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INFINITE DIMENSIONAL PERFECT SET THEOREMS

TAMÁS MÁTRAI

ABSTRACT. What largeness and structural assumptions on $A \subseteq [\mathbb{R}]^\omega$ can guarantee the existence of a non-empty perfect set $P \subseteq \mathbb{R}$ such that $[P]^\omega \subseteq A$? Such a set P is called A-homogeneous. We show that even if A is open, in general it is independent of ZFC whether for a cardinal κ, the existence of an A-homogeneous set $H \in [\mathbb{R}]^\kappa$ implies the existence of a non-empty perfect A-homogeneous set.

On the other hand, we prove an infinite dimensional analogue of Mycielski's Theorem: if A is large in the sense of a suitable Baire category-like notion then there exists a non-empty perfect A-homogeneous set. We introduce fusion games to prove this and other infinite dimensional perfect set theorems.

Finally we apply this theory to show that it is independent of ZFC whether Tukey reductions of the maximal analytic cofinal type can be witnessed by definable Tukey maps.

1. INTRODUCTION

The Perfect Set Theorem says that an analytic subset of a Polish space is either countable or has a non-empty perfect subset (see e.g. [11, Theorem 29.1 p. 226]). The complexity assumption in this result is consistently optimal: in L there exists an uncountable $\mathbf{\Pi}^1_1$ set without non-empty perfect subsets (see e.g. [9, Corollary 25.37]).

However, one is often obliged to quest for a perfect set which satisfies multidimensional relations. Let $N < \omega$ be fixed and let $[X]^N$ denote the set of N element subsets of X. Then an N dimensional perfect set theorem should address the following problem: what largeness and structural assumptions on $A \subseteq [X]^N$ can guarantee the existence of a non-empty perfect set $P \subseteq X$ such that $[P]^N \subseteq A$? Such a set P is called *A-homogeneous*.

In [12, Theorem 2.2 p. 620], W. Kubiś proved that if X is a Polish space, $A \subseteq [X]^N$ is G_δ and there exists an uncountable A-homogeneous set then there exists of a non-empty perfect A-homogeneous set. Obviously, as far as G_δ sets are concerned, this result is the exact multidimensional analogue of the Perfect Set Theorem. The surprising fact is that the complexity assumption in this result is also optimal: the Turing reducibility relation on 2^ω (see Definition 2.4) defines an F_σ set $A \subseteq [2^\omega]^2$ such that there exists an A-homogeneous set $H \subseteq 2^\omega$ with $|H| = \omega_1$ but there is no non-empty perfect A-homogeneous set.

For A analytic, W. Kubiś and S. Shelah [13] investigated a rank function which decides whether the existence of an A-homogeneous set with a given cardinality implies the existence of a non-empty perfect A-homogeneous set (see also [8], [22]). They obtain in particular that for every $\alpha < \omega_1$, it is

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consistent with ZFC that there exists an F_σ set $A \subseteq [2^\omega]^2$ such that there exists an A-homogeneous set of cardinality $\aleph_\alpha$ but there is no non-empty perfect A-homogeneous set.

Nevertheless, the most frequently applied perfect set theorem is a classical result of J. Mycielski, which says that if X is a non-empty perfect Polish space and $A_N \subseteq [X]^N$ $(N < \omega)$ are co-meager relations then there is a non-empty perfect set $P \subseteq X$ such that $[P]^N \subseteq A_N$ $(N < \omega)$. So in particular, if $A \subseteq [X]^N$ is co-meager then there exists a non-empty perfect A-homogeneous set. Obviously, the largeness assumption in this result is not optimal.

In the present paper we study the existence of non-empty perfect homogeneous sets for infinite dimensional relations $A \subseteq [X]^\omega$. Unlike in the finite dimensional case, it is not obvious how to topologize $[X]^\omega$, therefore we usually assume that $A \subseteq X^\omega$ is *symmetric*, i.e. it is invariant under permutations of coordinates in X^ω; and a set $H \subseteq X$ is called *A-homogeneous* if the injective sequences of H, $IS_\omega(H) = \{(x_n)_{n<\omega} \in H^\omega : x_n \neq x_m \ (n < m < \omega)\}$ are in A.

It is obvious that if $A \subseteq X^\omega$ is closed and $H \in [X]^{\omega_1}$ is A-homogeneous then $\mathrm{cl}_X(H)$ is also A-homogeneous hence there exists a non-empty perfect A-homogeneous set. In Section 2 we show that this complexity assumption is also optimal (see Theorem 2.1).

Theorem 1.1. *Let κ be a cardinal and suppose that there exists an F_σ set $A \subseteq [2^\omega]^2$ such that there exists an A-homogeneous set of cardinality κ but there is no non-empty perfect A-homogeneous set. Then there exists a symmetric open set $U \subseteq (2^\omega)^\omega$ such that there exists a U-homogeneous set of cardinality κ but there is no non-empty perfect U-homogeneous set.*

Thus in the infinite dimensional case, by the above mentioned result of W. Kubiś and S. Shelah, even for open relations, it is consistent that the existence of a homogeneous set of large cardinality does not imply the existence of a non-empty perfect homogeneous set (see Corollary 2.3).

On the other hand, Mycielski's Theorem has an infinite dimensional analogue. For a Polish space X, consider the σ-ideal $\mathbb{M}$ generated by the sets of the form $\bigcup_{n<\omega}(M_n \times X^{\omega \setminus (n+1)})$ where $M_n \subseteq X^{n+1}$ $(n < \omega)$ are meager. As we will see, an easy application of Mycielski's Theorem yields that if $A \subseteq X^\omega$ satisfies $X^\omega \setminus A \in \mathbb{M}$ then there exists a non-empty perfect A-homogeneous set (see Theorem 4.1). A more involved task is to find sufficient conditions for $X^\omega \setminus A \in \mathbb{M}$. In Section 4 we provide such sufficient conditions. In particular, we will show the following (see Corollary 4.7.2).

