowtail \mathbf{V}_i$ ($i = 1, 2$) such that $\boldsymbol{C} = (\boldsymbol{C}^1, \boldsymbol{C}^2)$. Then $\mathrm{D}\boldsymbol{C}(x_1, x_2)$ is an element of $(\mathbf{V}_1 \times \mathbf{V}_2) \otimes (\mathbf{V}_1 \times \mathbf{V}_2)^* \equiv (\mathbf{V}_1 \times \mathbf{V}_2) \otimes (\mathbf{V}_1^* \times \mathbf{V}_2^*)$. It is not hard to see that using a matrix form corresponding to the convention introduced in IV.3.7 we have

$$\mathrm{D}\boldsymbol{C}(x_1, x_2) = \begin{pmatrix} \mathrm{D}_1 \boldsymbol{C}^1 & \mathrm{D}_2 \boldsymbol{C}^1 \\ \mathrm{D}_1 \boldsymbol{C}^2 & \mathrm{D}_2 \boldsymbol{C}^2 \end{pmatrix} (x_1, x_2),$$

where the symbol (x_1, x_2) after the matrix means that every entry is to be taken at (x_1, x_2); shortly,

$$\mathrm{D}\boldsymbol{C} = \begin{pmatrix} \mathrm{D}_1 \boldsymbol{C}^1 & \mathrm{D}_2 \boldsymbol{C}^1 \\ \mathrm{D}_1 \boldsymbol{C}^2 & \mathrm{D}_2 \boldsymbol{C}^2 \end{pmatrix}.$$

Furthermore, we easily find that

$$\mathrm{D} \cdot \boldsymbol{C} = \mathrm{D}_1 \cdot \boldsymbol{C}^1 + \mathrm{D}_2 \cdot \boldsymbol{C}^2.$$

(*ii*) Similar notations for a covector field $\boldsymbol{S} = (\boldsymbol{S}_1, \boldsymbol{S}_2) : \mathrm{V}_1 \times \mathrm{V}_2 \rightarrowtail (\mathbf{V}_1 \times \mathbf{V}_2)^* \equiv \mathbf{V}_1^* \times \mathbf{V}_2^*$ yield

$$\mathrm{D}\boldsymbol{S} = \begin{pmatrix} \mathrm{D}_1 \boldsymbol{S}_1 & \mathrm{D}_2 \boldsymbol{S}_1 \\ \mathrm{D}_1 \boldsymbol{S}_2 & \mathrm{D}_2 \boldsymbol{S}_2 \end{pmatrix}$$

and

$$\mathrm{D} \wedge \boldsymbol{S} = \begin{pmatrix} \mathrm{D}_1 \wedge \boldsymbol{S}_1 & (\mathrm{D}_1 \boldsymbol{S}_2)^* - \mathrm{D}_2 \boldsymbol{S}_1 \\ (\mathrm{D}_2 \boldsymbol{S}_1)^* - \mathrm{D}_1 \boldsymbol{S}_2 & \mathrm{D}_2 \wedge \boldsymbol{S}_2 \end{pmatrix}.$$

3.9. A vector field $\boldsymbol{C} : \mathbb{R}^N \rightarrowtail \mathbb{R}^N$ is given by its components $C^i : \mathbb{R}^N \rightarrowtail \mathbb{R}$ $(i = 1, \dots, N)$, $\boldsymbol{C} = (C^1, \dots, C^N)$. Its derivative at ξ is a linear map $\mathbb{R}^N \to \mathbb{R}^N$; one easily finds for its matrix entries

$$\left(\mathrm{D}\boldsymbol{C}(\xi)\right)^i{}_k = \partial_k C^i(\xi) \qquad (i, k = 1, \dots, N)$$

where ∂_k denotes the k-th partial differentiation.

Then

$$\mathrm{D} \cdot \boldsymbol{C} = \sum_{i=1}^{N} \partial_i C^i.$$

A covector field $\boldsymbol{S} : \mathbb{R}^N \rightarrowtail \left(\mathbb{R}^N\right)^*$ is given by its components $S_i : \mathbb{R}^N \rightarrowtail \mathbb{R}$ $(i = 1, \dots, N)$, $\boldsymbol{S} = (S_1, \dots, S_N)$. We have

$$\left(\mathrm{D}\boldsymbol{S}(\xi)\right)_{ik} = \partial_k S_i(\xi)$$

and

$$\left(\mathrm{D} \wedge \boldsymbol{S}\right)_{ik} = \partial_i S_k - \partial_k S_i$$

for $i, k = 1, \dots, N$.

3.10. If I is a one-dimensional affine space, V is an affine space and $r : \mathrm{I} \rightarrowtail \mathrm{V}$ is differentiable, then, for $t \in \mathrm{Dom}\, r$, $\mathrm{D}r(t)$ is an element of $\mathbf{V} \otimes \mathbf{I}^* \equiv \frac{\mathbf{V}}{\mathbf{I}}$.

It is not hard to see that in this case

$$\frac{\mathrm{d}r(t)}{\mathrm{d}t} := \dot{r}(t) := \mathrm{D}r(t) = \lim_{\substack{\boldsymbol{t} \to 0 \\ \boldsymbol{t} \in \mathbf{I}}} \frac{r(t + \boldsymbol{t}) - r(t)}{\boldsymbol{t}}.$$

Similarly we arrive at $\frac{\mathrm{d}^2 r(t)}{\mathrm{d}t^2} := \ddot{r}(t) := \mathrm{D}^2 r(t) \in \frac{\mathbf{V}}{\mathbf{I} \otimes \mathbf{I}}$.

3.11. Let V and I as before and suppose I is real and oriented. Recall that then $\mathbf{I}^+$ and $\mathbf{I}^-$ denote the sets of positive and negative elements of $\mathbf{I}$, respectively.

Then $r : \mathrm{I} \rightarrowtail \mathrm{V}$ is called *differentiable on the right* at an interior point t of $\mathrm{Dom}\, r$ if there exists

$$\dot{r}^+(t) := \lim_{\substack{\boldsymbol{t} \to 0 \\ \boldsymbol{t} \in \mathbf{I}^+}} \frac{r(t + \boldsymbol{t}) - r(t)}{\boldsymbol{t}},$$

called the *right derivative* of r at t.

The differentiability on the left and the left derivative $\dot{r}^-(t)$ are defined similarly.

Definition. Let V and I be as before, I is oriented. A function $r : \mathrm{I} \rightarrowtail \mathrm{V}$ is called *piecewise differentiable* if it is

(i) continuous,
(ii) differentiable with the possible exception of finite points where r is differentiable both on the right and on the left.

r is called *piecewise twice differentiable* if

(i) it is piecewise differentiable,
(ii) it is twice differentiable where it is differentiable,
(iii) if a is a point where r is not differentiable then there exist

$$\lim_{\substack{\boldsymbol{t}\to 0\\ \boldsymbol{t}\in \mathbf{I}^+}} \frac{\dot r(a+\boldsymbol{t}) - \dot r^+(a)}{\boldsymbol{t}} \qquad \text{and} \qquad \lim_{\substack{\boldsymbol{t}\to 0\\ \boldsymbol{t}\in \mathbf{I}^-}} \frac{\dot r(a+\boldsymbol{t}) - \dot r^-(a)}{\boldsymbol{t}}.$$

3.12. Recall that for a finite dimensional vector space $\mathbf{V}$, $\mathrm{Lin}(\mathbf{V}) \equiv \mathbf{V}\otimes\mathbf{V}^*$ is a finite-dimensional vector space as well. Hence the differentiability of a function $\boldsymbol{R} : \mathrm{I} \rightarrowtail \mathrm{Lin}(\mathbf{V})$ makes sense. It can be shown without difficulty that $\boldsymbol{R}$ is differentiable (and then its derivative at t is $\dot{\boldsymbol{R}}(t) \in \frac{\mathbf{V}\otimes\mathbf{V}^*}{\mathbf{I}} \equiv \mathrm{Lin}\left(\frac{\mathbf{V}}{\mathbf{I}}, \mathbf{V}\right)$) if and only if $\mathrm{I} \rightarrowtail \mathbf{V}$, $t \mapsto \boldsymbol{R}(t)\cdot\boldsymbol{v}$ is differentiable for all $\boldsymbol{v}\in\mathbf{V}$ and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{R}(t)\cdot\boldsymbol{v}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}(t)\right)\cdot\boldsymbol{v}.$$

Moreover, if $\boldsymbol{r} : \mathrm{I} \rightarrowtail \mathbf{V}$ is a differentiable function then $\boldsymbol{R}\cdot\boldsymbol{r}$ is differentiable and

$$(\boldsymbol{R}\cdot\boldsymbol{r})^{\cdot} = \dot{\boldsymbol{R}}\cdot\boldsymbol{r} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}}.$$

4. Submanifolds in affine spaces

In this section the affine spaces are real and finite dimensional.

4.1. The inverse mapping theorem and the implicit mapping theorem are important and well-known results of analysis. Now we formulate them for affine spaces in a form convenient for our application.

The inverse mapping theorem. Let V and U be affine spaces, $\dim \mathrm{U} = \dim \mathrm{V}$. If $F : \mathrm{V} \rightarrowtail \mathrm{U}$ is $n \geq 1$ times continuously differentiable, $e \in \mathrm{Dom}\, F$ and $\mathrm{D}F(e) : \mathbf{V} \to \mathbf{U}$ is a linear bijection, then there is a neighbourhood $\mathcal{N}$ of e, $\mathcal{N} \subset \mathrm{Dom}\, F$, such that
(i) $F|_{\mathcal{N}}$ is injective,
(ii) $F[\mathcal{N}]$ is open in U,

(iii) $(F|_{\mathcal{N}})^{-1}$ is n times continuously differentiable.

The implicit mapping theorem. Let V and U be affine spaces, $\dim \mathrm{U} < \dim \mathrm{V}$. Suppose $S : \mathrm{V} \rightarrowtail \mathrm{U}$ is $n \geq 1$ times continuously differentiable, $e \in \mathrm{Dom}\ S$ and $\mathrm{D}S(e)$ is surjective.

Let $\mathbf{V}_1$ be a linear subspace of $\mathbf{V}$ such that the restriction of $\mathrm{D}S(e)$ onto $\mathbf{V}_1$ is a bijection between $\mathbf{V}_1$ and $\mathbf{U}$ and suppose $\mathbf{V}_0$ is a subspace complementary to $\mathbf{V}_1$.

Then there are

— neighbourhoods $\mathcal{N}_0$ and $\mathcal{N}_1$ of the zero in $\mathbf{V}_0$ and in $\mathbf{V}_1$, respectively, $e + \mathcal{N}_0 + \mathcal{N}_1 \subset \mathrm{Dom}\ S$,

— a uniquely determined, n times continuously differentiable mapping $G : \mathcal{N}_0 \to \mathcal{N}_1$such that

$$S(e + \boldsymbol{x}_0 + G(\boldsymbol{x}_0)) = S(e) \qquad (\boldsymbol{x}_0 \in \mathcal{N}_0). \quad \blacksquare$$

Observe that $\mathbf{V}_0 := \mathrm{Ker}\ \mathrm{D}S(e)$ and a subspace $\mathbf{V}_1$ complementary to $\mathbf{V}_0$ satisfy the above requirements.

4.2. Definition. Let V be an affine space, $\dim \mathrm{V} := N \geq 2$. Let M and n be natural numbers, $1 \leq M \leq N$, $n \geq 1$. A subset $\mathcal{H}$ of V is called an *M-dimensional n times differentiable simple submanifold* in V if there are

— an M-dimensional affine space D,

— a mapping $p : \mathrm{D} \rightarrowtail \mathrm{V}$, called a *parametrization* of $\mathcal{H}$, such that

(i) Dom p is open and connected, Ran $p = \mathcal{H}$,

(ii) p is n times continuously differentiable and $\mathrm{D}p(\xi)$ is injective for all $\xi \in$ Dom p,

(iii) p is injective and p^{-1} is continuous. $\blacksquare$

Recall that $\mathrm{D}p(\xi) \in \mathrm{Lin}(\mathbf{D}, \mathbf{V})$.

Since p is differentiable, it is continuous.

The parametrization of $\mathcal{H}$ is not unique. For instance, if E is an affine space and $L : \mathrm{E} \to \mathrm{D}$ is an affine bijection then $p \circ L$ is a parametrization, too. In particular, we can take $\mathrm{E} := \mathbb{R}^M$ (see Exercise 2.5.2); as a consequence, D can be replaced by $\mathbb{R}^M$ in the definition.

The inverse mapping theorem implies that the N-dimensional, n times differentiable simple submanifolds are the connected open subsets of V.

Evidently, an M-dimensional affine subspace of V is an M-dimensional n times differentiable simple manifold for all n.

4.3. Definition. Let $N \geq 2$. A subset $\mathcal{H}$ of the N-dimensional affine space V is called an *M-dimensional n times differentiable submanifold* if every $x \in \mathcal{H}$ has a neighbourhood $\mathcal{N}(x)$ in V such that $\mathcal{N}(x) \cap \mathcal{H}$ is an M-dimensional n times differentiable simple submanifold.

A subset which is an n times differentiable submanifold for all $n \in \mathcal{N}$ is a *smooth submanifold.*

A *submanifold* means an n times differentiable submanifold for some n.

A submanifold which is a closed subset of V is called a *closed submanifold.*

One-dimensional submanifolds, two-dimensional submanifolds and $(N-1)$-dimensional submanifolds are called *curves* or *lines, surfaces* and *hypersurfaces,* respectively. ■

By definition, every point of a submanifold has a neighbourhood in the submanifold that can be parametrized. A parametrization of such a neighbourhood is called a *local parametrization* of the manifold.

4.4. Proposition. Let $\mathcal{H}$ be an M-dimensional n times differentiable submanifold in V, $M < N$, and let $p : \mathbb{R}^M \rightarrowtail \mathrm{V}$ be a local parametrization of $\mathcal{H}$. If $e \in \mathrm{Ran}\ p$ then there are

— a neighbourhood $\mathcal{N}$ of e in V,

— continuously n times differentiable mappings

$$F : \mathcal{N} \to \mathbb{R}^M, \qquad S : \mathcal{N} \to \mathbb{R}^{N-M}$$

such that

(i) $\mathcal{N} \cap \mathcal{H} \subset \mathrm{Ran}\ p$;

(ii) $F(p(\xi)) = \xi$, $S(p(\xi)) = 0$ for all $\xi \in \mathrm{Dom}\ p$, $p(\xi) \in \mathcal{N}$;

(iii) $\mathrm{D}S(x)$ is surjective for all $x \in \mathcal{N}$.

Proof. There is a unique $\alpha \in \mathrm{Dom}\ p$ for which $p(\alpha) = e$. $\mathrm{D}p(\alpha) : \mathbb{R}^M \to \mathrm{V}$ is a linear injection, hence $\mathbf{V}_1 := \mathrm{Ran}\ \mathrm{D}p(\alpha)$ is an M-dimensional linear subspace. Let $\mathbf{V}_0$ be a linear subspace, complementary to $\mathbf{V}_1$. Evidently, $\dim \mathbf{V}_0 = N - M$.

Let $\boldsymbol{P} : \mathbf{V} \to \mathbf{V}$ be the projection onto $\mathbf{V}_1$ along $\mathbf{V}_0$ (i.e. $\boldsymbol{P}$ is linear and $\boldsymbol{P} \cdot \boldsymbol{x}_1 = \boldsymbol{x}_1$ for $\boldsymbol{x} \in \mathbf{V}_1$ and $\boldsymbol{P} \cdot \boldsymbol{x}_0 = \mathbf{0}$ for $\boldsymbol{x} \in \mathbf{V}_0$). Then

$$\boldsymbol{P} \cdot (p - e) : \mathbb{R}^M \rightarrowtail \mathbf{V}_1, \qquad \xi \mapsto \boldsymbol{P} \cdot (p(\xi) - e)$$

is n times continuously differentiable, its derivative at α equals $\boldsymbol{P} \cdot \mathrm{D}p(\alpha)$; it is a linear bijection from $\mathbb{R}^M$ onto $\mathbf{V}_1$. Thus, according to the inverse mapping theorem, there is a neighbourhood Ω of α such that $\boldsymbol{P} \cdot (p - e)|_\Omega$ is injective, its inverse is continuously differentiable, $(\boldsymbol{P} \cdot (p - e))[\Omega] = \boldsymbol{P}[p[\Omega] - e]$ is open in $\mathbf{V}_1$.

For the sake of simplicity and without loss of generality we can suppose $\Omega = \mathrm{Dom}\ p$ (considering $p|_\Omega$ instead of p).

Then the continuity of $\boldsymbol{P}$ involves that $\overset{-1}{\boldsymbol{P}}(\boldsymbol{P}[p[\Omega] - e])$ is an open subset of $\mathbf{V}$ and so $e + \overset{-1}{\boldsymbol{P}}(\boldsymbol{P}[p[\Omega] - e])$ is an open subset of V. Since p^{-1} is continuous, $p[\Omega]$ is open in $\operatorname{Ran} p$ and $p[\Omega] \subset e + \overset{-1}{\boldsymbol{P}}(\boldsymbol{P}[p[\Omega] - e])$; thus there is an open subset $\mathcal{N}$ in $e + \overset{-1}{\boldsymbol{P}}(\boldsymbol{P}[p[\Omega] - e]) \subset \mathrm{V}$ such that $p[\Omega] = \mathcal{H} \cap \mathcal{N}$.

Let $\boldsymbol{L} : \mathbf{V}_0 \to \mathbb{R}^{N-M}$ be a linear bijection and

$$F := (\boldsymbol{P} \cdot (p - e))^{-1} \circ \boldsymbol{P} \cdot (\mathrm{id}_V - e)\big|_{\mathcal{N}}, \qquad S := \boldsymbol{L} \circ (\mathrm{id}_V - p \circ F).$$

$\mathcal{N} \subset e + \overset{-1}{\boldsymbol{P}}(\boldsymbol{P}[p[\Omega] - e])$ implies $\boldsymbol{P}[e + \mathcal{N}] \subset \boldsymbol{P}[p[\Omega] - e] = \operatorname{Dom}(\boldsymbol{P} \cdot (p - e))^{-1}$, hence both F and S are defined on $\mathcal{N}$. It is left to the reader to prove that properties *(ii)* and *(iii)* in the proposition hold.

