Weak difference property of functions with the Baire property

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Abstract

We prove that the class of functions with the Baire property has the weak difference property in category sense. That is, every function for which $f(x+h)-f(x)$ has the Baire property for every $h \in \mathbf{R}$ can be written in the form $f = g+H+\phi$ where g has the Baire property, H is additive, and for every $h \in \mathbf{R}$ we have $\phi(x+h)-\phi(x) \neq 0$ only on a meager set. We also discuss the weak difference property of some subclasses of the class of functions with the Baire property and the consistency of the difference property of the class of functions with the Baire property.

1 Introduction

Let $\mathbf{R}$ denote the set of real numbers and let F be a class of real valued functions. We say that F has the *difference property* if every function for which

$$f(x+h)-f(x) \in F$$

holds for every $h \in \mathbf{R}$ can be written in the form

$$f = g + H$$

where $g \in F$ and H is additive, that is

$$H(x+y) = H(x) + H(y)$$

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holds for every $x, y \in \mathbf{R}$. For a real h we shall write

$$\Delta_h f(x) = f(x+h) - f(x)$$

for the difference functions.

Many function classes have the difference property, but the class of Lebesgue measurable functions does not if we assume the continuum hypothesis (see [5] or [6] for details). However, it was conjectured by Erdős that every function $f : \mathbf{R} \to \mathbf{R}$ for which $\Delta_h f$ is measurable for every $h \in \mathbf{R}$ is of the form $f = g + H + \phi$ where g is measurable, H is additive and for every $h \in \mathbf{R}$, $\phi(x+h) - \phi(x) = 0$ holds almost everywhere (according to Lebesgue measure). This led to the definition of weak difference property.

We say that a class F has the *weak difference property* if every function for which

$$\Delta_h f(x) \in F$$

holds for every $h \in \mathbf{R}$ can be written in the form

$$f = g + H + \phi$$

where $g \in F$, H is additive and for every $h \in \mathbf{R}$, $\Delta_h \phi = 0$ holds almost everywhere.

The conjecture of Erdős, namely the weak difference property of the class of Lebesgue measurable functions and many of its consequences, was proved by M. Laczkovich in [5].

The weak difference property in category sense was introduced in [1]. If a property $P(x)$ holds for every $x \in \mathbf{R}$ except a meager set of x's then we say that $P(x)$ holds $\mathcal{M}$-almost everywhere (in short $\mathcal{M}$-a.e.) or for $\mathcal{M}$-almost every x, where $\mathcal{M}$ stands for the class of meager subsets of $\mathbf{R}$. Analogously we say that a class F has the *weak difference property in category sense* if every function for which

$$\Delta_h f(x) \in F$$

holds for every $h \in \mathbf{R}$ can be written in the form

$$f = g + H + \phi$$

where $g \in F$, H is additive and for every $h \in \mathbf{R}$, $\Delta_h \phi = 0$ holds $\mathcal{M}$-almost everywhere. The functions ϕ of this kind will be called *null.*

In [1] some problems were formulated on the analogy of the classical weak difference property problems. One of these, the counterpart of the result of M. Laczkovich for Lebesgue measurable functions, the weak difference property in category sense of the class of functions with the Baire property is the subject of our work. (A real valued function has the *Baire property* if for every $b \in \mathbf{R}$ the set $\{x \in \mathbf{R} : f(x) < b\}$ has the Baire property, that is it can be obtained as the symmetric difference of an open and a meager set.) Once this is done, it will be a more simple task to establish the weak difference property of some subclasses of the class of functions having the Baire property. We will prove two theorems:

Theorem 1.1 *Let $f : \mathbf{R} \to \mathbf{R}$ be a function with uniformly essentially bounded difference functions, that is for a fixed $K \in \mathbf{R}$,*

$$|\Delta_h f(x)| < K \ \mathcal{M}\text{-a.e.}$$

holds for every $h \in \mathbf{R}$. If $\Delta_h f$ has the Baire property for every $h \in \mathbf{R}$ then

$$f = g + \phi$$

where g has the Baire property and ϕ is null.

This first theorem answers Problem 2.2 in [1].

Theorem 1.2 *Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary function. If $\Delta_h f$ has the Baire property for every $h \in \mathbf{R}$ then*

$$f = g + H + \phi$$

where g has the Baire property, H is additive and ϕ is null. That is, the class of functions with the Baire property has the weak difference property in category sense.

In [1] it was observed that in a similar way the weak difference property of the class of functions that are equal to a continuous function almost everywhere according to Lebesgue measure was proved in [3], the weak difference property in category sense of the class of functions with the Baire property would imply this for the class of functions that equal a continuous function $\mathcal{M}$-almost everywhere. Therefore Theorem 1.2 has the following corollary.

Corollary 1.3 *(Problem 2.1 in [1]) The class of functions that equal a continuous function $\mathcal{M}$-a.e. has the weak difference property in category sense.*

It was observed in [6] that the consistency of the difference property of the class of functions with the Baire property is also a corollary of Theorem 1.2.

Corollary 1.4 *(Problem 8.4 in [6]) It is consistent with ZFC that the class of functions with the Baire property has the difference property.*

Independently form our results this statement was proved recently in [2].

2 Preliminaries

During the proofs we will need two well known theorems. (See e.g. in [4] or [7].)

Theorem 2.1 *(Kuratowski-Ulam) Let* $k + l = n$ *and* $H \subset \mathbf{R}^n$ *be meager. Then there is an* $H_0 \subset \mathbf{R}^k$ *meager set such that*

$$\left(\{x\} \times \mathbf{R}^l\right) \cap H$$

is meager for every $x \in \mathbf{R}^k \setminus H_0$.

Conversely, if H *is of second category and has the Baire property then there is an* $H_0 \subset \mathbf{R}^k$ *set of second category such that*

$$\left(\{x\} \times \mathbf{R}^l\right) \cap H$$

is of second category for every $x \in H_0$. ■

Definition 2.2 Let H be a subset of $\mathbf{R}$ and let $I \subset \mathbf{R}$ be an open interval. We say that H is of second category everywhere in I if for every non-empty open interval $J \subset I$ the set

$$H \cap J$$

is of second category.

