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On perturbations preserving the immediate norm continuity of semigroups

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Abstract

We show that the Desch–Schappacher perturbation and the Miyadera–Voigt perturbation of an immediately norm continuous semigroup are immediately norm continuous. We also show that a perturbation theorem of C. Batty, C. Kaiser and L. Weis based on a generation theorem of A.M. Gomilko, D.-X. Feng and D.-H. Shi also preserves the immediate norm continuity of semigroups. The novelty of these results is that, contrary to the numerous related results, we obtain the immediate norm continuity of the perturbed semigroup without additional assumptions.

Keywords: Strongly continuous semigroup; Norm continuity; Perturbation

1. Introduction

If a strongly continuous semigroup $(T(t))_{t\geqslant 0}$, generated by A, turns out to be norm continuous, then several powerful tools become applicable: the Spectral Mapping Theorem holds, i.e. $\sigma(T(t))\setminus\{0\}=e^{t\sigma(A)}$ $(t\geqslant 0)$; as a consequence, the Liapunov stability theorem holds, that is the asymptotic behavior of $(T(t))_{t\geqslant 0}$ is completely described by the spectrum of A; the compactness of $(T(t))_{t\geqslant 0}$ is characterized by the compactness of the resolvent of A; etc. Therefore it is very useful to prove the norm continuity of a semigroup if it happens to be norm continuous.

This problem received particularly much attention in the context of perturbations. Since the typical motivation for the application of a perturbation technique is to handle a semigroup by splitting its generator into the sum or product of two simpler operators, it is an indispensable requirement to be able to transfer the regularity properties, if any, of the semigroups generated by these simpler operators to the initial semigroup. The importance of this concept is clearly indicated by the vast amount and diversity of results establishing the immediate or eventual norm continuity of numerous semigroups obtained by various perturbation methods in different contexts: see, e.g. [1–3] and [13] for

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regularity issues of partial differential equations with delay, [5] and [6] for spectral properties, [20] and references therein for positive perturbations.

The purpose of this paper is to show that three perturbation methods, the Desch–Schappacher perturbation, the Miyadera–Voigt perturbation and a perturbation of C. Batty, C. Kaiser and L. Weis based on a generation theorem of A.M. Gomilko, D.-X. Feng and D.-H. Shi, preserve the immediate norm continuity of semigroups. There are many results in the literature showing that in a given special situation some kind of norm continuity of the semigroup is preserved under some of these perturbations. However, even the works which aim to formulate general theorems on the preservation of regularity properties use superfluous additional assumptions. Such assumptions are usually of the form of a smallness requirement like [3, Hypothesis 1.45].

In our work we consider mainly the Desch–Schappacher and Miyadera–Voigt perturbations. The reason for this is that they are widely used in the literature, the theory of semigroups admitted already a canonical set of restrictions outlining their applicability and that norm continuity issues are usually present in the applications of these perturbation methods. The motivation for the third perturbation method we consider is its relation to the problem of characterizing immediate norm continuity via resolvent norm estimates (see Section 3). Note that the preservation of immediate norm continuity under such general perturbations is somewhat surprising: even a bounded perturbation can destroy both the slightly weaker eventual norm continuity property (see, e.g. [10, 1.15 Example]) and the stronger immediate differentiability property (see [7] and [19]). For the connection of our results with multiplicative perturbations we refer to [10, Chapter III.3.d].

It is important to point out that we do not aim to handle the different perturbation contexts by finding their common generalization. Instead, we would like to illustrate how the idea behind our results can be applied in different situations and we hope that it will help to eliminate needless technical assumptions also in cases which are not covered by the concrete theorems of this paper.

In the following section we prove first that the Desch–Schappacher and Miyadera–Voigt perturbations preserve immediate norm continuity. We close the paper with the analysis of the perturbation method of C. Batty, C. Kaiser and L. Weis and we also discuss the relation of our results to the problem of characterizing the immediate norm continuity of semigroups in arbitrary Banach spaces.

2. The perturbations

Our reference for the notions in semigroup theory is [10] with special emphasis on [10, Chapter III.3] treating perturbations. For a Banach space X, $\mathcal{L}(X)$ stands for the Banach space of bounded linear operators on X. We denote the reals, the nonnegative reals and the nonnegative integers by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively. We recall some definitions.

Definition 1. Let X be a Banach space. A strongly continuous function $T:\mathbb{R}^+ \to \mathcal{L}(X)$ is *immediately norm continuous* if $T:(0,\infty) \to \mathcal{L}(X)$ is continuous.

Definition 2. Let A generate a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X and let $\lambda \in \mathbb{C}\setminus\sigma(A)$ be fixed. Then $(X_1, \|.\|_1) = (D(A), \|.\|_1)$ with $\|x\|_1 = \|(\lambda - A)x\|$ $(x \in D(A))$ denotes the *Sobolev space of first order* associated to $T((t))_{t\geqslant 0}$. We set $T_1(t) = T(t)|_{X_1}$ $(t \geqslant 0)$.

Similarly, set $\|x\|_{-1} = \|(\lambda - A)^{-1}x\|$ $(x \in X)$; then $(X_{-1}, \|.\|_{-1})$ denotes the completion of X under the norm $\|.\|_{-1}$ and is called the *Sobolev space of order* -1 associated to $T((t))_{t\geqslant 0}$. For $t \geqslant 0$ we define $T_{-1}(t)$ as the continuous extension of $T(t)$ to the space $(X_{-1}, \|.\|_{-1})$.

We need the following properties of these Sobolev spaces (see, e.g. [10, 5.5 Theorem, p. 126 and 5.9 Exercises, p. 129]).

Proposition 3. *With the notation of Definition* 2 *we have the following*:

(1) *The spaces* $(X_1, \|.\|_1)$ *and* $(X_{-1}, \|.\|_{-1})$ *are independent of the choice of* λ.
(2) $(T_1(t))_{t\geqslant 0}$ *is a strongly continuous semigroup on the Banach space* $(X_1, \|.\|_1)$ *and we have* $\|T_1(t)\|_1 = \|T(t)\|$ $(t \geqslant 0)$.

(3) $(T_{-1}(t))_{t\geqslant 0}$ *is a strongly continuous semigroup on the Banach space* $(X_{-1}, \|.\|_{-1})$ *and we have* $\|T_{-1}(t)\|_{-1} = \|T(t)\|$ $(t \geqslant 0)$.

