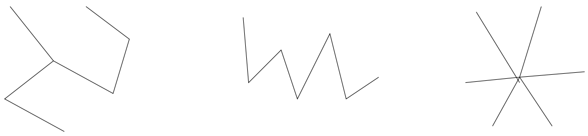
Trees



A *tree* is a graph which is

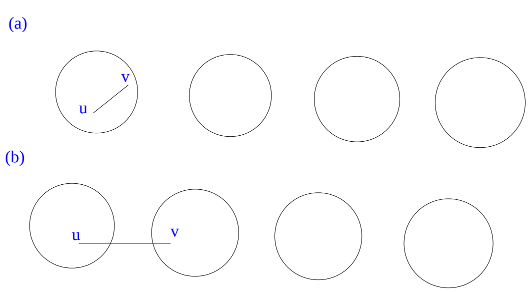
(a) Connected and

(b) has no cycles (*acyclic*).

Lemma 1 *Let the components of G be $C_1, C_2, \ldots, C_r$, Suppose $e = (u, v) \notin E$, $u \in C_i$, $v \in C_j$.*

(a) $i = j \Rightarrow \omega(G + e) = \omega(G)$.

(b) $i \neq j \Rightarrow \omega(G + e) = \omega(G) - 1$.



Proof Every path P in $G + e$ which is not in G must contain e. Also,

$$\omega(G + e) \leq \omega(G).$$

Suppose

$$(x = u_0, u_1, \ldots, u_k = u, u_{k+1} = v, \ldots, u_\ell = y)$$

is a path in $G + e$ that uses e. Then clearly $x \in C_i$ and $y \in C_j$.

(a) follows as now no new relations $x \sim y$ are added.

(b) Only possible new relations $x \sim y$ are for $x \in C_i$ and $y \in C_j$. But $u \sim v$ in $G + e$ and so $C_i \cup C_j$ becomes (only) new component. $\square$

Lemma 2 $G = (V, E)$ *is acyclic* (forest) *with* (tree) *components* $C_1, C_2, \ldots, C_k$. $|V| = n$. $e = (u, v) \notin E$, $u \in C_i$, $v \in C_j$.

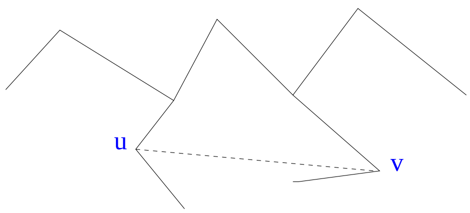
(a) $i = j \Rightarrow G + e$ *contains a cycle.*

(b) $i \neq j \Rightarrow G + e$ *is acyclic and has one less component.*

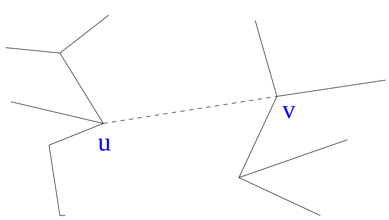
(c) G *has* $n - k$ *edges.*

(a) $u, v \in C_i$ implies there exists a path $(u = u_0, u_1, \ldots, u_\ell = v)$ in G.

So $G + e$ contains the cycle $u_0, u_1, \ldots, u_\ell, u_0$.



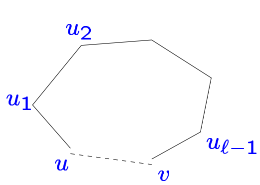
(a)



Suppose $G + e$ contains the cycle C. $e \in C$ else C is a cycle of G.

$$C = (u = u_0, u_1, \ldots, u_\ell = v, u_0).$$

But then G contains the path $(u_0, u_1, \ldots, u_\ell)$ from u to v – contradiction.



The drop in the number of components follows from Lemma 1.

The rest of the lemma follows from

(c) Suppose $E = \{e_1, e_2, \ldots, e_r\}$ and $G_i = (V, \{e_1, e_2, \ldots, e_i\})$ for $0 \leq i \leq r$.

Claim: G_i has $n - i$ components.
Induction on i.
$i = 0$: G_0 has no edges.
$i > 0$: G_{i-1} is acyclic and so is G_i. It follows from part (a) that e_i joins vertices in distinct components of G_{i-1}. It follows from (b) that G_i has one less component than G_{i-1}.
End of proof of claim

Thus $r = n - k$ (we assumed G had k components).

