The ω_1-limit of Baire-2 functions is Baire-2

Tamás Mátrai*

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Abstract

Let X be a Polish space, Y be a separable metric space and consider a family $f_\alpha : X \to Y$ ($\alpha < \omega_1$) of Baire-2 functions. Giving a partial answer to a question of Tomasz Natkaniec, we show that if for a function $f : X \to Y$, the set $\{\alpha < \omega_1 : \ f_\alpha(x) \neq f(x)\}$ is finite for every $x \in X$, then f itself is necessarily Baire-2. The proof is based on a characterization of Σ^0_3 sets which can be interesting on its own.

1 Introduction

Almost a century ago, W. Sierpiński [6] observed that the pointwise limit of a sequence with length ω_1 of continuous real functions is necessarily continuous (Theorem 1 on page 133), which may seem to be quite paradox compared to the behavior of ordinary pointwise convergence. In the same paper, Sierpiński has also proved this class preserving property of ω_1-convergence for the Baire-1 functions (Theorem 2 on page 137); and he pointed out that by assuming the Continuum Hypothesis, every real function can be obtained as the ω_1-limit of Baire-2 functions (for more details and discussions, see [6], Section 5, page 139 and [2], specially Theorem 3 on page 499).

In view of these facts, T. Natkaniec [5] introduced a stronger notion of pointwise convergence. We recall the precise setting in the following definition.

Definition 1. Let (X, τ) be a Polish space, (Y, d) be a separable metric space and consider an ideal $\mathcal{I}$ on ω_1. We say that a sequence of functions $f_\alpha : X \to Y$ ($\alpha < \omega_1$) *$\mathcal{I}$-converges* to the function $f : X \to Y$, in notation $f_\alpha \to_{\mathcal{I}} f$, if

$$\{\alpha < \omega_1 : f_\alpha(x) \neq f(x)\} \in \mathcal{I}$$

for every $x \in X$.

Similarly, we write $f_\alpha \to_{\mathcal{I}}^{d} f$ if for every $\varepsilon > 0$ and $x \in X$ we have

$$\{\alpha < \omega_1 : d(f(x), f_\alpha(x)) > \varepsilon\} \in \mathcal{I}.$$

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In case of the ordinary ω_1 convergence, as used in [2] and [6], we have $\mathcal{I} = [\omega_1]^\omega$, that is the ideal of countable subsets of ω_1. However, our motivating theorem, giving partial answer to Problem 1 in [5] on page 490, is related to the particular case when the ideal contains the finite subsets of ω_1, that is $\mathcal{I}_< = [\omega_1]^{<\omega}$.

Theorem 2. *Let (X, τ) be a Polish space, (Y, d) be a separable metric space, and consider a family $f_\alpha \colon X \to Y$ $(\alpha < \omega_1)$ of Baire-2 functions. If $f \colon X \to Y$ is such that $f_\alpha \to_{\frac{d}{\mathcal{I}_<}} f$, then f is Baire-2.*

We note here that the original question asked by T. Natkaniec refered to $\mathcal{I}_<$-convergence. However, it is easy to see that $\mathcal{I}_<$-convergence implies $\frac{d}{\mathcal{I}_<}$-convergence, so the result above is formally stronger than the required. The sufficiency of $\frac{d}{\mathcal{I}_<}$-convergence was pointed out to the author by Petr Holický. We also note that using more sophisticated techniques, this result has already been generalized to every Baire-ξ class (see [4]).

In the following section, we present the characterization of $\Sigma^0_3(\tau)$ sets which is the key element of the proof of Theorem 2. The last section contains the proof of Theorem 2.

Our reference for the basic notions in descriptive set theory is [1]; in particular, $\Pi^0_\xi(\tau)$ ($\Sigma^0_\xi(\tau)$ resp.) stands for the ξ^{th} multiplicative (additive resp.) Borel class in (X, τ), starting with $\Pi^0_1(\tau)$ = closed sets, $\Sigma^0_1(\tau)$ = open sets.

2 $\Sigma^0_3(\tau)$ sets in the Borel hierarchy

Let (C, τ_C) denote the Polish space 2^ω with its usual product topology. To distinguish $\Sigma^0_3(\tau)$ sets from $\Pi^0_3(\tau)$ sets, we construct a $\Pi^0_3(\tau_C)$ set $\mathcal{P} \subseteq C$ such that every $\Sigma^0_3(\tau)$ subset of X containing a suitable copy $\varphi(\mathcal{P})$ of $\mathcal{P}$ is "much bigger" in sense of Baire category than $\varphi(\mathcal{P})$ (for the precise statement, see Lemma 3).