Next we recall the Desch–Schappacher and the Miyadera–Voigt perturbation methods.

Definition 4. Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space X. For each $t_0 > 0$ let

$$\mathcal{X}_{t_0} = C\big([0, t_0], \mathcal{L}_s(X)\big)$$

denote the Banach space of strongly continuous $\mathcal{L}(X)$-valued functions equipped with the norm $\|F\|_\infty = \sup_{t\in[0,t_0]} \|F(t)\|$.

(1) We say that an operator $B \in \mathcal{L}(X, X_{-1})$ satisfies the conditions of the perturbation method of Desch and Schappacher if there is a $t_0 > 0$ such that for every $F \in \mathcal{X}_{t_0}$ the abstract Volterra operator

$$F \mapsto V_B(F), \quad (V_B F)(t) = \int_0^t T_{-1}(t-s) B F(s)\,\mathrm{d}s \in \mathcal{L}(X, X_{-1}) \quad (0 \leqslant t \leqslant t_0) \tag{1}$$

defines an $\mathcal{L}(X)$ valued strongly continuous function, that is $V_B : \mathcal{X}_{t_0} \to \mathcal{X}_{t_0}$ is bounded, and $\|V_B\| < 1$.

(2) We say that an operator $B \in \mathcal{L}(X_1, X)$ satisfies the conditions of the perturbation method of Miyadera and Voigt if there is a $t_0 > 0$ such that for every $F \in \mathcal{X}_{t_0}$ the abstract Volterra operator

$$F \mapsto V_B^*(F), \quad \big(V_B^* F\big)(t) = \int_0^t F(s) B T(t-s)\,\mathrm{d}s \in \mathcal{L}(X_1, X) \quad (0 \leqslant t \leqslant t_0), \tag{2}$$

can be extended to an $\mathcal{L}(X)$ valued strongly continuous function, that we also denote by V_B^*, i.e. $V_B^* : \mathcal{X}_{t_0} \to \mathcal{X}_{t_0}$ is bounded, and $\|V_B^*\| < 1$.

The relevant generation theorems are the following (see, e.g. [10, 3.1 Theorem, p. 183, 3.2 Corollary, p. 186], [10, 3.14 Theorem, p. 196, 3.15 Corollary, p. 198] and [18]).

Theorem 5. *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X.*

(1) *If an operator B satisfies Definition* 4(1), *then the operator $(A_{-1} + B)|_X$ with domain*

$$D\big((A_{-1} + B)|_X\big) = \{x \in X\colon\ A_{-1}x + Bx \in X\}$$

generates a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ on $(X, \|.\|)$. The perturbed semigroup $(S(t))_{t\geqslant 0}$ is given by the Dyson–Phillips series

$$S(t) = \sum_{n=0}^{\infty} S_n(t) \quad (t \geqslant 0),$$

where $S_0(t) = T(t)$ $(t \geqslant 0)$ and

$$S_n(t) = \int_0^t T_{-1}(t-s) B S_{n-1}(s)\,\mathrm{d}s \quad (t \geqslant 0,\ 1 \leqslant n < \infty).$$

(2) *If an operator B satisfies Definition* 4(2) *the operator $A + B$ with domain $D(A + B) = D(A)$ generates a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ on $(X, \|.\|)$. The perturbed semigroup $(S(t))_{t\geqslant 0}$ is given by the Dyson–Phillips series*

$$S(t) = \sum_{n=0}^{\infty} S_n(t) \quad (t \geqslant 0),$$

where $S_n(t) = [(V_B^*)^n T](t)$ $(t \geqslant 0,\ n \in \mathbb{N})$.

Moreover, for both 1 *and* 2*, if the functions* $S_n : (0, t_0) \to \mathcal{L}(X)$ $(n \in \mathbb{N})$ *are continuous, then* $(S(t))_{t \geqslant 0}$ *is immediately norm continuous.*

We turn to the proof of the preservation of immediate norm continuity under Desch–Schappacher and Miyadera–Voigt perturbations.

2.1. Desch–Schappacher setting

We prove the following theorem.

Theorem 6. *Let A generate an immediately norm continuous strongly continuous semigroup $(T(t))_{t \geqslant 0}$ on a Banach space X and let $B \in \mathcal{L}(X, X_{-1})$ satisfy Definition* 4(1)*, i.e. for a $t_0 > 0$ and $0 < q < 1$ the abstract Volterra operator in* (1) *satisfies that for every strongly continuous function $F : [0, t_0] \to \mathcal{L}(X)$,*

$$V_B(F) \in \mathcal{X}_{t_0} \quad \text{and} \quad \|V_B(F)\|_\infty \leqslant q\|F\|_\infty. \tag{3}$$

Then the perturbed semigroup $(S(t))_{t \geqslant 0}$ is immediately norm continuous.

First we need an auxiliary lemma.

Lemma 7. *Let $(T(t))_{t \geqslant 0}$ and B satisfy the assumptions of Theorem* 6 *and fix $0 < t_1 < t_2 < t_0$. Then for every $\varepsilon > 0$ there is a $0 < \delta < t_1$ such that for every $t_1 \leqslant t \leqslant t_2$ and $F \in \mathcal{X}_t$ satisfying $F(s) = 0$ $(s \in \{0\} \cup [\delta, t])$ we have*

$$\left\| \int_0^\delta T_{-1}(t-s)BF(s)\,\mathrm{d}s \right\| \leqslant \varepsilon \|F|_{[0,\delta]}\|_\infty.$$

Proof. Fix $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that

$$\frac{q}{N} \sup_{t \in [0,t_0]} \|T(t)\| \leqslant \frac{\varepsilon}{2}. \tag{4}$$

Let $0 < \delta < t_1$ satisfy $t_2 + N\delta \leqslant t_0$ and

$$q \sup\bigl\{\|T(t+\tau) - T(t)\|:\ t \in [t_1 - \delta, t_2],\ \tau \in [0, N\delta]\bigr\} \leqslant \frac{\varepsilon}{2}; \tag{5}$$

such a δ exists since $(T(t))_{t \geqslant 0}$ is immediately norm continuous. We show that this δ fulfills the requirements.

