On Perturbations of Eventually Compact Semigroups Preserveing Eventual Compactness

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Abstract

We show, within the context of Miyadera–Voigt perturbations on L^p spaces, that for partial differential equations with delay arising from an immediately compact undelayed equation the delay semigroup is eventually compact. We also indicate how this result can be applied to the spectral analysis and numerical treatment of these equations.

1 Introduction

In this paper we continue the investigations of András Bátkai and Susanna Piazzera in [1] on the semigroup approach to partial differential equations with delay in the state space of Banach space valued L^p functions. Their results in particular establish its equivalence to a corresponding abstract Cauchy problem under sufficiently mild assumptions. Just like in numerous similar cases (see e.g. [5], [10] and [14]), this allows the study of a wide class of delay equations through semigroup theory (see also [11] for a detailed study).

In this paper we prove the eventual compactness of the delay semigroup for partial differential equations with delay arising from immediately compact undelayed equations. The novelty of this result is in the choice of the state space L^p. Namely, the analogous result for Banach space valued continuous functions has been proven in [15] (Chapter 8.3, page 286). However, the state space L^p turns out to be more convenient for many applications (see e.g. [7], Chapter VI, Section 8 on control theory, or the Preface of [2]). In Section 5, following [1] we present a wide class of delay equations satisfying the assumptions of our abstract results.

The consequences of the eventual compactness are twofold. First, it opens the way to the application of the very sophisticated and elaborated methods of spectral analysis of compact operators, which in turn gives information about the stability properties of the semigroup ([7], Chapter

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V, Section 1). Second, it gives an a priori estimate on the distribution of the solution functions which can be helpful for the numerical treatment of partial differential equations with delay in L^p spaces. We will expose these applications in Section 5.

2 Posing the problem

On the Banach space X, we consider the equation

$$\begin{aligned} &u'(t) = Au(t) + \Phi u_t,\ t \geq 0,\\ &u(0) = x,\\ &u_0 = f, \end{aligned} \tag{DE}$$

where

- $x \in X$;
- $A : D(A) \subset X \to X$ is a closed and densely defined operator;
- $f \in L^p([-1,0], X)$ for a fixed $1 \leq p < \infty$;
- $\Phi : W^{1,p}([-1,0], X) \to X$ is a bounded linear operator;
- $u : [-1, \infty) \to X$, and $u_t : [-1,0] \to X$ is defined by $u_t(\sigma) = u(t+\sigma)$, $\sigma \in [-1,0]$.

We investigate the solutions of (DE) through the abstract Cauchy problem

$$\begin{aligned} &\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t),\\ &\mathcal{U}(0) = \begin{pmatrix} x \\ f \end{pmatrix}, \end{aligned} \tag{ACP}$$

defined on the Banach space

$$\mathcal{E} := X \times L^p([-1,0], X),$$

$$\left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{E}} := \max\{\|x\|_X, \|f\|_{L^p}\}$$

by the unbounded operator

$$\mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1,0], X) \;\middle|\; f(0) = x \right\}.$$

The "equivalence" of (DE) and (ACP) has been established in [1] (Proposition 2.1, Proposition 2.2).

Proposition 2.1. *Let* $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A})$. *Then* $u : [-1,\infty) \to X$ *is a solution of* (DE) *with initial values* x *and* f *if and only if* $\mathcal{U} : [0,\infty) \to \mathcal{E}$,

$$\mathcal{U}(t) = \begin{pmatrix} u(t) \\ u_t \end{pmatrix}$$

is a classical solution of (ACP) *with initial value* $\begin{pmatrix} x \\ f \end{pmatrix}$.

It turned out that it is worthwhile to examine the properties of (ACP) within the concept of Miyadera–Voigt perturbation by considering the unbounded operator

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix}$$

with domain $D(\mathcal{A}_0) = D(\mathcal{A})$ and the perturbing operator

$$\mathcal{B} := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(D(\mathcal{A}_0), \mathcal{E}). \tag{1}$$

By the general theory of Miyadera-Voigt Perturbation (see e.g. [7], Chapter VI, Section 6 or more specially [2], Part II, Section 3.3.2) one has the following. If $(A, D(A))$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X, then $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on $\mathcal{E}$ given by

$$\mathcal{T}_0(t) = \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix}, \tag{2}$$

where $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $L^p([-1,0],X)$ and $S_t : X \to L^p([-1,0],X)$ is defined by

$$(S_t x)(\sigma) = \begin{cases} S(t+\sigma)x, & -t \leq \sigma \leq 0, \\ 0, & -1 \leq \sigma < -t. \end{cases} \tag{3}$$

Moreover, if there exist $t_0 > 0$ and $q < 1$ such that

$$\int_0^{t_0} \|\Phi(S_s x + T_0(s) f)\|_X \,\mathrm{d}s \leq q \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{E}} \tag{4}$$

for every $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0)$, then the conditions of the Miyadera–Voigt perturbation theorem are satisfied, and the perturbed operator $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{E}$.

Smallness conditions like (4) will play an important role for our compactness results. Using the notations of [1], we recall two such conditions.

Definition 2.2. We say that Φ satisfies *condition* (M) if

$$\int_0^{t} \|\Phi(S_s x + T_0(s) f)\|_X \,\mathrm{d}s \leq q(t) \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{E}} \tag{M}$$

for every $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0)$, $t > 0$, and appropriate constants $q(t) \in \mathbb{R}$.

We say that Φ satisfies *condition* (K) if (M) holds and $q(t) \to 0$ as $t \to 0$.

Observe that (M) holds whenever the sufficient condition (4) for the existence of Miyadera–Voigt perturbation is satisfied.

Our investigations are concerned with the compactness of semigroups. We recall the relevant notions.

Definition 2.3. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Banach space X is called *eventually compact* if there is a $t_0 > 0$ such that the function $t \mapsto T(t)$ is compact operator valued as a mapping from (t_0, ∞) to $\mathcal{L}(X)$. The semigroup is *immediately compact* if $t_0 = 0$ can be chosen.

The present paper is devoted to prove that the immediate compactness of $(S(t))_{t\geq 0}$ on X implies the eventual compactness of $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{E}$. Here is our main result.

Theorem 2.4. *Let $1 \leq p < \infty$ be fixed and consider the abstract Cauchy problem (ACP). Suppose that $(A, D(A))$ generates an immediately compact strongly continuous semigroup $(S(t))_{t\geq 0}$ on X and Φ satisfies condition (M) with $q(t_0) < 1$ for some $t_0 > 0$. Then the strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{E}$ is compact for $t > 1$.*

On the way to obtain this result, we will prove several compactness theorems under various assumptions (see Theorem 4.1). The sufficiency of condition (M) in these results seems to be new; among other things, this is why one can prove Theorem 2.4 for $p = 1$, as well (compare to Proposition 5.2 in [1]). Lemma 4.2 contains the key estimate, which might be useful in other situations involving similar perturbation problems. We will also discuss the role of condition (K).

We remark here that it is known that the immediate compactness for $(S(t))_{t\geq 0}$ in Theorem 2.4 cannot be weakened to eventual compactness (see e.g. [3], Example 5.1, page 374). Note also that in general the Miyadera–Voigt Perturbation of eventually compact semigroups is not necessarily eventually compact. Therefore, since the unperturbed semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ is *not* immediately compact, our situation is quite peculiar.

