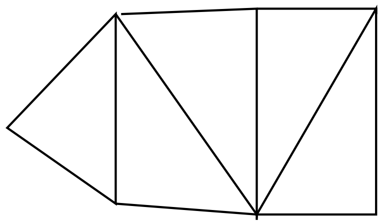
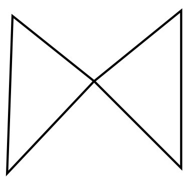
Connectivity

G is *k-connected* if $S \subseteq V, |S| < k$ implies $G - S$ is connected.

$$\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}.$$



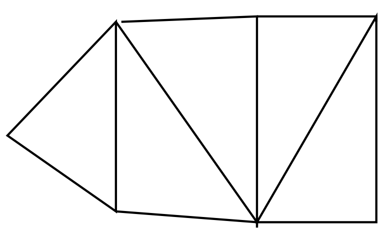
κ(G)=2



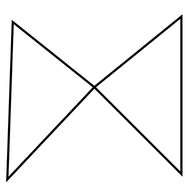
κ(G)=1

G is *k-edge connected* if $S \subseteq E, |S| < k$ implies $G - S$ is connected.

$$\kappa'(G) = \max\{k : G \text{ is } k\text{-edge connected}\}.$$



κ'(G)=2



κ'(G)=2

Assume G connected.

S is a k-vertex cutset if $S \subseteq V, |S| = k$ and $G - S$ is not connected.

A 1-vertex cutset is a *cutpoint*.

S is a k-edge cutset if $S \subseteq E, |S| = k$ and $G - E$ is not connected.

A 1-edge cutset is a *bridge* or cut-edge.

Lemma 1 *If G is connected and e is a bridge, then $H = G - e$ has exactly 2 components.*

Proof If H has components C_1, C_2, C_3 then $G = H + e$ has $\geq$ 2 components, since adding an edge decreases the number of components by at most 1. This contradicts the fact that G is connected. □

Complete Graphs

K_n has no vertex cutsets.

$\kappa(K_n) = n - 1$ by convention.

$\kappa'(K_n) = n - 1$.

So in general

$$\kappa(G) \leq \nu - 1.$$

G not complete. v, w not neighbours.
$V \setminus \{v, w\}$ is a $(\nu - 2)$-vertex cutset.

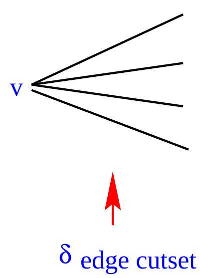
Theorem 1

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof If G has no edges then

$$\kappa' = 0 = \delta.$$

Otherwise the set of edges incident with a vertex v of minimum degree is a δ-edge cutset.



Therefore $\kappa' \leq \delta$.

We prove that $\kappa \leq \kappa'$ by induction on κ'.

True for $\kappa' = 0$.

Assume true for all graphs with $\kappa' < k$ and let G be a graph with $\kappa'(G) = k$.

Suppose $A \subseteq E$ is a k-edge cutset of G.

Let $e \in A$ and $H = G - e$. Then

$$H - (A \setminus e) = G - A \text{ is not connected}$$

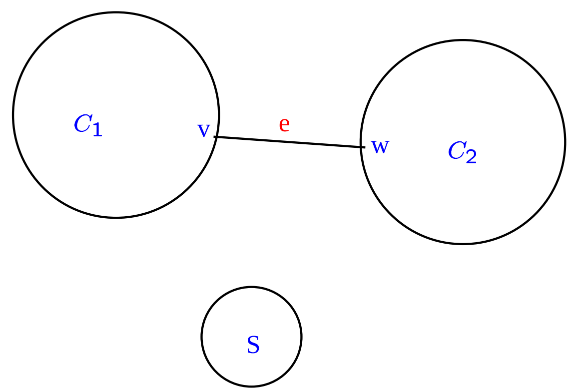
and so $\kappa'(H) < k$.

By the induction hypothesis $\kappa(H) \leq \kappa'(H) \leq k - 1$.

Let $S \subseteq V$ be a $k - 1$-vertex cutset of H.