Theorem 1.2. *Let X be a Polish space, let $A \subseteq X^\omega$ be co-analytic and suppose that there exists a non-meager A-homogeneous set. Then there exists a non-empty perfect A-homogeneous set.*

As a corollary, we obtain that in the iterated perfect set model, for every co-analytic set $A \subseteq X^\omega$ if there exists an A-homogeneous set of cardinality continuum then there exists a non-empty perfect A-homogeneous set (see Theorem 4.9). Moreover, we also obtain that in Cohen extensions the existence of a homogeneous set of sufficiently large cardinality implies the existence of a non-empty perfect homogeneous set (see Theorem 4.10). Thus by combining these result and Theorem 1.1, in Section 4 we will prove the following.

Theorem 1.3. *Let $1 < \alpha < \omega_1$ be an ordinal. Then it is independent of ZFC whether for an open set $A \subseteq (2^\omega)^\omega$, the existence of an A-homogeneous set of cardinality $\aleph_\alpha$ implies the existence of a non-empty perfect A-homogeneous set.*

The key result toward Theorem 1.2 is proved by using a game which is obtained as a fusion of Banach-Mazur games played on higher and higher dimensional powers of X (see Definition 3.2). In Section 3 we introduce this game and characterize the winning strategies of the players (see Theorem 3.4). It seems that our procedure of taking fusions is applicable to a wide class of games of descriptive set theory. As an illustration, in Section 3.2 we briefly study the fusion of Perfect Set Property games (see Definition 3.13 and Corollary 3.18). Independently of our work, M. Sabok [21] introduced and studied similar games. We discuss the relation between fusion games and some games of [21] and [29] in the introduction of Section 3.

Our study of infinite dimensional perfect set theorems was motivated by the problem whether Tukey reducibilities of the maximal analytic cofinal type $\mathcal{I}_{\max}$ can be witnessed by definable Tukey maps. In Section 5 we recall the relevant definitions and we show that this problem is independent of ZFC (see Theorem 5.7).

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2. Open relations

The main result of this section is the following slightly generalized version of Theorem 1.1. Recall that for every set H and $\alpha \le \omega$, $IS_\alpha(H) = \{(x_n)_{n<\alpha} \in H^\alpha \colon x_n \ne x_m \ (n < m < \alpha)\}$; and for $A \subseteq X^\alpha$, $H \subseteq X$ is called *A-homogeneous* if $IS_\alpha(H) \subseteq A$.

For every $\alpha \le \omega$, let S_α denote the permutation group on α. A set $A \subseteq X^\alpha$ is *symmetric* if for every $\pi \in S_\alpha$, $(a_k)_{k<\alpha} \in A$ implies $(a_{\pi(k)})_{k<\alpha} \in A$.

Theorem 2.1. *Let $A \subseteq (2^\omega)^2$ be a symmetric F_σ set such that there is no non-empty perfect A-homogeneous set. Then there is a symmetric open set $U \subseteq (2^\omega)^\omega$ such that*

(1) for every $(x_k)_{k<\omega} \in (2^\omega)^\omega$, if $|\{x_k \colon k < \omega\}| = \omega$ and there are $i, j < \omega$ such that $x_i \ne x_j$ and $(x_i, x_j) \in A$, then $(x_k)_{k<\omega} \in U$;

(2) there is no non-empty perfect U-homogeneous set.

In particular, by 1, if $H \subseteq 2^\omega$ is A-homogeneous then H is U-homogeneous, as well.

Before proving Theorem 2.1, we need some preparation. A well-known theorem of F. Galvin states that if X is a non-empty perfect Polish space and $B \subseteq [X]^2$ has the Baire property then either there is a non-empty perfect B-homogeneous set or there is a non-empty perfect $([X]^2 \setminus B)$-homogeneous set (see e.g. [11, Theorem 19.7 p. 130]). We will use the following corollary.

Corollary 2.2. *Let $A \subseteq (2^\omega)^2$ be a symmetric F_σ set such that there is no non-empty perfect A-homogeneous set. Then every non-empty perfect set $P \subseteq 2^\omega$ has a non-empty perfect subset $Q \subseteq P$ such that $IS_2(Q) \cap A = \emptyset$.*

Proof. Apply Galvin's Theorem with $X = P$ and $B = A \cap [P]^2$. □

We introduce some terminology and notation. Fix a metric d_1 on 2^ω. For every $0 < n < \omega$, let d_n denote the coordinate supremum metric on $(2^\omega)^n$ generated by d_1. With an abuse of notation, we extend d_n to the hyperspace by setting, for every $A, B \subseteq (2^\omega)^n$,

$$d_n(A, B) = \inf\{d_n(a, b) \colon a \in A,\ b \in B\}.$$

For every $0 < n < \omega$, $(x_k)_{k<n} \in (2^\omega)^n$ and $\delta > 0$, let

$$B^+((x_k)_{k<n}, \delta) = \{(y_k)_{k<\omega} \in (2^\omega)^\omega \colon \exists \pi \in S_\omega \ (d_n((x_k)_{k<n}, (y_{\pi(k)})_{k<n}) < \delta)\}.$$

Then the sets $B^+((x_k)_{k<n}, \delta)$ are symmetric open subsets of $(2^\omega)^\omega$.

For every $1 < n < \omega$ and $(x_k)_{k<n} \in (2^\omega)^n$, we define $\delta((x_k)_{k<n}) = \min\{|x_i - x_j| \colon i < j < n\}$. For every $0 < n < \omega$, $\Delta_n = \{(x, \dots, x) \in (2^\omega)^n \colon x \in 2^\omega\}$.