3 Preliminaries and Notations

For the norm unit ball of a Banach space X we use $B(X)$, while $B(F, \varepsilon)$ stands for the ε-neighborhood of a $F \subset X$ for $\varepsilon > 0$.

Integrals on $\mathbb{R}$ are to be understood in Bochner sense according to the canonical Lebesgue measure. The Banach space of p–integrable X–valued functions on the compact interval $I \subset \mathbb{R}$ is defined also with respect to the Lebesgue measure and is denoted by $(L^p(I, X), \|.\|_{L^p(I,X)})$. If a function f is defined on an interval $I \subset \mathbb{R}$, then we write $f\Big|_I$ for the restriction of f onto I; we will use this to simplify our notation while carrying out norm estimates of functions on parts of their domains. To shorten the expressions, we do not indicate the restriction onto $[-1, 0]$ for functions in the argument of Φ.

Our reference on Miyadera–Voigt Perturbation is [7] (Lemma III.1.13 and Corollary III.3.15). That is, if $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on the Banach space X and the operator $(C, D(C))$ satisfies the conditions of the perturbation theorem of Miyadera and Voigt, then the perturbed semigroup $(U(t))_{t\geq 0}$ is given by the Dyson-Phillips series

$$U(t)x = \sum_{n=0}^{\infty} (V^n T)(t)x, \tag{5}$$

where

$$(VF)(t)x = \int_0^t F(t-s)CT(s)x\mathrm{d}s \tag{6}$$

is defined for every $x \in D(G)$ and extended continuously onto X.

Moreover, if $((V^n T)(t))_{t\geq 0}$ is immediately compact for some $n \in \mathbb{N}$ and $((V^j T)(t))_{t\geq 0}$ is eventually compact for every $0 \leq j < n$, then $((V^m T)(t))_{t\geq 0}$ is immediately compact for every $m \geq n$, so $(U(t))_{t\geq 0}$ is eventually compact.

Since $\mathcal{E}$ is a product space, the definition of compactness of the appropriate operators will be checked coordinate wise. To verify relative compactness in $L^p([-1,0], X)$, we will make use of the following generalization of the well-known Compactness Theorem of Kolmogorov and Riesz ([12], Theorem 1, page 66).

Theorem 3.1. *Let $I \subset \mathbb{R}$ be a compact interval, X be a Banach space and consider a class of functions $\mathcal{F} \subset L^p(I, X)$ for some fixed $1 \leq p < \infty$. Then $\mathcal{F}$ is relatively compact if and only if*

(a) $\mathcal{F}$ is uniformly continuous, that is

$$\int_{I\cap(I-h)} \|f(s+h) - f(s)\|_X^p \, ds \to 0 \text{ if } h \to 0,$$

uniformly in $\mathcal{F}$;

(b) for every interval $D \subset I$, the set

$$\left\{ \int_D f(s)\, ds \;\middle|\; f \in \mathcal{F} \right\} \subset X$$

is relatively compact.

We need that the integral of an integrable compact operator valued map is a compact operator. This is contained in Theorem 1.3 of [13].

Theorem 3.2. *Let X be a Banach space, $[a,b] \subset \mathbb{R}$ be a compact interval. If $S : (a,b] \to \mathcal{L}(X)$ is a bounded measurable function, then the function $V : [a,b] \to \mathcal{L}(X)$,*

$$V(t) = \int_a^t S(a+t-s)\, ds$$

is norm continuous on $[a,b]$.

If, in addition, S is continuous and compact operator valued in (a,b), then for every measurable set $D \subset [a,b]$, the operator

$$\int_D S(s)\,ds \in \mathcal{L}(X)$$

is compact.

Finally we state here the important fact that immediate norm continuity follows from immediate compactness (see e.g. Lemma II.4.22 in [7]).

Theorem 3.3. *Immediately compact semigroups are immediately norm continuous.*

4 Eventual compactness

We will prove the following theorem. As usual, $\mathrm{Pr}_X : \mathcal{E} \to X$ and $\mathrm{Pr}_{L^p} : \mathcal{E} \to L^p([-1,0],X)$ stand for the projection from $\mathcal{E}$ onto its first and second coordinate, respectively.

Theorem 4.1. *Let X, $\mathcal{E}$, $(A, D(A))$, $(S(t))_{t\geq 0}$, $(\mathcal{T}_0(t))_{t\geq 0}$, $(\mathcal{T}(t))_{t\geq 0}$ and Φ be as in Theorem 2.4 for some fixed $1 \leq p < \infty$, with a Φ satisfying (M). Let V denote the operator defined by (6). If $(S(t))_{t\geq 0}$ is immediately compact, then*

1. *the operator $\mathcal{T}_0(t)$ is compact for $t > 1$;*
2. $\mathrm{Pr}_{L^p}(V\mathcal{T}_0)(t)$ *is compact for $t > 0$;*
3. *the operator $\mathrm{Pr}_X(V\mathcal{T}_0)(t)$ is compact for $t > 1$;*
4. *the operator $(V^2\mathcal{T}_0)(t)$ is compact for $t > 0$.*
5. *If Φ satisfies condition (K), then the operator $\mathrm{Pr}_X(V\mathcal{T}_0)(t)$ is compact for $t > 0$.*

From this, Theorem 2.4 immediately follows.

Proof of Theorem 2.4. By Corollary III.3.16 in [7], we can apply the Miyadera–Voigt perturbation theorem for

$$T(t) = \mathcal{T}_0(t) = \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix},$$

$$C = \mathcal{B} = \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}.$$

From Theorem 4.1, 1., 2. and 3. we have that $\mathcal{T}_0(t)$ and $(V\mathcal{T}_0)(t)$ are compact for $t > 1$, while Theorem 4.1, 4. gives that $(V^2\mathcal{T}_0)(t)$ is immediately compact. So the perturbed semigroup $(\mathcal{T}(t))_{t\geq 0}$ is compact for $t > 1$, which proves the statement.■

Before starting our quest for compactness, we show how (M) implies a weak version of (K). In short, the following lemma says that the $\|x\|_X$ on the right hand side of (M) can be replaced with $t\|x\|_X$.

Lemma 4.2. *If Φ satisfies (M), then*

$$\int_0^t \|\Phi(S_s x + T_0(s)f)\|_X \, ds \leq$$
$$\leq 2t\left((M+1)\|x\|_X + \|f\|_\infty\right)\|\Phi\|_{\mathcal{L}(W^{1,1},X)} +$$
$$+ q(t)\|f\|_{L^1} + q(t)2t\left(M\|x\|_X + \|f\|_\infty\right) \leq$$
$$\leq t\Lambda\left(\|x\|_X + \|f\|_\infty\right) + q(t)\|f\|_{L^1},$$

where M denotes the norm bound of $S(s)$ on $[0,t]$ and $q(t)$ is the function in the definition of condition (M).

Later on, we will use only the constant $\Lambda := \Lambda(t, M, \Phi)$ to shorten the resulting expressions.