Theorem 2.3 *(Banach) If* $H \subset \mathbf{R}$ *is of second category then it is of second category everywhere in a suitable nonempty open interval of* $\mathbf{R}$.

We use also a classical result on difference property. Its proof can be found e.g. in [6].

Theorem 2.4 *(Stability Theorem of Hyers) Let* $f : \mathbf{R} \to \mathbf{R}$ *be such that*

$$|f(x+y) - f(x) - f(y)| \leq K$$

for a nonnegative constant K*. Then there is an additive function* H *such that*

$$|f - H| \leq K.$$

In the proofs $\mathbf{Q}, \mathbf{N}$ and $\mathbf{Z}$ will stand for the set of rationals, positive integers and integers respectively and λ will denote the Lebesgue measure. For $A, B \subset \mathbf{R}$, $h \in \mathbf{R}$,

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

will denote the symmetric difference of A and B, while

$$A + h = \{x + h : x \in A\},$$

$$A + A = \{x + y : x, y \in A\}.$$

In order to avoid the use of an extreme number of parentheses we accept the convention that

$$A \cap B + h = (A \cap B) + h.$$

For an $x \in \mathbf{R}^n$ and a positive real r the open ball centered at x with radius r will be denoted by $B(x, r)$.

For a function $f : \mathbf{R} \to \mathbf{R}$ let

$$[f < b] = \{x \in \mathbf{R} : f(x) < b\}.$$

The sets $[f > b]$, $[f \leq b]$ and $[f \geq b]$ are defined analogously.

We will also use the following notation. Let $K : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a function. For an $y \in \mathbf{R}$,

$$K^y(x) = K(x, y)$$

denotes the horizontal section of K on $\mathbf{R} \times \{y\}$. The same is defined for sets, that is for an $S \subset \mathbf{R} \times \mathbf{R}$ we use

$$S^y = S \cap (\mathbf{R} \times \{y\}).$$

3 Bounded functions

In the proof of Theorem 1.1 we will follow some ideas of M. Laczkovich in [5]. We will need some kind of "norm" in order to measure the proximity of two functions having the Baire property.

Definition 3.1 For a set $H \subset \mathbf{R}$ with the Baire property let

$$\mathcal{N}(H) = \bigcup \{I \subset \mathbf{R} : I \text{ is an open interval}$$

$$\text{and } H \text{ is of second category everywhere in } I\}.$$

It is easy to check that $\mathcal{N}(H)$ is an open set. The Banach Theorem and the Baire property of H imply that $H \Delta \mathcal{N}(H)$ is meager.

Definition 3.2 Let $f : \mathbf{R} \to \mathbf{R}$ be a periodic function with period 1 having the Baire property. Let $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ be the positive and negative parts of f. Our "norm" will be the following.

$$||f|| = \int_0^1 \lambda\left(\mathcal{N}\left([f^+ > t]\right) \cap [0,1]\right) \mathrm{d}t +$$

$$+ \int_0^1 \lambda\left(\mathcal{N}\left([f^- > t]\right) \cap [0,1]\right) \mathrm{d}t.$$

The integrals exist since the integrated functions are non-increasing. It is easy to see that the triangle inequality does not hold for $||.||$. This is the major source of technical difficulties.

Lemma 3.3 *Let $\varepsilon > 0$ be fixed and $G_1, G_2, \ldots$ be a sequence of measurable subsets of $[0,1]$ such that*

$$\lambda(G_i) > \varepsilon \text{ for } i = 1, 2, \ldots,$$

and let (h_i) be a given sequence of reals converging to zero. Consider the following $T : \mathbf{R} \to \mathbf{N} \cup \{0\} \cup \{\infty\}$ function.

$$T(x) := \sup\{k \in \mathbf{N} \cup \{0\} : \exists (j_1, \ldots, j_k) \in \mathbf{N}^k :$$

$$x \in \underbrace{((\ldots((}_{k-1} G_{j_1} + h_{j_1}) \cap G_{j_2} + h_{j_2}) \cap \ldots) \cap G_{j_k} + h_{j_k}\}.$$

Then

$$\lambda([T = \infty]) \geq \varepsilon.$$

Proof. The function T is clearly measurable. First we prove by induction that $T \le K$ is impossible for any $K \in \mathbf{N}$.

The assumption $T \le 1$ would imply

$$\bigcup_{i=1}^{\infty} G_i \cap \bigcup_{i=1}^{\infty} (G_i + h_i) = \emptyset. \tag{1}$$

On the other hand, since the sequence (h_i) converges to zero, for an I sufficiently large we have

$$\lambda \left(\left(\bigcup_{i=1}^{\infty} G_i \right) \Delta \left(\left(\bigcup_{i=1}^{\infty} G_i \right) + h_I \right) \right) < \frac{\varepsilon}{2},$$

thus using that

$$\lambda(G_I + h_I) > \varepsilon$$

we get that

$$\lambda \left((G_I + h_I) \cap \left(\bigcup_{i=1}^{\infty} G_i \right) \right) > \frac{\varepsilon}{2},$$

contradicting (1).

Let now $K \ge 2$ and suppose that $T \le K - 1$ is impossible and that $T \le K$. The assumption $T \le K$ implies

$$\bigcup_{i=1}^{\infty} G_i \subset [T \le K - 1].$$

Again, for an I large enough, for any $j > I$ we have

$$\lambda \left([T \le K-1] \Delta \left([T \le K-1] + h_j \right) \right) < \frac{\varepsilon}{2},$$

so

$$\lambda \left((G_j + h_j) \cap [T \le K - 1] \right) > \frac{\varepsilon}{2}. \tag{2}$$

For $i = 1, 2, \ldots$ let

$$\tilde{G_i} = (G_{I+i} + h_{I+i}) \cap [T \le K-1] - h_{I+i}, \tag{3}$$

$$\tilde{h_i} = h_{I+i}.$$

Then $\tilde{G_i} \subset G_{I+i}$, $\tilde{G_i}$ is measurable and by (2)

$$\lambda \left(\tilde{G_i} \right) > \frac{\varepsilon}{2}.$$

Now one can define the function $\tilde{T}$ for this sequence of sets and reals the same way as T was defined. Since $\tilde{G}_i \subset G_{I+i}$ and $\tilde{h}_i = h_{I+i}$ it is easy to see that $\tilde{T} \leq T$. From (3) we get that

$$[\tilde{T} \neq 0] \subset [T \leq K-1].$$

Thus $\tilde{T} \leq K-1$ which contradicts the induction assumption. Therefore we have proved that T cannot be bounded.