□

Corollary 1 *If a tree T has n vertices then*

(a) *It has $n - 1$ edges.*

(b) *It has at least 2 vertices of degree 1, ($n \geq 2$).*

Proof (a) is part (c) of previous lemma. $k = 1$ since T is connnected.

(b) Let s be the number of vertices of degree 1 in T. There are no vertices of degree 0 – these would form separate components. Thus

$$2n - 2 = \sum_{v \in V} d_T(v) \geq 2(n - s) + s.$$

So $s \geq 2$. □

Theorem 1 *Suppose $|V| = n$ and $|E| = n - 1$. The following three statements become equivalent.*

(a) *G is connected.*

(b) *G is acyclic.*

(c) *G is a tree.*

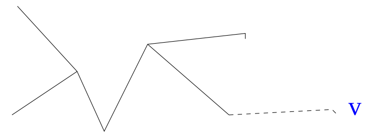
Proof Let $E = \{e_1, e_2, \ldots, e_{n-1}\}$ and $G_i = (V, \{e_1, e_2, \ldots, e_i\})$ for $0 \leq i \leq n - 1$.

$(a) \Rightarrow (b)$: G_0 has n components and G_{n-1} has 1 component. Addition of each edge e_i must reduce the number of components by 1 – Lemma 1(b). Thus G_{i-1} acyclic implies G_i is acyclic – Lemma 2(b). (b) follows as G_0 is acyclic.

$(b) \Rightarrow (c)$: We need to show that G is connected. Since G_{n-1} is acyclic, $\omega(G_i) = \omega(G_{i-1}) - 1$ for each i – Lemma 2(b). Thus $\omega(G_{n-1}) = 1$.

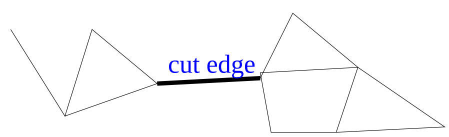
$(c) \Rightarrow (a)$: trivial.

Corollary 2 *If v is a vertex of degree 1 in a tree T then $T - v$ is also a tree.*



Proof Suppose T has n vertices and n edges. Then $T - v$ has $n - 1$ vertices and $n - 2$ edges. It acyclic and so must be a tree. □

Cut edges

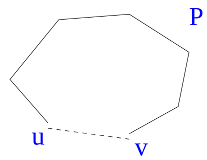


e is a *cut edge* of G if $\omega(G - e) > \omega(G)$.

Theorem 2 *$e = (u, v)$ is a cut edge iff e is not on any cycle of G.*

Proof ω increases iff there exist $x \sim y \in V$ such that all walks from x to y use e.

Suppose there is a cycle (u, P, v, u) containing e. Then if $W = x, W_1, u, v, W_2, y$ is a walk from x to y using e, x, W_1, P, W_2, y is a walk from x to y that doesn't use e. Thus e is not a cut edge.



If e is not a cut edge then $G-e$ contains a path P from u to v ($u \sim v$ in G and relations are maintained after deletion of e). So (v, u, P, v) is a cycle containing e. □

Corollary 3 *A connected graph is a tree iff every edge is a cut edge.*

Corollary 4 *Every finite connected graph G contains a spanning tree.*

Proof Consider the following process: starting with G,

1. If there are no cycles – **stop**.

2. If there is a cycle, delete an edge of a cycle.

Observe that (i) the graph remains connected – we delete edges of cycles. (ii) the process must terminate as the number of edges is assumed finite.

On termination there are no cycles and so we have a connected acyclic spanning subgraph i.e. we have a spanning tree. □

Alternative Construction

Let $E = \{e_1, e_2, \ldots, e_m\}$.

```
begin
    T := ∅
    for i = 1, 2, ..., m do
    begin
        if T + e_i does not contain a cycle
        then T ← T + e_i
    end
Output T
end
```

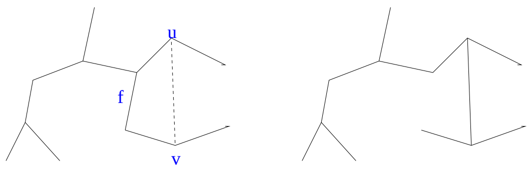

Lemma 3 *If G is connected then (V, T) is a spanning tree of G.*

Proof Clearly T is acyclic. Suppose it is not connected and has comppponents $C_1, C_2, \ldots, C_k$, $k \geq 2$. Let $D = C_2 \cup \cdots \cup C_k$. Then G has no edges joining C_1 and D – contradiction. (The first $C_1 : D$ edge found by the algorithm would have been added.)