Proof. Our strategy is to replace $S_s x + T_0(s)f$ by a function which is zero in 0. To this end, for a sufficiently large positive integer N (its value will be determined later), we define our cutoff function as

$$h(\sigma) := \begin{cases} 1, & \text{if } \sigma < -\frac{t}{N}; \\ -\frac{N}{t}\sigma, & \text{if } -\frac{t}{N} \leq \sigma \leq 0; \\ 0, & \text{if } 0 \leq \sigma. \end{cases}$$

To simplify the following calculations, we set

$$g_k := S_{\frac{kt}{N}} x + T_0\left(\frac{kt}{N}\right) f, \;\; k = 0, \ldots, N-1.$$

First we find our N. We have

$$\left\|\Phi\left(S_s x + T_0(s)f - T_0\left(s - \frac{kt}{N}\right)[hg_k]\right)\right\|_X \leq$$
$$\leq \|\Phi\|_{\mathcal{L}(W^{1,1},X)} \left\|\left[S_s x + T_0(s)f - T_0\left(s - \frac{kt}{N}\right)[hg_k]\right]\Big|_{[-1,0]}\right\|_{W^{1,1}}. \quad (7)$$

Since $h(\sigma) = 1$ for $\sigma \leq -\frac{t}{N}$, for every $\sigma \in \left[-1, -\frac{t}{N}\right]$

$$\left[S_{\frac{kt}{N}} x + T_0\left(\frac{kt}{N}\right) f\right](\sigma) = [hg_k](\sigma)$$

holds, so in particular

$$\left[S_s x + T_0(s)f - T_0\left(s - \frac{kt}{N}\right)[hg_k]\right](\sigma) = 0$$

for every $\sigma \in \left[-1, -\frac{2t}{N}\right]$ if $s \in \left[\frac{kt}{N}, \frac{(k+1)t}{N}\right]$. Thus for such s we have

$$\left\|\left(S_s x + T_0(s)f - T_0\left(s - \frac{kt}{N}\right)[hg_k]\right)\Big|_{[-1,0]}\right\|_{W^{1,1}} \leq$$
$$\leq \left\|[S_s x + T_0(s)f]\Big|_{[-\frac{2t}{N},0]}\right\|_{W^{1,1}} +$$
$$+ \left\|\left[T_0\left(s - \frac{kt}{N}\right)[hg_k]\right]\Big|_{[-\frac{2t}{N},0]}\right\|_{W^{1,1}}. \quad (8)$$

By the continuity of the norm, for an N sufficiently large we have

$$\left\| (S_s x + T_0(s) f) \Big|_{\left[-\frac{2t}{N}, 0\right]} \right\|_{W^{1,1}} \leq \|x\|_X$$

for every $s \in \left[\frac{kt}{N}, \frac{(k+1)t}{N}\right]$, $k = 0, \ldots, N-1$. Then, from the definition of $W^{1,1}$ norm, the second term of (8) can be estimated as

$$\begin{aligned}
&\left\| \left[T_0 \left(s - \frac{kt}{N} \right) [hg_k] \right] \Big|_{\left[-\frac{2t}{N}, 0\right]} \right\|_{W^{1,1}} = \\
&\qquad = \left\| \left[T_0 \left(s - \frac{kt}{N} \right) [hg_k] \right] \Big|_{\left[-\frac{2t}{N}, 0\right]} \right\|_{L^1} + \\
&\qquad + \left\| \left[T_0 \left(s - \frac{kt}{N} \right) [h' g_k + h g_k'] \right] \Big|_{\left[-\frac{2t}{N}, 0\right]} \right\|_{L^1} \leq \\
&\qquad \leq \|h\|_\infty \left\| \left[T_0 \left(s - \frac{kt}{N} \right) g_k \right] \Big|_{\left[-\frac{2t}{N}, 0\right]} \right\|_{W^{1,1}} + \\
&\qquad + \left\| \left[T_0 \left(s - \frac{kt}{N} \right) g_k \right] \Big|_{\left[-\frac{2t}{N}, 0\right]} \right\|_\infty \left\| h \Big|_{\left[-\frac{2t}{N}, 0\right]} \right\|_{W^{1,1}} \leq \\
&\qquad \leq \|x\|_X + 2 \left(M \|x\|_X + \|f\|_\infty \right).
\end{aligned}$$

So by (7) and (8), for an N sufficiently large we obtain

$$\begin{aligned}
&\left\| \Phi \left(S_s x + T_0(s) f - T_0 \left(s - \frac{kt}{N} \right) [hg_k] \right) \right\|_X \leq \\
&\qquad \leq 2 \left((M+1) \|x\|_X + \|f\|_\infty \right) \|\Phi\|_{\mathcal{L}(W^{1,1}, X)} \qquad (9)
\end{aligned}$$

for every $s \in \left[\frac{kt}{N}, \frac{(k+1)t}{N}\right]$, $k = 0, \ldots, N-1$.

Let us fix this N. Then by the triangle inequality and (9),

$$\begin{aligned}
&\int_{\frac{kt}{N}}^{\frac{(k+1)t}{N}} \|\Phi (S_s x + T_0(s) f)\|_X \, \mathrm{d}s \leq \\
&\qquad \leq \frac{2t}{N} \left((M+1) \|x\|_X + \|f\|_\infty \right) \|\Phi\|_{\mathcal{L}(W^{1,1}, X)} + \\
&\qquad\qquad + \int_{\frac{kt}{N}}^{\frac{(k+1)t}{N}} \left\| \Phi \left(T_0 \left(s - \frac{kt}{N} \right) [hg_k] \right) \right\|_X . \qquad (10)
\end{aligned}$$

We have to estimate the remaining integral in (10). By the triangle inequality,

$$\begin{aligned}
\|\Phi (T_0(s) [hg_k])\|_X &\leq \left\| \Phi \left(T_0 \left(\frac{kt}{N} + s \right) [hg_0] \right) \right\|_X + \\
&+ \sum_{l=0}^{k-1} \left\| \Phi \left(T_0 \left(\frac{(k-l-1)t}{N} + s \right) [hg_{l+1}] - \right. \right. \\
&\qquad \left. \left. - T_0 \left(\frac{(k-l)t}{N} + s \right) [hg_l] \right) \right\|_X , \qquad (11)
\end{aligned}$$

the sum to be understood empty for $k=0$.

Consider the functions on the right hand side,

$$e_{k,l} := T_0\left(\frac{(k-l-1)t}{N}\right)[hg_{l+1}] - T_0\left(\frac{(k-l)t}{N}\right)[hg_l],$$
$$k=1,\dots,N-1,\ l=0,\dots,k-1.$$

Observe that $e_{k,l}$ is zero outside the interval $\left[-\frac{(k-l+1)t}{N}, -\frac{(k-l-1)t}{N}\right]$. This is clear for $\sigma \geq -\frac{(k-l-1)t}{N}$, since both terms are zero, while for $\sigma < -\frac{(k-l+1)t}{N}$ it follows from the identities

$$\left[T_0\left(\frac{(k-l-1)t}{N}\right)[hg_{l+1}]\right](\sigma) = [hg_{l+1}]\left(\sigma + \frac{(k-l-1)t}{N}\right)$$
$$= g_{l+1}\left(\sigma + \frac{(k-l-1)t}{N}\right) = \left[S_{\frac{kt}{N}}x + T_0\left(\frac{kt}{N}\right)f\right](\sigma)$$

and

$$\left[T_0\left(\frac{(k-l)t}{N}\right)[hg_l]\right](\sigma) = [hg_l]\left(\sigma + \frac{(k-l)t}{N}\right)$$
$$= g_l\left(\sigma + \frac{(k-l)t}{N}\right) = \left[S_{\frac{kt}{N}}x + T_0\left(\frac{kt}{N}\right)f\right](\sigma).$$