Proof of Theorem 2.1. Let $F_n \subseteq (2^\omega)^2$ $(n < \omega)$ be symmetric closed sets such that

$$d_2(F_n, \Delta_2) > 0 \ (n < \omega) \text{ and } A \setminus \Delta_2 = \bigcup_{n<\omega} F_n. \tag{2.1}$$

For every $n < \omega$, set

$$U_n = \bigcup \{B^+((x_k)_{k<n+3}, \delta((x_k)_{k<n+3})/4) \colon (x_k)_{k<n+3} \in (2^\omega)^{n+3}, (x_0, x_1) \in F_n\}.$$

We show that $U = \bigcup_{n<\omega} U_n$ fulfills the requirements.

By definition, U is a symmetric open set. To see 1, let $(x_k)_{k<\omega} \in (2^\omega)^\omega$ be such that $|\{x_k \colon k < \omega\}| = \omega$ and there are $i, j < \omega$ such that $x_i \neq x_j$ and $(x_i, x_j) \in A$; say $(x_i, x_j) \in F_n$. Since U is symmetric, to have $(x_k)_{k<\omega} \in U$ it is enough to show that $(x_{\pi(k)})_{k<\omega} \in U$ for some $\pi \in S_\omega$. That is, we can assume $(x_0, x_1) \in F_n$ and $(x_k)_{k<n+3} \in IS_{n+3}(2^\omega)$. Then $\delta((x_k)_{k<n+3}) > 0$ hence $B^+((x_k)_{k<n+3}, \delta((x_k)_{k<n+3})/4) \neq \emptyset$. Thus $(x_k)_{k<\omega} \in U_n \subseteq U$, as required.

It remains to prove 2. Let $P \subseteq 2^\omega$ be an arbitrary non-empty perfect set. By Corollary 2.2, there is a non-empty perfect set $Q \subseteq P$ such that $IS_2(Q) \cap A = \emptyset$. We define inductively a sequence $(q_k)_{k<\omega} \in IS_\omega(Q)$ such that $(q_k)_{k<\omega} \notin U$; this will complete the proof.

By induction on $n < \omega$, we define $q_n \in Q$ $(n < \omega)$ and $\varepsilon_n > 0$ $(0 < n < \omega)$ such that

(i) for every $i, j < \omega$, $i \neq j$ implies $q_i \neq q_j$,
(ii) for every $0 < n < \omega$, $\varepsilon_n = \min\{d_2((q_i, q_j), F_k)) \colon i, j, k \leq n\}$,
(iii) for every $0 < i < n < \omega$, $d_1(q_n, q_i) < \varepsilon_i/2$.

Let $q_0, q_1 \in Q$ be arbitrary satisfying $q_0 \neq q_1$. Let $0 < n < \omega$ and suppose that q_i $(i \leq n)$ and ε_i $(0 < i < n)$ are defined such that (i-iii) hold. In accordance with (iii), set

$$\varepsilon_n = \min\{d_2((q_i, q_j), F_k)) \colon i, j, k \leq n\}.$$

Then $\varepsilon_n > 0$ by $IS_2(Q) \cap A = \emptyset$ and (2.1).

To satisfy (i) and (iii), let $q_{n+1} \in Q \setminus \{q_i \colon i \leq n\}$ be arbitrary satisfying $d_1(q_{n+1}, q_i) < \varepsilon_i/2$ $(i \leq n)$; by the inductive assumption (iii) for n, such a q_{n+1} exists, namely any $q_{n+1} \in Q \setminus \{q_i \colon i \leq n\}$ sufficiently close to q_n fulfills the requirements.

Suppose $(q_k)_{k<\omega} \in U$, say $(q_k)_{k<\omega} \in U_n$. Then $(q_k)_{k<\omega} \in B^+((x_k)_{k<n+3}, \delta((x_k)_{k<n+3})/4)$ for some $(x_k)_{k<n+3} \in (2^\omega)^{n+3}$ such that $(x_0, x_1) \in F_n$. Set $\delta = \delta((x_k)_{k<n+3})/4$. Then there are $k_0 < k_1 < \dots < k_{n+2} < \omega$ and an enumeration $n + 3 = \{l_i \colon i < n + 3\}$ such that

(iv) $d_1(q_{k_i}, x_{l_i}) < \delta/4$ $(i < n + 3)$;
(v) for some $a < b < n + 3$, $(x_{l_a}, x_{l_b}) \in F_n$.

We distinguish two cases. Suppose first $b < n + 2$. By (iv),

$$d_1(q_{k_a}, x_{l_a}) < \frac{\delta}{4}, \ d_1(q_{k_b}, x_{l_b}) < \frac{\delta}{4}. \tag{2.2}$$

We have $0 < k_{n+1} < k_{n+2}$. So by (iii), $d_1(q_{k_{n+2}}, q_{k_{n+1}}) < \varepsilon_{k_{n+1}}/2$. By $b < n+2$ we have $k_b \leq k_{n+1}$; in addition we have $n \leq k_{n+1}$, so $\varepsilon_{k_{n+1}} \leq d_2((q_{k_a}, q_{k_b}), F_n)$. By (2.2) and (v), $d_2((q_{k_a}, q_{k_b}), F_n) < \delta/4$. So to summarize,

$$d_1(q_{k_{n+2}}, q_{k_{n+1}}) < \frac{\varepsilon_{k_{n+1}}}{2} \leq \frac{d_2((q_{k_a}, q_{k_b}), F_n)}{2} < \frac{\delta}{8}.$$

Again by (iv),

$$d_1(q_{k_{n+1}}, x_{l_{n+1}}) < \frac{\delta}{4},\ d_1(q_{k_{n+2}}, x_{l_{n+2}}) < \frac{\delta}{4},$$

so the triangle inequality yields

$$d_1(x_{l_{n+1}}, x_{l_{n+2}}) \leq d_1(x_{l_{n+1}}, q_{k_{n+1}}) + d_1(q_{k_{n+1}}, q_{k_{n+2}}) + d_1(q_{k_{n+2}}, x_{l_{n+2}}) < 5\frac{\delta}{8}.$$

This contradicts the definition of δ.