4.5. Proposition. Let $p : \mathbb{R}^M \rightarrowtail \mathrm{V}$ and $q : \mathbb{R}^M \rightarrowtail \mathrm{V}$ be local parameterizations of the M-dimensional n times differentiable submanifold $\mathcal{H}$ such that $\operatorname{Ran} p \cap \operatorname{Ran} q \neq \emptyset$. Then $p^{-1} \circ q : \mathbb{R}^M \rightarrowtail \mathbb{R}^M$ is n times continuously differentiable and

$$\mathrm{D}(p^{-1} \circ q)(q^{-1}(x)) = \left[\mathrm{D}p(p^{-1}(x))\right]^{-1} \cdot \mathrm{D}q(q^{-1}(x)) \qquad (x \in \operatorname{Ran} p \cap \operatorname{Ran} q).$$

Proof. If $M = N$ then the inverse mapping theorem implies that p^{-1} is n times continuously differentiable; as a consequence, $p^{-1} \circ q$ is n times continuously differentiable as well and the above formula is valid in view of the well-known rule of differentiation of composite mappings.

If $M < N$, the differentiability of p^{-1} makes no sense because $\mathcal{H}$ contains no open subsets in V. Nevertheless, $p^{-1} \circ q$ is continuously differentiable as we shall see below.

Let e be an arbitrary point of $\operatorname{Ran} p \cap \operatorname{Ran} q$. According to the previous proposition, there are a neighbourhood $\mathcal{N}$ of e and an n times continuously differentiable mapping F for which $\mathcal{N} \cap \mathcal{H} \subset \operatorname{Ran} p$ and $F \circ p \subset \mathrm{id}_{\mathbb{R}^M}$ holds.

Then $\Omega := \overset{-1}{q}(\mathcal{N}) \subset \operatorname{Dom}(p^{-1} \circ q)$ is open and

$$(p^{-1} \circ q)\big|_{\Omega} = (F \circ p) \circ (p^{-1} \circ q)\big|_{\Omega} = F \circ q|_{\Omega};$$

the mapping on the right-hand side is n times continuously differentiable being a composition of two such mappings. Thus we have shown that each point of $\operatorname{Dom}(p^{-1} \circ q)$ has a neighbourhood Ω in which $p^{-1} \circ q$ is n times continuously differentiable.

Let x be an element of $\operatorname{Ran} p \cap \operatorname{Ran} q$, $\xi := q^{-1}(x)$ and $\Phi := p^{-1} \circ q$. Then $p \circ \Phi \subset q$ and $\xi \in \operatorname{Dom}(p \circ \Phi)$. Thus

$$\mathrm{D}q(q^{-1}(x)) = \mathrm{D}q(\xi) = \mathrm{D}p(\Phi(\xi)) \cdot \mathrm{D}\Phi(\xi) = \mathrm{D}p(p^{-1}(x)) \cdot \mathrm{D}\Phi(\xi), \qquad (*)$$

which gives immediately the desired equality. ■

Evidently, then $q^{-1} \circ p$ is n times continuously differentiable as well. Since $q^{-1} \circ p = (p^{-1} \circ q)^{-1}$, this means that the derivative of $p^{-1} \circ q$ at every point is a linear bijection $\mathbb{R}^M \to \mathbb{R}^M$.

As a consequence, the dimesion of a submanifold is uniquely determined. Supposing that a submanifold is both M-dimensional and M'-dimensional we get $M = M'$.

We have proved the statement for parametrizations from $\mathbb{R}^M$. Obviously, the same is true for parametrizations with domains in affine spaces.

4.6. Proposition. Let p and q be local parametrizations of a submanifold such that $\mathrm{Ran}\ p \cap \mathrm{Ran}\ q \neq \emptyset$. If $x \in \mathrm{Ran}\ p \cap \mathrm{Ran}\ q$ then

$$\mathrm{Ran}\ \left(\mathrm{D}p(p^{-1}(x))\right) = \mathrm{Ran}\ \left(\mathrm{D}q(q^{-1}(x))\right).$$

Proof. Equality $(*)$ in the preceding paragraph involves that the range of $\mathrm{D}q(q)^{-1}(x))$ is contained in the range of $\mathrm{D}p(p^{-1}(x))$. A similar argument yields that the range of $\mathrm{D}p(p^{-1}(x))$is contained in the range of $\mathrm{D}q(q)^{-1}(x))$.

Definition. Let $\mathcal{H}$ be an M-dimensional submanifold, $x \in \mathcal{H}$. Then

$$\mathrm{T}_x(\mathcal{H}) := \mathrm{Ran}\ \left(\mathrm{D}p(p^{-1}(x))\right)$$

is called the *tangent space* of $\mathcal{H}$ at x where p is a parametrization of $\mathcal{H}$ such that $x \in \mathrm{Ran}\ p$. The elements of $\mathrm{T}_x(\mathcal{H})$ are called *tangent vectors* of $\mathcal{H}$ at x. ■

The preceding proposition says that the tangent space, though it is defined by a parametrization, is independent of the parametrization.

The tangent space is an M-dimensional linear subspace of $\mathbf{V}$. $x + \mathrm{T}_x(\mathcal{H})$ is an affine subspace of V which we call the *geometric tangent space* of $\mathcal{H}$ at x.

4.7. Let $M < N$. We have seen in Proposition 4.4. that every point e of an M-dimensional n times differentiable submanifold $\mathcal{H}$ has a neighbourhood $\mathcal{N}$ in V and an n times continuously differentiable mapping $S : \mathcal{N} \to \mathbb{R}^{N-M}$ such that $\mathcal{N} \cap \mathcal{H} = \overset{-1}{S}(\{0\})$ and $\mathrm{D}S(x)$ is surjective. Evidently, $\mathbb{R}^{N-M}$ and $0 \in \mathbb{R}^{N-M}$ can be replaced by an arbitrary affine space U, $\dim \mathrm{U} = N - M$, and a point $o \in \mathrm{U}$, respectively.

Now we prove a converse statement.

Proposition. Let V and U be affine spaces, $\dim \mathrm{V} =: N$, $\dim \mathrm{U} =: N - M$, and $S : \mathrm{V} \rightarrowtail \mathrm{U}$ an n times continuously differentiable mapping. Suppose $o \in \mathrm{Ran}\ S$. Then

$$\mathcal{H} := \{x \in \overset{-1}{S}(\{o\}) \mid \mathrm{Ran}\ \mathrm{D}S(x) \quad \text{is} \quad (N-M)\text{-dimensional}\}$$

is either void or an M-dimensional n times differentiable submanifold of V.

Proof. Suppose $\mathcal{H}$ is not void and e belongs to it. Then $\mathbf{V}_0 := \mathrm{Ker}\ \mathrm{D}S(e)$ is an M-dimensional linear subspace of $\mathbf{V}$. Let $\mathbf{V}_1$ be a linear subspace, complementary to $\mathbf{V}_0$. Then we can apply the implicit mapping theorem: there are neighbourhoods $\mathcal{N}_0$ and $\mathcal{N}_1$ of the zero in $\mathbf{V}_0$ and in $\mathbf{V}_1$, respectively, an n times continuously differentiable mapping $G : \mathcal{N}_0 \to \mathcal{N}_1$ such that

$$S(e + \boldsymbol{x}_0 + G(\boldsymbol{x}_0)) = S(e) \qquad (\boldsymbol{x}_0 \in \mathcal{N}_0).$$

Let us define

$$p : \mathbf{V}_0 \rightarrowtail \mathrm{V}, \quad \boldsymbol{x}_0 \mapsto e + \boldsymbol{x}_0 + G(\boldsymbol{x}_0) \qquad (\boldsymbol{x}_0 \in \mathcal{N}_0).$$

Evidently, p is n times continuously differentiable and $\mathrm{Ran}\ p \subset \overset{-1}{S}(\{o\})$.

We can easily see that p is injective, its inverse is $x \mapsto \boldsymbol{P} \cdot (x - e)$ where $\boldsymbol{P}$ is the projection onto $\mathbf{V}_0$ along $\mathbf{V}_1$. Consequently, p^{-1} is continuous.

These mean that p is a parametrization of $\mathcal{H}$ in a neighbourhood of e.

4.8. Proposition. Let $\mathcal{H} \neq \emptyset$ be the submanifold described in the previous proposition. Then

$$\mathrm{T}_x(\mathcal{H}) = \mathrm{Ker}\ \mathrm{D}S(x) \qquad (x \in \mathcal{H}).$$

Proof. Let p be a local parametrization of $\mathcal{H}$. Then $S \circ p = \text{const.}$, thus for $x \in \mathrm{Ran}\ p$ we have $\mathrm{D}S(x) \cdot \mathrm{D}p(p^{-1}(x)) = 0$ from which we deduce immediately that $\mathrm{T}_x(\mathcal{H}) := \mathrm{Ran}\ \mathrm{D}p(p^{-1}(x)) \subset \mathrm{Ker}\ \mathrm{D}S(x)$. Since both linear subspaces on the two sides of $\subset$ are M-dimensional, equality occurs necessarily.

4.9. Definition. Let p and q be two local parametrizations of a submanifold, $\mathrm{Dom}\ p \subset \mathbb{R}^M$, $\mathrm{Dom}\ q \subset \mathbb{R}^M$ and $\mathrm{Ran}\ p \cap \mathrm{Ran}\ q \neq \emptyset$. Then p and q are said to be *equally oriented* if the determinant of $\mathrm{D}(p^{-1} \circ q)(\xi)$ is positive for all $\xi \in \mathrm{Dom}\ (p^{-1} \circ q)$.

A family $(p_i)_{i \in I}$ of local parametrizations of a submanifold $\mathcal{H}$ is *orienting* if $\mathcal{H} = \cup_{i \in I} \mathrm{Ran}\ p_i$ and, in the case $\mathrm{Ran}\ p_i \cap \mathrm{Ran}\ p_j \neq \emptyset$, p_i and p_j are equally oriented $(i, j \in I)$.

Two orienting parametrization families are called *equally orienting* if their union is orienting as well.

The submanifold is *orientable* if it has an orienting parametrization family. ■

To be equally orienting is an equivalence relation. If the submanifold is connected, there are exactly two equivalence classes.

An orientable submanifold together with one of the equivalence classes of the orienting local parametrization families is an *oriented submanifold.* A local parametrization of an oriented submanifold is called *positively oriented* if it belongs to a family of the chosen equivalence class.

A simple submanifold is obviously orientable.

Connected N-dimensional submanifolds — i.e. connected open subsets — are orientable.

4.10. Let p be a local parametrization of the submanifold $\mathcal{H}$, Dom $p \subset \mathbb{R}^M$. If $(\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_M)$ is the standard ordered basis of $\mathbb{R}^M$ then $\mathrm{D}p(\xi)\cdot\boldsymbol{\chi}_i = \partial_i p(\xi)$ $(i = 1, \ldots, M)$ for $\xi \in$ Dom p. This means that $((\partial_1 p(\xi), \ldots, \partial_M p(\xi))$ is an ordered basis in $\mathrm{T}_{p(\xi)}(\mathcal{H})$.

In other words, $\big(\partial_1 p(p^{-1}(x)), \ldots, \partial_M p(p^{-1}(x))\big)$ is an ordered basis in $\mathrm{T}_x(\mathcal{H})$ $(x \in \mathrm{Ran}\ p)$.

If q is another local parametrization, with domain in $\mathbb{R}^M$, and $x \in \mathrm{Ran}\ p \cap \mathrm{Ran}\ q \neq \emptyset$ then $\big(\partial_1 q(q)^{-1}(x)), \ldots, \partial_M q(q^1(x))\big)$ is another ordered basis in $\mathrm{T}_x(\mathcal{H})$.

Evidently,

$$\partial_i q(q^{-1}(x)) = \mathrm{D}q(q^{-1}(x)) \cdot \left[\mathrm{D}p(p^{-1}(x))\right]^{-1} \cdot \partial_i p(p^{-1}(x))$$

for all $i = 1, .., M$.

We know from 4.5 and IV.3.20 that

$$\begin{aligned}\det\big(\mathrm{D}(p^{-1}\circ q)(q^{-1}(x))\big) =&\det\Big(\left[\mathrm{D}p(p^{-1}(x))\right]^{-1}\cdot \mathrm{D}q(q^{-1}(x))\Big) = \\ =&\det\Big(\mathrm{D}q(q^{-1}(x))\cdot\left[\mathrm{D}p(p^{-1}(x))\right]^{-1}\Big).\end{aligned}$$

We have proved the following statement.

Proposition. Let p and q be local parametrizations of the submanifold $\mathcal{H}$, Dom $p \subset \mathbb{R}^M$, Dom $q \subset \mathbb{R}^M$ and Ran $p \cap$Ran $q \neq \emptyset$. p and q are equally oriented if and only if the ordered bases

$$\big(\partial_1 p(p^{-1}(x)), \ldots., \partial_M p(p^{-1}(x))\big)$$

and

$$\big(\partial_1 q(q^{-1}(x)), \ldots., \partial_M q(q^{-1}(x))\big)$$

in $\mathrm{T}_x(\mathcal{H})$ are equally oriented for all $x \in \mathrm{Ran}\ p \cap \mathrm{Ran}\ q$.

4.11 Observe that in the case $M = 1$, i.e. when the submanifold is a curve, instead of partial derivatives we have a single derivative of p, denoted usually by $\dot{p}$. Then $\dot{p}(p^{-1}(x))$ spans the (one-dimensional) tangent space at x.

Two local parametrizations p and q are equally oriented if and only if one of the following three conditions is fulfilled:

(i) $(p^{-1}\circ q)^{\cdot}(\alpha) > 0$ for all $\alpha \in$ Dom $(p^{-1}\circ q)$,

(ii) $p^{-1}\circ q : \mathbb{R} \rightarrowtail \mathbb{R}$ is strictly monotone increasing, (*iii*) $\dot{p}(p^{-1}(x))$ is a positive multiple of $\dot{q}(q^{-1}(x))$ for all $x \in \operatorname{Ran} p \cap \operatorname{Ran} q$.

4.12. The following notion concerning curves appears frequently in application.

Let x and y be different elements of V. We say that the curve $\mathcal{C}$ *connects* x and y if these points form the boundary of$\mathcal{C}$, i.e. $\{x, y\} = \overline{\mathcal{C}} \setminus \mathcal{C}$ where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$. We can conceive that x and y are the extremities of a curve connecting them.

4.13. Definition. Let $\mathcal{H}$ and $\mathcal{F}$ be M-dimensional and K-dimensional submanifolds of V and U, respectively. A mapping $F : \mathcal{H} \rightarrowtail \mathcal{F}$ is called *differentiable* at x if there are local parametrizations q of $\mathcal{H}$ and p of $\mathcal{F}$ for which $x \in \operatorname{Ran} q$, $F(x) \in \operatorname{Ran} p$, and the function $p^{-1} \circ F \circ q : \mathbb{R}^M \rightarrowtail \mathbb{R}^K$ is differentiable at $q^{-1}(x)$.

The derivative of F at x is defined to be the linear map $\mathrm{D}F(x) : \mathrm{T}_x(\mathcal{H}) \to \mathrm{T}_{F(x)}(\mathcal{F})$ that satisfies

$$\mathrm{D}p(F(x))^{-1} \cdot \mathrm{D}F(x) \cdot \mathrm{D}q(q^{-1}(x)) = \mathrm{D}(p^{-1} \circ F \circ q)(q^{-1}(x)).$$

F is *differentiable* if it is differentiable at each point of its domain. ∎

If $\mathcal{H}$ and $\mathcal{F}$ are n times differentiable submanifolds, we define F to be k times (continuously) differentiable, for $0 \leq k \leq n$, if $p^{-1} \circ F \circ q$ is k times (continuously) differentiable.

4.14. Exercises

1. Let V and U be affine spaces. The graph of an n times continuously differentiable mapping $F : \mathrm{V} \rightarrowtail \mathrm{U}$— i.e. the set $\{(x, F(x)|\ x \in \operatorname{Dom} F\}$— is a $(\dim V)$-dimensional n times differentiable submanifold in $\mathrm{V} \times \mathrm{U}$. Give its tangent space at an arbitrary point.

2. Prove that the mapping $(F, S) : \mathrm{V} \rightarrowtail \mathbb{R}^M \times \mathbb{R}^{N-M} \equiv \mathbb{R}^N$ described in 4.4 is injective and its inverse is $(\xi, \eta) \mapsto p(\xi) + \boldsymbol{L}^{-1}\eta$.

3. Let $(\mathbf{V}, \mathbf{B}, \mathbf{h})$ be a pseudo-Euclidean vector space, $\mathbf{0} \neq \boldsymbol{a} \in \mathbf{B}$. Prove that

$$\{\boldsymbol{x} \in \mathbf{V}|\ \boldsymbol{x} \cdot \boldsymbol{x} = \boldsymbol{a}^2\} \quad \text{and} \quad \{\boldsymbol{x} \in \mathbf{V}|\ \boldsymbol{x} \cdot \boldsymbol{x} = -\boldsymbol{a}^2\}$$

are either void or hypersurfaces in $\mathbf{V}$ whose tangent space at $\boldsymbol{x}$ equals

$$\{\boldsymbol{y} \in \mathbf{V}|\ \boldsymbol{x} \cdot \boldsymbol{y} = \mathbf{0}\}.$$

(The derivative of the map $\mathbf{V} \to \mathbf{B} \otimes \mathbf{B}$, $\boldsymbol{x} \mapsto \boldsymbol{x} \cdot \boldsymbol{x}$ at $\boldsymbol{x}$ is $2\boldsymbol{x}$ regarded as the linear map $\mathbf{V} \to \mathbf{B} \otimes \mathbf{B}$, $\quad \boldsymbol{y} \mapsto 2\boldsymbol{x} \cdot \boldsymbol{y}$.) Why is the statement not true for $\boldsymbol{a} = \mathbf{0}$?

4. A linear bijection $\mathbb{R}^M \to \mathbb{R}^M$ has a positive determinant if and only if it is orientation-preserving. On the basis of this remark define that two local parametrizations $p : \mathrm{D} \rightarrowtail \mathrm{V}$ and $q : \mathrm{E} \rightarrowtail \mathrm{V}$ of a submanifold are equally oriented where D and E are oriented affine spaces.

