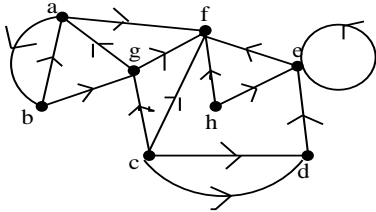


Directed graphs

Digraph $D = (V, A)$.
 $V = \{\text{vertices}\}$, $A = \{\text{arcs}\}$



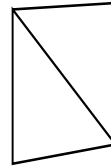
$V = \{a, b, \dots, h\}$, $A = \{(a, b), (b, a), \dots\}$

(2 arcs with endpoints (c, d))

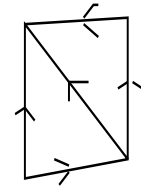
Thus a digraph is a graph with oriented edges.
 D is *strict* if there are no loops or repeated edges.

1

Graph G : an orientation of G is obtained by replacing each edge $\{a, b\}$ by (a, b) or (b, a) .



G

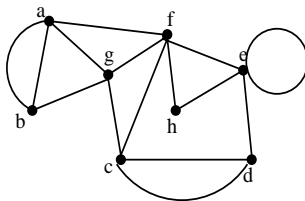


Orientation of G

There are $2^{|E|}$ distinct orientations of G .

3

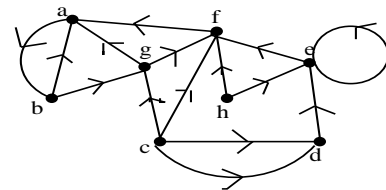
Digraph D : $G(D)$ is the *underlying* graph obtained by replacing each arc (a, b) by an edge $\{a, b\}$.



The graph underlying the digraph on previous slide

2

Walks, trails, paths, cycles now have directed counterparts.

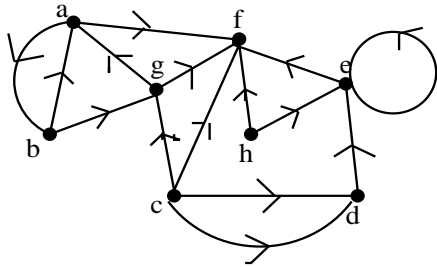


Directed Walk: (c, d, e, f, a, b, g, f) .
 Directed Path: (a, b, g, f) .
 Directed Cycle: (g, a, b, a)

(e, f, g, a) is not a directed walk -- there is no arc (f, g) .

4

The *indegree* $d_D^-(v)$ of vertex v is the number of arcs $(x, v), x \in V$. The *outdegree* $d_D^+(v)$ of vertex v is the number of arcs $(v, x), x \in V$.



	a	b	c	d	e	f	g	h
d^+	2	2	4	1	2	0	2	2
d^-	2	1	0	2	3	5	2	0

Note that since each arc contributes one to a vertex outdegree and one to a vertex indegree,

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |A|.$$

A graph is *strongly connected* if it has one strong component i.e. if there is a directed walk between each pair of vertices.

For a set $S \subseteq V$ let

$$N^+(S) = \{w \notin S : \exists v \in S \text{ s.t. } (v, w) \in A\}.$$

$$N^-(S) = \{w \notin S : \exists v \in S \text{ s.t. } (w, v) \in A\}.$$

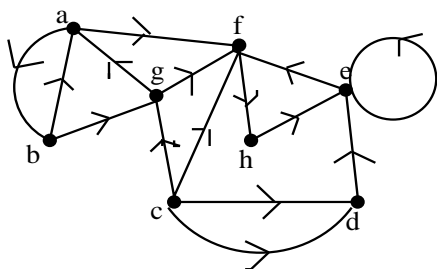
Theorem 1 D is strongly connected iff there does not exist $S \subseteq V, S \neq \emptyset, V$ such that $N^+(S) = \emptyset$.