Finally suppose $b = n+2$; as in the previous case, we estimate $d_1(x_{l_{n+1}}, x_{l_{n+2}})$. By (iv),

$$d_1(q_{k_a}, x_{l_a}) < \frac{\delta}{4},\ d_1(q_{k_{n+1}}, x_{l_{n+1}}) < \frac{\delta}{4},\ d_1(q_{k_{n+2}}, x_{l_{n+2}}) < \frac{\delta}{4}. \tag{2.3}$$

So by the triangle inequality,

$$d_1(x_{l_{n+1}}, x_{l_{n+2}}) \leq d_1(x_{l_{n+1}}, q_{k_{n+1}}) + d_1(q_{k_{n+1}}, q_{k_{n+2}}) + d_1(q_{k_{n+2}}, x_{l_{n+2}}) < \frac{\delta}{2} + d_1(q_{k_{n+1}}, q_{k_{n+2}}). \tag{2.4}$$

We have $0 < k_{n+1} < k_{n+2}$. So by (iii), $d_1(q_{k_{n+2}}, q_{k_{n+1}}) < \varepsilon_{k_{n+1}}/2$. Since $k_a \leq k_{n+1}$ and $n \leq k_{n+1}$, we get $\varepsilon_{k_{n+1}} \leq d_2((q_{k_a}, q_{k_{n+1}}), F_n)$. By (2.3),

$$d_2((q_{k_a}, q_{k_{n+1}}), F_n) \leq d_2((q_{k_a}, q_{k_{n+1}}), (x_{l_a}, x_{l_{n+2}})) \leq \max\left\{\frac{\delta}{4}, d_1(q_{k_{n+1}}, x_{l_{n+2}})\right\} \leq \frac{\delta}{4} + d_1(q_{k_{n+1}}, x_{l_{n+2}}).$$

By the triangle inequality and (2.3),

$$d_1(q_{k_{n+1}}, x_{l_{n+2}}) \leq d_1(q_{k_{n+1}}, q_{k_{n+2}}) + d_1(q_{k_{n+2}}, x_{l_{n+2}}) < d_1(q_{k_{n+1}}, q_{k_{n+2}}) + \frac{\delta}{4}.$$

To summarize,

$$d_1(q_{k_{n+2}}, q_{k_{n+1}}) < \frac{\varepsilon_{k_{n+1}}}{2} \leq \frac{d_2((q_{k_a}, q_{k_{n+1}}), F_n)}{2} \leq \frac{\delta}{8} + \frac{d_1(q_{k_{n+1}}, x_{l_{n+2}})}{2} < \frac{\delta}{8} + \frac{d_1(q_{k_{n+1}}, q_{k_{n+2}})}{2} + \frac{\delta}{8};$$

i.e. $d_1(q_{k_{n+2}}, q_{k_{n+1}}) < \delta/2$. By (2.4) we obtained $d_1(x_{l_{n+1}}, x_{l_{n+2}}) < \delta$, which again contradicts the definition of δ. □

Corollary 2.3. *For every $\alpha < \omega_1$, it is consistent with ZFC that there exists a symmetric open set $U \subseteq (2^\omega)^\omega$ such that there exists an U-homogeneous set of cardinality $\aleph_\alpha$ but there is no non-empty perfect U-homogeneous set.*

Proof. By [13, Corollary 5.13 p. 159] or [22, Theorem 1.13 p. 15], it is consistent with ZFC that there exists an F_σ set $A \subseteq [2^\omega]^2$ such that there exists an A-homogeneous set of cardinality $\aleph_\alpha$ but there is no non-empty perfect A-homogeneous set. So the statement follows from Theorem 2.1. □

As we mentioned in the introduction, the Turing reducibility relation is a ZFC example for an F_σ set $T \subseteq [2^\omega]^2$ such that there exists an uncountable T-homogeneous set but there is no non-empty perfect T-homogeneous set. For the sake of completeness, we recall (a simplified version of) this relation and prove its above-mentioned properties.

Definition 2.4. For every $j < \omega$, set $T_j^< = \{(x,y) \in 2^\omega \times 2^\omega \colon x(i) = y(2^{j+1} \cdot i + 2^j)\ (i < \omega)\}$ and $T_j^> = \{(x,y) \in 2^\omega \times 2^\omega \colon (y,x) \in T_j^<\}$. Then the *Turing reducibility relation* $T \subseteq 2^\omega \times 2^\omega$ is defined by $T = \bigcup_{j<\omega}(T_j^< \cup T_j^>)$.

Proposition 2.5. *The relation T is symmetric and F_σ such that there exists an uncountable T-homogeneous set but there is no non-empty perfect T-homogeneous set.*

Proof. It is obvious that $T_j^<$, $T_j^>$ $(j < \omega)$ are closed sets, so T is symmetric and F_σ. Observe that for given $(x_j)_{j<\omega} \subseteq 2^\omega$, the point $y \in 2^\omega$ defined by $y(2^{j+1} \cdot i + 2^j) = x_j(i)$ $(i, j < \omega)$ satisfies $(x_j, y) \in T_j^< \subseteq T$ $(j < \omega)$. Hence a straightforward transfinite recursion yields an uncountable T-homogeneous set.