First we have to construct $\mathcal{P}$. The method had already been used by Lusin to build a proper $\Pi^0_3(\tau_C)$ set and was communicated to the author by Petra Šindelářová. Following [1], for two finite sequences $s, t \in \omega^{<\omega}$, we write $s < t$ if t is a (proper) extension of s. The length of s is denoted by $|s|$. If $s = s_1 s_2 \dots s_n$ and $i \in \mathbb{N}$, then $s^\frown i$ stands for the sequence $s_1 s_2 \dots s_n i$.

For every finite sequence $s \in \omega^{<\omega}$, fix a nonempty perfect set $P_s \subseteq C$ with the following propreties:

$$P_\emptyset = C; \tag{1}$$

$$\text{for } t < s,\ P_s \subseteq P_t \text{ and } P_s \text{ is nowhere dense in } (P_t, \tau_C|_{P_t}); \tag{2}$$

$$\bigcup_{i \in \mathbb{N}} P_{s^\frown i} \text{ is dense in } (P_s, \tau_C|_{P_s}). \tag{3}$$

To have $P_{s^\frown i} \subseteq P_s$ $(i \in \mathbb{N})$, one simply has to take a countable dense subset $D_s = \{d_1, d_2, \dots\} \subseteq P_s$ and cover successively every d_i with a perfect set $P_{s^\frown i}$ which is nowhere dense in $(P_s, \tau_C|_{P_s})$. Then (1), (2) and

(3) are obviously satisfied. Once this done, let

$$\mathcal{P} = \bigcap_{n=0}^{\infty} \bigcup_{\substack{s \in \omega^{<\omega} \\ |s| = n}} P_s. \tag{4}$$

Now we can formulate our characterization.

Lemma 3. *Let (X, τ) be a Polish space, $A \subseteq (X, \tau)$ be a Borel set.*

1. *If A is $\Sigma^0_3(\tau)$, then whenever for a continuous one-to-one map*

$$\varphi\colon (C, \tau_C) \to (X, \tau)$$

we have $\varphi(\mathcal{P}) \subseteq A$, then there is an $s \in \omega^{<\omega}$ for which $A \cap \varphi(P_s)$ is of the second category in $(\varphi(P_s), \tau|_{\varphi(P_s)})$.

2. *If A is not $\Sigma^0_3(\tau)$, then there is a continuous one-to-one map*

$$\varphi\colon (C, \tau_C) \to (X, \tau)$$

such that $\varphi(\mathcal{P}) \subseteq A$ and $A \cap \varphi(P_s)$ is meager in $(\varphi(P_s), \tau|_{\varphi(P_s)})$ for every $s \in \omega^{<\omega}$.

The statements involving Baire category do make sense since $\varphi(P_s)$, as a continuous image of the compact set P_s, is closed in the Polish space (X, τ), so itself is Polish with the restricted topology $\tau|_{\varphi(P_s)}$ (see e.g. [1], Proposition 3.3.(ii) on page 13). To prove Lemma 3, we will use the following result (see e.g. [3], page 433). In some sense, Lemma 3 is a quantitative analogue of this result in the special $\xi = 3$ case.

Theorem 4. *(A. Louveau, J. Saint Raymond) Let $3 \leq \xi < \omega_1$ and (X, τ) be a Polish space. If $P_\xi \subseteq C$ is $\Pi^0_\xi(\tau_C)$ but not $\Sigma^0_\xi(\tau_C)$ and $A_0, A_1 \subseteq X$ is any pair of disjoint Borel sets, then either A_0 can be separated from A_1 by a $\Sigma^0_\xi(\tau)$ set or there is a continuous one-to-one map $\varphi\colon (C, \tau_C) \to (X, \tau)$ with $\varphi(P_\xi) \subseteq A_0$ and $\varphi(C \setminus P_\xi) \subseteq A_1$.*

Before giving the proof of Lemma 3, we make two easy observations.

Lemma 5.