Take t and F as prescribed, and define $f : [0, t_0] \to \mathcal{L}(X)$ by

$$f(k\delta + x) = F(x) \quad \bigl(x \in [0, \delta),\ 0 \leqslant k < N\bigr), \qquad f|_{[N\delta, t_0]} = 0.$$

Then f is strongly continuous on $[0, t_0]$ and

$$\|f\|_\infty \leqslant \|F|_{[0,\delta]}\|_\infty. \tag{6}$$

By assumption,

$$\int_0^{N\delta} T_{-1}(N\delta - s)Bf(s)\,\mathrm{d}s \in \mathcal{L}(X).$$

Since $T_{-1}(t)|_X = T(t)$ we have

$$\int_0^{N\delta} T_{-1}(t + N\delta - s)Bf(s)\,\mathrm{d}s = T(t)\left(\int_0^{N\delta} T_{-1}(N\delta - s)Bf(s)\,\mathrm{d}s\right) \in \mathcal{L}(X);$$

thus

$$\left\| \int_0^{N\delta} T_{-1}(t + N\delta - s)Bf(s)\,\mathrm{d}s \right\| \leqslant \|T(t)\| \left\| \int_0^{N\delta} T_{-1}(N\delta - s)Bf(s)\,\mathrm{d}s \right\|.$$

So by (3) and (6),

$$\left\| \int_0^{N\delta} T_{-1}(t + N\delta - s) B f(s)\,\mathrm{d}s \right\| \leqslant q \left\| T(t) \right\| \|f\|_\infty \leqslant q \left\| T(t) \right\| \|F|_{[0,\delta]}\|_\infty. \tag{7}$$

Similarly,

$$\int_0^{\delta} T_{-1}(\delta - s) B F(s)\,\mathrm{d}s \in \mathcal{L}(X),$$

so by $T_{-1}(t)|_X = T(t)$ for $1 \leqslant k \leqslant N$ we have

$$\int_0^{\delta} \bigl[T_{-1}(t + k\delta - s) - T_{-1}(t - s)\bigr] B F(s)\,\mathrm{d}s = \bigl[T\bigl(t + (k-1)\delta\bigr) - T(t-\delta)\bigr] \left(\int_0^{\delta} T_{-1}(\delta - s) B F(s)\,\mathrm{d}s \right)$$

hence by (3), (5) and (6),

$$\begin{aligned} \left\| \int_0^{\delta} \bigl[T_{-1}(t + k\delta - s) - T_{-1}(t - s)\bigr] B F(s)\,\mathrm{d}s \right\| &\leqslant q \left\| T\bigl(t + (k-1)\delta\bigr) - T(t - \delta) \right\| \|F|_{[0,\delta]}\| \\ &\leqslant \frac{\varepsilon}{2} \|F|_{[0,\delta]}\|. \end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned} \int_0^{N\delta} T_{-1}(t + N\delta - s) B f(s)\,\mathrm{d}s &= \sum_{k=1}^{N} \int_0^{\delta} T_{-1}(t + k\delta - s) B F(s)\,\mathrm{d}s \\ &= N \int_0^{\delta} T_{-1}(t - s) B F(s)\,\mathrm{d}s + \sum_{k=1}^{N} \int_0^{\delta} \bigl[T_{-1}(t + k\delta - s) - T_{-1}(t - s)\bigr] B F(s)\,\mathrm{d}s, \end{aligned}$$

i.e.

$$\begin{aligned} &\int_0^{\delta} T_{-1}(t - s) B F(s)\,\mathrm{d}s \\ &\quad = \frac{1}{N} \int_0^{N\delta} T_{-1}(t + N\delta - s) B f(s)\,\mathrm{d}s - \frac{1}{N} \sum_{k=1}^{N} \int_0^{\delta} \bigl[T_{-1}(t + k\delta - s) - T_{-1}(t - s)\bigr] B F(s)\,\mathrm{d}s. \end{aligned}$$

Thus by (7), (8) and (4),

$$\begin{aligned} &\left\| \int_0^{\delta} T_{-1}(t - s) B F(s)\,\mathrm{d}s \right\| \\ &\quad \leqslant \frac{1}{N} \left\| \int_0^{N\delta} T_{-1}(t + N\delta - s) B f(s)\,\mathrm{d}s \right\| + \frac{1}{N} \sum_{k=1}^{N} \left\| \int_0^{\delta} \bigl[T_{-1}(t + k\delta - s) - T_{-1}(t - s)\bigr] B F(s) g(s)\,\mathrm{d}s \right\| \\ &\quad \leqslant \left(\frac{q}{N} \left\| T(t) \right\| + \frac{\varepsilon}{2} \right) \|F|_{[0,\delta]}\|_\infty \\ &\quad \leqslant \varepsilon \|F|_{[0,\delta]}\|_\infty. \end{aligned}$$

This completes the proof. □

The key element of the proof of Theorem 6 is the following lemma.

Lemma 8. *Under the assumptions of Theorem* 6, *for every* $\varepsilon > 0$ *and* $0 < t_1 < t_2 < t_0$ *there is a* $0 < \delta < t_1$ *such that for every* $0 < \delta' \leqslant \delta$, $t \in [t_1, t_2]$ *and* $F \in \mathcal{X}_t$,

$$\left\| \int_0^{\delta'} T_{-1}(t-s)BF(s)\,\mathrm{d}s \right\| \leqslant \varepsilon \|F|_{[0,\delta']}\|_\infty. \tag{9}$$

Proof. Fix $\varepsilon > 0$ and $0 < t_1 < t_2 < t_0$. By Lemma 7 there is a $0 < \delta < t_1$ such that whenever $t_1 \leqslant t \leqslant t_2$, $F \in \mathcal{X}_t$ and $g \in C([0,t], \mathbb{C})$ with $\|g\|_\infty \leqslant 1$, $g(s) = 0$ $(s \in \{0\} \cup [\delta, t])$ then

$$\left\| \int_0^{t} T_{-1}(t-s)BF(s)g(s)\,\mathrm{d}s \right\| \leqslant \varepsilon \big\| (Fg)|_{[0,\delta]} \big\|_\infty. \tag{10}$$

We show that this δ fulfills the requirements.

Let $t \in [t_1, t_2]$, $F \in \mathcal{X}_t$ and $0 < \delta' \leqslant \delta$. For every $n \in \mathbb{N}$ take $g_n \in C([0,t], \mathbb{R})$ such that $0 \leqslant g_n \leqslant g_{n+1} \leqslant 1$, $g_n(s) = 0$ $(s \in \{0\} \cup [\delta', t])$ and $g_n(s) = 1$ $(s \in [1/n, \delta' - 1/n])$. Set

$$Z_n = \int_0^{\delta'} T_{-1}(t-s)BF(s)g_n(s)\,\mathrm{d}s \quad (n \in \mathbb{N}),$$

we show that $(Z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(X)$.