Proof Only if: suppose there is such an S and $x \in S, y \in V \setminus S$ and suppose there is a directed walk W from x to y . Let $(v_1 = x, v_2, \dots, v_k = y)$ be the sequence of vertices traversed by W . Let v_i be the first vertex of this sequence which is not in S . Then $v_i \in N^+(S)$, contradiction, since arc (v_{i-1}, v_i) exists.

Strong Connectivity or Diconnectivity

Given digraph D we define the relation \sim on V by $v \sim w$ iff there is a *directed* walk from v to w and a directed walk from w to v .

This is an equivalence relation (proof same as directed case) and the equivalence classes are called *strong components* or *dicomponents*.

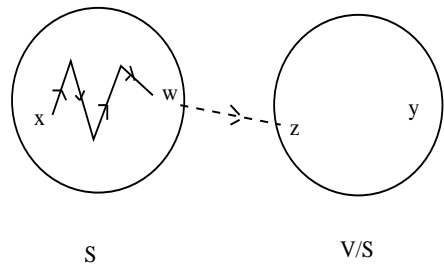


Here the strong components are

$$\{a, b, g\}, \{c\}, \{d\}, \{e, f, h\}.$$

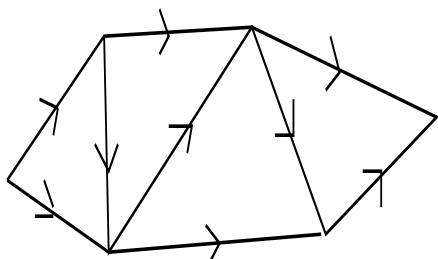
If: suppose that D is not strongly connected and that there is no directed walk from x to y . Let $S = \{v \in V : \exists \text{ a directed walk from } x \text{ to } v\}$.

$$S \neq \emptyset \text{ as } x \in S \text{ and } S \neq V \text{ as } y \notin S.$$



Then $N^+(S) = \emptyset$. If $z \in N^+(S)$ then there exists $w \in S$ such that $(w, z) \in A$. But then since $w \in S$ there is a directed walk from x to w which can be extended to z , contradicting the fact that $z \notin S$. \square

A Directed Acyclic Graph (DAG) is a digraph without any directed cycles.



Lemma 1 If D is a DAG then D has at least one source (vertex of indegree 0) and at least one sink (vertex of outdegree 0).

Proof Let $P = (v_1, v_2, \dots, v_k)$ be a directed path of maximum length in D . Then v_1 is a source and v_k is a sink.

Theorem 2 D has a topological ordering iff D is a DAG.

Proof Only if: Suppose there is a topological ordering and a directed cycle $v_{i_1}, v_{i_2}, \dots, v_{i_k}$. Then

$$i_1 < i_2 < \dots < i_k < i_1$$

which is absurd.

if: By induction on ν . Suppose that D is a DAG. The result is true for $\nu = 1$ since D has no loops. Suppose that $\nu > 1$, v_ν is any sink of D and let $D' = D - v_\nu$.

D' is a DAG and has a topological ordering $v_1, v_2, \dots, v_{\nu-1}$, induction. v_1, v_2, \dots, v_ν is a topological ordering of D . For if there is an edge $v_i v_j$ with $i > j$ then (i) it cannot be in D' and (ii) $i \neq \nu$ since v_ν is a sink. \square

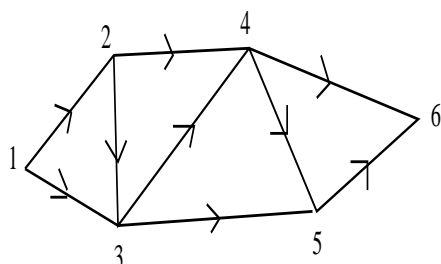
Suppose for example that there is an edge xv_1 . Then either

(a) $x \notin \{v_2, v_3, \dots, v_k\}$. But then (x, P) is a longer directed path than P – contradiction.

(b) $x = v_i$ for some $i \neq 1$ and D contains the cycle $v_1, v_2, \dots, v_i, v_1$. \square

A topological ordering v_1, v_2, \dots, v_ν of the vertex set of a digraph D is one in which

$$v_i v_j \in A \text{ implies } i < j.$$



Theorem 3 Let $G = G(D)$. Then D contains a directed path of length $\chi(G) - 1$.