Suppose now that for a $\delta > 0$ we have

$$\lambda\left([T=\infty]\right) < \varepsilon - \delta.$$

Let K be such that

$$\lambda\left([T \geq K]\right) < \varepsilon - \frac{\delta}{2},$$

and so

$$\lambda\left(G_i \cap [T < K]\right) > \frac{\delta}{2}.$$

Again, if I is sufficiently large then for any $j > I$ we have

$$\lambda\left([T<K]\Delta\left([T<K]+h_j\right)\right) < \frac{\delta}{4},$$

so

$$\lambda((G_j \cap [T<K]+h_j) \cap [T<K]) > \frac{\delta}{4}. \tag{4}$$

We continue as above. Let

$$\tilde{G}_i = (G_{I+i} \cap [T<K] + h_{I+i}) \cap [T<K] - h_{I+i},$$

$$\tilde{h}_i = h_{I+i}.$$

Since by (4) we have

$$\lambda\left(\tilde{G}_i\right) > \frac{\delta}{4},$$

and again $\tilde{G}_i \subset G_{I+i}$ is measurable, we can define the function $\tilde{T}$ for this sequence of open sets and reals the same way as before, and we get that

$$\tilde{T} < K.$$

This contradicts the impossibility of boundedness and proves the statement. ■

Lemma 3.4 *Let $\varepsilon > 0$ be fixed and $G_1, G_2, \ldots$ be a sequence of open subsets of $[0,1]$ such that*

$$\lambda(G_i) > \varepsilon \text{ for } i = 1, 2, \ldots,$$

and let (h_i) be a given sequence of reals converging to zero. Then for any $N \in \mathbf{N}$ one can find an I_N open interval and a sequence

$$(j_1, j_2, \ldots, j_{N+1}) \in \mathbf{N}^{N+1}$$

such that

$$(5) \quad I_N \subset \underbrace{((\ldots((}_{N} G_{j_1} + h_{j_1}) \cap G_{j_2} + h_{j_2}) \cap \ldots) \cap G_{j_{N+1}} + h_{j_{N+1}}.$$

Proof. Using the notations and the statement of Lemma 3.3 we get that $[T > N]$ is non-empty. Since

$$[T > N] = \bigcup \{ \underbrace{((\ldots((}_{N} G_{j_1} + h_{j_1}) \cap G_{j_2} + h_{j_2}) \cap \ldots) \cap G_{j_{N+1}} + h_{j_{N+1}} :$$

$$(j_1, j_2, \ldots, j_{N+1}) \in \mathbf{N}^{N+1} \},$$

we have that for a $(j_1, j_2, \ldots, j_{N+1})$ the open set

$$\underbrace{((\ldots((}_{N} G_{j_1} + h_{j_1}) \cap G_{j_2} + h_{j_2}) \cap \ldots) \cap G_{j_{N+1}} + h_{j_{N+1}}$$

is also non-empty. ■

Lemma 3.5 *Let the function $f : \mathbf{R} \to \mathbf{R}$ be periodic with period 1 and suppose that $\Delta_h f$ is uniformly essentially bounded and has the Baire property for every h. Then*

$$||\Delta_{h_n} f|| \to 0 \text{ if } h_n \to 0.$$

Proof. Let $|\Delta_h f| < K$ $\mathcal{M} - a.e.$ Suppose that there is an $\varepsilon > 0$ and a sequence (h_n) converging to zero such that

$$||\Delta_{h_n} f|| > \varepsilon \ \ (\forall n \in \mathbf{N}).$$

Using that for any non-increasing function $g : [0,1] \to [0,1]$ we have

$$\int_0^1 g \le a + g(a)$$

for every $a \in [0,1]$, we get that for any $n \in \mathbf{N}$ either

$$\lambda\left(\mathcal{N}\left(\left[(\Delta_{h_n} f)^+ > \frac{\varepsilon}{4}\right]\right)\right) > \frac{\varepsilon}{4}$$

or

$$\lambda\left(\mathcal{N}\left(\left[(\Delta_{h_n} f)^- > \frac{\varepsilon}{4}\right]\right)\right) > \frac{\varepsilon}{4}.$$

By choosing a subsequence we may suppose that the same case holds for every $n \in \mathbf{N}$. The two cases can be treated on the same way, so we consider only the first one.

So suppose that

$$\lambda\left(\mathcal{N}\left(\left[(\Delta_{h_n} f)^+ > \frac{\varepsilon}{4}\right]\right)\right) > \frac{\varepsilon}{4}$$

holds for every $n \in \mathbf{N}$. Let

$$G_n = \mathcal{N}\left(\left[(\Delta_{h_n} f)^+ > \frac{\varepsilon}{4}\right]\right) = \mathcal{N}\left(\left[\Delta_{h_n} f > \frac{\varepsilon}{4}\right]\right).$$