By Lemma 7, for every $\tilde{\varepsilon} > 0$ there is a $0 < \tilde{\delta} < \delta'$ such that for $\tilde{g} \in C([0,t], \mathbb{C})$ with $\|\tilde{g}\|_\infty \leqslant 1$, $\tilde{g}(s) = 0$ $(s \in \{0\} \cup [\tilde{\delta}, \delta' - \tilde{\delta}] \cup [\delta', t])$ we have

$$\left\| \int_0^{\tilde{\delta}} T_{-1}(t-s)BF(s)\tilde{g}(s)\,\mathrm{d}s \right\| \leqslant \frac{\tilde{\varepsilon}}{2} \|F|_{[0,\delta']}\|_\infty,$$

$$\left\| \int_{\delta'-\tilde{\delta}}^{\delta'} T_{-1}(t-s)BF(s)\tilde{g}(s)\,\mathrm{d}s \right\| = \left\| \int_0^{\tilde{\delta}} T_{-1}(t-\delta'+\tilde{\delta}-s)BF(\delta'-\tilde{\delta}+s)\tilde{g}(\delta'-\tilde{\delta}+s)\,\mathrm{d}s \right\| \leqslant \frac{\tilde{\varepsilon}}{2} \|F|_{[0,\delta']}\|_\infty.$$

Fix $\tilde{\varepsilon} > 0$ and let $n, m \in \mathbb{N}$ with $1/\tilde{\delta} \leqslant n, m$. Then with $\tilde{g} = g_n - g_m$ we get

$$\begin{aligned}\|Z_n - Z_m\| &\leqslant \left\| \int_0^{\tilde{\delta}} T_{-1}(t-s)BF(s)[g_n - g_m](s)\,\mathrm{d}s \right\| + \left\| \int_{\delta'-\tilde{\delta}}^{\delta'} T_{-1}(t-s)BF(s)[g_n - g_m](s)\,\mathrm{d}s \right\| \\ &\leqslant \tilde{\varepsilon} \|F|_{[0,\delta']}\|_\infty.\end{aligned}$$

Since $\tilde{\varepsilon} > 0$ was arbitrary, we obtained that $(Z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(X)$.

Hence $(Z_n)_{n\in\mathbb{N}}$ converges to some $Z \in \mathcal{L}(X)$, moreover by (10),

$$\|Z\| \leqslant \sup_{n\in\mathbb{N}} \|Z_n\| \leqslant \varepsilon \sup_{n\in\mathbb{N}} \big\| (Fg_n)|_{[0,\delta]} \big\|_\infty \leqslant \varepsilon \|F|_{[0,\delta']}\|_\infty. \tag{11}$$

On the other hand, for every $x \in X$, by Lebesgue's Dominated Convergence Theorem in $C([0,\delta'], (X_{-1}, \|.\|_{-1}))$ we have

$$Zx = \lim_{n\to\infty} \int_0^{\delta'} T_{-1}(t-s)BF(s)g_n(s)x\,\mathrm{d}s = \int_0^{\delta'} T_{-1}(t-s)BF(s)x\,\mathrm{d}s. \tag{12}$$

By putting (11) and (12) together we get (9), so the proof is complete. □

Proof of Theorem 6. Fix $t_0 > 0$ and $0 < q < 1$ such that (3) holds. It is enough to show that $V_B F : (0, t_0) \to \mathcal{L}(X)$ is norm continuous for every $F \in \mathcal{X}_{t_0}$ which is norm continuous on $(0, t_0)$; then by induction we have that

$$S_n = V_B S_{n-1} = \cdots = \underbrace{V_B\big[\ldots V_B[V_B}_{n} T]\ldots\big] = [\underbrace{V_B \circ \cdots \circ V_B}_{n}]T \quad (n \in \mathbb{N})$$

is immediately norm continuous, so the immediate norm continuity of the perturbed semigroup $(S(t))_{t\geqslant 0}$ follows from Theorem 5.

Let $F \in \mathcal{X}_{t_0}$ be norm continuous on $(0, t_0)$, let $0 < t_1 < t_2 < t_0$ and $\varepsilon > 0$ be arbitrary. We find a $\delta > 0$ such that $t, t' \in [t_1, t_2]$, $0 \leqslant t' - t < \delta$ imply $\|(V_B F)(t') - (V_B F)(t)\| \leqslant \varepsilon$. This will complete the proof.

We have

$$\begin{aligned}\big[(V_B F)(t') - (V_B F)(t)\big] &= \int_0^{t'} T_{-1}(t'-s)BF(s)\,\mathrm{d}s - \int_0^{t} T_{-1}(t-s)BF(s)\,\mathrm{d}s \\ &= \int_0^{t'-t} T_{-1}(t'-s)BF(s)\,\mathrm{d}s + \int_0^{t} T_{-1}(t-s)B\big[F(t'-t+s) - F(s)\big]\,\mathrm{d}s. \qquad (13)\end{aligned}$$

By the norm continuity of F on $(0, t_0)$ and by Lemma 8 we can find first a δ_1 and then a δ satisfying $0 < \delta \leqslant \delta_1 < \min\{t_1/2, t_0 - t_2\}$ such that $0 \leqslant t' - t < \delta$ implies

$$\big\|\big[F(t'-t+\delta_1+.) - F(\delta_1+.)\big]\big|_{[0,t-\delta_1]}\big\|_\infty \leqslant \frac{\varepsilon}{4q}, \qquad (14)$$

$$\left\|\int_0^{t'-t} T_{-1}(t'-s)BF(s)\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{4}, \qquad (15)$$

$$\left\|\int_0^{\delta_1} T_{-1}(t-s)BF(t'-t+s)\right\| \leqslant \frac{\varepsilon}{4}, \qquad \left\|\int_0^{\delta_1} T_{-1}(t-s)BF(s)\right\| \leqslant \frac{\varepsilon}{4}. \qquad (16)$$