Proof Let $D = (V, A)$ and $A' \subseteq A$ be a minimal set of edges such that $D' = D - A'$ is a DAG.

Let k be the length of the longest directed path in D' .

Define $c(v)$ = length of longest path from v in D' . $c(v) \in \{0, 1, 2, \dots, k\}$. We claim that $c(v)$ is a proper colouring of G , proving the theorem.

Note first that if D' contains a path $P = (x_1, x_2, \dots, x_k)$ then

$$c(x_1) \geq c(x_k) + k - 1. \quad (1)$$

(We can add the longest path Q from x_k to P to create a path (P, Q) . This uses the fact that D' is a DAG.)

Suppose c is not a proper colouring of G and there exists an edge $vw \in G$ with $c(v) = c(w)$. Suppose $vw \in A$ i.e. it is directed from v to w .

Case 1: $vw \notin A'$. (1) implies $c(v) \geq c(w) + 1$ - contradiction.

Case 2: $vw \in A'$. There is a cycle in $D' + vw$ which contains vw , by the minimality of A' . Suppose that C has $\ell \geq 2$ edges. Then (1) implies that $c(w) \geq c(v) + \ell - 1$. \square

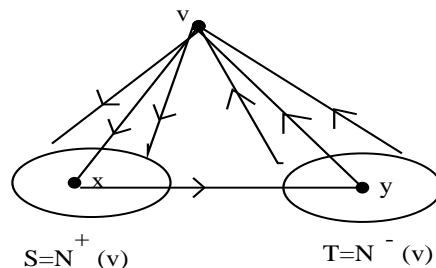
13

Theorem 4 If D is a strongly connected tournament with $\nu \geq 3$ then D contains a directed cycle of size k for all $3 \leq k \leq \nu$.

Proof By induction on k .

$k = 3$.

Choose $v \in V$ and let $S = N^+(v)$, $T = N^-(v) = V \setminus (S \cup \{v\})$.



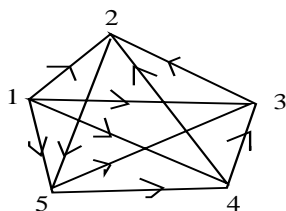
$S \neq \emptyset$ since D is strongly connected. Similarly, $T \neq \emptyset$ else $N^+(V \setminus \{v\}) = \emptyset$.

Thus $N^+(S) \neq \emptyset$. $v \notin N^+(S)$ and so $N^+(S) = T$. Thus $\exists x \in S, y \in T$ with $xy \in A$.

15

Tournaments

A tournament is an orientation of a complete graph K_n .



1,2,5,4,3 is a directed Hamilton Path

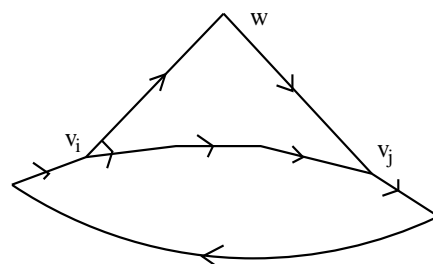
Corollary 1 A tournament T contains a directed Hamilton path.

Proof $\chi(G(T)) = n$. Now apply Theorem 3. \square

14

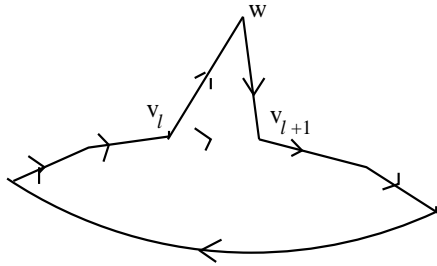
Suppose now that there exists a directed cycle $C = (v_1, v_2, \dots, v_k, v_1)$.

Case 1: $\exists w \notin C$ and $i \neq j$ such that $v_i w \in A$, $w v_j \in A$.