Finally let $P \subseteq 2^\omega$ be a non-empty perfect set. Observe that for every $j < \omega$, $T_j^>$ is the graph of a function, in particular $T_j^> \cap (P \times P)$ $(j < \omega)$ are meager in $P \times P$. By symmetry, this yields $T \cap (P \times P)$ is meager in $P \times P$. Hence P cannot be T-homogeneous, as required. □

From T, by Theorem 2.1, we get a symmetric open set $U \subseteq (2^\omega)^\omega$ with analogous properties. In particular, this yields an example of an open set $U \subseteq (2^\omega)^\omega$ which is dense even in the box topology, still there is no non-empty perfect U-homogeneous set. We refer to Section 6.1 for a further discussion of alternative topologies.

3. Fusion games

In this section we introduce the fusion of infinite sequences of games. The construction can be performed for most of the usual games of descriptive set theory. However, the method of the characterization of the winning strategies in a fusion game depends on the games whose fusion is taken. Therefore, here we study only the fusion game of Banach-Mazur games in detail, which is the most relevant for our perfect set theorems. In addition, in Section 3.2 we briefly discuss the fusion game of Perfect Set Property games.

Independently of our work, M. Sabok [21] introduced and studied games which are very similar to fusion games. The approach in [21], which originates from [29], makes explicit the connection between such games and iterated forcing. Fusion games, and the corresponding ideal $\mathbb{M}$ (see Definition 3.3) are also to be compared to the games and ideals of [29, Definition 5.1.1 p. 225].

3.1. Banach-Mazur games.

Let X be a Choquet space such that there is a metric d on X whose balls are open in X. The diameter of a set $A \subseteq X$ is denoted by $\mathrm{diam}_X(A)$. Recall that for every $A \subseteq X$, in the *Banach-Mazur game with payoff set A* (see e.g. [11, Section 21.D p. 153]) two players play

I:	$U(0)$		$U(2)$		$\dots$	$U(2n)$		$\dots$	
II:		$U(1)$		$U(3)$	$\dots$		$U(2n+1)$		$\dots$

where $U(n) \subseteq X$ $(n < \omega)$ are non-empty open sets such that $U(n) \supseteq U(n+1)$ and $\mathrm{diam}_X(U(n)) < 2^{-n}$ $(n < \omega)$, and player II wins the game if and only if $\bigcap_{i<\omega} U(i)$ is a singleton and $\bigcap_{i<\omega} U(i) \in A$.

The following is well-known (see e.g. [11, Theorem 8.33 p. 51]).

Theorem 3.1. *In the Banach-Mazur game with payoff set A,*

1. *player I has a winning strategy if and only if there is a non-empty open set $U \subseteq X$ such that $A \cap U$ is meager;*
2. *player II has a winning strategy if and only if $X \setminus A$ is meager.*

We define the fusion game of Banach-Mazur games. If Y is a set and $s, t \in Y^{<\omega}$, $|s|$ denotes the length of s and $s^\frown t$ stands for the sequence $s(0) \dots s(|s|-1)t(0) \dots t(|t|-1)$. We write $s \sqsubseteq t$ if $s = t|_{|s|}$. If $T \subseteq Y^{<\omega}$ is a tree and $n < \omega$, set $\mathrm{lev}_n(T) = \{t \in T \colon |t| = n\}$ and $[T] = \{\eta \in Y^\omega \colon \eta|_n \in$

$T\ (n<\omega)\}$. Recall also that the product space of Choquet spaces is Choquet (see e.g. [11, Exercise 8.13 p. 44]).

Definition 3.2. For every $k<\omega$, let X_k be a Choquet space such that there is a metric d_k on X_k whose balls are open in X_k. For every $A\subseteq\prod_{k<\omega}X_k$, $\mathcal{G}_\omega(A)$ denotes the *fusion game of the Banach-Mazur games with payoff set* A, in which players I and II play

$$\begin{array}{llllll} I\colon U_0(0) & (U_0(2),U_1(0)) & & \dots\ (U_i(2(n-i)))_{i\le n} & & \dots \\ II\colon & U_0(1) & (U_0(3),U_1(1))\ \dots & & (U_i(2(n-i)+1))_{i\le n} & \dots \end{array}$$

where for every $k<\omega$, $U_k(i)\subseteq X_k$ $(i<\omega)$ are non-empty open sets such that $U_k(i)\supseteq U_k(i+1)$ and $\operatorname{diam}_{X_k}(U_k(i))<2^{-i}$ $(i<\omega)$, and player II wins the game if and only if for every $k<\omega$, $\bigcap_{i<\omega}U_k(i)$ is a singleton and $(\bigcap_{i<\omega}U_k(i))_{k<\omega}\in A$.

We denote by $\mathcal{G}_\omega$ the tree of partial runs of this game, ordered by end-extension. We set

$$\mathcal{U}_\omega=\Big\{\prod_{k<n}U_k\colon n<\omega,\ U_k\subseteq X_k \text{ is non-empty open } (k<n)\Big\}.$$

A *quasi-strategy of player I* is a non-empty pruned tree $\sigma\subseteq\mathcal{G}_\omega$ such that for every $n<\omega$, $U\in\operatorname{lev}_{2n+1}(\sigma)$ and $u\in\mathcal{U}_\omega$, $U^\frown u\in\mathcal{G}_\omega$ implies $U^\frown u\in\sigma$.

Similarly, a *quasi-strategy of player II* is a non-empty pruned tree $\sigma\subseteq\mathcal{G}_\omega$ such that for every $n<\omega$, $U\in\operatorname{lev}_{2n}(\sigma)$ and $u\in\mathcal{U}_\omega$, $U^\frown u\in\mathcal{G}_\omega$ implies $U^\frown u\in\sigma$.