1. *$\mathcal{P} \subseteq C$ is a $\Pi^0_3(\tau_C)$ set.*
2. *$\mathcal{P} \cap P_s$ is dense and meager in $(P_s, \tau_C|_{P_s})$ for every $s \in \omega^{<\omega}$.*

Proof. The first statement follows immediately from (4). To see that $\mathcal{P} \subseteq (P_{s_0}, \tau_C|_{P_{s_0}})$ is dense for every $s_0 \in \omega^{<\omega}$, take any nonempty closed ball $B_0 \subseteq P_{s_0}$; we show that $B_0 \cap \mathcal{P} \neq \emptyset$. We construct finite sequences $s_i \in \omega^{<\omega}$ $(i \in \mathbb{N})$ and a sequence of nonempty closed balls $B_i \subseteq (P_{s_i}, \tau_C|_{P_{s_i}})$ $(i \in \mathbb{N})$ such that $s_i \leq s_j$ and $B_{s_j} \subseteq B_{s_i}$ for $0 \leq i \leq j$. This proves the statement since such a $(B_i)_{i \in \mathbb{N}}$ is a nested sequence of nonempty closed sets in (C, τ_C), so

$$\bigcap_{i \in \mathbb{N}} B_i \subseteq B_0 \cap \mathcal{P}$$

is nonempty.

Suppose that s_k and B_k have already already found. By (3), there is an $l \in \mathbb{N}$ such that for $s_{k+1} = s_k{}^\frown l$, $B_k \cap P_{s_{k+1}} \neq \emptyset$. Thus we can find a closed ball $B_{k+1} \subseteq (P_{s_{k+1}}, \tau_C|_{P_{s_{k+1}}})$ contained in $B_k \cap P_{s_{k+1}}$, which completes the construction.

Finally, for every $s \in \omega^{<\omega}$, $\mathcal{P} \cap P_s$ is meager in $(P_s, \tau_C|_{P_s})$ since

$$\mathcal{P} \cap P_s \subseteq \left(\bigcup_{i\in\mathbb{N}} P_{s^\frown i}\right) \cap P_s \tag{5}$$

and by (2), the union on the right hand side of (5) is meager in $(P_s, \tau_C|_{P_s})$.■

Proof of Lemma 3. For the first statement, let $A \subseteq X$ be $\Sigma^0_3(\tau)$, $\varphi\colon (C, \tau_C) \to (X, \tau)$ be continuous, one-to-one, and suppose that $A \cap \varphi(P_s)$ is meager in $(\varphi(P_s), \tau|_{\varphi(P_s)})$ for every $s \in \omega^{<\omega}$. Then

$$A = \bigcup_{n\in\mathbb{N}} A_n$$

where the sets A_n $(n \in \mathbb{N})$ are $\Pi^0_2(\tau)$, and since in Polish spaces a $\Pi^0_2(\tau)$ set is meager if and only if it is nowhere dense, $A_n \cap \varphi(P_s)$ is nowhere dense in $(\varphi(P_s), \tau|_{\varphi(P_s)})$ for every $n \in \mathbb{N}$ and $s \in \omega^{<\omega}$. We define by induction finite sequences $s_n \in \omega^{<\omega}$ $(n \in \mathbb{N})$ and a corresponding sequence of closed balls $B_n \subseteq (X, \tau)$ $(n \in \mathbb{N})$ such that

- (i) $|s_n| = n$ $(n \in \mathbb{N})$;
- (ii) $s_n \leq s_m$ if $n \leq m$;
- (iii) $B_m \subseteq B_n$ if $n \leq m$;
- (iv) $B_n \cap \varphi(P_{s_{n+1}})$ $(n \in \mathbb{N})$ is nonempty and perfect;
- (v) $B_n \cap \varphi(P_{s_n}) \cap A_n = \emptyset$ $(n \in \mathbb{N})$.

This completes the proof, since on one hand, by (2), (iii) and (iv), we have that $B_n \cap \varphi(P_{s_{n+1}})$ is a nested sequence of nonempty perfect sets, so

$$\mathcal{Q} = \bigcap_{n\in\mathbb{N}} B_n \cap \varphi(P_{s_{n+1}}) = \bigcap_{n\in\mathbb{N}} B_n \cap \varphi(P_{s_n})$$

is a nonempty subset of $\varphi(\mathcal{P})$, while on the other hand, by (v), $\mathcal{Q} \cap A = \emptyset$ since $\mathcal{Q} \cap A_n = \emptyset$ for every $n \in \mathbb{N}$, which contradicts $\varphi(\mathcal{P}) \subseteq A$.