These are open sets with $\lambda(G_n) > \frac{\varepsilon}{4}$ so by applying Lemma 3.4 for $N = \left[\frac{4K}{\varepsilon} + 1\right]$ (where $[.]$ stands for the integer part) we get a nonempty I_N open interval and a sequence $(j_1, j_2, \ldots, j_{N+1})$ with the properties in Lemma 3.4. Let

$$h = \sum_{i=1}^{N+1} h_{j_i}.$$

We claim that for $\mathcal{M}$-almost every $x \in I_N - h$ we have

$$\Delta_h f \geq N\frac{\varepsilon}{4} \geq K, \tag{6}$$

which is a contradiction. We have

$$\Delta_h f(x) = \Delta_{h_{j_{N+1}}} f(x + h_{j_1} + \ldots + h_{j_N}) +$$

$$+\Delta_{h_{j_N}} f(x + h_{j_1} + \ldots + h_{j_{N-1}}) + \ldots +$$

$$+\Delta_{h_{j_k}} f(x + h_{j_1} + \ldots + h_{j_{k-1}}) + \ldots + \Delta_{h_{j_2}} f(x + h_{j_1}) + \Delta_{h_{j_1}} f(x).$$

For every $x \in I_N - h$ we have $x + h \in I_N$, thus by (5) we get

$$x + h - h_{j_{N+1}} - \ldots - h_{j_k} \in G_{j_k}.$$

Since

$$x + h - h_{j_{N+1}} - \ldots - h_{j_k} = x + h_{j_1} + \ldots + h_{j_{k-1}},$$

by the definition of G_{j_k} we get that

$$\Delta_{h_{j_k}} f(x + h_{j_1} + \ldots + h_{j_{k-1}}) > \frac{\varepsilon}{4}\ \mathcal{M} - a.e. \text{ on } I_N - h$$

for every $1 \leq k \leq N + 1$, which implies (6). ■

The following lemma is a straightforward consequence of the Baire category theorem.

Lemma 3.6 *Let $S, T \subset [0,1] \times [0,1]$ be such that T^y and S^y are open for every $y \in [0,1]$. If $(T \setminus S)^y$ is of second category for every $y \in Y$ where Y is non-meager then there are $u, v \in \mathbf{Q}$ and an $Y' \subset Y$ of second category such that*

$$(u,v) \subset (T \setminus S)^y$$

holds for every $y \in Y'$.

Lemma 3.7 *Let $K : [0,1]^2 \to \mathbf{R}$ be a function with the following properties:*

1. *K^y has the Baire property for every $y \in [0,1]$;*
2. *$||K^{y_n} - K^y|| \to 0$ if $y_n \to y$ in $[0,1]$.*

Then there is a lower semi-continuous function $G : [0,1]^2 \to \mathbf{R}$ such that

$$K^y - G^y = 0\ \mathcal{M} - a.e.$$

holds for $\mathcal{M}$-almost every $y \in [0,1]$.

Proof. For a $q \in \mathbf{Q}$ let

$$N(q) = \bigcup_{y \in [0,1]} \mathcal{N}([K^y > q]) \times \{y\},$$

and let

$$M(q) = \bigcup \{B(x,r) : x \in \mathbf{Q} \times \mathbf{Q}, r \in \mathbf{Q},$$
$$(B(x,r) \setminus N(q))^y \text{ is meager for } \mathcal{M}\text{-almost every } y \in [0,1]\}.$$

Let

$$G(x,y) = \sup\{q \in \mathbf{Q} : (x,y) \in M(q)\}.$$

As a supremum of lower semi-continuous functions (M(q) is open) G itself is lower semi-continuous. In order to prove the statement of the Lemma it is enough to show that

$$K^y - G^y = 0\ \mathcal{M} - a.e.$$

holds for $\mathcal{M}$-almost every $y \in [0,1]$.

Consider the following relation on the subsets of $[0,1] \times [0,1]$. For $T, S \subset [0,1] \times [0,1]$ we write $T \sim S$ if

$$(T \Delta S)^y$$

is meager for $\mathcal{M}$-almost every $y \in [0,1]$. This is clearly an equivalence relation. Using this notation we shall prove that

$$[G > q] \sim [K > q]$$

for every $q \in \mathbf{Q}$. This will complete the proof, since

$$K^y - G^y \neq 0$$

on a set of second category for a non-meager set of y's would imply that

$$([G > q] \Delta [K > q])^y$$

is of second category for a non-meager set of y's with an appropriate $q \in \mathbf{Q}$.

We will prove the following equivalence chain:

$$[G > q] \sim M(q) \sim N(q) \sim [K > q].$$

We start with the middle one, that is $M(q) \sim N(q)$.

It follows easily from the definition of $M(q)$ that $(M(q) \setminus N(q))^y$ is meager for $\mathcal{M}$-almost every $y \in [0,1]$. So we have to show that $(N(q) \setminus M(q))^y$ is meager for $\mathcal{M}$-almost every $y \in [0,1]$.

Suppose that $(N(q) \setminus M(q))^y$ is not meager for a set of y's $Z_1 \subset [0,1]$ of second category. From the definition of $N(q)$ it is straightforward to see that for an ε sufficiently small even $(N(q+\varepsilon) \setminus M(q))^y$ is not meager for a set of y's $Z_2 \subset Z_1$ of second category. Using the fact that the horizontal sections of both $M(q)$ and $N(q+\varepsilon)$ are open, by Lemma 3.6 we get that there exists a $(u,v) \subset [0,1]$ open interval with $u, v \in \mathbf{Q}$ and a set $Z_3 \subset Z_2$ of second category such that

$$(u,v) \subset (N(q+\varepsilon) \setminus M(q))^y \quad \forall y \in Z_3. \tag{7}$$

By the Banach Theorem one can find an open interval U such that Z_3 is of second category everywhere in U. Let U' be the middle third of U and $Z_4 = Z_3 \cap U'$. Choose $\rho \in \mathbf{Q}$ such that

$$0 < \rho < \frac{1}{2} \min\left\{\frac{1}{2}|u-v|, \operatorname{diam}(U')\right\}.$$