If $G - S$ is not connected then $\kappa(G) \leq k - 1 < \kappa'(G)$.

Assume therefore that $G - S$ is connected.



Neither endpoint of e is in S, else $G - S = H - S$.

e is a bridge of $G - S$ since $(G - S) - e = H - S$ is not connected. It has 2 components C_1, C_2.

If $|C_1| \geq 2$ then $S + v$ is a k-vertex cutset of G and so $\kappa(G) \leq k$.

Similarly if $|C_2| \geq 2$.

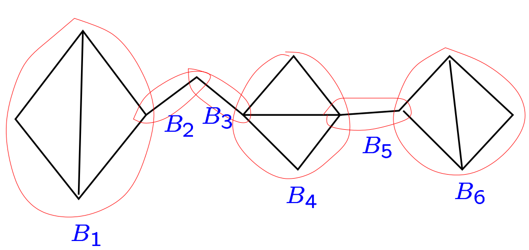
So assume that $G - S$ is the just the edge vw.

Then $\nu(G) = k + 1$ and so $\kappa(G) \leq k$.

A *block* is a connected graph with no cutpoints.

Thus a block is either a single vertex, a single edge or if $\nu \geq 3$ it is a 2-connected graph.

A block of a graph is a *maximal* connected subgraph with no cutpoints.



Note that blocks partition the edges of G, not the vertices.

Union and Intersection of Graphs

$G_i = (V_i, E_i), i = 1, 2.$

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2).$$

$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$$

provided $V_1 \cap V_2 \neq \emptyset$.

Theorem 2 *A connected graph G can be expressed as*

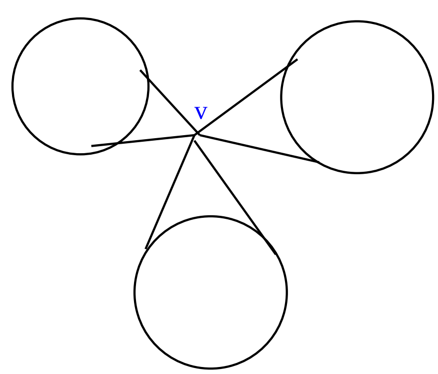
$$G = B_1 \cup B_2 \cup \cdots B_r$$

where $B_1, B_2, \ldots, B_r$ are the blocks of G.

By induction on ν. Trivial for $\nu = 1$.
Assume true for all connected graphs with $\nu < k$ and suppose that G has k vertices.

(a) G has no cutpoints – $G = B_1$.

(b) G has a cutpoint v.



Let $G - v$ have components $C_1, C_2, \ldots C_s$.

$C_i + v$ is connected for $1 \le i \le s$.

By induction

$$C_i + v = \bigcup_{j=1}^{k_i} B_{i,j}$$

where the $B_{i,j}$ are the blocks of $C_i + v$.

Thus

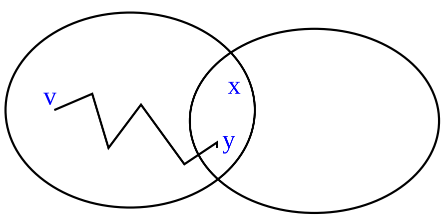
$$G = \bigcup_{i=1}^{r} \bigcup_{j=1}^{k_i} B_{i,j}.$$

We still have to check that the $B_{i,j}$ are maximal 2-connected subgraphs. But $B_{i,j}$ is not strictly contained in any 2-connected subgraph of $C_i + v$ since it is a block of $C_i + v$. Also, if $x \notin C_i + v$ then every path from x to C_i must go through v and so v is a cutpoint of any subgraph containing $B_{i,j}$ and x.