Let $s_0 = \emptyset$ and suppose that s_i and B_{i-1} are found for $0 \leq i \leq n$ satisfying (i)–(v); we define B_n and s_{n+1}. Since $A_n \cap \varphi(P_{s_n})$ is nowhere dense in $(\varphi(P_{s_n}), \tau|_{\varphi(P_{s_n})})$, we can find a closed ball $B_n \subseteq B_{n-1}$ for which $B_n \cap \varphi(P_{s_n})$ is a nonempty perfect set and $B_n \cap \varphi(P_{s_n}) \cap A_n = \emptyset$; thus (iii) and (v) hold. By (3), we can find an $i \in \mathbb{N}$ such that $B_n \cap \varphi(P_{{s_n}^\frown i})$ is also nonempty and perfect. With $s_{n+1} = {s_n}^\frown i$, (i), (ii) and (iv) are satisfied, which completes the proof.

For the second statement, let $A \subseteq X$ be Borel but not $\Sigma^0_3(\tau)$. By the first part of the lemma for $(X, \tau) = (C, \tau_C)$ and $\varphi = \mathrm{Id}_C$, $\mathcal{P}$ is not $\Sigma^0_3(\tau_C)$ since by Lemma 5.2, it is meager in P_s for every $s \in \omega^{<\omega}$. Since $\mathcal{P}$ is $\Pi^0_3(\tau_C)$ by Lemma 5.1, we can apply Theorem 4 for $\xi = 3$, $P_3 = \mathcal{P}$, $A_0 = A$ and $A_1 = X \setminus A$.

The set A is not $\Sigma^0_3(\tau)$, so A_0 cannot be separated from A_1 by a $\Sigma^0_3(\tau)$ set. Thus we have a continuous one-to-one map

$$\varphi\colon (C, \tau_C) \to (X, \tau)$$

such that $\varphi(C) \cap A = \varphi(\mathcal{P})$; hence $\varphi(\mathcal{P}) \subseteq A$. We show that $A \cap \varphi(P_s)$ is meager in $(\varphi(P_s), \tau|_{\varphi(P_s)})$ for every $s \in \omega^{<\omega}$.

Take an $s \in \omega^{<\omega}$. Since φ is a continuous one-to-one map on the compact set P_s, it is a homeomorphism between $(P_s, \tau|_{P_s})$ and $(\varphi(P_s), \tau|_{\varphi(P_s)})$. We have

$$A \cap \varphi(P_s) = A \cap (\varphi(C) \cap \varphi(P_s)) = (A \cap \varphi(C)) \cap \varphi(P_s) = \varphi(\mathcal{P}) \cap \varphi(P_s).$$

Since homeomorphism preserves category, $A \cap \varphi(P_s) = \varphi(\mathcal{P}) \cap \varphi(P_s)$ is meager in $(\varphi(P_s), \tau|_{\varphi(P_s)})$ by Lemma 5.2. The proof is complete.■

3 $\mathcal{I}$-convergent functions

We will have to establish connection between function classes and sublevel sets. For this, we will use the following classical result (see e.g. [1], Chapter II, Theorem 24.3 on page 190).

Theorem 6. *Let (X, τ) be a Polish space, (Y, d) be a separable metric space. Then for every $1 \leq \xi \leq \omega_1$, a function $f \colon X \to Y$ is Baire-ξ if and only if $f^{-1}(U) \subseteq X$ is $\Sigma^0_{\xi+1}(\tau)$ for every open set $U \subseteq Y$.*

In the metric space (Y, d), the open ball centered at $x \in Y$ with radius ρ is denoted by $B_d(x, \rho)$. Now we prove Theorem 2.

Proof of Theorem 2. Let $f_\alpha \to_{\frac{d}{\mathcal{I}_<}} f$ for a family $f_\alpha \colon X \to Y$ $(\alpha < \omega_1)$ of Baire-2 functions and suppose that $f : X \to Y$ is not Baire-2. As the pointwise limit of the functions $\{f_\alpha \colon \alpha < \omega\}$, f is clearly Borel, so by Theorem 6, there is an open ball $B_d(x, \rho) \subseteq Y$ such that the $f^{-1}(B_d(x, \rho))$ is Borel but not $\Sigma^0_3(\tau)$. Set

$$H(\varepsilon) = f^{-1}(B_d(x, \rho - \varepsilon)),\ H_\alpha(\varepsilon) = f_\alpha^{-1}(B_d(x, \rho - \varepsilon))$$

for every $\alpha < \omega_1$ and $0 < \varepsilon < \rho$. Note that by Theorem 6, $H_\alpha(\varepsilon)$ is $\Sigma^0_3(\tau)$ for every $\alpha < \omega_1$ and $0 < \varepsilon < \rho$.