A similar computation shows that

$$e_{k,l} = T_0\left(\frac{(k-l-1)t}{N}\right)e_{l+1,l},$$

while by the triangle inequality

$$\|e_{k,l}(\sigma)\|_X \leq \left\|\left[T_0\left(\frac{(k-l-1)t}{N}\right)[hg_{l+1}]\right](\sigma)\right\|_X +$$
$$+\left\|\left[T_0\left(\frac{(k-l)t}{N}\right)[hg_l]\right](\sigma)\right\|_X \leq 2\left(M\|x\|_X + \|f\|_\infty\right)$$

for every $\sigma \in \left[-\frac{(k-l+1)t}{N}, -\frac{(k-l-1)t}{N}\right]$, so in particular

$$\|e_{k,l}\|_{L^1} = \|e_{l+1,l}\|_{L^1} \leq \frac{4t}{N}\left(M\|x\|_X + \|f\|_\infty\right) \tag{12}$$

for every $k=1,\dots,N-1,\ l=0,\dots,k-1$. That is, by (M) for the function $e_{l+1,l}$, using (12) and $e_{l+1,l}(0)=0$ we obtain

$$\sum_{k=l+1}^{N-1}\int_0^{\frac{t}{N}} \|\Phi\left(T(s)e_{k,l}\right)\|_X \,\mathrm{d}s =$$
$$= \sum_{k=l+1}^{N-1}\int_0^{\frac{t}{N}} \left\|\Phi\left(T\left(\frac{(k-l-1)t}{N}+s\right)e_{l+1,l}\right)\right\|_X \mathrm{d}s =$$
$$= \int_0^{\frac{(N-l-2)t}{N}} \|\Phi\left(T(s)e_{l+1,l}\right)\|_X \,\mathrm{d}s \leq \int_0^t \|\Phi\left(T(s)e_{l+1,l}\right)\|_X \,\mathrm{d}s \leq$$
$$\leq q(t)\|e_{l+1,l}\|_{L^1} \leq q(t)\frac{4t}{N}\left(M\|x\|_X + \|f\|_\infty\right). \tag{13}$$

By summing on l, we have

$$\sum_{k=0}^{N-1}\int_0^{\frac{t}{N}}\sum_{l=0}^{k-1}\left\|\Phi\left(T_0(s)e_{k,l}\right)\right\|_X \mathrm{d}s =$$

$$= \sum_{l=0}^{N-2}\int_0^{\frac{t}{N}}\sum_{k=l+1}^{N-1}\left\|\Phi\left(T_0(s)e_{k,l}\right)\right\|_X \mathrm{d}s \leq$$

$$\leq \sum_{l=0}^{N-2} q(t)\frac{4t}{N}\left(M\|x\|_X + \|f\|_\infty\right) \leq q(t)4t\left(M\|x\|_X + \|f\|_\infty\right). \quad (14)$$

Similarly, for the first term of the right hand side of (11) we obtain

$$\sum_{k=0}^{N-1}\int_0^{\frac{t}{N}}\left\|\Phi\left(T_0\left(\frac{kt}{N}+s\right)[hg_0]\right)\right\|_X \mathrm{d}s =$$

$$= \int_0^t \left\|\Phi\left(T_0(s)[hg_0]\right)\right\|_X \mathrm{d}s \leq q(t)\|f\|_{L^1}. \quad (15)$$

Finally, by putting together (10), (11), (14) and (15), we arrive at

$$\int_0^t \|\Phi(S_s x + T_0(s)f)\|_X \mathrm{d}s = \sum_{k=0}^{N-1}\int_{\frac{kt}{N}}^{\frac{(k+1)t}{N}}\left\|\Phi\left(S_s x + T_0(s)f\right)\right\|_X \mathrm{d}s \leq$$

$$\leq \sum_{k=0}^{N-1}\frac{2t}{N}\left((M+1)\|x\|_X + \|f\|_\infty\right)\|\Phi\|_{\mathcal{L}(W^{1,1},X)} +$$

$$+ \sum_{k=0}^{N-1}\int_0^{\frac{t}{N}}\left\|\Phi\left(T_0\left(\frac{kt}{N}+s\right)[hg_0]\right)\right\|_X \mathrm{d}s +$$

$$+ \sum_{k=0}^{N-1}\int_0^{\frac{t}{N}}\sum_{l=0}^{k-1}\left\|\Phi\left(T_0(s)e_{k,l}\right)\right\|_X \mathrm{d}s \leq$$

$$\leq 2t\left((M+1)\|x\|_X + \|f\|_\infty\right)\|\Phi\|_{\mathcal{L}(W^{1,1},X)} +$$

$$+ q(t)\|f\|_{L^1} + q(t)4t\left(M\|x\|_X + \|f\|_\infty\right),$$

as stated. This finishes the proof.■

Next we prove a lemma which will help to check the relative compactness of sets appearing in the range of the perturbed semigroup.

Lemma 4.3. *Let X, Y be arbitrary Banach spaces, $a, b \in \mathbb{R}$ and $1 \leq p < \infty$ be fixed. Consider a bounded mapping $V : [a,b] \to \mathcal{L}(X,Y)$ and a bounded set $\Gamma \subset L^1([a,b],X)$.*

1. *Suppose that V is norm continuous and compact operator valued on $(a,b]$. If Γ satisfies*

$$\int_{b-\delta}^{b}\|\gamma(s)\|_X\, ds \to 0 \quad (16)$$

as $\delta \to 0$, uniformly in Γ, then the set

$$\mathcal{C} = \left\{\int_a^b V(a+b-s)\gamma(s)\, ds \,\middle|\, \gamma \in \Gamma\right\} \quad (17)$$

is relatively compact in Y.

2. *If V is norm continuous and compact operator valued on the closed interval $[a,b]$, then the conclusion of 1. holds for every bounded $\Gamma \subset L^1([a,b],X)$.*

Proof. For every $\varepsilon > 0$ we construct a relatively compact set $\mathcal{C}_\varepsilon \subset X$ such that $\mathcal{C}$ is in the ε-neighborhood of C_ε. This clearly proves the lemma.

Let Q be the norm bound of Γ in $L^1([a,b],X)$.

Suppose first that V is norm continuous and compact operator valued on $[a,b]$. Then

$$\{V(t) | \, t \in [a,b]\} \subset \left(\mathcal{L}(X,Y), \|.\|_{\mathcal{L}(X,Y)}\right)$$

is the continuous image of the compact interval $[a,b]$, so it is a compact set of operators. Hence there is finite set $\{b = t_0 > t_1 > \cdots > t_n > t_{n+1} = a\} \subset [a,b]$ such that

$$\|V(t) - V(t_j)\|_{\mathcal{L}(X,Y)} < \frac{\varepsilon}{Q} \tag{18}$$

whenever $t \in [t_{j+1}, t_j]$, $j = 0, \ldots n$. We show that

$$\text{abs conv } \left(\bigcup_{j=0}^{n} V(t_j)(QB(X))\right) \tag{19}$$

can be chosen for $\mathcal{C}_\varepsilon$, where abs conv indicates absolute convex hull.