For every pruned tree $\sigma\subseteq\mathcal{G}_\omega$, set

$$W(\sigma)=\Big\{(x_k)_{k<\omega}\in\prod_{k<\omega}X_k\colon \exists((U_i(2(k-i)))_{i\le k},(U_i(2(k-i)+1))_{i\le k})_{k<w}\in[\sigma]$$
$$\Big(x_k=\bigcap_{i<\omega}U_k(i)\ (k<\omega)\Big)\Big\}.$$

For every $P\in\{I,II\}$, $\Sigma(P)$ denotes the set of all quasi-strategies of player P in the game $\mathcal{G}_\omega$, and we set $\mathcal{W}(P)=\{A\subseteq\prod_{k\le n}X_k\colon\exists\sigma\in\Sigma(P)\ (W(\sigma)\subseteq A)\}$.

The winning strategies in $\mathcal{G}_\omega$ are characterized by the following Baire Category-like notion. For arbitrary $A\subseteq X\times Y$, we set $\operatorname{Pr}_X(A)=\{x\in X\colon\exists y\in Y\ ((x,y)\in A)\}$.

Definition 3.3. With the notation of Definition 3.2, for an arbitrary sequence $S_n\subseteq\prod_{k\le n}X_k$ $(n<\omega)$ we set

$$[(S_n)_{n<\omega}]=\bigcap_{n<\omega}\Big(S_n\times\prod_{n<k<\omega}X_k\Big).$$

We call $\mathbb{U}=(U_n)_{n<\omega}$ an *open tower* if for every $n<\omega$, $U_n\subseteq\prod_{k\le n}X_k$ is an open set and $U_n\Delta\operatorname{Pr}_{\prod_{k\le n}X_k}(U_{n+1})$ is meager in $\prod_{k\le n}X_k$. For two open towers $\mathbb{U}$ and $\mathbb{V}$, $\mathbb{U}$ is *dense in* $\mathbb{V}$ if U_n is dense in V_n $(n<\omega)$; and $\mathbb{U}$ is *dense* if U_n is dense in $\prod_{k\le n}X_k$ $(n<\omega)$. An open tower $\mathbb{U}$ is *non-empty* if $U_0\ne\emptyset$.

We define

$$\mathbb{M}=\Big\{M\subseteq\prod_{k<\omega}X_k\colon\exists M_n\subseteq\prod_{k\le n}X_k\ (n<\omega)$$
$$\Big(M_n \text{ is meager in } \prod_{k\le n}X_k\ (n<\omega),\ M\subseteq\bigcup_{n<\omega}\Big(M_n\times\prod_{n<k<\omega}X_k\Big)\Big)\Big\}.$$

We discuss the relation of these notions to usual topologies before Lemma 3.7. However, note that at this point it is not obvious that if $\mathbb{U}$ is a non-empty open tower then $[\mathbb{U}] \neq \emptyset$. We will point out this corollary of X_k $(k < \omega)$ being Choquet after the proof of Proposition 3.9.

We are in position to state the characterization of the winning strategies in $\mathcal{G}_\omega$. Notice its analogy with Theorem 3.1.

Theorem 3.4. *With the notation of Definition 3.3, for every* $A \subseteq \prod_{k<\omega} X_k$

(1) the following are equivalent:
  (a) $A \in \mathcal{W}(I)$;
  (b) there is a non-empty open tower $\mathbb{U}$ *such that* $[\mathbb{U}] \subseteq A$;
  (c) there is a non-empty open tower $\mathbb{U}$ *such that* $[\mathbb{U}] \setminus A \in \mathbb{M}$.

(2) the following are equivalent:
  (a) $A \in \mathcal{W}(II)$;
  (b) there is a dense open tower $\mathbb{U}$ *such that* $[\mathbb{U}] \subseteq A$;
  (c) $(\prod_{k<\omega} X_k) \setminus A \in \mathbb{M}$.

We will need that $\mathcal{G}_\omega(A)$ is determined for co-analytic A, as well. To this end, we introduce an unfolding of $\mathcal{G}_\omega$, as follows.

Definition 3.5. With the notation of Definition 3.2, for every $F \subseteq (\prod_{k<\omega} X_k) \times \omega^\omega$, $\mathcal{G}^\star_\omega(F)$ denotes the game with payoff set F in which players I and II play

$$\begin{array}{llllll} I: U_0(0), y_0 & & (U_0(2), U_1(0)), y_1 & & \ldots (U_i(2(n-i)))_{i\le n}, y_n & & \ldots \\ II: & U_0(1) & & (U_0(3), U_1(1)) \ldots & & (U_i(2(n-i)+1))_{i\le n} \ldots \end{array}$$

where for every $n < \omega$, $((U_i(2(k-i)))_{i\le k}, (U_i(2(k-i)+1))_{i\le k})_{k\le n} \in \mathcal{G}_\omega$ and $y_n \in \omega$, and player I wins the game if and only if for every $k < \omega$, $\bigcap_{i<\omega} U_k(i)$ is a singleton and

$$\Big(\Big(\bigcap_{i<\omega} U_k(i) \Big)_{k<\omega}, (y_k)_{k<\omega} \Big) \in F.$$

For every pruned tree $\sigma \subseteq \mathcal{G}^\star_\omega$, set

$$W(\sigma) = \Big\{ ((x_k)_{k<\omega}, (y_k)_{k<\omega}) \in \Big(\prod_{k<\omega} X_k \Big) \times \omega^\omega : \\ \exists ((U_i(2(k-i)))_{i\le k}, y_k, (U_i(2(k-i)+1))_{i\le k})_{k<w} \in [\sigma] \ \Big(x_k = \bigcap_{i<\omega} U_k(i) \ (k<\omega) \Big) \Big\}.$$

For every $P \in \{I, II\}$, $\Sigma^\star(P)$ denotes the set of all quasi-strategies of player P in the game $\mathcal{G}^\star_\omega$. We set

$$\mathcal{W}^\star(I) = \Big\{ F \subseteq \Big(\prod_{k<\omega} X_k \Big) \times \omega^\omega : \exists \sigma \in \Sigma^\star(I) \ (W(\sigma) \subseteq F) \Big\}$$

$$\mathcal{W}^\star(II) = \Big\{ F \subseteq \Big(\prod_{k<\omega} X_k \Big) \times \omega^\omega : \exists \sigma \in \Sigma^\star(II) \ (W(\sigma) \cap F = \emptyset) \Big\}.$$

For the winning strategies in the game $\mathcal{G}^\star_\omega$ we have the following.