The first term on the right-hand side of (13) is estimated by (15). For the second term we have

$$\begin{aligned}&\int_0^{t} T_{-1}(t-s)B\big[F(t'-t+s) - F(s)\big]\,\mathrm{d}s \\ &\quad = \int_0^{t-\delta_1} T_{-1}(t-\delta_1-s)B\big[F(t'-t+\delta_1+s) - F(\delta_1+s)\big]\,\mathrm{d}s + \int_0^{\delta_1} T_{-1}(t-s)B\big[F(t'-t+s) - F(s)\big]\,\mathrm{d}s.\end{aligned}$$

By (14) and (3),

$$\left\|\int_0^{t-\delta_1} T_{-1}(t-\delta_1-s)B\big[F(t'-t+\delta_1+s) - F(\delta_1+s)\big]\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{4};$$

and by (16),

$$\left\|\int_0^{\delta_1} T_{-1}(t-s)B\big[F(t'-t+s) - F(s)\big]\,\mathrm{d}s\right\| \leqslant \left\|\int_0^{\delta_1} T_{-1}(t-s)BF(t'-t+s)\right\| + \left\|\int_0^{\delta_1} T_{-1}(t-s)BF(s)\right\| \leqslant \frac{\varepsilon}{2}.$$

So we obtained $\|[(V_B F)(t') - (V_B F)(t)]\| \leqslant \varepsilon$, as required. □

2.2. Miyadera–Voigt setting

Given the notational similarity of the Desch–Schappacher and Miyadera–Voigt perturbations the argument what follows is also notationally similar to the one presented in the previous section. Notice however that the resemblance is only superficial: the same computation works for completely different reasons. In particular the Miyadera–Voigt setting is much simpler than the Desch–Schappacher was. We prove the following theorem.

Theorem 9. *Let A generate an immediately norm continuous strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X. Let $B \in \mathcal{L}(X_1, X)$ satisfy Definition 4(2), i.e. for a $t_0 > 0$ and $0 < q < 1$ the abstract Volterra operator in (2) satisfies that for every strongly continuous function $F : [0, t_0] \to \mathcal{L}(X)$, $[V_B^* F](t)$ $(0 \leqslant t \leqslant t_0)$ can be extended to an operator in $\mathcal{L}(X)$, that we also denote by $[V_B^* F](t)$, such that*

$$V_B^* F \in \mathcal{X}_{t_0} \quad \text{and} \quad \left\| V_B^* F \right\|_\infty \leqslant q \|F\|_\infty. \tag{17}$$

Then the perturbed semigroup $(S(t))_{t\geqslant 0}$ is immediately norm continuous.

We start with an analogue of Lemma 8.

Lemma 10. *Under the assumptions of Theorem 9, for every $\varepsilon > 0$ and $0 < t_1 < t_2 < t_0$ there is a $0 < \delta < t_1$ such that for every $0 < \delta' \leqslant \delta$, $t \in [t_1, t_2]$ and $F \in \mathcal{X}_t$,*

$$\left\| \int_0^{\delta'} F(s) BT(t-s)\, \mathrm{d}s \right\| \leqslant \varepsilon \|F|_{[0,\delta']}\|_\infty. \tag{18}$$

Proof. Fix $\varepsilon > 0$ and $0 < t_1 < t_2 < t_0$. There is an $N \in \mathbb{N}$ such that

$$\frac{q}{N} \sup_{t\in[0,t_0]} \left\| T(t) \right\| \leqslant \frac{\varepsilon}{3}. \tag{19}$$

Let $0 < \delta < t_1$ satisfy $t_2 + N\delta \leqslant t_0$ and

$$q \sup\left\{ \left\| T(t+\tau) - T(t) \right\| \colon\, t \in [t_1 - \delta, t_2],\ \tau \in [0, N\delta] \right\} \leqslant \frac{\varepsilon}{3}; \tag{20}$$

such a δ exists since $(T(t))_{t\geqslant 0}$ is immediately norm continuous. We show that this δ fulfills the requirements.

Let $0 < \delta' \leqslant \delta$ and $x \in D(A)$. For every $n \in \mathbb{N}$ take $g_n \in C([0, t_0], \mathbb{R})$ such that $0 \leqslant g_n \leqslant g_{n+1} \leqslant 1$, $g_n(s) = 0$ $(s \in \{0\} \cup [\delta', t_0])$ and $g_n(s) = 1$ $(s \in [1/n, \delta' - 1/n])$. For every $n \in \mathbb{N}$ let $f_n : [0, t_0] \to \mathcal{L}(X)$ be defined by

$$f_n(k\delta' + x) = F(x) g_n(x) \quad \big(x \in [0, \delta'),\ 0 \leqslant k < N\big), \qquad f|_{[N\delta', t_0]} = 0.$$

It is clear that f_n is strongly continuous on $[0, t_0]$ and

$$\|f_n\|_\infty = \left\| (F g_n)|_{[0,\delta']} \right\|_\infty \leqslant \|F|_{[0,\delta']}\|_\infty \quad (n \in \mathbb{N}).$$

So by (17),

$$\left\| \int_0^{N\delta'} f_n(s) BT(t + N\delta' - s) x\, \mathrm{d}s \right\| = \left\| \int_0^{N\delta'} f_n(s) BT(N\delta' - s)\big[T(t)x\big]\, \mathrm{d}s \right\| \leqslant q \|F|_{[0,\delta']}\|_\infty \left\| T(t)x \right\|. \tag{21}$$

On the other hand,

$$\int_0^{N\delta'} f_n(s) BT(t + N\delta' - s) x\, \mathrm{d}s = \sum_{k=1}^{N} \int_0^{\delta'} F(s) g_n(s) BT(t + k\delta' - s) x\, \mathrm{d}s$$

$$= N\int_0^{\delta'} F(s)BT(t-s)x\,\mathrm{d}s + \sum_{k=1}^{N}\int_0^{\delta'} F(s)\big[g_n(s)-1\big]BT(t+k\delta'-s)x\,\mathrm{d}s + \sum_{k=1}^{N}\int_0^{\delta'} F(s)B\big[T(t+k\delta'-s)-T(t-s)\big]x\,\mathrm{d}s,$$

i.e.

$$\int_0^{\delta'} F(s)BT(t-s)x\,\mathrm{d}s = \frac{1}{N}\int_0^{N\delta'} f_n(s)BT(t+N\delta'-s)x\,\mathrm{d}s - \frac{1}{N}\sum_{k=1}^{N}\int_0^{\delta'} F(s)\big[g_n(s)-1\big]BT(t+k\delta'-s)x\,\mathrm{d}s - \frac{1}{N}\sum_{k=1}^{N}\int_0^{\delta'} F(s)B\big[T(t+k\delta'-s)-T(t-s)\big]x\,\mathrm{d}s. \tag{22}$$