5. Use the notations of 4.13. Prove that

(i) $p^{-1} \circ F \circ q$ is differentiable for some p and q if and only if it is differentiable for all p and q;

(ii) the derivative of F is uniquely defined.

(iii) if F is the restriction of a k times (continuously) differentiable mapping $G : \mathrm{V} \rightarrowtail \mathrm{U}$ then F is k times (continuously) differentiable and $\mathrm{D}F(x)$ is the restriction of $\mathrm{D}G(x)$ onto $\mathrm{T}_x(\mathcal{H})$.

5. Coordinatization

5.1. Let V be an N-dimensional real affine space. Take an $o \in \mathrm{V}$ and an ordered basis $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ of $\mathbf{V}$. The affine map $K : \mathrm{V} \to \mathbb{R}^N$ determined by $K(o+\boldsymbol{x}_i) := \boldsymbol{\chi}_i$ $(i = 1, \ldots, N)$ where $(\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N)$ is the ordered standard basis of $\mathbb{R}^N$ is called the *coordinatization* of V corresponding to o and $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$.

The inverse of the coordinatization, $P := K^{-1}$, is called the corresponding *parametrization* of V. It is quite evident that

$$P(\xi) = o + \sum_{i=1}^{N} \xi^i \boldsymbol{x}_i \qquad (\xi \in \mathbb{R}^N).$$

Moreover, if $(\boldsymbol{p}^1, \ldots, \boldsymbol{p}^N)$ is the dual of the basis in question, then

$$K(x) = \big(\boldsymbol{p}^i \cdot (x - o) \big| \, i = 1, \ldots, N\big) \qquad (x \in \mathrm{V}).$$

Obviously, every affine bijection $K : \mathrm{V} \to \mathbb{R}^N$ is a coordinatization in the above sense: the one corresponding to $o := K^{-1}(0)$ and $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ where $\boldsymbol{x}_i := K^{-1}(\boldsymbol{\chi}_i) - o$ $(i = 1, \ldots, N)$.

Such a parametrization maps straight lines into straight lines. More closely, if $\alpha \in \mathbb{R}^N$ then P maps the straight line passing through α and parallel to $\boldsymbol{\chi}_i$ into the straight line passing through $P(\alpha)$ and parallel to $\boldsymbol{x}_i$:

$$P[\alpha + \mathbb{R}\boldsymbol{\chi}_i] = P(\alpha) + \mathbb{R}\boldsymbol{x}_i \qquad (i = 1, \ldots, N).$$

This is why affine coordinatizations are generally called *rectilinear.*

5.2. In application we often need non-affine coordinatizations as well. Coordinatization means in general that we represent the elements of the affine space by ordered N-tuples of real numbers (i.e. by elements of $\mathbb{R}^N$) in a smooth way.

Definition. Let V be an N-dimensional affine space. A mapping $K : \mathrm{V} \rightarrowtail \mathbb{R}^N$ is called a *local coordinatization* of V if

(i) K is injective,

(ii) K is smooth,

(iii) $\mathrm{D}K(x)$ is injective for all $x \in \mathrm{Dom}\ K$. ■

Evidently, $\mathrm{D}K(x)$ is bijective since the dimensions of its domain and range are equal; thus the inverse mapping theorem implies that also the inverse of K has the properties (i)–(ii)–$(iii;)$ $P := K^{-1}$ is called a *local parametrization* of V. We often omit the adjective "local".

5.3. If $\alpha \in \mathrm{Ran}\ K = \mathrm{Dom}\ P$ then $P[\alpha + \mathbb{R}\boldsymbol{\chi}_i]$ is a smooth curve in V; a parametrization of this curve is $p_i : \mathbb{R} \rightarrowtail \mathrm{V}$, $a \mapsto P(\alpha + a\boldsymbol{\chi}_i)$ $(i = 1, ..., N)$. The parametrization maps straight lines into curves, that is why such coordinatizations are often called *curvilinear.*

The curves corresponding to parallel straight lines do not intersect each other. The curves corresponding to meeting straight lines intersect each other *transversally,* i.e. their tangent spaces at the point of intersection do not coincide. For instance, using the previous notations we have that $\dot{p}_i(0) = \mathrm{D}P(\alpha)\cdot\boldsymbol{\chi}_i = \partial_i P(\alpha)$ is the tangent vector of the curve $P[\alpha+\mathbb{R}\boldsymbol{\chi}_i]$ at $P(\alpha)$; if $i \neq k$ then $\dot{p}_i(0) \neq \dot{p}_k(0)$.

If $x \in \mathrm{Dom}\ K$ then $P[K(x) + \mathbb{R}\boldsymbol{\chi}_i]$ is called the i-th *coordinate line* passing through x.

5.4. Recall Proposition 4.4: if $\mathcal{H}$ is an M-dimensional smooth submanifold of V then for every $e \in \mathcal{H}$ there is a coordinatization $K := (F, S)$ of V in a neighbourhood of e such that the first M coordinate lines run in $\mathcal{H}$. In other words, if P is the corresponding parametrization of V then $\mathbb{R}^M \rightarrowtail \mathrm{V}, \zeta \mapsto P(\zeta, 0)$ is a parametrization of $\mathcal{H}$.

5.5. The most frequently used curvilinear coordinatizations are the polar coordinatization, the cylindrical coordinatization and the spherical coordinatization. We give them as coordinatizations in $\mathbb{R}^2$ and $\mathbb{R}^3$; composed with affine coordinatizations they result in curvilinear coordinatizations of two- and three-dimensional affine spaces.

(i) *Polar coordinatization*

$$K : \mathbb{R}^2 \setminus \{(x_1, 0) |\ x_1 \leq 0\} \to \mathbb{R}^+ \times]-\pi, \pi[\ ,$$
$$x = (x_1, x_2) \mapsto \left(|x|,\ \mathrm{sign}(x_2)\ \arccos\frac{x_1}{|x|}\right);$$

its inverse is

$$P : \mathbb{R}^+ \times]-\pi, \pi[\to \mathbb{R}^2 \setminus \{(x_1, 0) |\ x_1 \leq 0\},$$

$$(r, \varphi) \mapsto (r \cos \varphi,\ r \sin \varphi),$$

for which

$$\mathrm{D}P(r, \varphi) = \begin{pmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{pmatrix},$$

$$\det (\mathrm{D}P(r, \varphi)) = r.$$

(ii) *Cylindrical coordinatization*

$$K : \mathbb{R}^3 \setminus \{(x_1, 0, x_3) |\ x_1 \leq 0,\ x_3 \in \mathbb{R}\} \to \mathbb{R}^+ \times] - \pi, \pi[\times \mathbb{R},$$

$$x = (x_1, x_2, x_3) \mapsto \left(\sqrt{x_1^2 + x_2^2},\ \mathrm{sign}(x_2) \arccos \frac{x_1}{\sqrt{x_1^2 + x_2^2}},\ x_3 \right);$$

its inverse is

$$P : \mathbb{R}^+ \times] - \pi, \pi[\times \mathbb{R} \to \mathbb{R}^3 \setminus \{(x_1, 0, x_3,) |\ x_1 \leq 0,\ x_3 \in \mathbb{R}\},$$

$$(\rho, \varphi, z) \mapsto (\rho \cos \varphi,\ \rho \sin \varphi,\ z),$$

for which

$$\mathrm{D}P(\rho, \varphi, z) = \begin{pmatrix} \cos \varphi & -\rho \sin \varphi & o \\ \sin \varphi & \rho \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\det (\mathrm{D}P(\rho, \varphi, z)) = \rho.$$

(iii) *Spherical coordinatization*

$$K : \mathbb{R}^3 \setminus \{(x_1, 0, x_3) |\ x_1 \leq 0,\ x_3 \in \mathbb{R}\} \to \mathbb{R}^+ \times]0, \pi[\times] - \pi, \pi[,$$

$$x = (x_1, x_2, x_3) \mapsto \left(|x|,\ \arccos \frac{x_3}{|x|},\ \mathrm{sign}(x_2) \arccos \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right);$$

its inverse is

$$P : \mathbb{R}^+ \times]0, \pi[\times [-\pi, \pi[\to \mathbb{R}^3 \setminus \{(x_1, 0, x_3) |\ x_1 \leq 0,\ x_3 \in \mathbb{R}\},$$

$$(r, \vartheta, \varphi) \mapsto (r \sin \vartheta \cos \varphi,\ r \sin \vartheta \sin \varphi,\ r \cos \vartheta),$$

for which

$$\mathrm{D}P(r, \vartheta, \varphi) = \begin{pmatrix} \sin \vartheta \cos \varphi & r \cos \vartheta \cos \varphi & -r \sin \vartheta \sin \varphi \\ \sin \vartheta \sin \varphi & r \cos \vartheta \sin \varphi & r \sin \vartheta \cos \varphi \\ \cos \vartheta & -r \sin \vartheta & 0 \end{pmatrix},$$

$$\det (\mathrm{D}P(r, \vartheta, \varphi)) = r^2 \sin \vartheta.$$

5.6. Let $K : \mathrm{V} \rightarrowtail \mathbb{R}^N$ be a coordinatization. Then for all $x \in \mathrm{Dom}\, K$ the tangent vectors of the coordinate lines passing through x form a basis in $\mathbf{V}$. More closely, if P is the corresponding parametrization then $\partial_i P(P^{-1}(x)) = \mathrm{D}P(P^{-1}(x)) \cdot \boldsymbol{\chi}_i$ $(i = 1, \dots, N)$ form a basis in $\mathbf{V}$ which is called the *local basis* at x corresponding to K.

Note that $\mathrm{D}K(x) : \mathbf{V} \to \mathbb{R}^N$ is the linear bijection that sends the local basis into the standard basis of $\mathbb{R}^N$, i.e. $\mathrm{D}K(x)$ is the coordinatization of $\mathbf{V}$ corresponding to the local basis at x.

We shall often use the relation

$$[\mathrm{D}K(P(\xi))]^{-1} = \mathrm{D}P(\xi) \qquad\qquad (\xi \in \mathrm{Dom}\, P)$$

which will be written in the form

$$\mathrm{D}K(P)^{-1} = \mathrm{D}P.$$

(i) A vector field $\boldsymbol{C} : \mathrm{V} \rightarrowtail \mathbf{V}$ is coordinatized in such a way that for $x \in \mathrm{Dom}\, \boldsymbol{C} \cap \mathrm{Dom}\, K$ the vector $\boldsymbol{C}(x)$ is given by its coordinates with respect to the local basis at x and x is represented by the coordinatization in question; the *coordinatized form* of $\boldsymbol{C}$ is the function

$$\mathrm{D}K(P) \cdot \boldsymbol{C}(P) : \mathbb{R}^N \rightarrowtail \mathbb{R}^N, \qquad \xi \mapsto \mathrm{D}K(P(\boldsymbol{\xi})) \cdot \boldsymbol{C}(P(\boldsymbol{\xi})).$$

(ii) A covector field $\boldsymbol{S} : \mathrm{V} \rightarrowtail \mathbf{V}^*$ is coordinatized similarly, with the aid of the dual of the local bases (see IV.2.2); the *coordinatized form* of $\boldsymbol{S}$ is the function

$$\mathrm{D}P^* \cdot \boldsymbol{S}(P) : \mathbb{R}^N \to \left(\mathbb{R}^N\right)^*, \qquad \xi \mapsto \mathrm{D}P(\xi)^* \cdot \boldsymbol{S}(P(\xi)) = \boldsymbol{S}(P(\xi)) \cdot \mathrm{D}P(\xi).$$

(iii) Accordingly (see IV.2.3), the *coordinatizated forms* of the tensor fields $\boldsymbol{L} : \mathrm{V} \rightarrowtail \mathbf{V} \otimes \mathbf{V}^* \equiv \mathrm{Lin}(\mathbf{V})$ and $\boldsymbol{F} : \mathrm{V} \rightarrowtail \mathbf{V}^* \otimes \mathbf{V}^* \equiv \mathrm{Lin}(\mathbf{V}, \mathbf{V}^*)$ are

$$\mathrm{D}K(P) \cdot \boldsymbol{L}(P) \cdot \mathrm{D}P : \ \mathbb{R}^N \rightarrowtail \mathbb{R}^N \otimes \left(\mathbb{R}^N\right)^*, \quad \xi \mapsto \mathrm{D}K(P(\xi)) \cdot \boldsymbol{L}(P(\xi)) \cdot \mathrm{D}P(\xi),$$
$$\mathrm{D}P^* \cdot \boldsymbol{F}(P) \cdot \mathrm{D}P : \ \mathbb{R}^N \rightarrowtail \left(\mathbb{R}^N\right)^* \otimes \left(\mathbb{R}^N\right)^*, \ \ \xi \mapsto \mathrm{D}P(\xi)^* \cdot \boldsymbol{F}(P(\xi)) \cdot \mathrm{D}P(\boldsymbol{\xi}).$$

5.7. If $K : \mathrm{V} \to \mathbb{R}^N$ is an affine coordinatization then $\mathrm{D}K(x) = \boldsymbol{K}$ for all $x \in \mathrm{V}$ where $\boldsymbol{K}$ is the linear map under K. Similarly, $\mathrm{D}P(\xi) = \boldsymbol{P}$ for all $\xi \in \mathbb{R}^N$.

In this case the vector field $\boldsymbol{C}$ and the covector field $\mathbf{S}$ have the coordinatized form

$$\xi \mapsto \boldsymbol{K} \cdot \boldsymbol{C}(P(\xi)), \tag{1}$$
$$\xi \mapsto \boldsymbol{P}^* \cdot \boldsymbol{S}(P(\xi)). \tag{2}$$

The derivative of $\boldsymbol{C}$ is the mixed tensor field $\mathrm{D}\boldsymbol{C} : \mathrm{V} \rightarrowtail \mathbf{V} \otimes \mathbf{V}^*$, $x \mapsto \mathrm{D}\boldsymbol{C}(x)$, and the derivative of $\boldsymbol{S}$ is the cotensor field $\mathrm{D}\boldsymbol{S} : \mathrm{V} \rightarrowtail \mathbf{V}^* \otimes \mathbf{V}^*$, $x \mapsto \mathrm{D}\boldsymbol{S}(x)$. Now they have the coordinatized forms

$$\xi \mapsto \boldsymbol{K} \cdot \mathrm{D}\boldsymbol{C}(P(\xi)) \cdot \boldsymbol{P}, \tag{3}$$

$$\xi \mapsto \boldsymbol{P}^* \cdot \mathrm{D}\boldsymbol{S}(P(\xi)) \cdot \boldsymbol{P}. \tag{4}$$

A glance at the previous formulae convinces us that (3) and (4) are the derivatives of (1) and (2), respectively.

Thus in the case of a rectilinear coordinatization the order of differentiation and coordinatization can be interchanged: taking coordinates first and then differentiating is the same as differentiating first and then taking coordinates.

5.8. In the case of curvilinear coordinates, in general, the order of differentiation and coordinatization cannot be interchanged.

To get a rule, how to compute the coordinatized form of the derivative of a vector field or a covector field from the coordinatized form of these fields, we introduce a new notation.

Without loss of generality we suppose that $\mathrm{V} = \mathbb{R}^N$, since every curvilinear coordinatization $\mathrm{V} \rightarrowtail \mathbb{R}^N$ can be obtained as the composition of a rectilinear coordinatization $\mathrm{V} \to \mathbb{R}^N$ and a curvilinear one $\mathbb{R}^N \rightarrowtail \mathbb{R}^N$.

For the components of elements in $\mathrm{V} = \mathbb{R}^N$, Latin subscripts and superscripts: $i, j, k, \ldots$; for the components of the curvililinear coordinates in $\mathbb{R}^N$, Greek subscripts and superscripts: $\alpha, \beta, \gamma, \ldots$ are used. Moreover, we agree that all indices run from 1 to N and we accept the Einstein summation rule: for equal subscripts and superscripts a summation is to be taken from 1 to N.

Thus for K we write K^α, for P we write P^i; moreover, for any function $\phi : \mathbb{R}^N \rightarrowtail \mathbb{R}$ we find it convenient to write $\phi(P)$ instead of $\phi \circ P$. The rule of differentiation of composite functions will be used frequently,

$$\partial_\alpha \left(\phi(P)\right) = (\partial_i \phi)(P) \partial_\alpha P^i,$$

as well as the relations

$$\begin{aligned} \partial_\gamma P^i \partial_j K^\gamma(P) &= \delta^i{}_j, \\ \partial_j K^\alpha(P) \partial P^j &= \delta^\alpha{}_\beta. \end{aligned} \tag{$*$}$$

The second one implies

$$\partial_i \partial_j K^\alpha(P) \partial_\gamma P^i \partial_\beta P^j + \partial_i K^\alpha(P) \partial_\gamma \partial_\beta P^i = 0.$$

We put

$$\Gamma^\alpha{}_{\beta\gamma} := \partial_\gamma \partial_\beta P^i \partial_i K^\alpha(P) = -\partial_i \partial_j K^\alpha(P) \partial_\gamma P^i \partial_\beta P^j$$

and we call it the *Christoffel symbol* of the coordinatization in question.

The Christoffel symbol is a mapping defined on Dom K; for $\xi \in$ Dom K, $\Gamma(\xi)$ is a bilinear map from $\mathbb{R}^N \times \mathbb{R}^N$ into $\left(\mathbb{R}^N\right)^*$:

$$(\boldsymbol{\zeta}, \boldsymbol{\eta}) \mapsto \left(\Gamma^\alpha{}_{\gamma\beta}(\xi)\zeta^\gamma\eta^\beta \mid \alpha = 1, \dots, N\right).$$

It is usually emphasized that the Christoffel symbol is not a tensor of third order though it has three indices. This means that in general there is no mapping $\mathrm{V} \rightarrowtail \mathrm{Bilin}(\mathbf{V} \times \mathbf{V}, \mathbf{V}^*)$ (third order tensor field) whose coordinatized form would be the Christoffel symbol.