Theorem 3 *If B_1, B_2 are blocks of the connected graph G then*

$$|V(B_1) \cap V(B_2)| \le 1.$$

Proof Suppose that $|V(B_1) \cap V(B_2)| \ge 2$. We obtain the contradiction that $B_1 \cup B_2$ is 2-connected. Let $x \in V(B_1) \cup V(B_2)$ and $y \in (V(B_1) \cap V(B_2)) - x$. Then there is a path in B_i from every vertex v of $B_i - x$ to y. Thus $B_1 \cup B_2 - x$ is connected. □

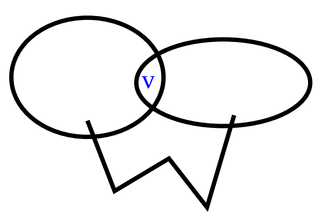


Lemma 2 *If v is a cutpoint of connected graph then there exist blocks B_1, B_2 such that $V(B_1) \cap V(B_2) = \{v\}$.*

Proof Let $G - v$ have components $C_1, C_2, \ldots, C_r$. Let B_i be the block of $C_i + v$ which contains v for $i = 1, 2$. B_1, B_2 are blocks of G, by the same argument as given in end of proof of Theorem 2. □

Lemma 3 *If $B_1 \ne B_2$ are blocks of G and $V(B_1) \cap V(B_2) = \{v\}$ then every path from a vertex of B_1 to a vertex of B_2 goes through v.*

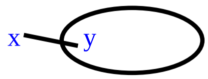
Proof If there is a path P from $x \in B_1$ to $y \in B_2$ which avoids v then $B_1 \cup B_2 \cup P$ is 2-connected.



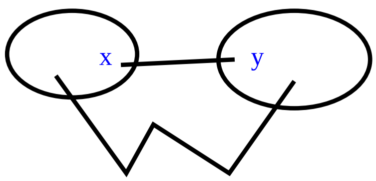
Corollary 1 *If $B_1 \ne B_2$ are blocks of G and $V(B_1) \cap V(B_2) = \{v\}$ then v is a cutpoint of G.*

Lemma 4 *Suppose $B_1 \neq B_2$ are vertex disjoint blocks of G and there is an edge xy with $x \in V(B_1), y \in V(B_2)$. Then x, y forms a block of G and both of x, y are cutpoints.*

Proof If y is of degree 1 then $B_1 = x$ is not a block.



Otherwise if y is not of degree 1 and is not a cutpoint then there is a path P from B_2 to $B_1 - x$ which implies $B_1 \cup B_2 \cup P$ is 2-connected – contradiction.

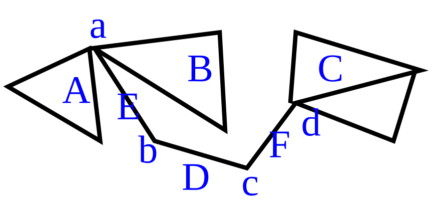


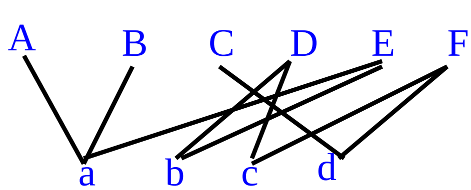
Thus xy is a bridge of G and so is a block.

Block Graph

Let G be a connected graph with blocks $B_1, B_2, \ldots, B_r$ and cutpoints $c_1, c_2, \ldots, c_s$. We define a bipartite graph H with $V(H) = \{b_1, b_2, \ldots, b_r\} \cup \{c_1, c_2, \ldots, c_s\}$ (*block vertices and cut vertices*) and $E(H) = \{b_i c_j : c_j \in B_i\}$.

Theorem 4 *H is a tree.*





Proof If G is 2-connected then H consists of a single vertex. Assume that G is not 2-connected.

(a) H is connected.
Suppose that H contains components $C_1, C_2, \ldots, C_r$, $r \geq 2$. Each component contains at least one block vertex and at least one cut vertex – each block contains a cutpoint and each cutpoint is contained in a block.

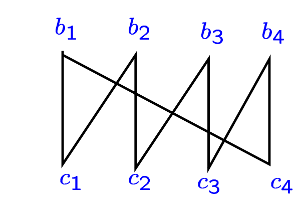
Since G is connected, there exist C_i, C_j and $b_i \in C_i$, $b_j \in C_j$ such that either

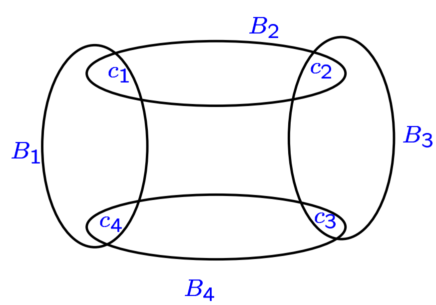
1. $V(B_i) \cap V(B_j) = \{c\}$: but then H contains the path b_i, c, b_j – contradiction.