We estimate the terms on the right-hand side of (22). By (21) and (19),

$$\frac{1}{N}\left\|\int_0^{N\delta'} f_n(s)BT(t+N\delta'-s)x\,\mathrm{d}s\right\| \leqslant \frac{q}{N}\|F|_{[0,\delta']}\|_\infty \big\|T(t)x\big\| \leqslant \frac{\varepsilon}{3}\|F|_{[0,\delta']}\|_\infty\|x\|.$$

The functions $F(.)BT(t+k\delta'-.)x:[0,\delta']\to X$ are continuous, we have that $|g_n(s)-1|\leqslant 1$ and $|g_n(s)-1|\to 0$ as $n\to\infty$ almost everywhere, so by Lebesgue's Dominated Convergence Theorem there is an $n>0$ such that

$$\frac{1}{N}\sum_{k=1}^{N}\left\|\int_0^{\delta'} F(s)\big[g_n(s)-1\big]BT(t+k\delta'-s)x\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{3}\|F|_{[0,\delta']}\|_\infty\|x\|.$$

By the semigroup property, for every $1\leqslant k\leqslant N$ we have

$$\int_0^{\delta'} F(s)B\big[T(t+k\delta'-s)-T(t-s)\big]x\,\mathrm{d}s = \int_0^{\delta'} F(s)BT(\delta'-s)\big[T\big(t+(k-1)\delta'\big)x - T(t-\delta')x\big]\,\mathrm{d}s,$$

so by (17) and (20),

$$\left\|\int_0^{\delta'} F(s)B\big[T(t+k\delta'-s)-T(t-s)\big]x\,\mathrm{d}s\right\| \leqslant q\|F|_{[0,\delta']}\|_\infty \big\|T\big(t+(k-1)\delta'\big)-T(t-\delta')\big\|\|x\| \leqslant \frac{\varepsilon}{3}\|F|_{[0,\delta']}\|_\infty\|x\|.$$

This leads to

$$\frac{1}{N}\sum_{k=1}^{N}\left\|\int_0^{\delta'} F(s)B\big[T(t+k\delta'-s)-T(t-s)\big]x\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{3}\|F|_{[0,\delta']}\|_\infty\|x\|.$$

Thus we conclude

$$\left\|\int_0^{\delta'} F(s)BT(t-s)x\,\mathrm{d}s\right\| \leqslant \varepsilon\|F|_{[0,\delta']}\|_\infty\|x\| \quad \big(x\in D(A)\big).$$

Since $D(A)$ is dense in X, the proof is complete. □

Proof of Theorem 9. Fix $t_0 > 0$ and $0 < q < 1$ such that (17) holds. As in the proof of Theorem 6, it is enough to show that $V_B^* F : (0, t_0) \to \mathcal{L}(X)$ is norm continuous for every $F \in \mathcal{X}_{t_0}$ which is norm continuous on $(0, t_0)$. Then by induction we have that

$$S_n = V_B^* S_{n-1} = \cdots = \underbrace{V_B^*\big[\dots V_B^*\big[V_B^*}_{n} T\big]\dots\big] = \big[\underbrace{V_B^* \circ \cdots \circ V_B^*}_{n}\big] T \quad (n \in \mathbb{N})$$

is immediately norm continuous, so the immediate norm continuity of the perturbed semigroup $(S(t))_{t \geqslant 0}$ follows from Theorem 5.

Let $F \in \mathcal{X}_{t_0}$ be norm continuous on $(0, t_0)$, let $0 < t_1 < t_2 < t_0$ and $\varepsilon > 0$ be arbitrary. We find a $\delta > 0$ such that $t, t' \in [t_1, t_2]$, $0 \leqslant t' - t < \delta$ imply $\|(V_B^* F)(t') - (V_B^* F)(t)\| \leqslant \varepsilon$. This will complete the proof.

By the norm continuity of F on $(0, t_0)$ and by Lemma 10 we can find first a δ_1 and then a δ satisfying $0 < \delta < \delta_1 < \min\{t_1/2, t_0 - t_2\}$ such that

$$\big\|F(t' - t + \delta_1 + s) - F(\delta_1 + s)\big\| \leqslant \frac{\varepsilon}{4q} \quad \big(s \in [0, t - \delta_1]\big), \tag{23}$$

$$\left\|\int_0^{\delta_1} F(t' - t + s) B T(t - s)\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{4}, \qquad \left\|\int_0^{\delta_1} F(s) B T(t - s)\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{4} \tag{24}$$

and

$$\left\|\int_0^{t'-t} F(s) B T(t' - s)\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{4}. \tag{25}$$

For every $x \in D(A)$ we have

$$\begin{aligned}\big(V_B^* F\big)(t')x - \big(V_B^* F\big)(t)x &= \int_0^{t'} F(s) B T(t' - s)x\,\mathrm{d}s - \int_0^{t} F(s) B T(t - s)x\,\mathrm{d}s \\ &= \int_0^{t'-t} F(s) B T(t' - s)x\,\mathrm{d}s + \int_0^{t} \big[F(t' - t + s) - F(s)\big] B T(t - s)x\,\mathrm{d}s.\end{aligned} \tag{26}$$

The first term on the right-hand side of (26) is estimated by (25). For the second term we have

$$\begin{aligned}&\int_0^{t} \big[F(t' - t + s) - F(s)\big] B T(t - s)x\,\mathrm{d}s \\ &\quad = \int_0^{\delta_1} \big[F(t' - t + s) - F(s)\big] B T(t - s)x\,\mathrm{d}s + \int_0^{t-\delta_1} \big[F(t' - t + \delta_1 + s) - F(\delta_1 + s)\big] B T(t - \delta_1 - s)x\,\mathrm{d}s.\end{aligned}$$

By (24),

$$\left\|\int_0^{\delta_1} \big[F(t' - t + s) - F(s)\big] B T(t - s)x\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{2}\|x\|$$

while by (23) and (17)

$$\left\|\int_0^{t-\delta_1} \big[F(t' - t + \delta_1 + s) - F(\delta_1 + s)\big] B T(t - \delta_1 - s)x\,\mathrm{d}s\right\| \leqslant \frac{\varepsilon}{4}\|x\|.$$