5.9. The coordinatized form of

$$\begin{array}{lll} f : V \rightarrowtail \mathbb{R} & \text{is} & f(P), \\ \boldsymbol{C} : \mathrm{V} \rightarrowtail \mathbf{V} & \text{is} & \partial_i K^\alpha(P) C^i(P) =: \mathcal{C}^\alpha, \\ \boldsymbol{S} : \mathrm{V} \rightarrowtail \mathbf{V}^* & \text{is} & \partial_\alpha P^i S_i(P) =: \mathcal{S}_\alpha, \\ \boldsymbol{L} : \mathrm{V} \rightarrowtail \mathbf{V} \otimes \mathbf{V}^* & \text{is} & \partial_i K^\alpha(P) L^i{}_k(P) \partial_\beta P^k =: \mathcal{L}^\alpha{}_\beta, \\ \boldsymbol{F} : \mathrm{V} \rightarrowtail \mathbf{V}^* \otimes \mathbf{V}^* & \text{is} & \partial_\alpha P^i T_{ik}(P) \partial_\beta P^k =: \mathcal{T}_{\alpha\beta}. \end{array}$$

(i) The coordinatized form of $\mathrm{D}f : \mathrm{V} \rightarrowtail \mathbf{V}^*$ is $\partial_\alpha P^i \partial_i f(P) = \partial_\alpha \left(f(P)\right)$; thus for a real-valued function the order of differentiation and coordinatization can be interchanged even in the case of curvilinear coordinatization.

(ii) The coordinatized form of $\mathrm{D}\boldsymbol{C} : \mathrm{V} \rightarrowtail \mathbf{V} \otimes \mathbf{V}^*$ is

$$(\mathcal{DC})^\alpha{}_\beta := \partial_i K^\alpha(P)(\partial_k C^i)(P)\partial_\beta P^k,$$

whereas the derivative of the coordinatized form of $\boldsymbol{C}$ reads

$$\partial_\beta \left(\partial_i K^\alpha(P) C^i(P)\right) = (\partial_i \partial_k K^\alpha)(P)\partial_\beta P^k C^i(P) + \partial_i K^\alpha(P)(\partial_k C^i)(P)\partial_\beta P^k.$$

The second term equals the coordinatized form of $\mathrm{D}\boldsymbol{C}$; with the aid of relation $(*)$ in 5.8, the first term is transformed into an expression containing the Christoffel symbol and the coordinatized form of $\boldsymbol{C}$. In this way we get

$$(\mathcal{DC})^\alpha{}_\beta = \partial_\beta \mathcal{C}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\mathcal{C}^\gamma.$$

(iii) Similarly, if $(\mathcal{DS})_{\alpha\beta}$ denotes the coordinatized form of $\mathrm{D}\boldsymbol{S}$ then

$$(\mathcal{DS})_{\alpha\beta} = \partial_\beta \mathcal{S}_\alpha - \Gamma^\gamma{}_{\alpha\beta}\mathcal{S}_\gamma.$$

5.10. Now we shall examine the coordinatizated form of two-times differentiable functions $\mathrm{I} \rightarrowtail \mathrm{V}$ where I is a one-dimensional affine space.

A useful notation will be applied: functions $\mathrm{I} \rightarrowtail \mathrm{V}$ and elements of V will be denoted by the same letter. If necessary, supplementary remarks rule out ambiguity.

For the sake of simplicity and without loss of generality we suppose that $\mathrm{I} = \mathbb{R}$.

Let $K : \mathrm{V} \rightarrowtail \mathbb{R}^N$ be a coordinatization, $P := K^{-1}$.

For $x \in \mathrm{V}$ let $\xi := K(x)$; then $x = P(\xi)$.

For $x : \mathrm{I} \rightarrowtail \mathrm{V}$ we put $\xi := K(x) := K \circ x$; then $x = P(\xi) := P \circ \xi$.

Denoting the differentiation by a dot we deduce

$$\dot{\xi} = \mathrm{D}K(x) \cdot \dot{x}, \qquad \dot{x} = \mathrm{D}P(\xi) \cdot \dot{\xi}, \tag{$**$}$$

$$\ddot{x} = \mathrm{D}^2P(\xi)(\dot{\xi}, \dot{\xi}) + \mathrm{D}P(\xi) \cdot \ddot{\xi},$$

$$\ddot{\xi} = \mathrm{D}^2K(x)(\dot{x}, \dot{x}) + \mathrm{D}K(x) \cdot \ddot{x},$$

from which we obtain

$$\mathrm{D}K(x) \cdot \ddot{x} = \ddot{\xi} - \Gamma(\xi)(\dot{\xi}, \dot{\xi}) \tag{$***$}$$

where

$$\Gamma(\xi) := \mathrm{D}^2K(P(\xi)) \circ (\mathrm{D}P(\xi) \times \mathrm{D}P(\xi))$$

is exactly the Christoffel symbol of the coordinatization.

In view of physical application, x, $\dot{x}$ and $\ddot{x}$ will be called *position, velocity* and *acceleration,* respectively.

The velocity at $t \in \mathbb{R}$, $\dot{x}(t)$ is in $\mathbf{V}$; it is represented by its coordinates corresponding to the local basis at $x(t)$, i.e. by $\mathrm{D}K(x(t)) \cdot \dot{x}(t)$. Thus $(**)$ tells us that the coordinatized form of velocity coincides with the derivative of the coordinatization of position.

Similarly, $\mathrm{D}K(x(t)) \cdot \ddot{x}(t)$ gives the coordinates of acceleration in the local basis at $x(t)$. Thus $(***)$ shows that the coordinatized form of acceleration does not coincide with the second derivative of the coordinatization of position.

5.11. Now we consider the coordinatizations treated in 5.5. They are *orthogonal* which means that every local basis is orthogonal with respect to the usual inner product in $\mathbb{R}^N$ $(N = 2, 3)$; in other words, if $\{\boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_N\}$ is the standard basis in $\mathbb{R}^N$ then $\{\mathrm{D}P(\xi) \cdot \boldsymbol{\chi}_i |\ i = 1, \dots, N\}$ is an orthogonal basis (the local basis at $P(\xi)$).

Introducing the notation

$$\alpha_i(\xi) := |\mathrm{D}P(\xi) \cdot \boldsymbol{\chi}_i| \qquad (i = 1, \dots, N)$$

we define the linear map $T(\xi) : \mathbb{R}^N \to \mathbb{R}^N$ by

$$T(\xi) \cdot \boldsymbol{\chi}_i := \alpha_i(\xi)\boldsymbol{\chi}_i \qquad (i = 1, \dots, N)$$

and then

$$\mathrm{D}P(\xi) = R(\xi) \cdot T(\xi)$$

where $R(\xi) : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal linear map.

In usual physical applications one prefers orthonormal local bases, i.e. one takes $\frac{\mathrm{D}P(\xi)\cdot\boldsymbol{\chi}_i}{\alpha_i(\xi)} = \mathrm{D}P(\xi)\cdot T(\xi)^{-1}\cdot\boldsymbol{\chi}_i = R(\xi)\cdot\boldsymbol{\chi}_i$ instead of $\mathrm{D}P(\xi)\cdot\boldsymbol{\chi}_i$ $(i = 1, .., N)$.

The vector $\boldsymbol{y} \in \mathbb{R}^N$ at $P(\xi)$ has the coordinates $R(\xi)^{-1} \cdot \boldsymbol{y}$ in the local basis $\{R(\xi) \cdot \boldsymbol{\chi}_i | \ i = 1, \ldots, N\}$.

Take $x : \mathbb{R} \rightarrowtail \mathbb{R}^N$, $\xi := K(x)$, $x = P(\xi)$ as in the previous paragraph. Then

$$\dot{x} = R(\xi) \cdot T(\xi) \cdot \dot{\xi}$$

from which we derive

$$\begin{aligned}\ddot{x} =& R(\xi)^{\cdot} \cdot T(\xi) \cdot \dot{\xi} + R(\xi) \cdot T(\xi)^{\cdot} \cdot \dot{\xi} + R(\xi) \cdot T(\xi) \cdot \ddot{\xi} = \\ =& R(\xi) \cdot \Big(\Big(R(\xi)^{-1} \cdot R(\xi)^{\cdot} \cdot T(\xi) + T(\xi)^{\cdot} \Big) \cdot \dot{\xi} + T(\xi) \cdot \ddot{\xi} \Big) .\end{aligned}$$

According to the foregoings, the coordinates of velocity in the orthonormal local basis at $P(\xi)$ are

$$T(\xi) \cdot \dot{\xi}$$

and the coordinates of acceleration in the orthonormal local basis at $P(\xi)$ are

$$\Big(R(\xi)^{-1} \cdot R(\xi)^{\cdot} \cdot T(\xi) + T(\xi)^{\cdot} \Big) \cdot \dot{\xi} + T(\xi) \cdot \ddot{\xi}.$$

5.12. (i) For polar coordinates $\xi = (r, \varphi)$,

$$R(r, \varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} =: \mathrm{R}(\varphi)$$

$$T(r, \varphi) = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} =: T(r).$$

Furthermore

$$R(\varphi)^{\cdot} = \dot{\varphi} R(\varphi) \cdot R(\pi/2), \qquad T(r)^{\cdot} = T(\dot{r}),$$

and so velocity and acceleration in the local orthonormal basis at (r, φ) are

$$(\dot{r}, r\dot{\varphi}) \qquad \text{and} \qquad (\ddot{r} - r\dot{\varphi}^2,\ r\ddot{\varphi} + 2\dot{r}\dot{\varphi}),$$

respectively.

(ii) For cylindrical coordinates $\xi = (\rho, \varphi, z)$,

$$R(\rho, \varphi, z) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} =: R(\varphi),$$

$$T(\rho, \varphi, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} =: T(\rho)$$

and we deduce as previously that velocity and acceleration in the local orthonormal basis at (ρ, φ, z) are

$$(\dot\rho,\ \rho\dot\varphi,\ \dot z) \qquad \text{and} \qquad (\ddot\rho - \rho\dot\varphi^2,\ \rho\ddot\varphi + 2\dot\rho\dot\varphi,\ \ddot z),$$

respectively.
(iii) For spherical coordinates $\xi = (r, \vartheta, \varphi)$,

$$R(r,\vartheta,\varphi) = \begin{pmatrix} \sin\vartheta\cos\varphi & \cos\vartheta\cos\varphi & -\sin\varphi \\ \sin\vartheta\sin\varphi & \cos\varphi\sin\varphi & \cos\varphi \\ \cos\vartheta & -\sin\vartheta & 0 \end{pmatrix} =: R(\vartheta,\varphi),$$

$$T(r,\vartheta,\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\vartheta \end{pmatrix} =: T(r,\vartheta).$$

The components of velocity in the local orthonormal basis at (r, ϑ, φ) are

$$(\dot r,\ r\dot\vartheta,\ r\sin\vartheta\ \dot\varphi).$$

The components of acceleration are given by rather complicated formulae; the ambitious reader is asked to perform the calculations.

5.13. Exercises

1. Give the polar coordinatized form of the linear map (vector field) $\mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

2. Give the cylindrical and the spherical coordinates of the following vector fields:
(i) $\boldsymbol{L} : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map;
(ii) $\mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto |x|\boldsymbol{v}$ where $\boldsymbol{v}$ is a given non-zero element of $\mathbb{R}^3$.

3. Find the coordinatized form of
(i) the divergence of a vector field,
(ii) the curl of a covector field.

6. Differential equations

6.1. Definition. Let V be a finite-dimensional affine space over the vector space $\mathbf{V}$.

Suppose $\boldsymbol{C} : \mathrm{V} \rightarrowtail \mathbf{V}$ is a differentiable vector field, Dom $\boldsymbol{C}$ is connected.

Then a *solution* of the *differential equation*

$$(x : \mathbb{R} \rightarrowtail \mathrm{V})? \qquad \dot{x} = \boldsymbol{C}(x)$$

is a differentiable function $r : \mathbb{R} \rightarrowtail \mathrm{V}$ such that
(i) Dom r is an interval,
(ii) Ran $r \subset$ Dom $\boldsymbol{C}$,
(iii) $\dot{r}(t) = \boldsymbol{C}(r(t))$ for $t \in$ Dom r.

The range of a solution is called an *integral curve* of $\boldsymbol{C}$. An integral curve is *maximal* if it is not contained properly in an integral curve. ■

An integral curve, in general, is not a curve in the sense of our definition in 4.3, i.e. it is not necessarily a submanifold.

6.2. Definition. Let $\boldsymbol{C}$ be as before and let x_o be an element of Dom $\boldsymbol{C}$. A *solution* of the *initial value problem*

$$(x : \mathbb{R} \rightarrowtail \mathrm{V})? \qquad \dot{x} = \boldsymbol{C}(x), \qquad x(t_o) = x_o \tag{$*$}$$

is a solution r of the corresponding differential equation such that

$$t_o \in \mathrm{Dom}\, r \qquad \text{and} \qquad r(t_o) = x_o.$$

The range of the solution of the initial value problem is called the *integral curve* of $\boldsymbol{C}$ *passing through* x_o. ■

The well-known existence and local uniqueness theorem asserts that solutions of the initial value problem exist and two solutions coincide on the intersection of their domain; consequently there is a single maximal integral curve of $\boldsymbol{C}$ passing through x_o.

6.3. Let U be another affine space over the vector space $\boldsymbol{U}$, $\dim \mathrm{U} = \dim \mathrm{V}$. Suppose $L : \mathrm{V} \rightarrowtail \mathrm{U}$ is a continuously differentiable injection whose inverse is continuously differentiable as well, and Dom $\boldsymbol{C} \subset$ Dom L.

Put

$$\boldsymbol{G} : \mathrm{U} \rightarrowtail \mathbf{U}, \qquad y \mapsto \mathrm{D}L\left(L^{-1}(y)\right) \cdot \boldsymbol{C}(L^{-1}(y)).$$

Then r is a solution of the initial value problem $(*)$ if and only if $L \circ r$ is a solution of the initial value problem

$$(y : \mathbb{R} \rightarrowtail \mathrm{U})? \qquad \dot{y} = \boldsymbol{G}(y), \qquad y(t_o) = L(x_o) \qquad\qquad (**).$$

That is why we call $(**)$ the *transformation of* $(*)$ by L.

6.4. Proposition. Let $\boldsymbol{C}$ be a differentiable vector field in V and let $\mathcal{H}$ be a submanifold in the domain of $\boldsymbol{C}$. If $\boldsymbol{C}(x) \in \mathrm{T}_x(\mathcal{H})$ for all $x \in \mathcal{H}$ and $x_o \in \mathcal{H}$ then every solution r of the initial value problem $(*)$ runs in $\mathcal{H}$, i.e. Ran $r \subset \mathcal{H}$.

Proof. The element x_o has a neighbourhood $\mathcal{N}$ in V and there are continuously differentiable functions $F : \mathcal{N} \to \mathbb{R}^M$, $S : \mathcal{N} \to \mathbb{R}^{N-M}$ such that $S(x) = 0$ for $x \in \mathcal{H} \cap \mathcal{N}$, and $K := (F, S) : \mathrm{V} \rightarrowtail \mathbb{R}^M \times \mathbb{R}^{N-M} \equiv \mathbb{R}^N$ is a local coordinatization of V. For $P := K^{-1}$ (the corresponding local parametrization of V) $\zeta \mapsto P(\zeta, 0)$ is a parametrization of $\mathcal{H} \cap \mathcal{N}$. Thus the tangent space of $\mathcal{H}$ at $P(\zeta, 0)$ is Ker $\mathrm{D}S(P(\zeta, 0))$ (see 4.4 and 4.8).

The coordinatized form of $\boldsymbol{C}$ becomes

$$(\Phi, \Psi) : \mathbb{R}^M \times \mathbb{R}^{N-M} \rightarrowtail \mathbb{R}^M \times \mathbb{R}^{N-M}$$

where

$$\begin{aligned} \Phi(\zeta, \eta) :=& \mathrm{D}F(P(\zeta, \eta)) \cdot \boldsymbol{C}(P(\zeta, \eta)), \\ \Psi(\zeta, \eta) :=& \mathrm{D}S(P(\zeta, \eta)) \cdot \boldsymbol{C}(P(\zeta, \eta)). \end{aligned}$$

Then the coordinatization transforms the initial value problem $(*)$ into the following one:

$$(\dot{\zeta}, \dot{\eta}) = (\Phi(\zeta, \eta), \Psi(\zeta, \eta)), \qquad \zeta(t_o) = F(x_o), \qquad \eta(t_o) = 0. \qquad (***)$$

This means that r is a solution of $(*)$ if and only if $(F \circ r, S \circ r)$ is a solution of $(***)$, or (ρ, σ) is a solution of $(***)$ if and only if $P \circ (\rho, \sigma)$ is a solution of $(*)$.

Since $\boldsymbol{C}(x) \in T_x(\mathcal{H})$ for $x \in \mathcal{H}$, $\boldsymbol{C}(P(\zeta, 0))$ is in the kernel of $\mathrm{D}S(P(\zeta, 0))$, i.e. $\Psi(\zeta, 0) = 0$ for all possible $\zeta \in \mathbb{R}^M$. Then if ρ is a solution of the initial value problem

$$\dot{\zeta} = \Phi(\zeta, 0), \qquad \zeta(t_o) = F(x_o)$$

then $(\rho, 0)$ is a solution of $(***)$. Then the uniqueness of solutions of initial value problems implies that every solution of $(***)$ has the form $(\rho, 0)$. Consequently, $t \mapsto P(\rho(t), 0)$, a solution of $(*)$, takes values in $\mathcal{H}$.

6.5. Physical application requires differential equations for functions $\mathrm{I} \rightarrowtail \mathrm{V}$ where I is a one-dimensional real affine space. Since the derivative of such functions takes values in $\frac{\mathbf{V}}{\mathbf{I}}$, we start with a differentiable mapping $\boldsymbol{C} : \mathrm{V} \rightarrowtail \frac{\mathbf{V}}{\mathbf{I}}$. A solution of the differential equation

$$(x : \mathrm{I} \rightarrowtail \mathrm{V})? \qquad \dot{x} = \boldsymbol{C}(x)$$

is a differentiable function $r : \mathrm{I} \rightarrowtail \mathrm{V}$ for which (i)–(ii)–(iii) of definition 6.1 holds.

Integral curves, solutions of initial value problems etc. are formulated as previously.