Since $V(t_j)(B(X))$ is relatively compact for every $0 \le j \le n$, C_ε is relatively compact, as well. Let now $\gamma \in \Gamma$. Then

$$\begin{aligned}
\int_a^b V(a+b-s)\gamma(s)\mathrm{d}s &= \sum_{j=0}^{n} \int_{a+b-t_j}^{a+b-t_{j+1}} V(a+b-s)\gamma(s)\mathrm{d}s = \\
&= \sum_{j=0}^{n} \int_{a+b-t_j}^{a+b-t_{j+1}} V(t_j)\gamma(s)\mathrm{d}s + \\
&+ \sum_{j=0}^{n} \int_{a+b-t_j}^{a+b-t_{j+1}} (V(a+b-s) - V(t_j))\gamma(s)\mathrm{d}s.
\end{aligned}$$

The first sum of the right hand side is clearly in $\mathcal{C}_\varepsilon$, while by (18) for the second term we have

$$\begin{aligned}
&\left\| \sum_{j=0}^{n} \int_{a+b-t_j}^{a+b-t_{j+1}} (V(a+b-s) - V(t_j))\gamma(s)\mathrm{d}s \right\|_X \le \\
&\quad \le \sum_{j=0}^{n} \int_{a+b-t_j}^{a+b-t_{j+1}} \|V(a+b-s) - V(t_j)\|_{\mathcal{L}(X)} \, \|\gamma(s)\|_X \, \mathrm{d}s < \\
&\quad < \frac{\varepsilon}{Q} \|\gamma\|_{L^1([a,b],X)} \le \varepsilon.
\end{aligned}$$

This proves the second part of the lemma.

To prove 1., let M denote the norm bound of $V(t)$ on $[a,b]$. By (16), there is a $\delta > 0$ such that

$$\int_{b-\delta}^{b} \|\gamma(s)\|_X \mathrm{d}s < \frac{\varepsilon}{M} \tag{20}$$

for every $\gamma \in \Gamma$. According to the second part of the lemma the set

$$\mathcal{C}_\varepsilon = \left\{ \int_a^{b-\delta} V(a+b-s)\gamma(s)\mathrm{d}s \,\middle|\, \gamma \in \Gamma \right\}$$

is relatively compact in Y. By (20),

$$\left\| \int_a^b V(a+b-s)\gamma(s)\mathrm{d}s - \int_a^{b-\delta} V(a+b-s)\gamma(s)\mathrm{d}s \right\|_Y \le$$
$$\le \int_{b-\delta}^b \|V(a+b-s)\gamma(s)\|_Y \,\mathrm{d}s \le M \int_{b-\delta}^b \|\gamma(s)\|_X \mathrm{d}s < \varepsilon,$$

so $\mathcal{C}_\varepsilon$ fulfills the requirements.■

Now we return to the setting of Section 2 and study the compactness properties of the operator family S_t.

Lemma 4.4. *Let $(S(t))_{t\ge 0}$ be an immediately compact semigroup on the Banach space X and fix $1 \le p < \infty$.*

1. *For every $t \ge 0$, the set*
$$\{S_t x | \ \|x\|_X \le 1\} \tag{21}$$
is relatively compact in $L^p([-1,0],X)$.
2. *The mapping $s \mapsto S_s$ is norm continuous as a*
$$[0,\infty) \to \mathcal{L}(X, L^p([-1,0]))$$
function.
3. *If the set $\Gamma \subset L^1([0,t],X)$ is bounded, then the set*
$$\left\{ \int_0^t S_{t-s}\gamma(s)\mathrm{d}s \,\middle|\, \gamma \in \Gamma \right\}$$
is relatively compact in $L^p([-1,0],X)$.

Proof. To prove 1, we apply Theorem 3.1 for $\mathcal{F} := \{S_t x | \ \|x\|_X \le 1\}$. By Theorem 3.2, for every measurable set $D \subset [0,t]$ the operator

$$W = \int_{D\cap(-t,0]} S(t+\sigma)\mathrm{d}\sigma$$

is compact, so the set

$$\left\{ \int_D f \,\middle|\, f \in \mathcal{F} \right\} = \{Wx | \ \|x\|_X \le 1\}$$

is relatively compact.

To prove the uniform continuity of $\mathcal{F}$, let M denote the norm bound of $S(s)$ on $[0,t]$. Again, to simplify the calculations, we set $S(s) = 0$ for

$s < 0$. Take a $\delta > 0$ and an $0 < h < \delta$. By definition,

$$\int_{-1}^{-h} \|[S_t x](\sigma + h) - [S_t x](\sigma)\|_X^p \, d\sigma =$$

$$= \int_{1}^{-h} \|S(t+\sigma+h)x - S(t+\sigma)x\|_X^p \, d\sigma \leq$$

$$\leq \int_{-t-h}^{-t} \|S(t+\sigma+h)x\|_X^p + \int_{-t}^{-t+\delta} \|S(t+\sigma+h)x - S(t+\sigma)x\|_X^p \, d\sigma +$$

$$+ \int_{-t+\delta}^{-h} \|S(t+\sigma+h)x - S(t+\sigma)x\|_X^p \, d\sigma.$$

It is clear that

$$\int_{-t-h}^{-t} \|S(t+\sigma+h)x\|_X^p \leq hM^p \|x\|_X^p,$$

$$\int_{-t}^{-t+\delta} \|S(t+\sigma+h)x - S(t+\sigma)x\|_X^p \, d\sigma \leq \delta(2M)^p \|x\|_X^p,$$

and that

$$\int_{-t+\delta}^{-h} \|S(t+\sigma+h)x - S(t+\sigma)x\|_X^p \, d\sigma \leq$$

$$\leq \sup_{s \in [\delta, t]} \|S(s+h) - S(s)\|_{\mathcal{L}(X)}^p \|x\|_X^p.$$

The immediate norm continuity of $(S(t))_{t \geq 0}$ implies that

$$\sup_{s \in [\delta, t]} \|S(s+h) - S(s)\|_{\mathcal{L}(X)}^p < \delta \tag{22}$$

for every h sufficiently small, and then

$$\int_{-1}^{-h} \|[S_t x](\sigma + h) - [S_t x](\sigma)\|_X^p \, d\sigma \leq \delta(3M)^p \|x\|_X^p.$$

Since δ was arbitrary, this proves uniform continuity in the sense of Theorem 3.1 and so relative compactness.

The second statement of the lemma has been implicitly proven in Proposition 4.2 of [2], so we omit the simple proof.

The third statement of the lemma follows immediately from the preceding ones using Lemma 4.3.2 for $Y = L^p([-1,0], X)$, $V(s) = S_s$, $a = 0$, $b = 1$ and our given bounded set Γ. The needed compactness and norm continuity of V have been shown in 1 and 2, respectively. This completes the proof.■

In our last lemma we prove the missing ingredient for the proof of Theorem 4.1.

Lemma 4.5. *Let* $(S(t))_{t\geq 0}$ *be immediately compact, and define*

$$\Gamma_0 := \left\{ \gamma_{x,f} \;\middle|\; \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0) \cap B(\mathcal{E}) \right\} \subset L^p([0,t],X), \tag{23}$$

where

$$\gamma_{x,f}(s) = \Phi\left(S_s x + T_0(s) f\right), \;\; s \in [0,t] \tag{24}$$

for every $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0)$

If Φ *satisfies condition* (K)*, then* Γ_0 *satisfies the condition of Lemma 4.3.1, that is for every* $t > 0$,

$$\int_{t-\delta}^{t} \|\gamma(s)_{x,f}\|_X \, ds \to 0 \tag{25}$$

as $\delta \to 0$*, uniformly in* Γ_0.

Proof. Let M be the norm bound of $S(s)$ on $[0,t]$, consider a $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0)$ and take a $\delta > 0$.