By putting these estimates together we conclude

$$\left\|\left(V_B^* F\right)(t')x - \left(V_B^* F\right)(t)x\right\| \leqslant \varepsilon \|x\| \quad \left(x \in D(A)\right).$$

Since $D(A)$ is dense in X, the proof is complete. □

3. Immediate norm continuity and norm estimates for the resolvent

The present paper is motivated by the question whether for a given semigroup the decay of the norm of the resolvent of its generator along some imaginary axis implies the immediate norm continuity of the semigroup. In *Hilbert spaces* the answer to this question is affirmative: it is a fundamental result in the theory of immediate norm continuity that if $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup on a Hilbert space H and its generator A satisfies $\sigma(A) \subset \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant \xi_0\}$ for some $\xi_0 \in \mathbb{R}$, then $(T(t))_{t\geqslant 0}$ is immediately norm continuous if and only if for some $\xi > \xi_0$,

$$\lim_{\eta\in\mathbb{R},\,|\eta|\to\infty} \left\|R(\xi + \mathbf{i}\eta, A)\right\| = 0 \tag{27}$$

(see, e.g. [8] or [10]). It was an open problem whether condition (27) characterizes the immediate norm continuity in arbitrary Banach spaces (see, e.g. [9]); recently it turned out that it does not do so (see [17]). However, there are situations where (27) characterizes the immediate norm continuity, e.g. by a remarkable result of V. Goersmeyer and L. Weis [11] this happens for positive semigroups in L^p spaces ($1 < p < \infty$). So for special semigroups, condition (27) can imply immediate norm continuity in arbitrary Banach spaces; in particular, it can be of interest to decide whether the perturbation methods which appear in applications and keep (27) can destroy norm continuity or not. This is the approach of [22] to the perturbation of eventually norm continuous semigroups in Hilbert spaces, i.e. where an analogue of (27) characterizes eventual norm continuity.

Recall that the Desch–Schappacher and Miyadera–Voigt perturbations keep the decay of the norm of the resolvent along some imaginary axis, as follows. If B is a Desch–Schappacher perturbation of A, then with $C = (A_{-1} + B)|_X$ for the resolvents we have

$$R(\lambda, C) = R(\lambda, A)\left(I - R(\lambda, A_{-1})B\right)^{-1}$$

for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda$ sufficiently large (see, e.g. [10, (3.6) and (3.9), p. 186]). Moreover, $(I - R(\lambda, A_{-1})B)^{-1}$ is uniformly bounded in norm so for a $\xi \in \mathbb{R}$ sufficiently large,

$$\lim_{\eta\in\mathbb{R},\,|\eta|\to\infty} \left\|R(\xi + \mathbf{i}\eta, C)\right\| = 0.$$

Since a similar computation holds for the Miyadera–Voigt perturbation, as well (see, e.g. [10, (v), p. 198]), we can summarize that the Desch–Schappacher and Miyadera–Voigt perturbations keep the decay of the norm of the resolvent along some imaginary axis. From Theorems 6 and 9 we know that, in accordance with our expectations, these perturbation methods also keep the immediate norm continuity of semigroups.

In this section we would like to examine a third perturbation method which exhibits a similar behavior. Completing a result of C. Kaiser and L. Weis in [15] and [16], C. Batty proved the following perturbation theorem. It is based on a generation theorem of D.-X. Feng and D.-H. Shi [21], which in turn extends a result of A.M. Gomilko [12] characterizing semigroup generators in Hilbert spaces. We recall the theorems following C. Batty (see [4, Theorems 1 and 2]).

Theorem 11. *Let A be a closed densely defined operator on a Banach space X such that for a $\xi_0 \in \mathbb{R}$,*

$$\sigma(A) \subset \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant \xi_0\} \tag{28}$$

and for a $0 < K < \infty$,

$$\sup_{\xi>\xi_0} (\xi - \xi_0) \int_{\mathbb{R}} \left\|R(\xi + \mathbf{i}\eta, A)x\right\|^2 \mathrm{d}\eta \leqslant K\|x\|^2 \quad (x \in X), \tag{29}$$

$$\sup_{\xi>\xi_0} (\xi - \xi_0) \int_{\mathbb{R}} \left\|R(\xi + \mathbf{i}\eta, A)^* x^*\right\|^2 \mathrm{d}\eta \leqslant K\left\|x^*\right\|^2 \quad \left(x^* \in X^*\right). \tag{30}$$

Then A generates a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on X.

Suppose that B is a closed operator on X with $D(A) \subset D(B)$, and there exists $0 < M < 1$ such that for every $\operatorname{Re}\lambda \geqslant \xi_0$, $x \in X$ and $y \in D(B)$,

$$\|BR(\lambda, A)x\| \leqslant M\|x\| \quad \text{and} \quad \|R(\lambda, A)By\| \leqslant M\|y\|.$$

Then $A + B$ satisfies (28) *and* (29), (30) *with constant $K/(1-M)^2$ hence $A + B$ generates a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ on X.*

It is important to note that in Theorem 2 of [4] instead of (29) and (30) one finds

$$\sup_{\xi>\xi_0} (\xi - \xi_0) \int_{\mathbb{R}} \|R(\xi + \mathbf{i}\eta, A)x\|^2 \,\mathrm{d}\eta < \infty \quad (x \in X), \tag{31}$$

$$\sup_{\xi>\xi_0} (\xi - \xi_0) \int_{\mathbb{R}} \|R(\xi + \mathbf{i}\eta, A)^* x^*\|^2 \,\mathrm{d}\eta < \infty \quad \bigl(x^* \in X^*\bigr). \tag{32}$$

But as observed in the proof of [14, Theorem 5.1, p. 392], requiring (31) and (32) is equivalent with (29) and (30) by the Banach–Steinhaus Theorem or the Closed Graph Theorem.

We aim to prove the following extension of Theorem 11. Informally it says that for semigroups obtained via Theorem 11, immediate norm continuity and (27) are equivalent.

Theorem 12. *In Theorem* 11*, suppose in addition that*

$$\lim_{\eta\in\mathbb{R},\,|\eta|\to\infty} \|R(\xi + \mathbf{i}\eta, A)\| = 0 \quad (\xi > \xi_0).$$

Then $(T(t))_{t\geqslant 0}$ is immediately norm continuous.

Theorem 12 easily implies the following.