16

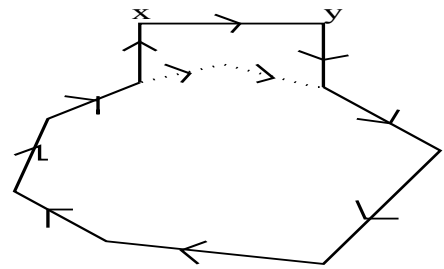
It follows that there exists ℓ with $v_\ell w \in A$, $wv_{\ell+1} \in A$.
 $C' = (w, v_{\ell+1}, \dots, v_\ell, v_1, \dots, v_\ell, w)$ is a cycle of length $k + 1$.



$S = \emptyset$ implies $T = \emptyset$ (and C is a Hamilton cycle) or $N^+(T) = \emptyset$.
 $T = \emptyset$ implies $N^+(C) = \emptyset$.

Thus we can assume $S, T \neq \emptyset$ and $N^+(T) \neq \emptyset$.
 $N^+(T) \cap C = \emptyset$ and so $N^+(T) \cap S \neq \emptyset$.

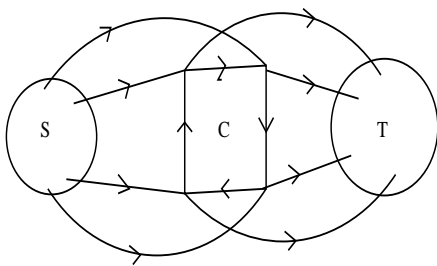
Thus $\exists x \in T, y \in S$ such that $xy \in A$.



The cycle $(v_1, x, y, v_3, \dots, v_k, v_1)$ is a cycle of length $k + 1$.

Case 2 $V \setminus C = S \cup T$ where

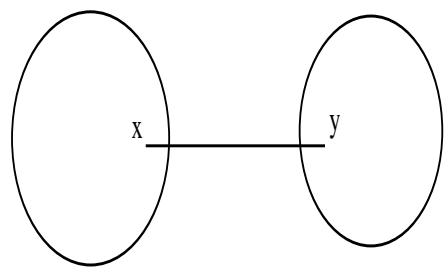
$w \in S$ implies $wv_i \in A, 1 \leq i \leq k$.
 $w \in T$ implies $v_iw \in A, 1 \leq i \leq k$.



Robbin's Theorem

Theorem 5 A connected graph G has an orientation which is strongly connected iff G is 2-edge connected.

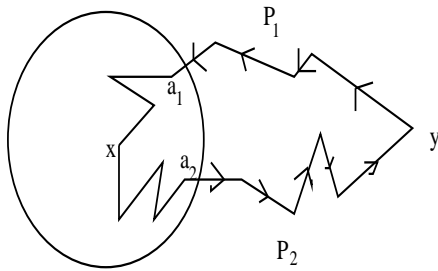
Only if: Suppose that G has a cut edge $e = xy$.



If we orient e from x to y (resp. y to x) then there is no directed path from y to x (resp. x to y).

If: Suppose G is 2-edge connected. It contains a cycle C which we can orient to produce a directed cycle.

At a general stage of the process we have a set of vertices $S \supseteq C$ and an orientation of the edges of $G[S]$ which is strongly connected.



If $S \neq V$ choose $x \in S, y \notin S$.
 There are 2 edge disjoint paths P_1, P_2 joining y to x .
 Let a_i be the first vertex of P_i which is in S .
 Orient $P_1[y, a_1]$ from y to a_1 .
 Orient $P_2[y, a_2]$ from a_2 to y .

Directed Euler Tours

An Euler tour of a digraph D is a directed walk which traverses each arc of D exactly once.

Theorem 6 A digraph D has an Euler tour iff $G(D)$ is connected and $d^+(v) = d^-(v)$ for all $v \in V$.

Proof This is similar to the undirected case.