It follows immediately from the semigroup property that for every $s \in [0,\delta]$,

$$\begin{aligned} &S_{t-\delta+s}x + T_0(t-\delta+s)f = \\ &\qquad = S_s\left(S(t-\delta)x\right) + T_0(s)\left[S_{t-\delta}x + T_0(t-\delta)f\right]. \end{aligned} \tag{26}$$

By condition (K), for every $\delta > 0$ there is a constant $q(\delta) > 0$ for which

$$\begin{aligned} &\int_0^{\delta} \|\Phi(S_s\left(S(t-\delta)x\right) + T_0(s)\left[S_{t-\delta}x + T_0(t-\delta)f\right])\|_X \mathrm{d}s \leq \\ &\qquad \leq q(\delta) \left\| \begin{pmatrix} S(t-\delta)x \\ S_{t-\delta}x + T_0(t-\delta)f \end{pmatrix} \right\|_{\mathcal{E}} \leq \\ &\qquad \leq q(\delta) \max\{M\|x\|_X, M\|x\|_X + \|f\|_{L^p}\} \leq \\ &\qquad \leq q(\delta)(M+1) \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{E}}. \end{aligned} \tag{27}$$

Since $q(\delta) \to 0$ as $\delta \to 0$, by (26) and (27) for every $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0) \cap B(\mathcal{E})$ we have

$$\begin{aligned} &\int_{t-\delta}^{t} \|\Phi(S_s x + T_0(s)f)\|_X \mathrm{d}s = \\ &\quad = \int_0^{\delta} \|\Phi(S_{t-\delta+s}x + T_0(t-\delta+s)f)\|_X \mathrm{d}s \leq (M+1)q(\delta) \to 0 \end{aligned} \tag{28}$$

if $\delta \to 0$, uniformly in Γ_0, as stated.■

Proof of Theorem 4.1. First we show that $\mathcal{T}_0(t)$ is compact for $t > 1$. By (2), for fixed $t > 1$ and $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{E}$,

$$\mathcal{T}_0(t) \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} S(t)x \\ S_t x \end{pmatrix}.$$

Since $(S(t))_{t\geq 0}$ is immediately compact, $\mathcal{T}_0(t)$ is compact in the first coordinate. The compactness in the second coordinate has been proven in Lemma 4.4.1.

Now we prove assertions 2 and 3. By (1), (2) and (6), for every $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A})$,

$$
\begin{aligned}
(V\mathcal{T}_0)(t)\begin{pmatrix} x \\ f \end{pmatrix} &= \int_0^t \mathcal{T}_0(t-s)\mathcal{B}\mathcal{T}_0(s)\begin{pmatrix} x \\ f \end{pmatrix} \mathrm{d}s = \\
&= \int_0^t \mathcal{T}_0(t-s)\begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S(s)x \\ S_s x + T_0(s)f \end{pmatrix} \mathrm{d}s = \\
&= \int_0^t \begin{pmatrix} S(t-s) & 0 \\ S_{t-s} & T_0(t-s) \end{pmatrix}\begin{pmatrix} \Phi\left(S_s x + T_0(s)f\right) \\ 0 \end{pmatrix} \mathrm{d}s = \\
&= \int_0^t \begin{pmatrix} S(t-s)\Phi\left(S_s x + T_0(s)f\right) \\ S_{t-s}\Phi\left(S_s x + T_0(s)f\right) \end{pmatrix} \mathrm{d}s. \qquad (29)
\end{aligned}
$$

Take now Γ_0 of (23), which is bounded in $L^1([-1,0],X)$ by condition (M) . By Lemma 4.4.3 we have that

$$
\begin{aligned}
\mathcal{F} &= \left\{ \int_0^t S_{t-s}\gamma(s)\mathrm{d}s \,\middle|\, \gamma \in \Gamma_0 \right\} = \\
&= \left\{ \int_0^t S_{t-s}\Phi\left(S_s x + T_0(s)f\right)\mathrm{d}s \,\middle|\, \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0) \cap B(\mathcal{E}) \right\}
\end{aligned}
$$

is relatively compact in $L^p([-1,0],X)$, which proves 2.

We prove 3 by showing that

$$
\left\{ \int_0^t S(t-s)\Phi\left(S_s x + T_0(s)f\right)\mathrm{d}s \,\middle|\, \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0) \cap B(\mathcal{E}) \right\} \qquad (30)
$$

is in the ε-neighborhood of a relatively compact set for every $\varepsilon > 0$ and $t > 1$.

Since by Lemma 4.4.1 and 2,

$$
\{S_t x \mid \|x\|_X \leq 1\}
$$

is relatively compact in $L^p([-1,0],X)$ for every $t > 0$ and $s \mapsto S_s$ is norm continuous, we can fix a finite set

$$
\{x_j \mid j \in J\} \subset B(X)
$$

such that for every $x \in B(X)$ there exists a $j(x) \in J$ for which

$$
\|S_s x_{j(x)} - S_s x\|_{L^1} \leq \frac{\varepsilon}{2q(t)} \qquad (31)
$$

for every $0 \leq s \leq t$. With the notation of Lemma 4.2, we also set

$$
\Delta := \min\left\{t-1, \frac{\varepsilon}{8M^2\Lambda}\right\},
$$

where M is the norm bound of $S(s)$ on $[0,t]$.

For every $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0) \cap B(\mathcal{E})$ and $\delta < \Delta$ we have

$$\begin{aligned}\int_0^t & S(t-s)\Phi\left(S_s x + T_0(s)f\right) \mathrm{d}s = \\ &= \int_0^1 S(t-s)\Phi\left(S_s x + T_0(s)f\right) \mathrm{d}s + \\ &+ \int_1^t S(t-s)\Phi\left(S_s x_{j(x)}\right) \mathrm{d}s + \\ &+ \int_1^{t-\delta} S(t-s)\Phi\left(S_s\left(x - x_{j(x)}\right)\right) \mathrm{d}s + \\ &+ \int_{t-\delta}^t S(t-s)\Phi\left(S_s\left(x - x_{j(x)}\right)\right) \mathrm{d}s. \end{aligned} \tag{32}$$

By Lemma 4.3.2, the first and third term of the right hand side of (32) maps to a relatively compact set, while the second term can have only finitely many values. To estimate the fourth term, observe that similarly to (26) we have

$$\begin{aligned} &S_s\left(x - x_{j(x)}\right) + T_0(s)f = \\ &\quad = S_{s-t+\delta}\left(S(t-\delta)\left(x - x_{j(x)}\right)\right) + T_0(s-t+\delta)\left[S_{t-\delta}\left(x - x_{j(x)}\right)\right] \end{aligned}$$

for $s \geq t-\delta \geq 1$. So after a change of variable, by Lemma 4.2,

$$\begin{aligned} &\left\|\int_{t-\delta}^t S(t-s)\Phi\left(S_s\left(x - x_{j(x)}\right)\right) \mathrm{d}s\right\|_X = \\ &\quad = \left\|\int_0^\delta S(\delta-s)\Phi\left(S_s\left(S(t-\delta)\left(x - x_{j(x)}\right)\right) + \right.\right. \\ &\quad \left.\left. + T_0(s)\left[S_{t-\delta}\left(x - x_{j(x)}\right)\right]\right) \mathrm{d}s\right\|_X \leq \\ &\quad \leq M\int_0^\delta \left\|\Phi\left(S_s\left(S(t-\delta)\left(x - x_{j(x)}\right)\right) + \right.\right. \\ &\quad \left.\left. + T_0(s)\left[S_{t-\delta}\left(x - x_{j(x)}\right)\right]\right)\right\|_X \mathrm{d}s \leq \\ &\leq \delta M\Lambda\left(\left\|S(t-\delta)\left(x - x_{j(x)}\right)\right\|_X + \left\|S_{t-\delta}\left(x - x_{j(x)}\right)\right\|_\infty\right) + \\ &\quad + q(t)\left\|S_{t-\delta}\left(x - x_{j(x)}\right)\right\|_{L^1} \leq \delta 4M^2\Lambda + q(t)\frac{\varepsilon}{2q(t)} \leq \varepsilon. \end{aligned}$$

This proves 3.