7. Integration on curves

7.1. Let I be an oriented one-dimensional affine space over the vector space $\mathbf{I}$. Suppose $\mathbf{A}$ is a one-dimensional vector space and $f : \mathrm{I} \rightarrowtail \mathbf{A}$ is a continuous function defined on an interval (see Exercise 1.7.5). If $a, b \in \mathrm{Dom}\ f$, $a < b$, then

$$\int_a^b f(t)\mathrm{d}t \in \mathbf{A} \otimes \mathbf{I}$$

is defined by some limit procedure, in the way well-known in standard analysis of real functions, using the integral approximation sums of the form

$$\sum_{k=1}^{n} f(t_k)(t_{k+1} - t_k).$$

7.2. Let V be an affine space over the vector space $\mathbf{V}$ and let $\mathbf{A}$ be a one-dimensional vector space. Suppose $F : \mathrm{V} \times \mathbf{V} \to \mathbf{A}$ is a continuous function, positively homogeneous in the second variable, i.e.

$$F(x, \lambda \boldsymbol{x}) = \lambda F(x, \boldsymbol{x}) \qquad (x \in \mathrm{V}, \lambda \in \mathbb{R}_0^+, \boldsymbol{x} \in \mathbf{V}).$$

Let $\mathcal{C}$ be a connected curve in V.

Proposition. Let $p, q : \mathbb{R} \rightarrowtail \mathrm{V}$ be equally oriented parametrizations of $\mathcal{C}$, $x, y \in \mathrm{Ran}\ p \cap \mathrm{Ran}\ q$. Then

$$\int_{p^{-1}(x)}^{p^{-1}(y)} F\left(p(t), \dot{p}(t)\right) \mathrm{d}t = \int_{q^{-1}(x)}^{q^{-1}(y)} F\left(q(s), \dot{q}(s)\right) \mathrm{d}s.$$

Proof. We know that $\Phi := p^{-1} \circ q : \mathbb{R} \to \mathbb{R}$ is differentiable and $\dot{\Phi} > 0$ (see 4.11). Consequently, $q = p \circ \Phi$, $\dot{q}(s) = \dot{p}(\Phi(s)) \cdot \dot{\Phi}(s)$ and

$$\int_{q^{-1}(x)}^{q^{-1}(y)} F\left(q(s), \dot{q}(s)\right) \mathrm{d}s = \int_{\Phi^{-1}(p^{-1}(x))}^{\Phi^{-1}(p^{-1}(y))} F\left(p(\Phi(s)),\ \dot{p}(\Phi(s))\right) \dot{\Phi}(s)\mathrm{d}s,$$

which gives the desired result by the well-known formula of integration by substitution. ■

7.3. Suppose $\mathcal{C}$ is oriented. Then, according to the previous result, we introduce the notation

$$\int_x^y F(\cdot, \mathrm{d}\mathcal{C}) := \int_{p^{-1}(x)}^{p^{-1}(y)} F(p(t), \dot{p}(t))\mathrm{d}t$$

where p is an arbitrary positively oriented parametrization of $\mathcal{C}$ such that $x, y \in$ Ran p.

Note that according to the definition we have

$$\int_y^x F(\cdot, \mathrm{d}\mathcal{C}) = -\int_x^y F(\cdot, \mathrm{d}\mathcal{C}).$$

If $\mathcal{C}$ is not oriented, we shall use the symbol

$$\int_{[x,y]} F(\cdot, \mathrm{d}\mathcal{C}) := \left| \int_{p^{-1}(x)}^{p^{-1}(y)} F(p(t), \dot{p}(t))\mathrm{d}t \right|$$

where p is an arbitrary parametrization.

We frequently meet the particular case when F does not depend on the elements of V, i.e. there is a positively homogeneous $f : \mathbf{V} \to \mathbf{A}$ such that $F(x, \boldsymbol{x}) = f(\boldsymbol{x})$ for all $x \in \mathrm{V}$, $\boldsymbol{x} \in \mathbf{V}$. Then we use the symbol

$$\int_x^y f(\mathrm{d}\mathcal{C}) \qquad \text{and} \qquad \int_{[x,y]} f(\mathrm{d}\mathcal{C})$$

for the corresponding integrals.

7.4. We can generalize the previous result for a parametrization $r : \mathrm{I} \rightarrowtail \mathrm{V}$ where I is an oriented one-dimensional affine space over the vector space $\mathbf{I}$. Then $\dot{r}(t)$ is in $\frac{\mathbf{V}}{\mathbf{I}}$ and accepting the definition $F\left(x, \frac{\boldsymbol{x}}{\boldsymbol{t}}\right) := \frac{F(x,\boldsymbol{x})}{|\boldsymbol{t}|}$ $(x \in \mathrm{V}, \boldsymbol{x} \in \mathbf{V}, \mathbf{0} \neq \boldsymbol{t} \in \mathbf{I})$ we have

$$\int_x^y F(\cdot, \mathrm{d}\mathcal{C}) = \int_{r^{-1}(x)}^{r^{-1}(y)} F\left(r(t), \dot{r}(t)\right) \mathrm{d}t$$

if r is positively oriented.

7.5. Let $(\mathrm{V}, \mathbf{B}, \mathbf{h})$ be a pseudo-Euclidean affine space (i.e. V is an affine space over $\mathbf{V}$ and $(\mathbf{V}, \mathbf{B}, \mathbf{h})$ is a pseudo-Euclidean vector space). Supposing $\mathbf{B}$ is oriented, we have the square root mapping $(\mathbf{B} \otimes \mathbf{B})_{\mathbf{0}}^{+} \to \mathbf{B}_{\mathbf{0}}^{+}$ and

$$\mathbf{V} \to \mathbf{B}, \qquad \boldsymbol{x} \mapsto |\boldsymbol{x}| := \sqrt{|\boldsymbol{x} \cdot \boldsymbol{x}|}$$

is a positively homogeneous function. Thus if $\mathcal{C}$ is an oriented curve in V, then

$$\int_x^y |\mathrm{d}\mathcal{C}|$$

is meaningful for all $x, y \in \mathcal{C}$. In the Euclidean case it is regarded as the signed *length* of the curve segment between x and y; in the non-Euclidean case it is interpreted as the *pseudo-length* of the curve segment.

Proposition. Suppose that $|\boldsymbol{x}| \neq 0$ for all non-zero tangent vectors $\boldsymbol{x}$ of $\mathcal{C}$. Then for all $x_o \in \mathcal{C}$,

$$\mathcal{C} \to \mathbf{B}, \qquad x \mapsto \int_{x_o}^x |\mathrm{d}\mathcal{C}|$$

is a continuous injection whose inverse is a positively oriented parametrization of $\mathcal{C}$.

Proof. Let Z denote the above mapping and choose a positively oriented parametrization $p : \mathbb{R} \rightarrowtail \mathrm{V}$ and put $t_o := p^{-1}(x_o)$. Then

$$(Z \circ p)(t) = \int_{t_o}^t |\dot{p}(s)|\mathrm{d}s \qquad\qquad (t \in \mathrm{Dom}\ p);$$

consequently, $Z \circ p : \mathbb{R} \rightarrowtail \mathbf{B}$ is continuously differentiable and $(Z \circ p)^{\cdot}(t) = |\dot{p}(t)| > 0$ for all $t \in \mathrm{Dom}\ p$. Thus $Z \circ p$ is strictly monotone increasing: it is injective and its inverse $(Z \circ p)^{-1}$ is continuously differentiable as well, and according to the well-known rule,

$$\left((Z \circ p)^{-1}\right)^{\cdot} = \frac{1}{(Z \circ p)^{\cdot}\left((Z \circ p)^{-1}\right)} > 0.$$

As a consequence, introducing the notation $r := Z^{-1}$, we have that $r = p \circ (Z \circ p)^{-1}$ is continuously differentiable, too, and

$$\dot{r} \circ r^{-1} = \frac{\dot{p}}{|\dot{p}|} \circ p^{-1}.$$

This means that r is a parametrization of $\mathcal{C}$ and $r^{-1} \circ p$ $(= Z \circ p)$ has everywhere positive derivative, i.e. r and p are equally oriented. ■

It is worth noting that $|\dot{r}| = 1$.

VII. LIE GROUPS

We treat only a special type of Lie groups appearing in physics; so we avoid the application of the theory of smooth manifolds.

1. Groups of linear bijections

1.1. Let $\mathbf{V}$ be an N-dimensional real vector space, $N \neq 0$.

Then $\mathrm{Lin}(\mathbf{V})$ is an N^2-dimensional real vector space.

Now the symbol of composition between elements of $\mathrm{Lin}(\mathbf{V})$ will be omitted, i.e. we write $\boldsymbol{AB} := \boldsymbol{A} \circ \boldsymbol{B}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}(\mathbf{V})$.

Since $\mathbf{V}$ is finite dimensional, all norms on it are equivalent, i.e. all norms give the same open subsets. Given a norm $\|\ \|$ on $\mathbf{V}$, a norm is defined on $\mathrm{Lin}(\mathbf{V})$ by

$$\|\boldsymbol{A}\| := \sup_{\|\boldsymbol{v}\|=1} \|\boldsymbol{A} \cdot \boldsymbol{v}\|$$

for which $\|\boldsymbol{AB}\| \leq \|\boldsymbol{A}\|\ \|\boldsymbol{B}\|$ holds $(\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}(\mathbf{V}))$.

We introduce the notation

$$\mathcal{G}\ell(\mathbf{V}) := \{\boldsymbol{F} \in \mathrm{Lin}(\mathbf{V}) \mid \boldsymbol{F} \text{ is bijective}\}.$$

Endowed with the multiplication $(\boldsymbol{F}, \boldsymbol{G}) \mapsto \boldsymbol{FG}$ (composition), $\mathcal{G}\ell(\mathbf{V})$ is a group whose identity (neutral element) is

$$\boldsymbol{I} := \mathrm{id}_{\mathbf{V}}.$$

1.2. One can prove without difficulty that if $\boldsymbol{A} \in \mathrm{Lin}(\mathbf{V})$, $\|\boldsymbol{A}\| < 1$, then $\boldsymbol{I} - \boldsymbol{A} \in \mathcal{G}\ell(\mathbf{V})$ and

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{n=0}^{\infty} \boldsymbol{A}^n.$$

In other words, if $\boldsymbol{K} \in \operatorname{Lin}(\mathbf{V})$, $\|\boldsymbol{I}-\boldsymbol{K}\|<1$, then $\boldsymbol{K} \in \mathcal{G}\ell(\mathbf{V})$ and

$$\boldsymbol{K}^{-1}=\sum_{n=0}^{\infty}(\boldsymbol{I}-\boldsymbol{K})^{n}.$$

Proposition. Let $\boldsymbol{F} \in \mathcal{G}\ell(\mathbf{V})$. If $\boldsymbol{L} \in \operatorname{Lin}(\mathbf{V})$ and $\|\boldsymbol{F}-\boldsymbol{L}\|<\frac{1}{\|\boldsymbol{F}^{-1}\|}$ then $\boldsymbol{L} \in \mathcal{G}\ell(\mathbf{V})$.

Proof. $\|\boldsymbol{I}-\boldsymbol{F}^{-1}\boldsymbol{L}\|=\|\boldsymbol{F}^{-1}(\boldsymbol{F}-\boldsymbol{L})\| \leq \|\boldsymbol{F}^{-1}\|\ \|\boldsymbol{F}-\boldsymbol{L}\|<1$, thus $\boldsymbol{F}^{-1}\cdot\boldsymbol{L}$ is bijective. $\boldsymbol{F}$ is bijective by assumption, hence $\boldsymbol{F}(\boldsymbol{F}^{-1}\boldsymbol{L})=\boldsymbol{L}$ is bijective as well. ■

As a corollary of this result we have that $\mathcal{G}\ell(\mathbf{V})$ is an open subset of $\operatorname{Lin}(\mathbf{V})$.

1.3. The proof of the following statement is elementary.
The mappings

$$\mathrm{m}: \mathcal{G}\ell(\mathbf{V}) \times \mathcal{G}\ell(\mathbf{V}) \rightarrow \mathcal{G}\ell(\mathbf{V}), \qquad (\boldsymbol{F}, \boldsymbol{G}) \mapsto \boldsymbol{F}\boldsymbol{G},$$

$$\mathrm{j}: \mathcal{G}\ell(\mathbf{V}) \rightarrow \mathcal{G}\ell(\mathbf{V}), \qquad \boldsymbol{F} \mapsto \boldsymbol{F}^{-1}$$

are smooth and

$$\mathrm{Dm}(\boldsymbol{F}, \boldsymbol{G}): \operatorname{Lin}(\mathbf{V}) \times \operatorname{Lin}(\mathbf{V}) \rightarrow \operatorname{Lin}(\mathbf{V}), \qquad (\boldsymbol{A}, \boldsymbol{B}) \mapsto \boldsymbol{A}\boldsymbol{G}+\boldsymbol{F}\boldsymbol{B},$$

$$\mathrm{Dj}(\boldsymbol{F}): \operatorname{Lin}(\mathbf{V}) \mapsto \operatorname{Lin}(\mathbf{V}), \qquad \boldsymbol{A} \mapsto -\boldsymbol{F}^{-1}\boldsymbol{A}\boldsymbol{F}^{-1}.$$

1.4. It is a well-known fact, too, that for $\boldsymbol{A} \in \operatorname{Lin}(\mathbf{V})$

$$\exp \boldsymbol{A}:=e^{\boldsymbol{A}}:=\sum_{n=0}^{\infty} \frac{\boldsymbol{A}^{n}}{n!}$$

is meaningful, it is an element of $\mathcal{G}\ell(\mathbf{V})$,

$$e^{\mathbf{0}}=\boldsymbol{I}, \qquad \left(e^{\boldsymbol{A}}\right)^{-1}=e^{-\boldsymbol{A}}.$$

Moreover, the *exponential mapping,*

$$\operatorname{Lin}(\mathbf{V}) \rightarrow \mathcal{G}\ell(\mathbf{V}), \qquad \boldsymbol{A} \mapsto e^{\boldsymbol{A}}$$

is smooth, its derivative at $\mathbf{0} \in \operatorname{Lin}(\mathbf{V})$ is the identity map $\operatorname{Lin}(\mathbf{V}) \rightarrow \operatorname{Lin}(\mathbf{V})$.

The inverse mapping theorem implies that the exponential mapping is injective in a neighbourhood of $\mathbf{0}$, its inverse regarding this neighbourhood is smooth as well.

If $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}(\mathbf{V})$ and $\boldsymbol{AB} = \boldsymbol{BA}$ then $e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{B}}e^{\boldsymbol{A}} = e^{\boldsymbol{A}+\boldsymbol{B}}$. In particular, $e^{t\boldsymbol{A}}e^{s\boldsymbol{A}} = e^{s\boldsymbol{A}}e^{t\boldsymbol{A}} = e^{(t+s)\boldsymbol{A}}$ for $t, s \in \mathbb{R}$.

1.5. For $\boldsymbol{A} \in \mathrm{Lin}(\mathbf{V})$, the function $\mathbb{R} \to \mathcal{G}\ell(\mathbf{V})$, $t \mapsto e^{t\boldsymbol{A}}$ is smooth and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t\boldsymbol{A}}\right) = \boldsymbol{A}e^{t\boldsymbol{A}} = e^{t\boldsymbol{A}}\boldsymbol{A}.$$

As a consequence, the initial value problem

$$(\boldsymbol{X} : \mathbb{R} \rightarrowtail \mathrm{Lin}(\mathbf{V}))? \qquad \dot{\boldsymbol{X}} = \boldsymbol{X}\boldsymbol{A}, \qquad \boldsymbol{X}(0) = \boldsymbol{I}$$

has the unique maximal solution

$$\boldsymbol{R}(t) = e^{t\boldsymbol{A}} \qquad (t \in \mathbb{R}).$$

2. Groups of affine bijections

2.1. Let V be an affine space over the N-dimensional real vector space $\mathbf{V}$. Then

$$\mathrm{Aff}(\mathrm{V}, \mathbf{V}) := \{A : \mathrm{V} \to \mathbf{V} \mid A \quad \text{is affine}\},$$

endowed with the pointwise operations, is a real vector space.

Given $o \in \mathrm{V}$, the correspondence

$$\mathrm{Aff}(\mathrm{V}, \mathbf{V}) \to \mathbf{V} \times \mathrm{Lin}(\mathbf{V}), \qquad A \mapsto (A(o), \boldsymbol{A})$$

(where $\boldsymbol{A}$ is the linear map under A) is a linear bijection; it is evidently linear and injective and it is surjective because the affine map $\mathrm{V} \to \mathbf{V}$, $x \mapsto \boldsymbol{A} \cdot (x - o) + \boldsymbol{a}$ corresponds to $(\boldsymbol{a}, \boldsymbol{A}) \in \mathbf{V} \times \mathrm{Lin}(\mathbf{V})$.

As a consequence, $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$ is an $(N + N^2)$-dimensional vector space.

2.2. We easily find that

$$\mathrm{Aff}(\mathrm{V}) := \{L : \mathrm{V} \to \mathrm{V} \mid L \quad \text{is affine}\},$$

endowed with the pointwise subtraction (see VI.2.3(iv)), is an affine space over $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$. Thus, according to the previous paragraph, $\mathrm{Aff}(\mathrm{V})$ is $(N + N^2)$-dimensional.

Two elements K and L of $\mathrm{Aff}(\mathrm{V})$, as well as an element A of $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$ and an element L of $\mathrm{Aff}(\mathrm{V})$ can be composed; the symbol of compositions will be ommitted, i.e. $KL := K \circ L$ and $AL := A \circ L$.

We introduce

$$\mathcal{G}a(\mathrm{V}) := \{F \in \mathrm{Aff}(\mathrm{V}) |\ F \quad \text{is bijective}\}.$$

Endowed with the multiplication $(F, G) \mapsto FG$ (composition), $\mathcal{G}a(\mathrm{V})$ is a group whose identity (neutral element) is

$$I := \mathrm{id}_{\mathrm{V}}.$$

2.3. Given $o \in \mathrm{V}$, the mapping

$$\mathrm{Aff}(\mathrm{V}) \to \mathbf{V} \times \mathrm{Lin}(\mathbf{V}), \qquad L \mapsto (L(o) - o, \boldsymbol{L})$$

is an affine bijection over the linear bijection given in 2.1. Evidently, this bijection maps $\mathcal{G}a(\mathrm{V})$ onto $\mathbf{V} \times \mathcal{G}\ell(\mathbf{V})$. As a consequence, $\mathcal{G}a(\mathrm{V})$ is an open subset of $\mathrm{Aff}(\mathrm{V})$.