Theorem 3.6. *With the notation of Definition 3.5, for every* $F \subseteq (\prod_{k<\omega} X_k) \times \omega^\omega$,

(1) $F \in \mathcal{W}^\star(I)$ *implies* $\mathrm{Pr}_{\prod_{k<\omega} X_k}(F) \in \mathcal{W}(I)$;
(2) $F \in \mathcal{W}^\star(II)$ *implies* $\mathrm{Pr}_{\prod_{k<\omega} X_k}(F) \in \mathbb{M}$.

It remains to prove our theorems. We start with the analysis of open towers vs. $\mathbb{M}$. It is obvious that every $M \in \mathbb{M}$ is meager in $\prod_{k<\omega} X_k$ even in the box topology. However, for an arbitrary non-empty open tower $\mathbb{U}$, $[\mathbb{U}]$ may also be nowhere dense in $\prod_{k<\omega} X_k$ in the box topology. Also, it is easy to construct two non-empty open towers $\mathbb{U}$ and $\mathbb{V}$ such that $[\mathbb{U}] \cap [\mathbb{V}]$ is a singleton, i.e. such an intersection may not contain a non-empty open tower. This indicates that open towers are not the open sets of a carefully chosen topology. As the next lemma shows, "meager" and "nowhere dense" coincide for $\mathbb{M}$. In particular, once we obtained $[\mathbb{U}] \neq \emptyset$ for every non-empty open tower $\mathbb{U}$, from Corollary 3.8 we get $[\mathbb{U}] \notin \mathbb{M}$ for every non-empty open tower $\mathbb{U}$.

Lemma 3.7. *With the notation of Definition 3.3, let $\mathbb{U}$ be a non-empty open tower and for every $n < \omega$, let $B_n \subseteq U_n$ be co-meager in U_n. Then there is a non-empty open tower $\mathbb{V} = (V_n)_{n<\omega}$ such that $\mathbb{V}$ is dense in $\mathbb{U}$ and $[\mathbb{V}] \subseteq [(B_n)_{n<\omega}]$.*

Proof. For every $n < \omega$, let $R_n(i) \subseteq \prod_{k\le n} X_k$ $(i < \omega)$ be closed nowhere dense sets such that $U_n \setminus B_n \subseteq \bigcup_{i<\omega} R_n(i)$. Set

$$V_n = U_n \setminus \bigcup_{i\le n} \Big(R_{n-i}(i) \times \Big(\prod_{n-i<k\le n} X_k\Big)\Big) \ (n < \omega).$$

It is obvious that $V_n \subseteq U_n$ is an open set which is dense in U_n $(n < \omega)$. We have $(x_i)_{i<\omega} \in [\mathbb{V}]$ if and only if $(x_i)_{i\le n} \in V_n$ $(n < \omega)$. So for every $n < \omega$, by $(x_i)_{i\le n} \in V_n$ we have $(x_i)_{i\le n} \in U_n$, and by $(x_i)_{i\le n+k} \in V_{n+k}$ $(k < \omega)$ we have $(x_i)_{i\le n} \notin R_n(k)$ $(k < \omega)$. Thus $(x_i)_{i\le n} \in B_n$ $(n < \omega)$, which completes the proof.■

Corollary 3.8. *With the notation of Definition 3.3, let $A \subseteq \prod_{k<\omega} X_k$ be arbitrary and let $\mathbb{U}$ be a non-empty open tower. Then*

(1) $[\mathbb{U}] \setminus A \in \mathbb{M}$ if and only if there is a non-empty open tower $\mathbb{V}$ such that $\mathbb{V}$ is dense in $\mathbb{U}$ and $[\mathbb{V}] \subseteq A$.
(2) $(\prod_{k<\omega} X_k) \setminus A \in \mathbb{M}$ if and only if there is a dense open tower $\mathbb{U}$ such that $[\mathbb{U}] \subseteq A$.

Proof. For 1, let first $[\mathbb{U}] \setminus A \subseteq \bigcup_{n<\omega}(M_n \times \prod_{n<k<\omega} X_k)$ where $M_n \subseteq \prod_{k\le n} X_k$ $(n < \omega)$ are meager. Then the statement follows from Lemma 3.7 applied with $B_n = U_n \setminus M_n$ $(n < \omega)$. The converse is obvious.

Statement 2 is a special case of 1, so the proof is complete. □

Proposition 3.9. *With the notation of Definition 3.3, let $A \subseteq \prod_{k<\omega} X_k$ be arbitrary.*

(1) If there is a non-empty open tower $\mathbb{U}$ such that $[\mathbb{U}] \subseteq A$ then $A \in \mathcal{W}(I)$.
(2) If there is a dense open tower $\mathbb{U}$ such that $[\mathbb{U}] \subseteq A$ then $A \in \mathcal{W}(II)$.

Proof. Let $\mathbb{U} = (U_n)_{n<\omega}$. For 1, it is enough to construct a strategy $\sigma \in \Sigma(I)$ such that $W(\sigma) \subseteq [\mathbb{U}]$. We define $\sigma \subseteq \mathcal{G}_\omega$ by induction, as follows. Let $(U_0(0)) \in \sigma$ if and only $(U_0(0)) \in \mathcal{G}_\omega$ and $U_0(0) \subseteq U_0$.