If: Suppose $W = (v_1, v_2, \dots, v_m, v_1)$ ($m = |A|$) is an Euler Tour. Fix $v \in V$. Whenever W visits v it enters through a new arc and leaves through a new arc. Thus each visit requires one entering arc and one leaving arc. Thus $d^+(v) = d^-(v)$.

Only if: We use induction on the number of arcs. D is not a DAG as it has no sources or sinks. Thus it must have a directed cycle C . Now remove the edges of C . Each component C_i of $G(D - C)$ satisfies the degree conditions and so contains an Euler tour W_i . Now, as in the undirected case, go round the cycle C and the first time you visit C_i add the tour W_i . This produces an Euler tour of the whole digraph D .

Claim: The subgraph $G[S \cup P_1 \cup P_2]$ is strongly connected.

Let $S' = S \cup P_1 \cup P_2$. We must show that there is a directed path from α to β for all $\alpha, \beta \in S'$.

- (i) $\alpha, \beta \in S$: \exists a directed path from α to β in S .
- (ii) $\alpha \in S, \beta \in P_1 \setminus S$: Go from α to a_2 in S , from a_2 to y on P_2 , from y to β along P_1 .
- (iii) $\alpha \in S, \beta \in P_2 \setminus S$: Go from α to a_2 in S , from a_2 to β on P_2 .
- (iv) $\alpha \in P_1 \setminus S, \beta \in S$: Go from α to a_1 on P_1 , from a_1 to β in S .
- (v) $\alpha \in P_2 \setminus S, \beta \in S$: Go from α to y on P_1 , from y to a_1 on P_2 , from a_1 to β in S .

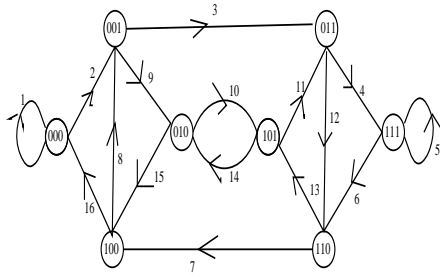
Continuing in this way we can orient the whole graph. \square

As a simple application of the previous theorem we consider the following problem. A 0-1 sequence $x = (x_1, x_2, \dots, x_m)$ has property P_n if for every 0-1 sequence $y = (y_1, y_2, \dots, y_n)$ there is an index k such that $x_k = y_1, x_{k+1} = y_2, \dots, x_{k+n-1} = y_n$. Here $x_t = x_{m+1-t}$ if $t > m$.

Note that we must have $m \geq 2^n$ in order to have a distinct k for each possible x .

Theorem 7 There exists a sequence of length 2^n with property P_n .

Proof Define the digraph D_n with vertex set $\{0, 1\}^{n-1}$ and 2^n directed arcs of the form $((p_1, p_2, \dots, p_{n-1}), (p_2, p_3, \dots, p_n))$. $G(D_n)$ is connected as we can join $(p_1, p_2, \dots, p_{n-1})$ to $(q_1, q_2, \dots, q_{n-1})$ by the path $(p_1, p_2, \dots, p_{n-1}), (p_2, p_3, \dots, p_{n-1}, q_1), (p_3, p_4, \dots, p_{n-1}, q_1, q_2), \dots, (q_1, q_2, \dots, q_{n-1})$. Each vertex of D_n has indegree and outdegree 2 and so it has an Euler tour W .



Suppose that W visits the vertices of D_n in the sequence $(v_1, v_2, \dots, v_{2^n})$. Let x_i be the first bit of v_i . We claim that x_1, x_2, \dots, x_{2^n} has property P_n . Give arc $((p_1, p_2, \dots, p_{n-1}), (p_2, p_3, \dots, p_n))$ the label (p_1, p_2, \dots, p_n) . No other arc has this label.

Given (y_1, y_2, \dots, y_n) let k be such that (v_k, v_{k+1}) has this label. Then $v_k = (y_1, y_2, \dots, y_{n-1})$ and $v_{k+1} = (y_2, y_3, \dots, y_n)$ and then $x_k = y_1, x_{k+1} = y_2, \dots, x_{k+n-1} = y_n$. \square