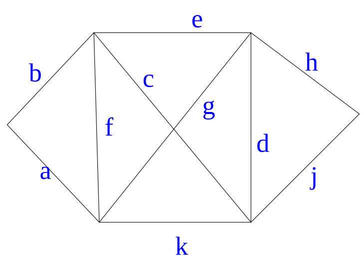
Eulerian Graphs

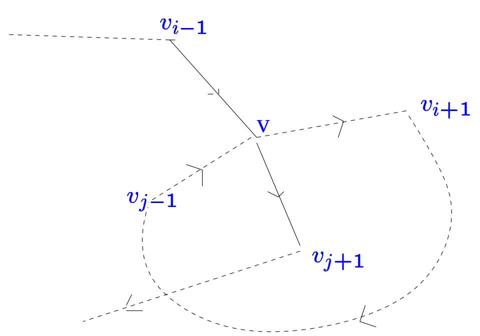
An *Eulerian cycle* of a graph $G = (V, E)$ is a closed walk which uses each edge $e \in E$ exactly once.



The walk using edges a,b,c,d,e,f,g,h,j,k in this order is an Eulerian cycle.

Theorem 1 *A connected graph is Eulerian i.e. has an Eulerian cycle, iff it has no vertex of odd degree.*

Proof Suppose $W = (v_1, v_2, \dots, v_m, v_1)$ ($m = |E|$) is an Eulerian cycle. Fix $v \in V$. Whenever W visits v it enters through a new edge and leaves through a new edge. Thus each visit requires 2 new edges. Thus the degree of v is even.



The converse is proved by induction on $|E|$. The result is true for $|E| = 3$. The only possible graph is a triangle.

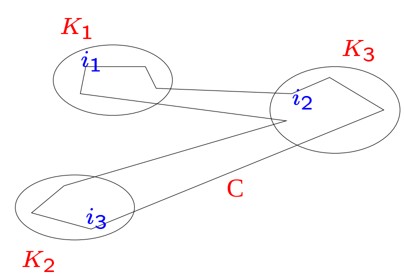
Assume $|E| \geq 4$. G is not a tree, since it has no vertex of degree 1. Therefore it contains a cycle C. Delete the edges of C. The remaining graph has components $K_1, K_2, \dots, K_r$.

Each K_i is connected and is of even degree – deleting C removes 0 or 2 edges incident with a given $v \in V$. Also, each K_i has strictly less than $|E|$ edges. So, by induction, each K_i has an Eulerian cycle, C_i say.

We create an Eulerian cycle of G as follows: let $C = (v_1, v_2, \dots, v_s, v_1)$. Let v_{i_t} be the first vertex of C which is in K_t. Assume w.l.o.g. that $i_1 < i_2 < \cdots < i_r$.

$$W = (v_1, v_2, \dots, v_{i_1}, C_1, , v_{i_1}, \dots, v_{i_2}, C_2, v_{i_2}, \dots, v_{i_r}, C_r, v_{i_r}, \dots, v_1)$$

is an Eulerian cycle of G. □



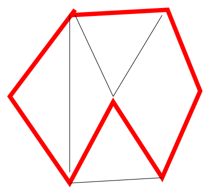
Corollary 1 *A connected graph has an Eulerian Walk i.e. a walk which uses each edge exactly once, iff it has exactly 2 vertices of odd degree.*

Proof If a walk exists then the endpoints have odd degree and the interior vertices have even degree.
Conversely, if there are two odd degree vertices x, y add an extra edge $e = xy$ to create a connected graph G' with only even vertices. This has an Eulerian cycle C. Delete e from C to create the required path. □

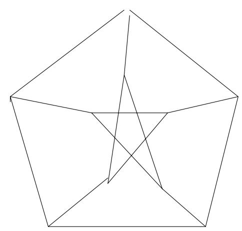
Hamilton Cycles

A *Hamilton Cycle* of a graph $G = (V, E)$ is a cycle which goes through each vertex (once).

A graph is called *Hamiltonian* if it contains a Hamilton cycle.



Hamiltonian Graph



Non-Hamiltonian Graph
Petersen Graph

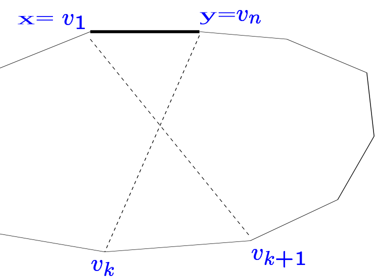
Lemma 1 *Let $G = (V, E)$ and $|V| = n$. Suppose $x, y \in V$, $e = (x, y) \notin E$ and $d(x) + d(y) \geq n$. Then*

$$G + e \text{ is Hamiltonian } \leftrightarrow G \text{ is Hamiltonian.}$$

Proof
$\leftarrow$ Trivial.