Fix $x_0 = \frac{u+v}{2}$ and any $y_0 \in Z_4$. Since $(x_0, y_0) \notin M(q)$, we have that for any $y_0' \in \mathbf{Q}$ with $|y_0 - y_0'| < \rho$,

$$B((x_0, y_0'), \rho) \not\subset M(q).$$

This implies that

$$\left(B\left(\left(x_0, y_0'\right), \rho\right) \setminus N(q)\right)^y$$

is not meager for a set of y's of second category, so

$$\left(B\left(\left(x_0, y_0\right), 2\rho\right) \setminus N(q)\right)^y$$

is not meager for a set of y's $V_1 \subset U$ of second category either, since

$$\left(B\left(\left(x_0, y_0'\right), \rho\right) \setminus N(q)\right)^y \subset \left(B\left(\left(x_0, y_0\right), 2\rho\right) \setminus N(q)\right)^y.$$

Again by Lemma 3.6, since the horizontal sections of $B((x_0, y_0), 2\rho)$ and $N(q)$ are open, we get that there is an open interval $(u', v') \subset (u, v)$ and a set $V_2 \subset V_1$ of second category such that

$$(u', v') \times V_2 \subset B((x_0, y_0), 2\rho) \setminus N(q). \tag{8}$$

Now we have a set Z_3 dense in U and a set V_2 somewhere dense in U. So one can take a sequence $(y_n) \subset V_2$ and an $y \in Z_3$ with $y_n \to y$. From (7) we have that $K^y > q + \varepsilon$ holds $\mathcal{M} - a.e.$ on $(u', v') \subset (u, v)$ and from (8) we get that $K^{y_n} \leq q$ holds $\mathcal{M} - a.e.$ on (u', v'). Hence

$$[K^y - K^{y_n} < \varepsilon] \cap (u', v')$$

is meager, so

$$\lambda(\mathcal{N}([K^y - K^{y_n}] \geq \delta)) \geq \lambda((u', v'))$$

for every $0 < \delta < \varepsilon$, thus

$$||K^y - K^{y_n}|| \geq \varepsilon\lambda\left((u', v')\right)$$

for every $n \in \mathbf{N}$. This is a contradiction, hence $(N(q) \setminus M(q))^y$ is meager for $\mathcal{M}$-almost every $y \in [0, 1]$ and so $M(q) \sim N(q)$.

The equivalence

$$N(q) \sim [K > q]$$

easily follows form the definitions.

Finally we prove $[G > q] \sim M(q)$. For any $q \in \mathbf{Q}$ we have

$$[G > q] = \{(x, y) \in [0, 1]^2 | \exists r > q : (x, y) \in M(r)\} = \bigcup_{r>q} M(r).$$

Since

$$M(r) \sim N(r)$$

for every $r \in \mathbf{Q}$, we get

$$\bigcup_{r>q} M(r) \sim \bigcup_{r>q} N(r).$$

Since

$$N(q) \sim [K > q]$$

for every $q \in \mathbf{Q}$ and

$$[K > q] = \bigcup_{r>q} [K > r],$$

we have that

$$N(q) \sim [K > q] \sim \bigcup_{r>q} [K > r] \sim \bigcup_{r>q} N(r).$$

So we get

$$M(q) \sim N(q) \sim \bigcup_{r>q} N(r) \sim \bigcup_{r>q} M(r) \sim [G > q],$$

as stated.

The proof of the lemma is complete.■

Proof of Theorem 1.1 We can suppose that f is periodic with period 1. Indeed, let $f_1(x) = f(\{x\})$ where $\{x\}$ stands for the fractional part of x. Then $f - f_1$ is essentially bounded and has the Baire property since the difference functions of f are uniformly essentially bounded and for $n \le x < n + 1$ we have

$$f_1 - f = f(x - n) - f(x) = \Delta_{-n} f.$$

This implies that the difference functions of f_1 are also uniformly essentially bounded, while

$$\Delta_h f_1(x) = f_1(x + h) - f_1(x) =$$

$$= [f_1(x+h) - f(x+h)] + [f(x+h) - f(x)] + [f(x) - f_1(x)]$$

shows the Baire property for every h. If $f_1 = g_1 + \phi$ where g_1 has Baire property and ϕ is null then we have $f = g + \phi$, where $g = (f - f_1) + g_1$ also has the Baire property.

Suppose now that f is periodic with period 1 and let

$$K(x, y) = f(x+y) - f(x).$$

The equality

$$K^{y_n}(x) - K^y(x) = f(x + y_n) - f(x+y) = \Delta_{y_n - y} f(x+y)$$

and Lemma 3.5 imply that

$$||K^{y_n} - K^y|| \to 0 \text{ if } y_n \to y.$$

Thus using the periodicity of K and Lemma 3.7 we get a

$$G : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

Borel function such that $K^y - G^y = 0 \ \mathcal{M} - a.e.$ holds for $\mathcal{M}$-almost every $y \in \mathbf{R}$.

Let

$$S(x, y) = K(x, y) - G(x, y) = f(x+y) - f(x) - G(x, y).$$

We have that $S(x, y) = 0$ holds $\mathcal{M}$-a.e. in x for $\mathcal{M}$-almost every fixed $y \in \mathbf{R}$. By the definition of S we have

$$S(x, y+z) = f(x+y+z) - f(x) - G(x, y+z),$$

$$-S(x+y, z) = -f(x+y+z) + f(x+y) + G(x+y, z)$$

and

$$-S(x, y) = -f(x+y) + f(x) + G(x, y).$$

By adding and using

$$L(x, y, z) = S(x, y+z) - S(x+y, z) - S(x, y)$$

we get that

$$L(x, y, z) = -G(x, y+z) + G(x+y, z) + G(x, y),$$

so L, as a difference of Borel functions, has the Baire property. Hence the fact that for $\mathcal{M}$-almost every fixed $z \in \mathbf{R}$ and for $\mathcal{M}$-almost every fixed $y \in \mathbf{R}$ we have $L(x, y, z) = 0$ $\mathcal{M}$-a.e. in x implies by the converse part of the Kuratowski-Ulam Theorem that

$$L(x, y, z) = 0 \ \mathcal{M} - a.e. \text{ in } \mathbf{R}^3.$$

The Kuratowski-Ulam Theorem tells us that there exists a point x_0 such that for $\mathcal{M}$-almost every $z \in \mathbf{R}$ we have

$$L(x_0, y, z) = 0$$

for $\mathcal{M}$-almost every $y \in \mathbf{R}$. However, for $\mathcal{M}$-almost every $z \in \mathbf{R}$ also $S(x_0 + y, z) = 0$ holds $\mathcal{M}$-a.e. in y, so for $\mathcal{M}$-almost every fixed $z \in \mathbf{R}$ we have

$$L(x_0, y, z) + S(x_0 + y, z) =$$

$$= S(x_0, y + z) - S(x_0, y) = 0 \text{ for } \mathcal{M} - a.e. \ y \in \mathbf{R}. \tag{9}$$