Theorem 3 *Let T be a spanning tree of $G = (V, E)$, $|V| = n$. Suppose $e = (u, v) \in E \setminus T$.*

(a) *$T + e$ contains a* unique *cycle $C(T, e)$.*

(b) *$f \in C(T, e)$ implies that $T + e - f$ is a spanning tree of G.*



Proof (a) Lemma 2(a) implies that $T + e$ has a cycle C. Suppose that $T + e$ contains another cycle $C' \neq C$. Let edge $g \in C' \setminus C$. $T' = T + e - g$ is connected, has $n - 1$ edges. But T' contains a cycle C, contradictng Theorem 1.

(b) $T + e - f$ is connected and has $n - 1$ edges. Therefore it is a tree. □

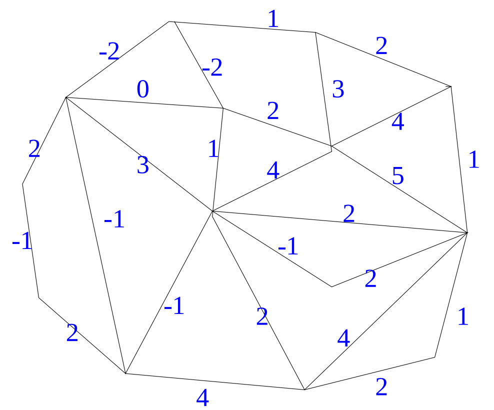
Maximum weight trees

$G = (V, E)$ is a connected graph.

$w : E \to \boldsymbol{R}$. $w(e)$ is the *weight* of edge e.

For spanning tree T, $w(T) = \sum_{e \in T} w(e)$.

Problem: find a spanning tree of maximum weight.



Greedy Algorithm

Sort edges so that $E = \{e_1, e_2, \ldots, e_m\}$ where

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m).$$

```
begin
    T := ∅
    for i = 1, 2, ..., m do
    begin
        if T + e_i does not contain a cycle
        then T ← T + e_i
    end
Output T
end
```

Greedy always adds the maximum weight edge which does not make a cycle with previously chosen edges.

Theorem 4 *Let G be a connected weighted graph. The tree constructed by GREEDY is a maximum weight spanning tree.*

Proof Lemma 3 implies that T is a spanning tree of G.

Let the edges of the *greedy tree* be $e_1^\star, e_2^\star, \ldots, e_{n-1}^\star$, in order of choice. Note that $w(e_i^\star) \geq w(e_{i+1}^\star)$ since neither makes a cycle with $e_1^\star, e_2^\star, \ldots, e_{i-1}^\star$.

Let $f_1, f_2, \ldots, f_{n-1}$ be the edges of any other tree where $w(f_1) \geq w(f_2) \geq \cdots w(f_{n-1})$.

We show that

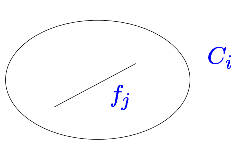
$$w(e_i^\star) \geq w(f_i) \qquad 1 \leq i \leq n-1. \qquad (1)$$

Suppose (1) is false. There exists $k > 0$ such that

$$w(e_i^\star) \geq w(f_i),\ 1 \leq i < k \text{ and } w(e_k^\star) < w(f_k).$$

Each f_i, $1 \leq i \leq k$ is either one of or makes a cycle with $e_1^\star, e_2^\star, \ldots, e_{k-1}^\star$. Otherwise one of the f_i would have been chosen in preference to $e_k^\star$.

Let components of forest $(V, \{e_1^\star, e_2^\star, \ldots, e_{k-1}^\star\})$ be $C_1, C_2, \ldots, C_{n-k+1}$. Each f_i, $1 \leq i \leq k$ has both of its endpoints in the same component.