$\rightarrow$ Suppose $G + e$ has a Hamilton cycle H. If $e \notin H$ then $H \subseteq G$ and G is Hamiltonian.

Suppose $e \in H$. We show that we can find another Hamilton cycle in $G + e$ which does not use e.



$H = (x = v_1, v_2, \ldots, v_n = y, x)$.
$S = \{i : (x, v_{i+1}) \in E\}$ and
$T = \{i : (y, v_i) \in E\}$.

$S \subseteq \{1, 2, \ldots, n-2\}$, $T \subseteq \{2, 3, \ldots, n-1\}$.
$|S| + |T| \geq n$ and $|S \cup T| \leq n - 1$.

Thus

$$|S \cap T| = |S| + |T| - |S \cup T| \geq 1$$

and so $\exists 1 \neq k \in S \cap T$ and then

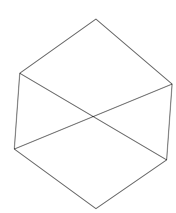
$$H' = (v_1, v_2, \ldots, v_k, v_n, v_{n-1}, \ldots, v_{k+1}, v_1)$$

is a Hamilton cycle of G.

Bondy-Chvatál Closure of a graph

begin
  $c(G) := G$
  while $\exists (x, y) \notin E$ with $d_{c(G)}(x) + d_{c(G)}(y) \geq n$ **do**
  begin
    $c(G) := c(G) + (x, y)$
  end
  Output $c(G)$
end

The graph $c(G)$ is called the closure of G.



Lemma 2 *$c(G)$ is independent of the order in which edges are added i.e. it depends only on G.*

Proof Suppose algorithm is run twice to obtain
$G_1 = G + e_1 + e_2 + \cdots + e_k$ and
$G_2 = G + f_1 + f_2 + \cdots + f_\ell$.
We show that $\{e_1, e_2, \ldots, e_k\} = \{f_1, f_2, \ldots, f_\ell\}$.

Suppose not. Let $t = \min\{i : e_i \notin G_2\}$, $e_t = (x, y)$ and $G' = G + e_1 + e_2 + \cdots + e_{t-1}$. Then

$$\begin{aligned} d_{G_2}(x) + d_{G_2}(y) &\geq d_{G'}(x) + d_{G'}(y) \\ &\geq n \end{aligned}$$

since e_t was added to G'.

But then e_t should have been added to G_2 – contradiction.

- $c(G)$ Hamiltonian $\Rightarrow$ G is Hamiltonian.
- $c(G)$ complete $\Rightarrow$ G is Hamiltonian.
- $\delta(G) \geq n/2$ $\Rightarrow$ G is Hamiltonian.

Second statement is due to Bondy and Murty.
Third statement is due to Dirac.

Theorem 2 *Let G be a simple graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_\nu$, $\nu \geq 3$. Suppose that there does* **not** *exist $m < \nu/2$ such that*

$$d_m \leq m \text{ and } d_{\nu-m} < \nu - m.$$

Then G is Hamiltonian.

Proof We prove that $c(G)$ is complete. Let d' denote degree in $c(G)$. Suppose $c(G)$ is not complete. Among all pairs of vertices u, v which are not adjacent in $c(G)$ choose a pair which maximise $d'(u) + d'(v)$ and assume $m = d'(u) \leq d'(v)$. Note that

$$d'(u) + d'(v) \leq \nu - 1.$$

$S = \{w \in V \setminus \{v\} : v, w$ not adjacent in $c(G)\}$.
$T = \{w \in V \setminus \{u\} : u, w$ not adjacent in $c(G)\}$.

$$\begin{aligned} |S| &= \nu - 1 - d'(v) \geq d'(u) = m \quad (1) \\ |T \cup \{u\}| &= \nu - m. \quad (2) \end{aligned}$$

The choice of u, v means that

$$\begin{aligned} d'(w) \leq d'(u) \quad &\text{for} \quad w \in S \quad (3) \\ d'(w) \leq d'(v) < \nu - m \quad &\text{for} \quad w \in T \quad (4) \end{aligned}$$

Now $d(w) \leq d'(w)$ for $w \in V$ and so

(1) and (3) imply that $d_m \leq m$.

(2) and (4) imply that $d_{\nu-m} < \nu - m$.

Contradiction. □