For 4, we only have to prove that $Pr_X(V^2\mathcal{T}_0)(t)$ is compact for every $t > 0$, since $Pr_{L^p}(V^2\mathcal{T}_0)(t)$ is compact by 3 (see Lemma III.1.13 in [7]).

First we show that $Pr_X(V\mathcal{T}_0|_X)(t)$ is compact for every $t > 0$. As in the proof of 3, for a $\varepsilon > 0$ we fix a finite set $\{x_j |\, j \in J\} \subset B(X)$ satisfying (31), and for every $\delta > 0$ a function $f_\delta : [-1,0] \to [0,1]$ such that $f(0) = 1$ and $\|f\|_{L^1} \leq \delta$. Observe that

$$\mathcal{F}_\delta := \left\{ \begin{pmatrix} x \\ xf_\delta \end{pmatrix} \middle|\, x \in B(X) \right\} \subset D(\mathcal{A}_0),$$

and since $(V\mathcal{T}_0)(t)$ extends continuously onto X, for every $\rho > 0$,

$$(Pr_X(V\mathcal{T}_0)(t))\, B(X) \subset B\left(((V\mathcal{T}_0)(t))\, \mathcal{F}_\delta, \rho\right),$$

if δ is sufficiently small. So to prove that $(Pr_X(V\mathcal{T}_0)(t))\, B(X)$ is relatively compact, it is enough to show that $(V\mathcal{T}_0)\, \mathcal{F}_\delta$ is in the ε-neighborhood of a relatively compact set whenever δ is sufficiently small.

Let

$$\Delta := \frac{\varepsilon}{4M^2\Lambda + 2q(t)M}.$$

Then, with $\delta < \Delta$, we have

$$\begin{aligned}
Pr_X(V\mathcal{T}_0)(t)) \begin{pmatrix} x \\ x f_\delta \end{pmatrix} = \\
&= Pr_X(V\mathcal{T}_0)(t)) \begin{pmatrix} x_{j(x)} \\ x_{j(x)} f_\delta \end{pmatrix} + \\
&+ \int_0^t S(t-s)\Phi(S_s\left(x - x_{j(x)}\right) + T_0(s)\left[\left(x - x_{j(x)}\right) f_\delta\right])\mathrm{d}s = \\
&= Pr_X(V\mathcal{T}_0)(t)) \begin{pmatrix} x_{j(x)} \\ x_{j(x)} f_\delta \end{pmatrix} + \\
&+ \int_0^{t-\delta} S(t-s)\Phi(S_s\left(x - x_{j(x)}\right) + T_0(s)\left[\left(x - x_{j(x)}\right) f_\delta\right])\mathrm{d}s + \\
&\int_{t-\delta}^t S(t-s)\Phi(S_s\left(x - x_{j(x)}\right) + T_0(s)\left[\left(x - x_{j(x)}\right) f_\delta\right])\mathrm{d}s. \qquad (33)
\end{aligned}$$

Just as in the proof of 3, we only have to estimate the third term of the right hand side of (33).

We have

$$\begin{aligned}
&S_s\left(x - x_{j(x)}\right) + T_0(s)\left[\left(x - x_{j(x)}\right) f_\delta\right] = \\
&= S_{s-t+\delta}\left(S(t-\delta)\left(x - x_{j(x)}\right)\right) + T_0(s-t+\delta)\left[T_0(t-\delta)\left(x - x_{j(x)}\right) f_\delta\right].
\end{aligned}$$

By Lemma 4.2,

$$\begin{aligned}
&\left\| \int_{t-\delta}^t S(t-s)\Phi(S_s\left(x - x_{j(x)}\right) + T_0(s)\left[\left(x - x_{j(x)}\right) f_\delta\right])\mathrm{d}s \right\|_X \le \\
&\le \delta M\Lambda\left(\|S(t-\delta)\left(x - x_{j(x)}\right)\|_X + \|T_0(t-\delta)\left(x - x_{j(x)}\right) f_\delta\|_\infty\right) + \\
&+ q(t)\|T_0(t-\delta)\left(x - x_{j(x)}\right) f_\delta\|_{L^1} \le \delta 4M^2\Lambda + 2q(t)M\delta \le \varepsilon,
\end{aligned}$$

as required. So $Pr_X(V\mathcal{T}_0|_X)(t)$ is compact for every $t > 0$.

Next we show that

$$\|Pr_X(V\mathcal{T}_0|_X)(t)\|_{\mathcal{L}(X)} \to 0 \qquad (34)$$

as $t \to 0$. Since by Lemma 4.2,

$$\left\| Pr_X(V\mathcal{T}_0)(t)) \begin{pmatrix} x \\ xf_\delta \end{pmatrix} \right\|_X =$$

$$= \left\| \int_0^t S(t-s)\Phi\left(S_s x + T_0(s)\left[xf_\delta\right]\right) \mathrm{d}s \right\|_X \leq$$

$$\leq tM\Lambda\left(\|x\|_X + \|xf_\delta\|_\infty\right) + q(t)M\|xf_\delta\|_{L^1} \leq$$

$$\leq \left(2Mt\Lambda + q(t)M\delta\right)\|x\|_X,$$

for $\delta \to 0$ we obtain

$$\|Pr_X(V\mathcal{T}_0|_X)(t)\|_{\mathcal{L}(X)} \leq 2Mt\Lambda,$$

which proves (34).

Since $Pr_X(V\mathcal{T}_0|_X)(t)$ is clearly norm continuous for $t > 0$, we have that $Pr_X(V\mathcal{T}_0|_X)(t)$ is compact and norm continuous on the whole interval $[0, t]$.

Similarly to (29),

$$Pr_X(V^2\mathcal{T}_0)(t) \begin{pmatrix} x \\ f \end{pmatrix} =$$

$$= \int Pr_X(V\mathcal{T}_0)(t) \begin{pmatrix} \Phi\left(S_s x + T_0(s)f\right) \\ 0 \end{pmatrix} \mathrm{d}s =$$

$$= \int Pr_X(V\mathcal{T}_0)(t)|_X \Phi\left(S_s x + T_0(s)f\right) \mathrm{d}s.$$

As pointed out above, Lemma 4.3.2 can be applied for $a = 0$, $b = t$, $V(s) = Pr_X(V\mathcal{T}_0|_X)(s)$ and $\Gamma = \Gamma_0$ of (23). This proves the compactness of $Pr_X(V^2\mathcal{T}_0)(t)$ for every $t > 0$.