Let μ_i be the number of f_j which have both endpoints in C_i and let ν_i be the number of vertices of C_i. Then

$$\mu_1 + \mu_2 + \cdots \mu_{n-k+1} = k \qquad (2)$$
$$\nu_1 + \nu_2 + \cdots \nu_{n-k+1} = n \qquad (3)$$

It follows from (2) and (3) that there exists t such that

$$\mu_t \geq \nu_t. \qquad (4)$$

[Otherwise

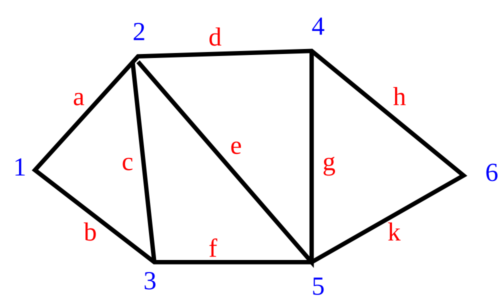
$$\sum_{i=1}^{n-k+1} \mu_i \leq \sum_{i=1}^{n-k+1} (\nu_i - 1) = \sum_{i=1}^{n-k+1} \nu_i - (n-k+1) = k-1.$$]

But (4) implies that the edges f_j such that $f_j \subseteq C_t$ contain a cycle. □

Cut Sets and Bonds

If $S \subseteq V$, $S \neq \emptyset, V$ then the **cut-set**

$$S : \bar{S} = \{e = vw \in E : \ v \in S, w \in \bar{S} = V \setminus S\}$$



$$S = \{1, 2, 3\} \qquad S : \bar{S} = \{d, e, f\}.$$

Lemma 4 *Let G be connected and $X \subseteq E$. Then $G[E \setminus X]$ is not connected iff X contains a cutset.*

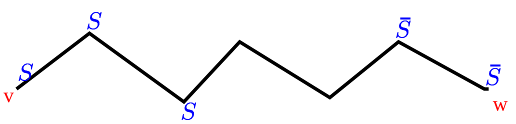
Proof

Only if

$G[E \setminus X]$ contains components $C_1, C_2, \ldots, C_k$, $k \geq 2$ and so $X \supseteq C_1 : \bar{C}_1$ and $C_1 \neq \emptyset, V$.

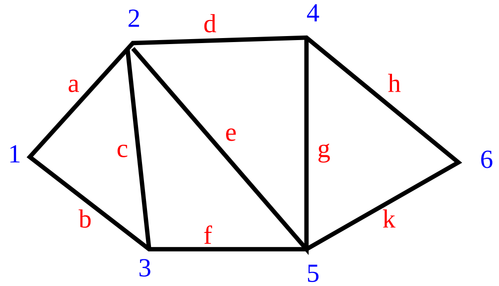
If

Suppose $X = S : \bar{S}$ and $v \in S, w \in \bar{S}$. Then every walk from v to w in G contains an edge of X.



So $G[E \setminus X]$ contains no walk from v to w. $\square$

A **Bond** B is a *minimal* cut-set. I.e. $B = S : \bar{S}$ and if $T : \bar{T} \subseteq B$ then $B = T : \bar{T}$.



$S_1 = \{1, 2, 3\}$ $B_1 = S_1 : \bar{S}_1$ is a bond
$S_2 = \{2, 3, 4, 5\}$ $B_2 = S_2 : \bar{S}_2$ is not a bond

since $B_2 \supset S_3 : \bar{S}_3$ and $B_2 \neq S_3 : \bar{S}_3$.

Theorem 5 *G is connected and B is a a bond $\leftrightarrow G \setminus B$ contains exactly 2 components.*

Proof $\rightarrow$: $G \setminus B$ contains components $C_1, C_2, \ldots, C_k$. Assume w.l.o.g. that there is at least one edge e in G joining C_1 and C_2. If $k \geq 3$ then $B \supseteq C_3 : \bar{C}_3$ and $B \neq C_3 : \bar{C}_3$ since B contains e.