2.4. The mappings

$$\mathrm{m} : \mathcal{G}a(\mathrm{V}) \times \mathcal{G}a(\mathrm{V}) \to \mathcal{G}a(\mathrm{V}), \qquad (F, G) \mapsto FG,$$

$$\mathrm{j} : \mathcal{G}a(\mathrm{V}) \to \mathcal{G}a(\mathrm{V}), \qquad F \mapsto F^{-1}$$

are smooth and

$$\mathrm{Dm}(F, G) : \mathrm{Aff}(\mathrm{V}, \mathbf{V}) \times \mathrm{Aff}(\mathrm{V}, \mathbf{V}) \to \mathrm{Aff}(\mathrm{V}, \mathbf{V}), \qquad (A, B) \mapsto \boldsymbol{A}G + \boldsymbol{F}B,$$

$$\mathrm{Dj}(F) : \mathrm{Aff}(\mathrm{V}, \mathbf{V}) \to \mathrm{Aff}(\mathrm{V}, \mathbf{V}), \qquad A \mapsto -\boldsymbol{F}^{-1} A F^{-1}.$$

2.5. If $\boldsymbol{P} \in \mathcal{G}\ell(\mathbf{V})$ then

$$\ell_{\boldsymbol{P}} : \mathrm{Aff}(\mathrm{V}, \mathbf{V}) \to \mathrm{Aff}(\mathrm{V}, \mathbf{V}), \qquad A \to \boldsymbol{P}A$$

is a linear bijection, $(\ell_{\boldsymbol{P}})^{-1} = \ell_{\boldsymbol{P}^{-1}}$.

If $P \in \mathcal{G}a(\mathrm{V})$ then

$$\ell_P : \mathrm{Aff}(\mathrm{V}) \to \mathrm{Aff}(\mathrm{V}), \qquad L \mapsto PL$$

is an affine bijection over $\ell_{\boldsymbol{P}}$, where $\boldsymbol{P}$ is the linear map under P; moreover, $(\ell_P)^{-1} = \ell_{P^{-1}}$.

2.6. If $A \in \mathrm{Aff}(\mathrm{V}, \mathbf{V})$ and $\boldsymbol{A} \in \mathrm{Lin}(\mathbf{V})$ is the linear map under A then

$$\exp A := e^A := I + \sum_{n=1}^{\infty} \frac{\boldsymbol{A}^{n-1}A}{n!}$$

is meaningful, it is an element of $\mathcal{G}a(\mathrm{V})$,

$$e^0 = I, \qquad \left(e^A\right)^{-1} = e^{-A}$$

and the linear map under e^A is $e^{\boldsymbol{A}}$.

Moreover, the *exponential mapping*

$$\mathrm{Aff}(\mathrm{V}, \mathbf{V}) \to \mathcal{G}a(\mathrm{V}), \qquad A \mapsto e^A$$

is smooth, its derivative at $0 \in \mathrm{Aff}(\mathrm{V}, \mathbf{V})$ is the identity map $\mathrm{Aff}(\mathrm{V}, \mathbf{V}) \to \mathrm{Aff}(\mathrm{V}, \mathbf{V})$.

The inverse mapping theorem implies that the exponential mapping is injective in a neighbourhood of 0, its inverse regarding this neighbourhood is smooth as well.

If $A, B \in \mathrm{Aff}(\mathrm{V}, \mathbf{V})$ and $\boldsymbol{A}B = \boldsymbol{B}A$ then $e^A e^B = e^B e^A = e^{A+B}$. In particular, $e^{tA}e^{sA} = e^{sA}e^{tA} = e^{(t+s)A}$ for $t, s \in \mathbb{R}$.

2.7. For $A \in \mathrm{Aff}(\mathrm{V}, \mathbf{V})$, the function $\mathbb{R} \to \mathcal{G}a(\mathrm{V})$, $t \mapsto e^{tA}$ is smooth and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right) = e^{t\boldsymbol{A}}A.$$

(Note that $Ae^{t\boldsymbol{A}}$ makes no sense!).

As a consequence, the initial value problem

$$(X : \mathbb{R} \rightarrowtail \mathrm{Aff}(\mathrm{V}, \mathbf{V}))? \qquad \dot{X} = \boldsymbol{X}A, \qquad X(0) = I$$

($\boldsymbol{X} : \mathbb{R} \rightarrowtail \mathrm{Lin}(\mathbf{V})$, $\boldsymbol{X}(t)$ is the linear map under $X(t)$) has the unique maximal solution

$$R(t) = e^{tA} \qquad (t \in \mathbb{R}).$$

3. Lie groups

3.1. Definition Let V be an N-dimensional real affine space. A subgroup $\mathcal{G}$ of $\mathcal{G}a(\mathrm{V})$ which is an M-dimensional smooth submanifold of $\mathcal{G}a(\mathrm{V})$ is called an *M-dimensional plain Lie group.* ■

The group multiplication $\mathcal{G}\times\mathcal{G}\to\mathcal{G}$, $(F,G)\mapsto FG$ and the inversion $\mathcal{G}\to\mathcal{G}$, $F\mapsto F^{-1}$ are smooth mappings (see 2.4 and VI.4.13, Exercise VI.4.14.5(*iii*)).

Observe that by definition $0<M\leq N+N^2$. $(N+N^2)$-dimensional plain Lie groups are $\mathcal{G}a(\mathrm{V})$ and its open subgroups.

Remark. In general, a Lie group is defined to be a group endowed with a smooth structure in such a way that the group multiplication and the inversion are smooth mappings.

Since we shall deal only with plain Lie groups, we shall omit the adjective "plain". By the way, all the results we shall derive for plain Lie groups are valid for arbitrary Lie groups as well.

3.2. (*i*) For $\boldsymbol{x}\in\mathbf{V}$ we defined the affine bijection $T_{\boldsymbol{x}}:\mathrm{V}\to\mathrm{V}$, $x\mapsto x+\boldsymbol{x}$ (VI.2.4.3), the *translation by* $\boldsymbol{x}$. It is quite evident that $T_{\boldsymbol{x}}=T_{\boldsymbol{y}}$ if and only if $\boldsymbol{x}=\boldsymbol{y}$ and so

$$\mathcal{T}n(\mathbf{V}):=\{T_{\boldsymbol{x}}|\ \boldsymbol{x}\in\mathbf{V}\},$$

called the *translation group* of V, is an N-dimensional Lie group. The group multiplication in $\mathcal{T}n(\mathbf{V})$ corresponds exactly to the addition in $\mathbf{V}$ that is why one often says that $\mathbf{V}$ — in particular $\mathbb{R}^N$ — endowed with the addition as a group multiplication is an N-dimensional Lie group.

(*ii*) If the vector space $\mathbf{V}$ is considered to be an affine space then $\mathcal{G}\ell(\mathbf{V})$ is a subgroup and an N^2-dimensional submanifold of $\mathcal{G}a(\mathbf{V})$, thus $\mathcal{G}\ell(\mathbf{V})$ is an N^2-dimensional Lie group.

3.3. It is obvious that

$$\mathcal{G}a(\mathrm{V})\to\mathcal{G}\ell(\mathbf{V}),\qquad L\mapsto\boldsymbol{L}\qquad(\boldsymbol{L}\ \text{ is the linear map under } L)$$

is a smooth group homomorphism whose kernel is $\mathcal{T}n(\mathbf{V})$ ($\boldsymbol{L}=\boldsymbol{I}$ if and only if $L\in\mathcal{T}n(\mathbf{V})$, see VI.2.5.6).

(*i*) Take a Lie group $\mathcal{G}\subset\mathcal{G}a(\mathrm{V})$. Then

$$\mathrm{under}(\mathcal{G}):=\{\boldsymbol{F}\in\mathcal{G}\ell(\mathbf{V})|\ \boldsymbol{F}\ \text{ is under an }\ F\in\mathcal{G}\},$$

i.e. the image of $\mathcal{G}$ by the above group homomorphism is a Lie group.

(*ii*) Conversely, if $\mathcal{G}\subset\mathcal{G}\ell(\mathbf{V})$ is an M-dimensional Lie group, then

$$\mathrm{over}(\mathcal{G}):=\{F\in\mathcal{G}a(\mathrm{V})|\ F\ \text{ is over an }\ \boldsymbol{F}\in\mathcal{G}\},$$

the pre-image of $\mathcal{G}$ by the above group homomorphism, is an $(M+N)$-dimensional Lie group.

3.4. Recall that the tangent spaces of $\mathcal{G}$ are linear subspaces of $\mathrm{Aff}(\mathrm{V},\mathbf{V})$. Every tangent space of $\mathcal{G}$ is obtained quite simply from the tangent space at I : $\mathrm{T}_F(\mathcal{G})$ is the "translation" by F of $\mathrm{T}_I(\mathcal{G})$.

Proposition. Let $\mathcal{G} \subset \mathcal{G}a(\mathrm{V})$ be a Lie group. Then

$$\mathrm{T}_F(\mathcal{G}) = \boldsymbol{F}[\mathrm{T}_I(\mathcal{G}] = \{\boldsymbol{F}A \mid A \in \mathrm{T}_I(\mathcal{G})\} \qquad (F \in \mathcal{G}).$$

Proof. Let $\mathcal{G}$ be M-dimensional. There is a neighbourhood $\mathcal{N}$ of I in $\mathcal{G}a(\mathrm{V})$, a smooth mapping $S : \mathcal{N} \to \mathbb{R}^{N+N^2-M}$ such that $\mathcal{G} \cap \mathcal{N} = \overset{-1}{S}(\{0\})$ (see VI.4.4), and $\mathrm{T}_I(\mathcal{G}) = \mathrm{Ker}\ \mathrm{D}S(I)$ (see VI.4.8).

Let F be an arbitrary element of $\mathcal{G}$. Then $\mathcal{G}$ is invariant under the affine bijection $\ell_{F^{-1}} = \ell_F{}^{-1}$, thus $S \circ \ell_F{}^{-1}\big|_{\mathcal{G}} = 0$. Consequently, if P is in the domain of $S \circ \ell_F{}^{-1}$, i.e. $\ell_F{}^{-1}P = F^{-1}P$ is in $\mathcal{N}$, recalling that $\ell_F{}^{-1}$ is an affine map over $\ell_{\boldsymbol{F}}{}^{-1}$, hence $\mathrm{D}\ell_F{}^{-1}(P) = \ell_{\boldsymbol{F}}{}^{-1}$, we have

$$\begin{aligned}\mathrm{T}_P(\mathcal{G}) =& \mathrm{Ker}\ \mathrm{D}(S \circ \ell_F{}^{-1})(P) = \mathrm{Ker}\ \left(\mathrm{D}S(\ell_F{}^{-1}P) \cdot \mathrm{D}\ell_F{}^{-1}(P)\right) = \\ =& \mathrm{Ker}\ \left(\mathrm{D}S(F^{-1}P)\ell_{\boldsymbol{F}}{}^{-1}\right) = \{A \in \mathrm{Aff}(\mathrm{V}, \mathbf{V}) \mid \mathrm{D}S(F^{-1}P)\boldsymbol{F}^{-1}A = \mathbf{0}\} = \\ =& \{\boldsymbol{F}B \mid B \in \mathrm{Ker}\ \mathrm{D}S(F^{-1}P)\} = \boldsymbol{F}\left(\mathrm{Ker}\ \mathrm{D}S(F^{-1}P)\right).\end{aligned}$$

We can take $P := F$ to have the desired result. ■

The tangent space of $\mathcal{G}$ at I plays an important role; for convenience we introduce the notation

$$\mathbf{La}(\mathcal{G}) := \mathrm{T}_I(\mathcal{G}).$$

Note that $\mathbf{La}(\mathcal{G}a(\mathrm{V})) = \mathrm{Aff}(\mathrm{V}, \mathbf{V})$, $\mathbf{La}(\mathcal{G}\ell(\mathbf{V}) = \mathrm{Lin}(\mathbf{V})$.

Moreover, $\mathbf{La}(\mathcal{T}n(\mathbf{V})) = \mathbf{V}$ where $\mathbf{V}$ is identified with the constant maps $\mathrm{V} \to \mathbf{V}$.

3.5. Definition. A smooth function $R : \mathbb{R} \to \mathcal{G} \subset \mathcal{G}a(\mathrm{V})$ is called a *one-parameter subgroup* in the Lie group $\mathcal{G}$ if

$$R(t + s) = R(t)R(s) \qquad (t, s \in \mathbb{R}). \quad ■$$

In other words, a one-parameter subgroup is a smooth group homomorphism $R : \mathcal{T}n(\mathbb{R}) \to \mathcal{G}$. Evidently, $R(0) = I$ and $R(-t) = R(t)^{-1}$.

There are three possibilites.

(i) There is a neighbourhood of $0 \in \mathbb{R}$ such that $R(t) = I$ for all t in that neighbourhood; then R is a constant function, $R(t) = I$ for all $t \in \mathbb{R}$.

(ii) There is a $T \in \mathbb{R}^+$ such that $R(T) = I$ but $R(t) \neq I$ for $0 < t < T$; then R is periodic, $R(t + T) = R(t)$ for all $t \in \mathbb{R}$.

(iii) $R(t) \neq I$ for all $0 \neq t \in \mathbb{R}$.

3.6. If $\boldsymbol{R}(t)$ denotes the linear map under $R(t)$ then $\boldsymbol{R} : \mathbb{R} \to \mathrm{under}(\mathcal{G})$ is a one-parameter subgroup; $\boldsymbol{R}(0) = \boldsymbol{I}$.

Differentiating with respect to s in the defining equality of R and then putting $s = 0$ we get

$$\dot{R}(t) = \boldsymbol{R}(t)\dot{R}(0) \qquad (t \in \mathbb{R})$$

which shows that if Ran R is not a single point (if R is not constant) then it is a one-dimensional submanifold and a subgroup in $\mathcal{G}a(V)$. Thus Ran R is either the singleton $\{I\}$ or a one-dimensional Lie group. In the case (*ii*) treated in the preceding paragraph, the restriction of R to an interval shorter than T is a local parametrization of Ran R; in the case (*iii*) R is a parametrization of Ran R.

3.7. Proposition. Every one-parameter subgroup R in $\mathcal{G}$ has the form

$$R(t) = e^{tA} \qquad (t \in \mathbb{R})$$

where $A = \dot{R}(0) \in \mathbf{La}(\mathcal{G})$.

Conversely, if $A \in \mathbf{La}(\mathcal{G}) \subset \mathrm{Aff}(\mathrm{V}, \mathbf{V})$ then $t \mapsto e^{tA}$ is a one-parameter subgroup in $\mathcal{G}$.

Proof. According to the previous paragraph, the one-parameter subgroup R is the solution of the initial value problem

$$(X : \mathbb{R} \rightarrowtail \mathcal{G}a(\mathrm{V}))? \qquad \dot{X} = \boldsymbol{X}A, \qquad X(0) = I$$

where $A := \dot{R}(0)$. Apply 2.7 to obtain the first statement.

Conversely, $t \mapsto e^{tA}$ is a one-parameter subgroup in $\mathcal{G}a(\mathrm{V})$; we have to show only that $e^{tA} \in \mathcal{G}$ for all $t \in \mathbb{R}$ which follows from VI.6.4. ■

The assertions are true for *local one-parameter subgroups* as well, i.e. for smooth functions $R : \mathbb{R} \rightarrow \mathcal{G}$ defined on an interval around $0 \in \mathbb{R}$ such that $R(t + s) = R(t)R(s)$ whenever $t, s, t + s$ are in Dom R.

3.8. The previous result involves that $e^A \in \mathcal{G}$ for $A \in \mathbf{La}(\mathcal{G})$, i.e. the restriction of the exponential mapping onto $\mathbf{La}(\mathcal{G})$ takes values in $\mathcal{G}$. Since the exponential mapping is smooth and injective in a neighbourhood of 0, its inverse regarding this neighbourhood is smooth as well (in particular continuous), we can state:

Proposition. Let $\mathcal{G}$ be a Lie group. Then

$$\mathbf{La}(\mathcal{G}) \rightarrow \mathcal{G}, \qquad A \mapsto e^A$$

is a parametrization of $\mathcal{G}$ in a neighbourhood of the identity I.

In particular, every element in a neighbourhood of I belongs to a one-parameter subgroup.

3.9. Proposition. Every element of $\mathcal{G}$ in a neighbourhood of the identity is a product of elements taken from one-parameter subgroups corresponding to a basis of $\mathbf{La}(\mathcal{G})$.

Proof. Let $A_1, \dots A_M$ be a basis of $\mathbf{La}(\mathcal{G})$ and complete it to a basis $A_1, \dots, A_P$ of $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$ where $P := N + N^2$. Then

$$\Phi : \mathbb{R}^P \to \mathcal{G}a(\mathrm{V}), \qquad (t_1, t_2, \dots, t_P) \mapsto \exp(t_1 A_1) \exp(t_2 A_2) \dots \exp(t_P A_P)$$

is a smooth map, $\Phi(0, 0, \dots, 0) = I$, $\partial_k \Phi(0, 0, \dots, 0) = A_k$ $(k = 1, \dots, P)$. We can state on the basis of the inverse mapping theorem that Φ is injective in a neighbourhood of $(0, 0, \dots, 0)$, its inverse regarding this neighbourhood is smooth as well.

Thus the restriction of Φ onto $\mathbb{R}^M$ regarded as the subspace of $\mathbb{R}^P$ consisting of elements whose i-th components are zero for $i = M+1, \dots, P$ is a parametrization of $\mathcal{G}$ in a neighbourhood of I. ■

Note that in general

$$\exp(t_1 A_1) \exp(t_2 A_2) \dots \exp(t_P A_P) \neq \exp\left(\sum_{k=1}^{P} t_k A_k\right).$$

3.10. If $\mathcal{G}$ is connected, every element of $\mathcal{G}$ is a product of elements in a neighbourhood of I, hence every element is a product of elements taken from one-parameter subgroups corresponding to a basis of $\mathbf{La}(\mathcal{G})$, since the following proposition is true.