Let Z denote the residual set from where z can be chosen in (9). Since for every $h \in \mathbf{R}$ we have $Z \cap (h - Z) \neq \emptyset$, there are $z_1, z_2 \in Z$ such that $h = z_1 + z_2$. Therefore for every $h \in \mathbf{R}$ we have that

$$S(x_0, y + h) - S(x_0, y) = \tag{10}$$

$$= [S(x_0, y + z_1 + z_2) - S(x_0, y + z_2)] + [S(x_0, y + z_2) - S(x_0, y)] = 0$$

holds for $\mathcal{M}$-almost every $y \in \mathbf{R}$.

Now we can define g and ϕ. By the definition of S we have

$$S(x_0, y) = f(x_0 + y) - f(x_0) - G(x_0, y),$$

so

$$f(y) = S(x_0, y - x_0) + f(x_0) + G(x_0, y - x_0).$$

Let

$$g(y) = f(x_0) + G(x_0, y - x_0)$$

and

$$\phi(y) = S(x_0, y - x_0).$$

The function g – as a section of the Borel function G – obviously has the Baire property. The function ϕ is null, since by (10)

$$\Delta_h \phi(y) = S(x_0, y + h - x_0) - S(x_0, y - x_0) = 0$$

holds for $\mathcal{M}$-a.e. $y \in \mathbf{R}$. ■

4 Unbounded functions

Theorem 1.1 allows us to find a decomposition $f = g + \phi$ if it is guaranteed on some way that f contains no additive function. In the following our goal is to find the additive function in a general f.

Definition 4.1 Let $g : \mathbf{R} \to \mathbf{R}$ have the Baire property. For an $I \subset \mathbf{R}$ open interval we say that *the induced oscillation of g is less than D in I*, if there exists an $a \in \mathbf{R}$ such that $I \subset \mathcal{N}([|g-a| < D])$. For an $x \in \mathbf{R}$ we say that *the induced oscillation of g is less than D in x*, if there is an I open interval with $x \in I$ such that the induced oscillation of g is less than D in I. In this case we say that the fact that the induced oscillation of g in x is less than D is *witnessed by I*.

We will use

$$X_D(g) = \{x \in \mathbf{R} : \text{ the induced oscillation of } g \text{ in } x \text{ is less than } D\}.$$

Lemma 4.2 *For every $g : \mathbf{R} \to \mathbf{R}$ with the Baire property and $D > 0$ the set*

$$\mathbf{R} \setminus X_D(g)$$

is nowhere dense.

Proof. The open set

$$\bigcup_{a \in \mathbf{Q}} \mathcal{N}([|g-a| < D])$$

is contained in $X_D(g)$, so it is enough to prove that it is dense. Consider an I open interval. By the Baire category theorem for some $a \in \mathbf{Q}$ the set

$$[|g-a| < D] \cap I$$

is of second category, which – using the Baire property of g – implies

$$\mathcal{N}([|g-a| < D]) \cap I \neq \emptyset. \blacksquare$$

Definition 4.3 Let $f : \mathbf{R} \to \mathbf{R}$ be such that $\Delta_h f$ has the Baire property for every $h \in \mathbf{R}$, let $D > 0$ and consider the following relation on $\mathbf{R}$.

$$y \sim_D z \text{ if } y \in X_D(\Delta_{z-y} f).$$

For an open interval I with $y \in I$ we say that $y \sim_D z$ is *witnessed by I* if $y \in X_D(\Delta_{z-y} f)$ is witnessed by I.

That is, $y \sim_D z$ if the induced oscillation of

$$f(t+z) - f(t+y)$$

is less than D in $t = 0$. This relation is symmetric and reflexive, but not transitive. It is easy to check that instead of transitivity we have the following property.

Lemma 4.4 *If $t \sim_D y, t \sim_D z$ are witnessed by $B(t, \varepsilon)$ then $y \sim_{2D} z$ and this is witnessed by $B(y, \varepsilon)$.*

We also have the following property.

Lemma 4.5 *If $y \sim_D t, y \sim_D \tau$ are witnessed by $B(y, \varepsilon)$ and $|t - \tau| < \delta < \varepsilon$ then $y \sim_{2D} y + t - \tau$ and this is witnessed by $B(y, \varepsilon - \delta)$.*

Proof. By definition, $y \sim_D t$, $y \sim_D \tau$ witnessed by $B(y, \varepsilon)$ means that the induced oscillation of $\Delta_{t-y} f(x)$ and $\Delta_{\tau-y} f(x)$ is less than D in $B(y, \varepsilon)$. So from $|t - \tau| < \delta$ we get that the induced oscillation of $\Delta_{t-y} f(x + \tau - t)$ in $B(y, \varepsilon - \delta)$ is also less than D. Using

$$\Delta_{\tau-y} f(x) - \Delta_{t-y} f(\tau - t + x) = \Delta_{\tau-t} f(x)$$

we get that the induced oscillation of $\Delta_{\tau-t} f(x)$ is less than $2D$ in $B(y, \varepsilon - \delta)$, as required. ■

Definition 4.6 Let $f : \mathbf{R} \to \mathbf{R}$ be such that $\Delta_h f$ has the Baire property for every $h \in \mathbf{R}$ and let $D > 0$. For a $t \in \mathbf{R}$ let

$$E_{f,D}(t) = \{y \in \mathbf{R} : t \sim_D y\}.$$

Lemma 4.7 *Let $(x_i) \subset \mathbf{R}$. If*

$$E = \bigcup_{i=1}^{\infty} E_{f,D}(x_i)$$

is dense in an interval then

$$F = \bigcup_{i=1}^{\infty} E_{f,2D}(x_i)$$

is of second category everywhere in $\mathbf{R}$.