2. $V(B_i) \cap V(B_j) = \emptyset$ and there is an edge x, y joining B_i to B_j: but then x, y is a block B_k, say, and H contains the path b_i, x, b_k, y, b_j – contradiction.

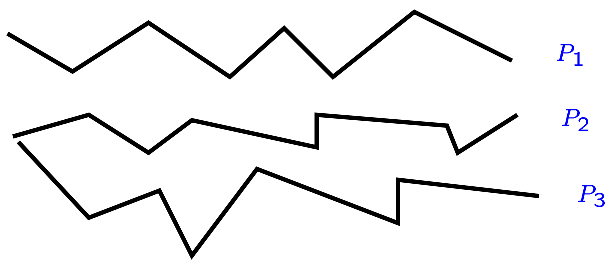
(b) H contains no cycles.

If H contains the cycle $(b_1, c_1, b_2, c_2, \ldots, b_k, c_k, b_1)$ then $B_1 \cup B_2 \cup \cdots B_k$ is 2-connected – contradiction.





A family of paths is said to be *internally disjoint* if no vertex of G is an internal vertex of more than one path.



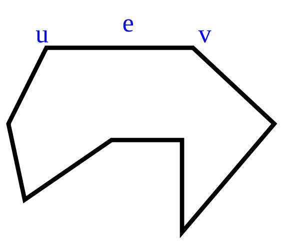
Theorem 5 *A graph G with $\nu \geq 3$ is 2-connected iff any two vertices of G are connected by at least two internally disjoint paths.*

Proof
If: $G - v$ is connected for any v since $x, y \in V - v$ must contain at least one path which avoids v.

Only if: assume G is 2-connected. We show by induction on $d(u, v)$ that every pair of vertices u, v are joined by two internally disjoint paths.

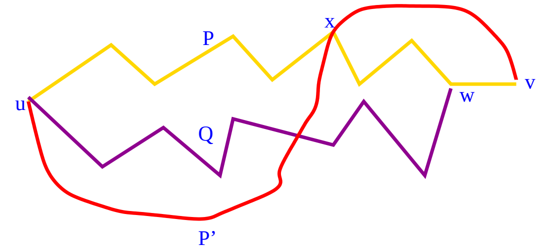
Base case: $d(u, v) = 1$.

$e = uv$ is not a cut-edge and so is contained in a cycle.



Assume true for $d(u, v) < k$ and consider a pair u, v with $d(u, v) = k$. Let w be the penultimate vertex on some path of length k from u to v. Thus $d(u, w) = k - 1$ and there are two internally disjoint paths P, Q joining u and w.

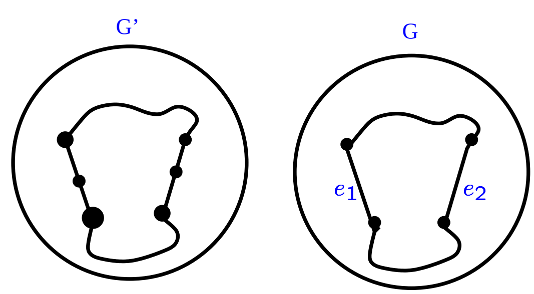
$G - w$ is connected and so there exists a u, v-path P' in $G - w$. Let x be the last vertex of P' which is not in $P \cup Q$.



If $x \in P$ take $P(u, x) + P'(x, v)$ and Q as the two internally disjoint paths. □

Corollary 2 *If G is 2-connected and $\nu \geq 3$ then every pair of vertices are contained in a cycle.*

Corollary 3 *If G is 2-connected and $\nu \geq 3$ then every pair of edges e_1, e_2 are contained in a cycle.*



G' is obtained from G by *dividing* e_1, e_2. G' is 2-connected. Apply Theorem 5.