$\leftarrow$: Assume that $G \setminus B$ contains exactly two components $C_1 = G[S], C_2 = G[\bar{S}]$. Let $e \in B$. Adding e to the graph $C_1 \cup C_2$ clearly produces a connected graph and so $B \setminus e$ is not a cutset. $\square$

A **co-tree** $\bar{T}$ of a connected graph G is the edge complement of a spanning tree of G i.e. $\bar{T} = E \setminus T$ for some spanning tree T.

Theorem 6 *Let T be a spanning tree of G and $e \in T$. Then*

(a) *$\bar{T}$ contains no bond of G.*

(b) *$\bar{T} + e$ contains a unique bond $B(\bar{T}, e)$ of G.*

(c) *$f \in B(\bar{T}, e)$ implies that $\bar{T} + e - f$ is a co-tree of G.*

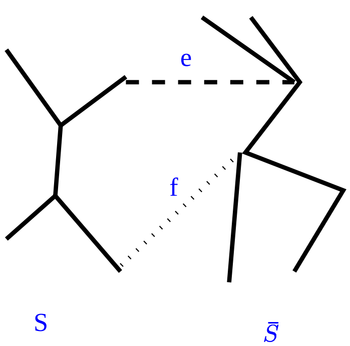
[Compare with Tree + edge $\supseteq$ cycle.]

Proof (a) $X \subseteq \bar{T} \leftrightarrow G \setminus X \supseteq T$ which implies that $G \setminus X$ is connected. So X is not a bond.

(b)&(c) $G \setminus (\bar{T} + e) = T \setminus e$ contains exactly two components and so by Theorem 5 $\bar{T} + e$ contains a bond $B = S : \bar{S}$ where $S, \bar{S}$ are the 2 components of $T \setminus e$.

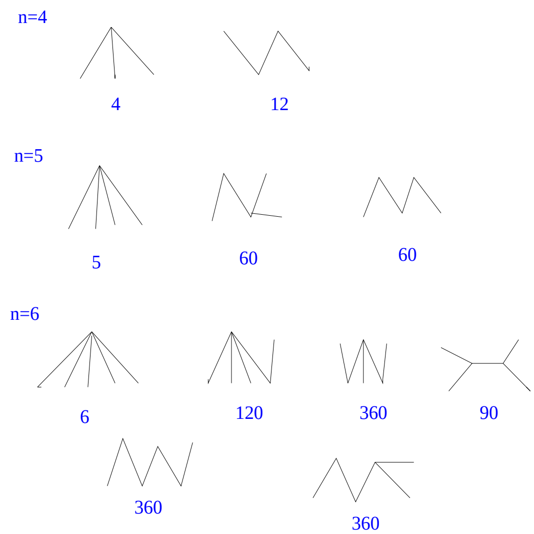
$$\begin{aligned} f \in B \quad &\Rightarrow \quad e \in C(T, f) \\ &\Rightarrow \quad T + f - e \text{ is a tree} \\ &\Rightarrow \quad \bar{T} + e - f \text{ is a co-tree} \qquad \text{proving (c)} \end{aligned}$$

Hence every bond of $\bar{T} + e$ contains f – otherwise $\bar{T} + e - f$ contains a bond, contradicting (a) and proving (b). □



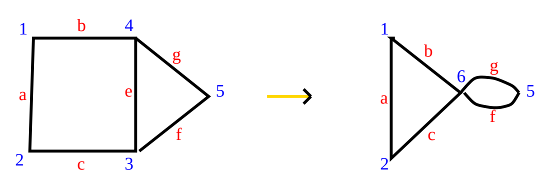
$B = (S : \bar{S})$

How many trees? – Cayley's Formula



Contracting edges

If $e = vw \in E$, $v \neq w$ then we can **contract** e to get $G \cdot e$ by (i) deleting e, (ii) identifying v, w i.e. make them into a single new vertex.



$G - e$ is obtained by deleting e.

$\tau(G)$ is the number of spanning trees of G.

Theorem 7 *If $e \in E$ is not a loop then*

$$\tau(G) = \tau(G \cdot e) + \tau(G - e).$$

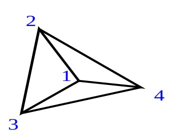
Proof

- $\tau(G - e)$ = the number of spanning trees of G which do not contain e.
- $\tau(G \cdot e)$ = the number of spanning trees of G which contain e.