Proof. Suppose that E is dense in an open interval I but F is meager in an open interval J. We can suppose $\lambda(I) = \frac{\lambda(J)}{2}$. Let $(e_j) \subset E$ be also dense in I. Then one can take a translation with an $h \in \mathbf{R}$ such that

$$(e_j) + h \subset (J \setminus F),$$

that is $(E+h) \cap (J \setminus F)$ is dense in J. We claim that for every $t \in (E+h) \cap (J \setminus F) - h$ the induced oscillation of $\Delta_h f$ is not less than D in t. Indeed, since an induced oscillation of $\Delta_h f$ less than D in t would imply $t \sim_D (t+h)$ so if $t \sim_D x_i$ (such an x_i exists by $t \in E$) then by Lemma 4.4 we would have $x_i \sim_{2D} t+h$ and $t+h \in F$ which is not true. Since $(E+h) \cap (J \setminus F) - h$ is dense in an interval this contradicts Lemma 4.2. ■

Lemma 4.8 *Let $g : \mathbf{R} \to \mathbf{R}$ have the Baire property. Suppose that for a $\delta > 0$, a fixed nonnegative constant K and a dense subset Z of $B(0, 2\delta)$ we have that for every $z \in Z$*

$$|\Delta_z g| < K$$

holds $\mathcal{M}-a.e.$ in $B(q_0, \delta)$ with a $q_0 \in \mathbf{R}$. Then the induced oscillation of g in $B(q_0, \delta)$ is less than K.

Proof. Suppose that the induced oscillation of g in $B(q_0, \delta)$ is not less than K. Then there is an $a \in \mathbf{R}$ such that both $[g - a > \frac{3K}{4}]$ and $[g - a < -\frac{3K}{4}]$ is of second category in $B(q_0, \delta)$. The Baire property of g implies that

$$\mathcal{N}\left(\left[g - a > \frac{3K}{4}\right]\right) \cap B(q_0, \delta)$$

and

$$\mathcal{N}\left(\left[g - a < -\frac{3K}{4}\right]\right) \cap B(q_0, \delta)$$

are nonempty, so by the density of Z in $B(0, 2\delta)$ there is a $z \in Z$ and an $x \in \mathcal{N}([g - a < -\frac{3K}{4}])$ such that $x + z \in \mathcal{N}([g - a > \frac{3K}{4}])$. From this and the Baire property of g we have that

$$|\Delta_z g| > \frac{3K}{2}$$

holds $\mathcal{M} - a.e.$ in the non-empty open set

$$\left(\mathcal{N}\left(\left[g - a > \frac{3K}{4}\right]\right) - z\right) \cap \mathcal{N}\left(\left[g - a < -\frac{3K}{4}\right]\right).$$

This is a contradiction. ■

Proof of Theorem 1.2 Let $f : \mathbf{R} \to \mathbf{R}$ be such that $\Delta_h f$ has the Baire property for every $h \in \mathbf{R}$. By the reflexivity of $\sim_D$, the set

$$\bigcup_{q \in \mathbf{Q}} E_{f,1}(q)$$

contains $\mathbf{Q}$, so it is dense. Then, by Lemma 4.7 for $(x_i) = \mathbf{Q}$, $D = 1$, there is a $q_0 \in \mathbf{Q}$ such that $E_{f,2}(q_0)$ is of second category.

Since

$$E_{f,2}(q_0) = \bigcup_{n \in \mathbf{N}} \left\{ x \in \mathbf{R} : q_0 \sim_2 x \text{ is witnessed by } B\left(q_0, \frac{1}{n}\right)\right\},$$

there exists an $n \in \mathbf{N}$ such that the set

$$E^n = \left\{ x \in \mathbf{R} : q_0 \sim_2 x \text{ is witnessed by } B\left(q_0, \frac{1}{n}\right)\right\}$$

is of second category. By the Banach Theorem one can find an open interval $B(x_0, 24\rho)$ in which E^n is of second category everywhere. We can suppose $24\rho < \frac{1}{n}$. Let $E = E^n \cap B(x_0, 24\rho)$.

From Lemma 4.5 for every $t, \tau \in E$ with $|t - \tau| < 8\rho$ we have $q_0 \sim_4 q_0 + \tau - t$ witnessed by $B(q_0, 16\rho)$. So the induced oscillation of $\Delta_{\tau - t} f(x)$ is less than 4 on $B(q_0, 16\rho)$. This implies that for every $y \in B(q_0, 8\rho)$ we have

$$|\Delta_{y-q_0} \Delta_{\tau-t} f(x)| < 8 \quad \mathcal{M} - a.e.$$

on $B(q_0, 8\rho)$. Since

$$\Delta_{y-q_0} \Delta_{\tau-t} f(x) = \Delta_{\tau-t} \Delta_{y-q_0} f(x),$$

we have

$$|\Delta_{\tau-t} \Delta_{y-q_0} f(x)| < 8 \quad \mathcal{M} - a.e. \tag{11}$$

on $B(q_0, 8\rho)$ for every $t, \tau \in E$ with $|t - \tau| < 8\rho$. Since $\Delta_{y-q_0} f$ has the Baire property, $E - E$ is dense in $B(0, 8\rho)$ and (11) shows that

$$|\Delta_z \Delta_{y-q_0} f(x)| < 8$$

holds $\mathcal{M}$-a.e. in $B(q_0, 4\rho)$ for every $z \in (E - E) \cap B(0, 8\rho)$, we can apply Lemma 4.8 for $Z = (E - E) \cap B(0, 8\rho)$, $g = \Delta_{y-q_0} f(x)$,

$\delta = 4\rho$ and $K = 8$. We get that the induced oscillation of $\Delta_{y-q_0} f(x)$ on $B(q_0, 4\rho)$ is less than 8, that is for every $y \in B(q_0, 4\rho)$ we have $q_0 \sim_8 y$ witnessed by $B(q_0, 4\rho)$.