Proposition. If $\mathcal{G}$ is connected and $\mathcal{V}$ is a neighbourhood of the identity I in $\mathcal{G}$, then

$$\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{V}^n$$

where $\mathcal{V}^n := \{F_1 F_2 \dots F_n |\ F_k \in \mathcal{V},\ \ k = 1, \dots, n\}$.

Proof. Given $F \in \mathcal{G}$, the mapping $\mathcal{G} \to \mathcal{G}$, $G \mapsto FG$ is bijective, continuous, its inverse is continuous as well. Thus for all $F \in \mathcal{G}$, $F\mathcal{V} := \{FG |\ G \in \mathcal{V}\}$ is open, so $\mathcal{V}^2 = \bigcup_{F \in \mathcal{V}} F\mathcal{V}$ is open as well. Consequently, $\mathcal{V}^n$ is open for all n and thus $\mathcal{H} := \bigcup_{n \in \mathcal{N}} \mathcal{V}^n$ is open, too. We shall show that the closure of $\mathcal{H}$ in $\mathcal{G}$ equals $\mathcal{H}$; thus $\mathcal{H}$, being open and closed, equals $\mathcal{G}$.

Let L be an element of the closure of $\mathcal{H}$ in $\mathcal{G}$. Since $L\mathcal{V}^{-1}$ is a neighbourhood of L, there is an $F \in \mathcal{H}$ such that $F \in L\mathcal{V}^{-1}$ which implies $L \in F\mathcal{V}$; since $F\mathcal{V} \subset \mathcal{H}\mathcal{V} = \mathcal{H}$, the proof is complete.

4. The Lie algebra of a Lie group

4.1. Recall that if $\mathcal{G}$ is a Lie group in $\mathcal{G}a(\mathrm{V})$ then $\mathbf{La}(\mathcal{G})$, the tangent space of $\mathcal{G}$ at $I = \mathrm{id}_{\mathrm{V}}$ is a linear subspace of $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$. If $A \in \mathrm{Aff}(\mathrm{V}, \mathbf{V})$ then $\boldsymbol{A}$ denotes the underlying linear map $\mathbf{V} \to \mathbf{V}$.

Proposition. Let $\mathcal{G}$ be a Lie group. If $A, B \in \mathbf{La}(\mathcal{G})$ then

$$\boldsymbol{A}B - \boldsymbol{B}A \in \mathbf{La}(\mathcal{G}).$$

Proof. Take a neighbourhood $\mathcal{N}$ of I in $\mathcal{G}a(\mathrm{V})$ and a smooth map S defined on $\mathcal{N}$ such that $\overset{-1}{S}(\{0\}) = \mathcal{G} \cap \mathcal{N}$ and $\mathbf{La}(\mathcal{G}) = \mathrm{Ker}\, \mathrm{D}S(I)$ (see the proof of 3.3). Then

$$t \mapsto S\left(e^{tA}e^{tB}\right) = 0 \qquad \text{and} \qquad t \mapsto S\left(e^{tB}e^{tA}\right) = 0$$

for t in a neighbourhood of $0 \in \mathbb{R}$. Differentiating the first function with respect to t we get

$$t \mapsto \mathrm{D}S\left(e^{tA}e^{tB}\right) \cdot \left(e^{t\boldsymbol{A}}Ae^{t\boldsymbol{B}} + e^{t\boldsymbol{A}}e^{t\boldsymbol{B}}B\right) = 0.$$

Again differentiating and then taking $t = 0$ we deduce

$$\mathrm{D}^2S(I)\,(A + B, A + B) + \mathrm{D}S(I) \cdot (\boldsymbol{A}A + 2\boldsymbol{A}B + \boldsymbol{B}B) = 0.$$

Similarly we derive from the second function that

$$\mathrm{D}^2S(I)\,(B + A, B + A) + \mathrm{D}S(I) \cdot (\boldsymbol{B}B + 2\boldsymbol{B}A + \boldsymbol{A}A) = 0.$$

Let us subtract the equalities from each other to have

$$\mathrm{D}S(I) \cdot (\boldsymbol{A}B - \boldsymbol{B}A) = 0$$

which ends the proof.

4.2. According to the previous proposition we are given the *commutator mapping*

$$\mathbf{La}(\mathcal{G}) \times \mathbf{La}(\mathcal{G}) \to \mathbf{La}(\mathcal{G}), \qquad (A, B) \mapsto \boldsymbol{A}B - \boldsymbol{B}A =: [A, B].$$

Proposition. The commutator mapping
(*i*) is bilinear,
(*ii*) is antisymmetric,
(*iii*) satisfies the *Jacobian identity:*

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \qquad (A, B, C \in \mathbf{La}(\mathcal{G})).$$

Definition. $\mathbf{La}(\mathcal{G})$ endowed with the commutator mapping is called the *Lie algebra of* $\mathcal{G}$. ■

We deduce without difficulty that for $A, B \in \mathbf{La}(\mathcal{G})$

$$[A, B] = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(e^{tA}e^{tB} - e^{tB}e^{tA}\right)\right)_{t=0}.$$

4.3. The Lie algebra of $\mathcal{G}a(\mathrm{V})$ is $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$. We have seen that if a linear subspace $\mathbf{L}$ of $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$ is the tangent space at I of a Lie group then the commutator of elements from $\mathbf{L}$ belongs to $\mathbf{L}$, too; in other words, $\mathbf{L}$ is a Lie subalgebra of $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$.

Conversely, if $\mathbf{L}$ is a Lie subalgebra of $\mathrm{Aff}(\mathrm{V}, \mathbf{V})$ then there is a Lie group $\mathcal{G}$ such that $\mathbf{La}(\mathcal{G}) = \mathbf{L}$: the subgroup generated by $\{e^A \mid A \in \mathbf{L}\}$. It is not so easy to verify that this subgroup is a submanifold.

4.4. Definition. Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groups. A mapping $\Phi : \mathcal{G} \rightarrowtail \mathcal{H}$ is called a local Lie group homomorphism if

(i) $\mathrm{Dom}\,\Phi$ is a neighbourhood of the identity of $\mathcal{G}$,

(ii) Φ is smooth,

(iii) $\Phi(FG) = \Phi(F)\Phi(G)$ whenever $F, G, FG \in Dom\Phi$.

If Φ is injective and Φ^{-1} is smooth as well, then Φ is a *local Lie group isomorphism*. ■

4.5. For a local Lie group homomorphism $\Phi : \mathcal{G} \rightarrowtail \mathcal{H}$ we put

$$\mathbf{\Phi} := \mathrm{D}\Phi(I) \in \mathrm{Lin}\left(\mathbf{La}(\mathcal{G}), \mathbf{La}(\mathcal{H})\right).$$

If $A \in \mathbf{La}(\mathcal{G})$, then $t \mapsto \Phi(e^{tA})$ is a local one-parameter subgroup in $\mathcal{H}$ and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tA})\right)_{t=0} = \mathbf{\Phi}(A),$$

which implies

$$\Phi(e^{tA}) = e^{t\mathbf{\Phi}(A)}$$

for t in a neighbourhood of $0 \in \mathbb{R}$.

Proposition. $\mathbf{\Phi} : \mathbf{La}(\mathcal{G}) \to \mathbf{La}(\mathcal{H})$ is a Lie algebra homomorphism, i.e. it is linear and

$$[\mathbf{\Phi}(A), \mathbf{\Phi}(B)] = \mathbf{\Phi}([A, B]) \qquad (A, B \in \mathbf{La}(\mathcal{G})).$$

Proof. Start with

$$\begin{aligned}[\mathbf{\Phi}(A),\mathbf{\Phi}(B)] =&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(e^{t\mathbf{\Phi}(A)}e^{t\mathbf{\Phi}(B)}-e^{t\mathbf{\Phi}(B)}e^{t\mathbf{\Phi}(A)}\right)\right)_{t=0} = \\ =&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\Phi(e^{tA}e^{tB})-\Phi(e^{tB}e^{tA})\right)\right)_{t=0}\end{aligned}$$

and then apply the formulae in the proof of 4.1 putting Φ in place of S.

4.6. The previous proposition involves that locally isomorphic Lie groups have isomorphic Lie algebras. One can prove the converse, too, a fundamental theorem of the theory of Lie groups: if the Lie algebras of two Lie groups are isomorphic then the Lie groups are locally isomorphic.

5. Pseudo-orthogonal groups

Let $(\mathbf{V},\mathbf{B},\mathbf{h})$ be a pseudo-Euclidean vector space. Recall the notations (see V.2.7)

$$\begin{aligned}\mathcal{O}(\mathbf{h}) :=&\{\boldsymbol{L}\in\mathcal{G}\ell(\mathbf{V})|\ \boldsymbol{L}^*\cdot\boldsymbol{L}=\boldsymbol{I}\},\\ \mathrm{A}(\mathbf{h}) :=&\{\boldsymbol{A}\in\mathrm{Lin}(\mathbf{V})|\ \boldsymbol{A}^*=-\boldsymbol{A}\}.\end{aligned}$$

Proposition. If $\dim\mathbf{V}=N$ then $\mathcal{O}(\mathbf{h})$ is an $\frac{N(N-1)}{2}$-dimensional Lie group having $\mathrm{A}(\mathbf{h})$ as its Lie algebra.

Proof. It is evident that $\mathcal{O}(\mathbf{h})$ is a subgroup of $\mathcal{G}\ell(\mathbf{V})$.

We know that $\mathrm{A}(\mathbf{h})$ and $\mathrm{S}(\mathbf{h}) := \{\boldsymbol{S}\in\mathrm{Lin}(\mathbf{V})|\ \boldsymbol{S}^*=\boldsymbol{S}\}$ are complementary subspaces, $\dim\mathrm{S}(\mathbf{h})=\frac{N(N+1)}{2}$, $\dim\mathrm{A}(\mathbf{h})=\frac{N(N-1)}{2}$ (see V.2.9).

Let us consider the mapping

$$\varphi:\mathcal{G}\ell(\mathbf{V})\to\mathrm{S}(\mathbf{h}),\qquad \boldsymbol{L}\mapsto\boldsymbol{L}^*\cdot\boldsymbol{L}.$$

Since the $\mathbf{h}$-adjunction $\boldsymbol{L}\mapsto\boldsymbol{L}^*$ is linear and the multiplication in $\mathrm{Lin}(\mathbf{V})$ is bilinear, φ is smooth. Moreover, the equality

$$(\boldsymbol{L}+\boldsymbol{H})^*\cdot(\boldsymbol{L}+\boldsymbol{H})-\boldsymbol{L}^*\cdot\boldsymbol{L}=\boldsymbol{L}^*\cdot\boldsymbol{H}+\boldsymbol{H}^*\cdot\boldsymbol{L}+\boldsymbol{H}^*\cdot\boldsymbol{H}$$

shows that

$$\mathrm{D}\varphi(\boldsymbol{L})\cdot\boldsymbol{H}=\boldsymbol{L}^*\cdot\boldsymbol{H}+\boldsymbol{H}^*\cdot\boldsymbol{L}\qquad\qquad(\boldsymbol{L}\in\mathcal{G}\ell(\mathbf{V}),\ \boldsymbol{H}\in\mathrm{Lin}(\mathbf{V}).$$

We have $\mathcal{O}(\mathbf{h})=\{\boldsymbol{L}\in\mathcal{G}\ell(\mathbf{V})|\ \varphi(\boldsymbol{L})=\boldsymbol{I}\}$ and $\mathrm{D}\varphi(\boldsymbol{L})$ is surjective if $\boldsymbol{L}$ is in $\mathcal{O}(\mathbf{h})$: if $\boldsymbol{S}\in\mathrm{S}(\mathbf{h})$ then $\mathrm{D}\varphi(\boldsymbol{L})\cdot\frac{\boldsymbol{L}\cdot\boldsymbol{S}}{2}=\boldsymbol{S}$. Consequently, $\mathcal{O}(\mathbf{h})$ is a smooth submanifold of $\mathcal{G}\ell(\mathbf{V})$ (see VI.4.7).

Finally, $\mathrm{D}\varphi(\boldsymbol{I}) \cdot \boldsymbol{H} = \mathbf{0}$ if and only if $\boldsymbol{H} \in \mathrm{A}(\mathbf{h})$, hence

$$\mathbf{La}(\mathcal{O}(\mathbf{h})) = \mathrm{Ker}\ \mathrm{D}\varphi(\boldsymbol{I}) = \mathrm{A}(\mathbf{h}).$$

6. Exercises

1. Let $\mathcal{G}$ be a Lie group, $A, B \in \mathbf{La}(\mathcal{G})$. Prove that $[A, B] = 0$ if and only if $e^{tA}e^{tB} = e^{tB}e^{tA}$ for all t in an interval around $0 \in \mathbb{R}$.

Consequently, $\mathcal{G}$ is commutative (Abelian) if and only if $\mathbf{La}(\mathcal{G})$ is commutative (the commutator mapping on $\mathbf{La}(\mathcal{G})$ is zero).

2. Using the definition of exponentials (see 2.6) demonstrate that

$$[A, B] = \lim_{t \to 0} \frac{1}{t^2}\left(e^{tA}e^{tB} - e^{tB}e^{tA}\right) = \lim_{t \to 0} \frac{1}{t^2}\left(e^{tA}e^{tB}e^{-tA}e^{-tB} - I\right).$$

3. Let $\mathbf{V}$ be a finite dimensional real vector space and make the identification

$$\mathrm{Aff}(\mathbf{V}) \equiv \mathbf{V} \times \mathrm{Lin}(\mathbf{V}), \qquad A \equiv (A(0), \boldsymbol{A}),$$

i.e. $(\boldsymbol{a}, \boldsymbol{A}) \in \mathbf{V} \times \mathrm{Lin}(\mathbf{V})$ is considered to be the affine map

$$\mathbf{V} \to \mathbf{V}, \qquad \boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} + \boldsymbol{a}.$$

Then the composition of such affine maps is

$$(\boldsymbol{a}, \boldsymbol{A})(\boldsymbol{b}, \boldsymbol{B}) = (\boldsymbol{a} + \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}\boldsymbol{B}).$$

In this way we have $\mathcal{G}a(\mathbf{V}) \equiv \mathbf{V} \times \mathrm{GL}(\mathbf{V})$.

Prove that

$$e^{(\boldsymbol{a}, \mathbf{0})} = (\boldsymbol{a}, \boldsymbol{I}), \qquad e^{(\mathbf{0}, \boldsymbol{A})} = (\mathbf{0}, e^{\boldsymbol{A}}).$$

4. Let n be a positive integer. Prove that

$$\mathcal{O}(n) := \{\boldsymbol{L} \in \mathrm{Lin}(\mathbb{R}^n) |\ \boldsymbol{L}^*\boldsymbol{L} = \boldsymbol{I}\}$$
$$\mathcal{SO}(n) := \{\boldsymbol{L} \in \mathcal{O}(n) |\ \det\boldsymbol{L} = 1\}$$

are $\frac{n(n-1)}{2}$-dimensional Lie groups having the same Lie algebra:

$$\{\boldsymbol{A} \in \mathrm{Lin}(\mathbb{R}^n) |\ \boldsymbol{A}^* = -\boldsymbol{A}\}$$

(*cf.* Proposition in Section 5).

Give a local Lie group isomorphism between $\mathcal{O}(n)$ and $\mathcal{SO}(n)$.

5. A complex vector space and its complex linear maps can be considered to be a real vector space and real linear maps.

Demonstrate that

$$\mathcal{S}\ell(2,\mathbb{C}) := \{\boldsymbol{L} \in \mathrm{Lin}(\mathbb{C}^2) |\ \det\boldsymbol{L} = 1\}$$

is a six-dimensional Lie group having

$$\{\boldsymbol{A} \in \mathrm{Lin}(\mathbb{C}^2) |\ \mathrm{Tr}\boldsymbol{A} = \boldsymbol{0}\}$$

as its Lie algebra.

6. Let n be a positive integer. Prove that

$$\begin{aligned}\mathcal{U}(n) :=& \{\boldsymbol{L} \in \mathrm{Lin}(\mathbb{C}^n) |\ \boldsymbol{L}^*\boldsymbol{L} = \boldsymbol{I}\},\\ \mathcal{SU}(n) :=& \{\boldsymbol{L} \in \mathcal{U}(n) |\ \det\boldsymbol{L} = 1\}\end{aligned}$$

are an n^2-dimensional and an (n^2-1)-dimensional Lie group, respectively. (The star denotes adjoint with respect to the usual complex inner product; in other words, if $\boldsymbol{L}$ is regarded as a matrix then $\mathbf{L}^*$ is the conjugate of the transpose of $\boldsymbol{L}$.) Verify that they have the Lie algebras

$$\begin{gathered}\{\boldsymbol{A} \in \mathrm{Lin}(\mathbb{C}^n) |\ \boldsymbol{A}^* = -\boldsymbol{A}\},\\ \{\boldsymbol{A} \in \mathrm{Lin}(\mathbb{C}^n) |\ \boldsymbol{A}^* = -\boldsymbol{A},\ \mathrm{Tr}\boldsymbol{A} = \boldsymbol{0}\},\end{gathered}$$

respectively.

7. Prove that

$$\mathcal{U}(1) := \{\boldsymbol{L} \in \mathrm{Lin}(\mathbb{C}) |\ \boldsymbol{L}^*\boldsymbol{L} = \boldsymbol{I}\} \equiv \{\alpha \in \mathbb{C} |\ \ |\alpha| = 1\}$$

is a one-dimensional Lie group, locally isomorphic but not isomorphic to $\mathcal{T}n(\mathbb{R})$.

8. Let $\mathcal{G} \subset \mathcal{G}a(\mathrm{V})$ be a Lie group. An *orbit* of $\mathcal{G}$ is a non-void subset P of V such that $\{L(x) |\ L \in \mathcal{G}\} = \mathrm{P}$ for some — hence for all — $x \in \mathrm{P}$.