[Bijection $T \to T \cdot e$ maps spanning trees of G which contain e to spanning trees of $G \cdot e$.]

Matrix Tree Theorem

Define the $V \times V$ matrix $L = D - A$ where A is the adjacency matrix of G and D is the diagonal matrix with $D(v, v)$ = degree of v.



$$L = \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{vmatrix}$$

$$L_1 = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}$$

Determinant $L_1 = 16$

Let L_1 be obtained by deleting the first row and column of L.

Theorem 8

$$\tau(G) = \text{determinant } L_1.$$

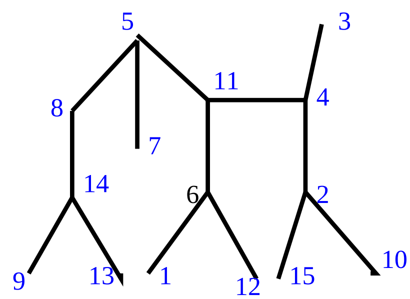
Pfuffer's Correspondence

There is a 1-1 correspondence ϕ_V between spanning trees of K_V (the complete graph with vertex set V) and sequences V^{n-2}. Thus for $n \geq 2$

$\tau(K_n) = n^{n-2}$ Cayley's Formula.

Assume some arbitrary ordering $V = \{v_1 < v_2 < \cdots < v_n\}$.

$\phi_V(T)$:
begin
  $T_1 := T$;
  for $i = 1$ **to** $n - 2$ **do**
  begin
    $s_i :=$ neighbour of least leaf ℓ_i of T_i.
    $T_{i+1} = T_i - \ell_i$.
  end $\phi_V(T) = s_1 s_2 \dots s_{n-2}$
end

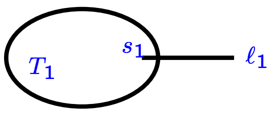


6,4,5,14,2,6,11,14,8,5,11,4,2

Lemma 5 *$v \in V(T)$ appears exactly $d_T(v)-1$ times in $\phi_V(T)$.*

Proof Assume $n = |V(T)| \geq 2$. By induction on n.
$n = 2$: $\phi_V(T) = \Lambda$ = empty string.

Assume $n \geq 3$:



$\phi_V(T) = s_1\phi_{V_1}(T_1)$ where $V_1 = V - \{\ell_1\}$.

s_1 appears $d_{T_1}(s_1) - 1 + 1 = d_T(s_1) - 1$ times – induction.
$v \neq s_1$ appears $d_{T_1}(v) - 1 = d_T(v) - 1$ times – induction. □

Construction of ϕ_V^{-1}

Inductively assume that for all $|X| < n$ there is an inverse function ϕ_X^{-1}. (True for $n = 2$).
Now define ϕ_V^{-1} by

$\phi_V^{-1}(s_1s_2\ldots s_{n-2}) = \phi_{V_1}^{-1}(s_2\ldots s_{n-2})$ plus edge $s_1\ell_1$,

where $\ell_1 = \min\{s : s \notin s_1, s_2, \ldots s_{n-2}\}$ and $V_1 = V - \{\ell_1\}$.

Then

$$\begin{aligned}\phi_V(\phi_V^{-1}(s_1s_2\ldots s_{n-2})) &= \\ &= \phi_V(\phi_{V_1}^{-1}(s_2\ldots s_{n-2}) \text{ plus edge } s_1\ell_1) \\ &= s_1\phi_{V_1}(\phi_{V_1}^{-1}(s_2\ldots s_{n-2})) \\ &= s_1s_2\ldots s_{n-2}.\end{aligned}$$

Thus ϕ_V has an inverse and the correspondence is established.

Number of trees with a given degree sequence

Corollary 5 *If $d_1 + d_2 + \cdots + d_n = 2n - 2$ then he number of spanning trees of K_n with degree sequence $d_1, d_2, \ldots, d_n$ is*

$$\binom{n-2}{d_1-1, d_2-1, \ldots, d_n-1} = \frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}.$$

Proof From Pfuffer's correspondence and Lemma 5 this is the number of sequences of length $n-2$ in which 1 appears $d_1 - 1$ times, 2 appears $d_2 - 1$ times and so on. □