Thus by Lemma 4.4 we get that

$$\forall x, y \in B(q_0, 4\rho)\ x \sim_{16} y \text{ and witnessed by } B(x, 4\rho). \tag{12}$$

Let f_1 be the periodic extension of $f|_{[q_0-\frac{\rho}{2}, q_0+\frac{\rho}{2})}$, that is for an $x \in [q_0 - \frac{\rho}{2}, q_0 + \frac{\rho}{2})$, $l \in \mathbf{Z}$ let

$$f_1 (x + l\rho) = f (x) .$$

We show that for every $h \in \mathbf{R}$ the induced oscillation of $\Delta_h f_1$ is essentially bounded with a fixed nonnegative constant K.

Let

$$W = \{z\rho : z \in \mathbf{Z}\},$$

and let $h \in \mathbf{R}$. Since $\Delta_h f_1(t)$ is also periodic with period ρ, it is enough to verify its boundedness while $t \in [q_0 - \frac{\rho}{2}, q_0 + \frac{\rho}{2})$.

For $h \in W$ we have $\Delta_h f_1 = 0$, which is clearly bounded. For $h \notin W$ let

$$\tilde{h} = h - \rho \left[\frac{h}{\rho}\right]$$

where [.] stands for the integer part. For $q_0 - \frac{\rho}{2} \le t < q_0 + \frac{\rho}{2} - \tilde{h}$ we have

$$\Delta_h f_1(t) = \Delta_{\tilde{h}} f(t).$$

By (12) for $x = t + \tilde{h}$ and $y = t$ we have that in $[q_0 - \frac{\rho}{2}, q_0 + \frac{\rho}{2})$ the induced oscillation of $\Delta_{\tilde{h}} f(t)$ is less than 16. For $q_0 + \frac{\rho}{2} - \tilde{h} \le t < q_0 + \frac{\rho}{2}$ we have

$$\Delta_h f_1(t) = \Delta_{\tilde{h}-\rho} f(t)$$

and by (12) for $x = t + \tilde{h} - \rho$ and $y = t$ we get that the induced oscillation of $\Delta_{\tilde{h}-\rho} f(t)$ in $[q_0 - \frac{\rho}{2}, q_0 + \frac{\rho}{2})$ is also less than 16.

So if we show that $\Delta_{\tilde{h}} f(t) - \Delta_{\tilde{h}-\rho} f(t)$ is essentially bounded for $t \in [q_0 - \frac{\rho}{2}, q_0 + \frac{\rho}{2})$ by a fixed nonnegative constant R not depepnding on h, then we get that the induced oscillation of $\Delta_h f_1$ on $[q_0 - \frac{\rho}{2}, q_0 + \frac{\rho}{2})$ is less than $K = R + 16 + 16$.

To see this, observe first that

$$\Delta_{\tilde{h}} f(t) - \Delta_{\tilde{h}-\rho} f(t) = \Delta_\rho f(t + \tilde{h} - \rho), \tag{13}$$

a difference function of f with the fixed difference ρ not depending on h. By applying (12) for $x = t + \tilde{h}$ and $y = t + \tilde{h} - \rho$ we have that this difference function $\Delta_\rho f$ has bounded oscillation on $B(q_0, \frac{\varrho}{2})$, so it is essentially bounded here by a fixed nonnegative constant R not depending on h. Thus by (13),

$$\Delta_{\tilde{h}} f(t) - \Delta_{\tilde{h}-\rho} f(t)$$

is also essentially bounded on $B(q_0, \frac{\varrho}{2})$ by the fixed nonnegative constant R not depending on h. This implies that the induced oscillation of $\Delta_h f_1$ is essentially bounded with a fixed nonnegative constant K.

It is an easy computation that $f_1 - f$ and $\Delta_h f_1$ has the Baire property for every $h \in \mathbf{R}$ (see the beginning of the proof of Theorem 1.1).

Since the induced oscillation of $\Delta_h f_1$ is at most K, one can find a $D\,(h)$ "mean value" of $\Delta_h f_1$, that is a real such that

$$\mathcal{N}\left(\left[\left|\Delta_h f_1 - D\,(h)\right| < K\right]\right) = \mathbf{R}.$$

Thus for any $h \in \mathbf{R}$ the set

$$[|\Delta_h f_1 - D(h)| > K]$$

is meager, so for any fixed $u, v \in \mathbf{R}$ the set

$$\Big[\,|\,(D\,(u+v) - \Delta_{u+v} f_1) - (D(u) - \Delta_u f_1\,(Id + v)) -$$

$$- (D\,(v) - \Delta_v f_1)\,| > 3K]$$

is also meager.

On the other hand,

$$D\,(u+v) - D\,(u) - D\,(v) =$$

$$= (D\,(u+v) - \Delta_{u+v} f_1) - (D(u) - \Delta_u f_1\,(Id + v)) - (D\,(v) - \Delta_v f_1)$$

is constant for every fixed $u, v \in \mathbf{R}$, so we have

$$|D\,(u+v) - D\,(u) - D\,(v)\,| \leq 3K.$$

According to the Stability Theorem of Hyers this implies

$$D = d + H$$

where d is a bounded function and H is additive. Let $l = f_1 - H$. Since

$$\Delta_h l = \Delta_h f_1 - H(h) =$$
$$= \Delta_h f_1 - D(h) + d(h),$$

$\Delta_h l$ is uniformly essentially bounded and has the Baire property for every $h \in \mathbf{R}$, so by Theorem 1.1

$$l = k + \phi$$

where k has the Baire property and ϕ is null.

With $g = f - f_1 + k$ finally we have $f = g + H + \phi$ where g has the Baire property, H is additive and ϕ is null. This completes the proof. ■

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