Since $H(0)$ is not $\Sigma^0_3(\tau)$, by Lemma 3.2 there is a continuous one-to-one map $\varphi \colon (C, \tau_C) \to (X, \tau)$ such that

(*a*) $\varphi(\mathcal{P}) \subseteq H(0)$, and

(*b*) $H(0) \cap \varphi(P_s)$ is meager in $(\varphi(P_s), \tau|_{\varphi(P_s)})$ for every $s \in \omega^{<\omega}$.

For $\varepsilon > 0$, let $J_1(\varepsilon)$ denote the set of those indices $\alpha < \omega_1$ for which $H_\alpha(\varepsilon)$ is of the second category in $(\varphi(P_s), \tau|_{\varphi(P_s)})$ for some $s \in \omega^{<\omega}$. We prove that $\omega_1 \setminus J_1(\varepsilon)$ is finite for every ε sufficiently small. Suppose that this is not true; take a positive sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ with $\varepsilon_i \to 0$ and a countably infinite set $J'(\varepsilon_i) \subseteq \omega_1 \setminus J_1(\varepsilon_i)$ for every $i \in \mathbb{N}$. By the definition of $\frac{d}{\mathcal{I}_<}-$ convergence,

$$H(\varepsilon_i) \subseteq H'(\varepsilon_i) := \bigcup_{\alpha \in J'(\varepsilon_i)} H_\alpha(\varepsilon_i), \tag{6}$$

so by (a), we have that

$$\varphi(\mathcal{P}) \subseteq H(0) \subseteq \bigcup_{i \in \mathbb{N}} H(\varepsilon_i) \subseteq \bigcup_{i \in \mathbb{N}} H'(\varepsilon_i). \tag{7}$$

By (6), $H'(\varepsilon_i)$ $(i \in \mathbb{N})$ is $\Sigma^0_3(\tau)$, so by (7) we can apply Lemma 3.1 for $A = \bigcup_{i\in\mathbb{N}} H'(\varepsilon_i)$. We obtain that A is of the second category in $(\varphi(P_s), \tau|_{\varphi(P_s)})$ for some $s \in \omega^{<\omega}$, which contradicts to the definition of $J_1(\varepsilon)$.

So there is an $\varepsilon_0 > 0$ such that $J_1(\varepsilon)$ is of cardinality ω_1 for every $\varepsilon < \varepsilon_0$. In particular, given that $\omega^{<\omega}$ is countable and $(\varphi(P_s), \tau|_{\varphi(P_s)})$ has countable base for every $s \in \omega^{<\omega}$, there is an $s \in \omega^{<\omega}$ and an open set $U \subseteq (\varphi(P_s), \tau|_{\varphi(P_s)})$ such that for a countably infinite set of indices $J'' \subseteq J_1(\varepsilon_0/2)$ we have that $H_\alpha(\varepsilon_0/2)) \cap \varphi(P_s)$ is comeager in U in the $\tau|_{\varphi(P_s)}$ topology whenever $\alpha \in J''$. Hence by the Baire Category Theorem for

$$H'' = \bigcap_{\alpha \in J''} H_\alpha(\varepsilon_0/2),$$

H'' is also comeager in U in the $\tau|_{\varphi(P_s)}$ topology, so by (b) we can find a point $x_0 \in H'' \setminus H(0)$. Thus f_α $(\alpha < \omega_1)$ is not $\frac{d}{\mathcal{I}_<}$-convergent since

$$J'' \subseteq \left\{\alpha < \lambda \colon d(f(x_0), f_\alpha(x_0)) > \frac{\varepsilon_0}{2}\right\}$$

is infinite; a contradiction. This completes the proof.■

As we have mentioned above, Theorem 2 is true for every Baire class (see [4]). The proof of the general theorem uses a characterization of $\Sigma^0_\xi(\tau)$ sets for every $\xi < \omega_1$ involving Baire category, similarly to Lemma 3. Finally we note that this approach makes also possible to treat the pointwise convergence of sequences of Borel functions with length λ where $\omega_1 < \lambda < 2^{\aleph_0}$ is a cardinal.

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