Corollary 13. *With the hypotheses of Theorem* 11*, if $(T(t))_{t\geqslant 0}$ is immediately norm continuous, then $(S(t))_{t\geqslant 0}$ is also immediately norm continuous.*

Proof. Suppose that $(T(t))_{t\geqslant 0}$ is immediately norm continuous. By [10, Corollary 4.19, p. 114] we have

$$\lim_{\eta\in\mathbb{R},\,|\eta|\to\infty} \|R(\xi + \mathbf{i}\eta, A)\| = 0 \quad (\xi > \xi_0).$$

As it was observed in the proof of Theorem 1 of [4], for every $\xi > \xi_0$ and $\eta \in \mathbb{R}$ we have $\xi + \mathbf{i}\eta \in \rho(A + B)$ and

$$R(\xi + \mathbf{i}\eta, A + B)x = \sum_{n=0}^{\infty} \bigl(R(\xi + \mathbf{i}\eta, A)B\bigr)^n R(\xi + \mathbf{i}\eta, A)x \quad (x \in X).$$

In particular,

$$\|R(\xi + \mathbf{i}\eta, A + B)x\| \leqslant \sum_{n=0}^{\infty} M^n \|R(\xi + \mathbf{i}\eta, A)x\| \leqslant \frac{1}{1-M} \|R(\xi + \mathbf{i}\eta, A)x\| \quad (x \in X)$$

thus

$$\lim_{\eta\in\mathbb{R},\,|\eta|\to\infty} \|R(\xi + \mathbf{i}\eta, A + B)\| = 0 \quad (\xi > \xi_0). \tag{33}$$

So by Theorem 12, $(S(t))_{t\geqslant 0}$ is immediately norm continuous. □

Proof of Theorem 12. We prove the statement by repeating the proof of Theorem in [10, p. 115] (see also [8] and [9]). It is enough to show that for every $\varepsilon > 0$ there is a $\delta \in (0, 1)$ such that for every $s, t > 0$, $|t - s| < \delta$ implies $\|t^2 T(t) - s^2 T(s)\| \leqslant \varepsilon$. Fix $\varepsilon > 0$, let $\xi > \xi_0$, $s, t > 0$; for every $x \in X$ with $\|x\| \leqslant 1$ we have

$$\begin{aligned}\|t^2T(t)x - s^2T(s)x\| &= \frac{1}{\pi}\left\|\int_{\mathbb{R}}\left(e^{t(\xi+\mathbf{i}\eta)} - e^{s(\xi+\mathbf{i}\eta)}\right)R(\xi+\mathbf{i}\eta, A)^3x\,\mathrm{d}\eta\right\| \\ &\leqslant \frac{1}{\pi}\left\|\int_{|\eta|\geqslant N}\left(e^{t(\xi+\mathbf{i}\eta)} - e^{s(\xi+\mathbf{i}\eta)}\right)R(\xi+\mathbf{i}\eta, A)^3x\,\mathrm{d}\eta\right\| \\ &\quad + \frac{1}{\pi}\int_{|\eta|\leqslant N}\left|e^{t(\xi+\mathbf{i}\eta)} - e^{s(\xi+\mathbf{i}\eta)}\right|\left\|R(\xi+\mathbf{i}\eta, A)^3x\right\|\mathrm{d}\eta. \end{aligned} \tag{34}$$

Let $x^* \in X^*$ with $\|x^*\| \leqslant 1$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}&\left|\left(\int_{|\eta|\geqslant N}\left(e^{t(\xi+\mathbf{i}\eta)} - e^{s(\xi+\mathbf{i}\eta)}\right)R(\xi+\mathbf{i}\eta, A)^3x\,\mathrm{d}\eta, x^*\right)\right| \\ &\quad\leqslant \left(e^{t\xi}+e^{s\xi}\right)\int_{|\eta|\geqslant N}\left|\left(R(\xi+\mathbf{i}\eta, A)^2x, R(\xi+\mathbf{i}\eta, A)^*x^*\right)\right|\mathrm{d}\eta \\ &\quad\leqslant 2e^{(t+1)|\xi|}\left(\int_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)^2x\right\|^2\mathrm{d}\eta\right)^{1/2}\left(\int_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)^*x^*\right\|^2\mathrm{d}\eta\right)^{1/2}.\end{aligned}$$

For the first term in the last product we have

$$\left(\int_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)^2x\right\|^2\mathrm{d}\eta\right)^{1/2} \leqslant \sup_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)\right\|\left(\int_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)x\right\|^2\mathrm{d}\eta\right)^{1/2}$$

and by the hypothesis of Theorem 11,

$$\left(\int_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)x\right\|^2\mathrm{d}\eta\right)^{1/2} \leqslant \frac{K^{1/2}}{(\xi-\xi_0)^{1/2}}$$

and

$$\left(\int_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)^*x^*\right\|^2\mathrm{d}\eta\right)^{1/2} \leqslant \frac{K^{1/2}}{(\xi-\xi_0)^{1/2}}.$$

Thus

$$\left|\left(\int_{|\eta|\geqslant N}\left(e^{t(\xi+\mathbf{i}\eta)} - e^{s(\xi+\mathbf{i}\eta)}\right)R(\xi+\mathbf{i}\eta, A)^3x\,\mathrm{d}\eta, x^*\right)\right| \leqslant 2e^{(t+1)|\xi|}\sup_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)\right\|\frac{K}{(\xi-\xi_0)}.$$

Choose N such that

$$2e^{(t+1)|\xi|}\sup_{|\eta|\geqslant N}\left\|R(\xi+\mathbf{i}\eta, A)\right\|\frac{K}{(\xi-\xi_0)} \leqslant \frac{\varepsilon}{2}.$$

There is a $\delta \in (0,1)$ such that for every $s, t > 0$ with $|t-s| < \delta$ we have

$$\frac{1}{\pi}\int_{|\eta|\leqslant N}\left|e^{t(\xi+\mathbf{i}\eta)} - e^{s(\xi+\mathbf{i}\eta)}\right|\left\|R(\xi+\mathbf{i}\eta, A)^3x\right\|\mathrm{d}\eta \leqslant \frac{\varepsilon}{2}.$$

Hence we obtained $\|t^2T(t)x - s^2T(s)x\| \leqslant \varepsilon$ for every $s, t > 0$ with $|t-s| < \delta$ and $x \in X$ with $\|x\| \leqslant 1$, so the proof is complete. □

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