Prove that distinct orbits are disjoint. V is the union of orbits of $\mathcal{G}$. In other words, the $\sim$ relation on V defined by $x \sim y$ if x and y are in the same orbit of $\mathcal{G}$ is an equivalence relation.

9. Find the orbits of $\mathcal{G}a(\mathrm{V})$, $\mathcal{G}\ell(\mathbf{V})$, $\mathcal{T}n(\mathbf{V})$, $\mathcal{O}(n)$, $\mathcal{SO}(n)$, $\mathcal{U}(n)$, $\mathcal{SU}(n)$.

SUBJECT INDEX

LIST OF SYMBOLS

1. Basic notation

$\blacksquare$	marks the end of a proposition, a proof or a definition, if necessary
$:=\ =:$	defining equalities; the symbol on the side of the colon is defined to equal the other one
$\emptyset$	the void set
$\mathbb{N}$	the set of non-negative integers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}_0^+$	the set of non-negative real numbers
X^n	the n-fold Cartesian product of the set X with itself ($n \in \mathbb{N}$)
$\mathrm{Dom}\ f$	the *domain* of the map f
$\mathrm{Ran}\ f$	the *range* of the map f
$f : \mathrm{X} \to \mathrm{Y}$	f is a map with $\mathrm{Dom}\ f = \mathrm{X}$, $\mathrm{Ran}\ f \subset Y$
$f : \mathrm{X} \rightarrowtail \mathrm{Y}$	f is a map with $\mathrm{Dom}\ f \subset \mathrm{X}$, $\mathrm{Ran}\ f \subset \mathrm{Y}$
$\mapsto$	the symbol showing a mapping rule
$f\vert_{\mathrm{A}}$	the *restriction* of the map f onto $\mathrm{A} \cap \mathrm{Dom}\ f$
$f \subset g$	the map g is an *extension* of f, i.e. $\mathrm{Dom}\ f \subset \mathrm{Dom}\ g$, $g\vert_{\mathrm{Dom}\ f} = f$
$\overset{-1}{f}$	the *total inverse* of the map $f : \mathrm{X} \rightarrowtail \mathrm{Y}$, i.e. if $\mathrm{H} \subset \mathrm{Y}$ then $\overset{-1}{f}(\mathrm{H}) = \{x \in \mathrm{Dom}\ f \vert\, f(x) \in \mathrm{H}\}$
$g \circ f$	the *composition* of the maps $g : \mathrm{Y} \rightarrowtail \mathrm{Z}$ and $f : \mathrm{X} \rightarrowtail \mathrm{Y}$, $\mathrm{Dom}\ (g \circ f) := \overset{-1}{f}(\mathrm{Dom}\ g) \cap \mathrm{Dom}\ f$, $x \mapsto g(f(x))$
$\underset{i\in\mathrm{I}}{\times} f_i$	the *Cartesian product* of the maps $f_i : \mathrm{X}_i \mapsto \mathrm{Y}_i$ $(i \in \mathrm{I})$: $\left(\underset{i\in\mathrm{I}}{\times} \mathrm{X}_i\right) \to \left(\underset{i\in\mathrm{I}}{\times} \mathrm{Y}_i\right)$, $(x_i)_{i\in\mathrm{I}} \mapsto \big(f_i(x_i)\big)_{i\in\mathrm{I}}$
$\overset{n}{\times} f$	the n-fold Cartesian product of f with itself ($n \in \mathbb{N}$)

$(f_i)_{i\in\mathrm{I}}$	the *joint* of the maps $f_i : \mathrm{X} \to \mathrm{Y}_i$, $X \to \underset{i\in\mathrm{I}}{\times} \mathrm{Y}_i$, $x \mapsto \big(f_i(x)\big)_{i\in\mathrm{I}}$
Ker L	$:= \overset{-1}{\boldsymbol{L}}\{\mathbf{0}\}$, the *kernel* of the linear map $\boldsymbol{L}$
pr^k	$\mathbb{R}^N \to \mathbb{R}$, the k-th coordinate projection
id_X	the identity map $\mathrm{X} \to \mathrm{X}$, $x \mapsto x$

2. Other notations

*	marks the dual of vector spaces and the transpose of linear maps, IV.1.1, IV.1.4
$^{\circledast}$	marks adjoints of linear maps, V.1.5
$\otimes$	tensor product, IV.3.2
$\wedge$	antisymmetric tensor product, IV.3.14
$\vee$	symmetric tensor product, IV.3.14
$\star$	$q \star t$ is the unique element in the intersection of q and t, I.2.2, II.3.4.2
ar	the arrow of spacetime transformations, I.11.3.1, II.10.1.1
$\mathrm{A}(\mathbf{h})$	the set of $\mathbf{h}$-antisymmetric maps, V.2.7 in particular, $\mathrm{A}(\mathbf{b})$: V.3.8 $\mathrm{A}(\mathbf{g})$: V.4.15
$\mathbf{b}$	Euclidean form, V.3.1, I.1.1.2
$\mathbf{b}_{\boldsymbol{u}}$	Euclidean form on $\mathbf{E}_{\boldsymbol{u}}$, II.1.3.3
$\mathrm{B}_{\boldsymbol{u}}$	the set of relative velocities with respect to $\boldsymbol{u}$, II.4.2.5
$C_{\boldsymbol{U}}$	$C_{\boldsymbol{U}}(x)$ is the $\boldsymbol{U}$-line passing through x ($\boldsymbol{U}$-space point that x is incident with), I.3.2.2, II.3.1.3
$\mathbf{E}$	the set of spacelike vectors, I.1.2.2
$\mathbf{E}_{\boldsymbol{u}}$	the set of vectors $\mathbf{g}$-orthogonal to $\boldsymbol{u}$, II.1.3.2
$\mathrm{E}_{\boldsymbol{U}}$	$\boldsymbol{U}$-space, I.3.2.1, II.3.1.3
$\mathbf{E}_{\boldsymbol{U}}$	the set of space vectors of a rigid observer $\boldsymbol{U}$, I.4.3.4
det	determinant, IV.3.18
$\mathrm{D}F$	derivative of F, VI.3.3
$\mathbf{g}$	Lorentz form, V.4.1, II.1.2.1 $:= \mathrm{id}_\mathbf{M}$, II.1.3.6
$\mathcal{G}$	Galilean group, I.11.3.1
$H_{\boldsymbol{U}}$	splitting according to $\boldsymbol{U}$, I.3.2.2, II.3.4.2
$H_{\boldsymbol{U},o}$	splitting according to $(\boldsymbol{U},o)$, I.4.1.4, I.4.4.2, II.3.4.3
$\boldsymbol{h}_{\boldsymbol{u}}$	vector splitting, I.8.2.1, II.7.1.1

$h_{\boldsymbol{u},o}$	$= H_{\boldsymbol{U},o}$ for inertial $\boldsymbol{U}$ with value $\boldsymbol{u}$, I.11.8.1, II.10.5.1
$\boldsymbol{H}_{\boldsymbol{u}'\boldsymbol{u}}$	the vector transformation rule, II.8.2.4, II.7.1.4
$\mathbf{i}$	embedding of $\mathbf{E}$ into $\mathbf{M}$, I.1.2.1
$\mathbf{i}_{\boldsymbol{u}}$	embedding of $\mathbf{E}_{\boldsymbol{u}}$ into $\mathbf{M}$, II.1.3.2
$\mathrm{I}_{\mathcal{S}}$	time according to the simultaneity $\mathcal{S}$, II.6.2.3
$\mathrm{I}_{\boldsymbol{U}}$	$\boldsymbol{U}$-time, II.3.2.2, II.6.2.4
$\mathcal{L}$	Lorentz group, II.10.1.1
$\mathbf{La}(\ )$	Lie algebra of a Lie group, VII.3.3, VII.4.2
$\boldsymbol{L}(\boldsymbol{u}', \boldsymbol{u})$	Lorentz boost, II.1.3.8
	special Galilean transformation, I.11.3.7
$\mathcal{N}$	Noether group, I.11.6.1
$\mathbf{N}$	$:= \frac{\mathbf{E}}{\mathbf{D}}$, I.1.2.5
O_o	vectorization with origin o, VI.1.1
$\mathcal{O}(\mathbf{b})$	group of orthogonal transformations, I.11.1.1
$\mathcal{O}(\mathbf{b}_{\boldsymbol{u}})$	group of orthogonal transformations, II.10.1.4
$\pi_{\boldsymbol{u}}$	projection along $\boldsymbol{u}$, I.1.2.8, II.1.3.2
$\mathcal{P}$	Poincaré group, II.10.3.1
$\boldsymbol{P}_{\boldsymbol{u}}$	$\boldsymbol{u}$-spacelike inversion I.11.3.4, II.10.1.3
$R_{\boldsymbol{U}}(t, t_o)$	rotation of a rigid observer, I.4.2.2
$\boldsymbol{r}_{\boldsymbol{u}}$	covector splitting, I.8.3.1, II.7.2.1
$\boldsymbol{R}_{\boldsymbol{u}'\boldsymbol{u}}$	the covector transformation rule, I.8.3.2, II.7.2.2
sign	sign of permutations, IV.3.14
	sign of spacetime transformations, I.11.3.1,II.10.1.1
$\mathcal{SO}(\mathbf{b})$	group of rotations, I.11.1.2
$\boldsymbol{\tau}$	time evaluation, I.1.2.2
$\boldsymbol{\tau}_{\mathcal{S}}$	time evaluation of a simultaneity, II.6.2.3
$\boldsymbol{\tau}_{\boldsymbol{U}}$	$\boldsymbol{U}$-time evaluation, II.3.4.1, II.6.2.4
$\mathrm{T}_x(\ )$	tangent space at x, VI.4.6
$\mathcal{T}n(\ )$	translation group, VII.3.1
$T_{\boldsymbol{x}}$	translation by $\boldsymbol{x}$, VII.3.1
$\boldsymbol{T}_{\boldsymbol{u}}$	$\boldsymbol{u}$-timelike inversion, I.11.3.4, II.10.1.3
$\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$	relative velocity, I.6.2.2, II.4.2.2
$\mathrm{V}(1)$	the set of velocities, I.1.2.7, II.1.2.1
$\mathrm{V}(0)$	the set of lightlike velocities, II.4.7.1

COMMENTS AND BIBLIOGRAPHY

The fundamental notions of space and time appear in all branches of physics, giving a general background of phenomena. Nowadays the mathematical way of thinking and speaking becomes general in physics; that is why it is indispensable to construct mathematically exact models of spacetime.

Since 17 years an educational and research programme has been in progress at the Department of Applied Analysis, Eötvös Loránd University, Budapest, to build up a mathematical theory of physics in which only mathematically defined notions appear. In this way we can rule out tacit assumptions and the danger of confusions, and physics can be put on a firm basis.

The first results of this work were published in two books:

[1] Matolcsi, T.: *A Concept of Mathematical Physics, Models for Spacetime,* Akadémiai Kiadó, 1984;

[2] Matolcsi, T.: *A Concept of Mathematical Physics, Models in Mechanics,* Akadémiai Kiadó, 1986.

Since that time our teaching experience revealed that a mathematical treatment of spacetime could claim more interest than we had thought it earlier. The notions of the spacetime models throw new light on the whole physics, a number of relations become clearer, simpler and more understandable; e.g. the old problem of material objectivity in continuum physics has been completely solved, as discussed in:

[3] Matolcsi, T.: *On Material Frame-Indifference,* Archive for Rational Mechanics and Analysis, **91** (1986), 99–118.

That is why it seems necessary that spacetime models be formulated in a way more familiar to physicists; so they can acquire and apply the notions and results more easily. The present work is an enlarged and more detailed version of [1]. The notations (due to the dot product) became simpler. The amount of applied mathematical tools decreased (by omitting some marginal facts, the theory of smooth manifolds could be eliminated), the material, the explanations and the number of the illustrative examples increased.

There is only one point where the new version contradicts the former one because of the following reason. In the literature one usually distingushes between the Lorentz group (a group of linear transformations of $\mathbb{R}^4$) and the Poincaré group, called also the "inhomogeneous Lorentz group" (the Lorentz group together with the translations of $\mathbb{R}^4$). In our terminology, one considers the arithmetic Lorentz group which is a subgroup of the arithmetic Poincaré group. However, we know that in the absolute treatment the Poincaré group consists of transformations of the affine space M, whereas the Lorentz group consists of transformations of the vector space **M**; the Lorentz group is not a subgroup of the Poincaré group. Special Lorentz transformations play a fundamental role in usual treatments in connection with transformation rules.

The counterpart of the Poincaré group in the non-relativistic case is usually called the Galilean group and one does not determine its vectorial subgroup that corresponds to the Lorentz group. The special Galilean transformations play a fundamental role in connection with transformation rules. In the absolute treatment we must distinguish between the transformation group of the affine space M and the transformation group of the vector space **M** which is not a subgroup of the former group. The special Galilean transformations turn to be transformations of **M**; that is why I found it convenient to call the corresponding linear transformation group the *Galilean group* and to introduce the name *Noether group* for the group of affine transformations.

In the former version I used these names interchanged because then group representations (applied in mechanical models) were in my mind and it escaped my attention that from the point of view of transformation rules — which have a fundamental importance — the present names are more natural.

The present treatment of spacetime is somewhat different from the usual ones; of course, there are works in which elements of the present models appear. First of all, in

[4] Weyl, H.: *Space–Time–Matter,* Dover publ. 1922

spacetime is stated to be a four-dimensional affine space, the bundle structure of non-relativistic spacetime (i.e. spacetime, time and time evaluation) and the Euclidean structure on a hyperplane of simultaneous world points appear as well. However, all these are not collected to form a clear mathematical structure; moreover, the advantages of affine spaces are not used, immediately coordinates and indices are taken; thus the possibility of an absolute description is not utilized.

A similar structure ("neoclassical spacetime": spacetime and time elapse) is expounded in

[5] Noll, W.: *Lectures on the foundation of continuum mechanics and thermodynamics,* Arch.Rat.Mech. **52** (1973) 62–92.

In these works time periods and distances are considered to be real numbers. The notion of observer remains undefined; even this undefined notion is used to introduce e.g. differentiability in the "neoclassical spacetime".

When comparing our notions, results and formulae with those of other treatments, using the phrases "in most of the textbooks", "in conventional treatments" we refer e.g. to the following books:

[6] French, A.P. *Special Relativity,* Norton, New York, 1968

[7] Essen, L.: *The Special Theory of Relativity,* Clarendon, Oxford, 1971

[8] Møller, C.: *The Theory of Relativity,* Oxford University Press, 1972

[9] Taylor, J.G.: *Special Relativity,* Clarendon, Oxford, 1975

[10] Bergmann, P.G.: *Introduction to the Theory of Relativity,* Dover publ., New York, 1976

General relativity, i.e. the theory of gravitation is one of the most beautiful and mathematically well elaborated area of physics which is treated in a number of excellent books, e.g.

[11] Misner, C.W.–Thorne, K.S.–Wheeler, J.A.: *Gravitation,* W.H.Freeman & Co., 1973

[12] Adler, R.–Bazin, M.–Schiffer, M.: *Introduction to General Relativity,* McGraw-Hill, 1975

[13] Ohanian, H.C.: *Gravitation and Spacetime,* W.W.Norton & Co., 1976

[14] Rindler, W.: *Essential Relativity. Special, General and Cosmological,* McGraw-Hill, 1977

[15] Wald, R.: *Space, Time and Gravity,* Chicago Press, 1977

To understand the non-relativistic and special relativistic spacetime models, it is sufficient to have some elementary knowledge in linear algebra and analysis. Tensors and tensorial operations are the main mathematical tools used throughout the present book. Those familiar with tensors will have no difficulty in reading the book. The necessary mathematical tools are summarized in its second part where the reader can find a long and detailed chapter on tensors.

The book uses the basic notions and theorems of linear algebra (linear combination, linear independence, linear subspace etc.) without explanation. There are many excellent books on linear algebra from which the reader can acquire the necessary knowledge, e.g.

[16] Halmos, P.R.: *Finite Dimensional Vector Spaces,* Springer, 1974

[17] Smith, L.: *Linear Algebra, Springer,* 1978

[18] Grittel, D.H.: *Linear Algebra and its Applications,* Harwood, 1989

[19] Fraleigh, J.B.–Beauregard, R.A.: *Linear Algebra,* Addison–Wesley, 1990

Some notions and theorems of elementary analysis (limit of functions, continuity, Lagrange's mean value theorem, implicit mapping theorem, etc.) are used without any reference; the following books are recommended to be consulted

[20] Zamansky, M.: *Linear Algebra and Analysis,* Van Nostrand, 1969

[21] Rudin, W.: *Principles of Mathematical Analysis,* McGraw Hill, 1976

[22] Aliprantis, C.D.–Burkinshow, O.: *Principles of Real Analysis,* Arnold, 1981

[23] Haggarty, R.: *Fundamentals of Mathematical Analysis,* Addison-Wesley, 1989

[24] Adams, R.A.: *Calculus: a Complete Course,* Addison–Wesley, 1991

From the theory of differential equations only the well-known existence and uniqueness theorem is used which can be found e.g. in

[25] Hyint-U Tyn: *Ordinary Differential Equations,* North-Holland, 1978

[26] Birkhoff, G.–Rota, G.C.: *Ordinary Differential Equations,* Wiley, 1989

The present book avoids the theory of smooth manifolds though it would be useful for the investigation of the space of general observers and necessary for the treatment of general relativistic spacetime models. The following books are recommended to the reader interested in this area:

[27] Boothby, W.M.: *An Introduction to Differentiable Manifolds and Riemannian Geometry,* Academic Press, 1975

[28] Choquet-Bruhat, Y.–Dewitt-Morette, C.: *Analysis, Manifolds and Physics,* North-Holland, 1982

[29] Abraham, R–Marsden, J.E.–Ratiu, T.: *Manifolds, Tensor Analysis, and Applications,* Springer, 1988

Non-relativistic and special relativistic spacetime models involve some elementary facts about certain Lie groups. Those who want to get more knowledge on Lie groups can study, e.g. the following books:

[30] Warner, F.W.: *Foundations of Differentiable Manifolds and Lie Groups,* Springer, 1983

[31] Sattinger, R.H.–Weawer, O.L.: *Lie groups and Lie algebras with Applications to Physics, Geometry and Mechanics,* Springer, 1986