If Φ satisfies condition (K), then by Lemma 4.5 we can apply Lemma 4.3.1 for $a = 0$, $b = t$, the immediately compact $V(t) = S(t)$ and the bounded set $\Gamma = \Gamma_0$. We obtain that $(V\mathcal{T}_0)(t)$ is compact in the first coordinate for every $t > 0$. This proves assertion 5 and finishes the proof.■

5 Applications

Our main example for delay terms satisfying (M) and (K) has been discussed in details in [2], Section 3.3.3. It is shown that for every $1 \leq p < \infty$ and $\eta : [-1, 0] \to \mathcal{L}(X)$ of bounded variation the delay term given by the Riemann-Stieltjes integral

$$\Phi(f) = \int_{-1}^{0} \mathrm{d}\eta f, \;\; f \in C([-1, 0], X) \tag{35}$$

satisfies condition (M), while condition (K) holds if $1 < p < \infty$. So if (4) holds, then the conditions of Theorem 2.4 are satisfied, that is $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{E}$, which is eventually compact if the semigroup $(S(t))_{t\geq 0}$ generated by $(A, D(A))$ is immediately compact.

More specially we can consider the case when η is a finite sum of Dirac measures, that is

$$\Phi(f) = \sum_{i=1}^{n} B_i f(h_i),$$

where $B_i \in \mathcal{L}(X)$, $h_i \in [-1, 0]$ for every $i = 1, \dots, n$. This type of delay equation often appears in applications (see e.g. [4], Chapter 2.4.).

5.1 Spectral analysis

By virtue of the Spectral Mapping Theorem for eventually norm continuous semigroups ([7], Theorem IV.3.10, page 280), the growth bound and the spectral bound of our perturbed semigroup coincide. The non-zero spectrum of a compact operator consists of a countable number of isolated points in $\mathbb{C}$. These points are all eigenvalues belonging to finite dimensional eigenspaces. Since the spectrum of the generator of an eventually norm continuous operator is bounded in every half-plane $\{\lambda \in \mathbb{C} | \ \Re\lambda > \lambda_0\}$, $\lambda_0 \in \mathbb{R}$, we obtain the following decomposition theorem (see also Theorem 4.15 in [2] for a different proof).

Theorem 5.1. *Let X, $\mathcal{E}$, $(A, D(A))$, Φ, $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{T}(t))_{t\geq 0}$ be as in Theorem 2.4. Then there exists a decomposition $\mathcal{E} = \mathcal{E}_S + \mathcal{E}_C + \mathcal{E}_U$ of $\mathcal{E}$ into subspaces invariant under the semigroup $(\mathcal{T}(t))_{t\geq 0}$ such that $\mathcal{E}_C$ and $\mathcal{E}_U$ are finite dimensional, and*

- *the semigroup*
$$\mathcal{T}_S(t) = \mathcal{T}(t)\big|_{\mathcal{E}_S}$$
is uniformly exponentially stable;
- *the semigroup*
$$\mathcal{T}_U(t) = \mathcal{T}(t)\big|_{\mathcal{E}_U}$$
is invertible and the semigroup $\mathcal{T}_U^{-1}(t)$ is uniformly exponentially stable;
- *the semigroup*
$$\mathcal{T}_C(t) = \mathcal{T}(t)\big|_{\mathcal{E}_C}$$
is a group, it is polynomially bounded, hence has growth bound 0, in both time directions.

This decomposition result is crucial in many situation. As an example, see e.g. [4], Chapter 5.2 and many other delay equations arising in control theory.

5.2 Numerical aspects

Finally we would like to point out the quantitative nature of Theorem 2.4, namely that $Pr_X \mathcal{T}(t)$ is "as compact as" $S(t)$. To do so we consider a delay equation with numerically well-understood operators A and Φ, the latter satisfying condition (K) with a "computable" $q(t)$: one may simply take the Dirichlet Laplacian for A on $X = L^2(\Omega)$ for a bounded domain Ω and a point evaluation for Φ, as in Example 3.14 of [2] (for the value of $q(t)$ see (3.49) in Theorem 3.34 of [2]).

The solution of the equation is completely described by $Pr_X\mathcal{T}(t)$, so we examine only this coordinate. Let t_0 be such that $q(t_0) \leq 1/2$; let M denote the norm bound of $S(s)$ on $[0, t_0]$. We analyze the projection $Pr_Z\mathcal{T}(t)$ for every subspace $Z \leq X$.

For $t < t_0$ the general estimate of the Dyson-Phillips series gives

$$\left\| \sum_{n=1}^{\infty} (V^n F)(t)x \right\|_X \leq \left\| F(s)\big|_{[0,t_0]} \right\|_\infty \|x\|_X, \tag{36}$$

(see the proof of Corollary III.3.16 in [7]). By (29) and (28) we have

$$\begin{aligned} &\|\mathrm{Pr}_Z(VS)(t)\|_{\mathcal{L}(X)} \leq \\ &\qquad \leq \left\| \mathrm{Pr}_Z S(s)\big|_{[0,\delta]} \right\|_\infty (M+1)q(\delta) + \left\| \mathrm{Pr}_Z S(s)\big|_{[\delta,t_0]} \right\|_\infty q(t_0) \end{aligned} \tag{37}$$

for every $\delta > 0$.

Let now $\varepsilon > 0$ be arbitrary and choose $\delta > 0$ satisfying $M(M+1)q(\delta) \leq \varepsilon$. Then if Z is chosen such that $\|\mathrm{Pr}_Z S(s)\|_{\mathcal{L}(X)} \leq \varepsilon$ for every $s \in [\delta, t_0]$, then by (37),

$$\|\mathrm{Pr}_Z(VS)(t)\|_{\mathcal{L}(X)} \leq 2\varepsilon.$$

So by (36) for the function $F := \mathrm{Pr}_Z(V\mathcal{T}_0)(t)$ we have

$$\begin{aligned} &\|\mathrm{Pr}_Z\mathcal{T}(t)\|_{\mathcal{L}(X)} \leq \\ &\qquad \leq \|\mathrm{Pr}_Z S(t)\|_{\mathcal{L}(X)} + \|\mathrm{Pr}_Z(VS)(t)\|_{\mathcal{L}(X)} + \\ &\qquad\qquad + \left\| \mathrm{Pr}_Z \sum_{n=1}^{\infty} V^n(VS)(t) \right\|_{\mathcal{L}(X)} \leq 3\varepsilon. \end{aligned}$$

The corresponding estimate for $t > t_0$ can be directly obtained from the semigroup property. Note that the dependence of t_0, δ and Z on ε is not "wild"; e.g., in Example 3.14 of [2], $(S(t))_{t\geq 0}$ is contractive and analytic, so we have $t_0 = 1/4$, while δ and Z depends directly on the eigenvalue structure of A.

To summarize, $Pr_X\mathcal{T}(t)$ inherits the approximability properties of $S(t)$ on a quantitative way. In a Hilbert space setting, whenever a (finite dimensional) subspace W provides a good approximation to the image of $S(t)$, it also approximates $\mathrm{Pr}_X\mathcal{T}(t)$ well (above Z plays the role of $W^\perp$). So the numerical treatment of our delay equations may be made by applying the methods developed for smooth functions by approximating the initial data (L^p functions on bounded domains), since not only the error in L^p norm of the approximation is preserved but also its distribution in the space X (for the basic numerical techniques for delay equations involving smooth functions see e.g. [8], Chapter II.15 and [9], Chapter IV.5). This procedure may give a method with good convergence properties and with control on the L^p structure, as well.

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