Let $n < \omega$ be arbitrary and suppose that $\sigma \cap \mathrm{lev}_{2n+1}(\mathcal{G}_\omega)$ is already defined. For every $U \in \mathrm{lev}_{2n+1}(\sigma)$, $U_i(2(n-i)+1)$ $(i \le n)$ and $U_i(2(n+1-i))$ $(i \le n+1)$, let

$$U^\frown((U_i(2(n-i)+1))_{i\le n}) \in \sigma \text{ and } U^\frown((U_i(2(n-i)+1))_{i\le n}, (U_i(2(n+1-i)))_{i\le n+1}) \in \sigma$$

if and only if $U^\frown((U_i(2(n-i)+1))_{i\le n}, (U_i(2(n+1-i)))_{i\le n+1}) \in \mathcal{G}_\omega$ and $\prod_{i\le n+1} U_i(2(n+1-i)) \subseteq U_{n+1}$. This completes the inductive step of the definition of σ.

It is obvious that $\sigma \subseteq \mathcal{G}_\omega$ and that for every $n < \omega$, $U \in \mathrm{lev}_{2n+1}(\sigma)$ and $u \in \mathcal{U}_\omega$, $U^\frown u \in \mathcal{G}_\omega$ implies $U^\frown u \in \sigma$. To see that σ is pruned, let $n < \omega$, $U \in \mathrm{lev}_{2n+1}(\sigma)$ and $(U_i(2(n-i)+$

$1))_{i\leq n}$ be arbitrary such that $U^\frown((U_i(2(n-i)+1))_{i\leq n})\in\sigma$. Since $\prod_{i\leq n}U_i(2(n-i))\subseteq U_n$ and $U_n\Delta\operatorname{Pr}_{\prod_{k\leq n}X_k}(U_{n+1})$ is meager,

$$\Big(\Big(\prod_{i\leq n}U_i(2(n-i)+1)\Big)\times X_{n+1}\Big)\cap U_{n+1}\neq\emptyset.$$

In particular there are $U_i(2(n+1-i))$ $(i\leq n+1)$ such that $U_i(2(n+1-i))\subseteq U_i(2(n-i)+1)$ $(i\leq n)$, $\operatorname{diam}_{X_i}(U_i(2(n+1-i)))<2^{-(n+1-i)}$ $(i\leq n+1)$ and $\prod_{i\leq n+1}U_i(2(n+1-i))\subseteq U_{n+1}$. Thus $U^\frown((U_i(2(n-i)+1))_{i\leq n},(U_i(2(n+1-i)))_{i\leq n+1})\in\sigma$, which concludes $\sigma\in\Sigma(I)$.

It remains to prove $W(\sigma)\subseteq[\mathbb{U}]$. Let $(x_k)_{k<\omega}\in W(\sigma)$ be arbitrary, say $x_k=\bigcap_{i<\omega}U_k(i)$ $(k<\omega)$ for some $U\in[\sigma]$. Then for every $n<\omega$, $(x_k)_{k\leq n}\in\prod_{i\leq n}U_i(2(n-i))\subseteq U_n$. This shows $(x_k)_{k<\omega}\in U_n\times(\prod_{n<k<\omega}X_k)$ $(n<\omega)$, so the statement follows.

Statement 2 follows by an analogous argument. □

Since the spaces X_k $(k<\omega)$ are Choquet, in $\mathcal{G}_\omega$ both players can refine their quasi-strategies in such a way that the resulting sequence $((U_i(2(k-i)))_{i\leq k},(U_i(2(k-i)+1))_{i\leq k})_{k<\omega}$ satisfies $\bigcap_{i<\omega}U_k(i)$ is a singleton for every $k<\omega$. In particular, for every $P\in\{I,II\}$, $A\in\mathcal{W}(P)$ implies $A\neq\emptyset$. So by Proposition 3.9.1, if $\mathbb{U}$ is a non-empty open tower then $[\mathbb{U}]\neq\emptyset$.

The proof of the following proposition closely follows the proofs of [11, Theorem 8.33 p. 51] and [11, Theorem 21.8 p. 153].

Proposition 3.10. *With the notation of Definition 3.3 and Definition 3.5,*

(1) *if $A\in\mathcal{W}(I)$ then there is a non-empty open tower $\mathbb{U}$ such that $[\mathbb{U}]\subseteq A$;*
(2) *if $A\in\mathcal{W}(II)$ then there is a dense open tower $\mathbb{U}$ such that $[\mathbb{U}]\subseteq A$;*
(3) *if $F\in\mathcal{W}^\star(II)$ then $\operatorname{Pr}_{\prod_{k<\omega}X_k}(F)\in\mathbb{M}$.*

Proof. To see 1, let $\sigma\in\Sigma(I)$ be such that $W(\sigma)\subseteq A$. We define a tree $\tau\subseteq\sigma$ by induction, as follows. Let $\mathfrak{U}_0\subseteq\operatorname{lev}_1(\sigma)$ be a maximal family of pairwise disjoint open sets, and set $\operatorname{lev}_1(\tau)=\mathfrak{U}_0$. Let $n<\omega$ and suppose that $\operatorname{lev}_{2n+1}(\tau)$ is already defined. Let

$$\mathfrak{U}_{n+1}\subseteq\{u\in\mathcal{U}_\omega\colon\exists U\in\operatorname{lev}_{2n+1}(\tau)\ \exists v\in\mathcal{U}_\omega\ (U^\frown(v,u)\in\sigma)\}$$

be a maximal family of pairwise disjoint open sets. For every $u\in\mathfrak{U}_{n+1}$ fix one $v(u)\in\mathcal{U}_\omega$ such that $U^\frown(v(u),u)\in\sigma$ and set

$$\operatorname{lev}_{2n+3}(\tau)=\{U^\frown(v(u),u)\in\sigma\colon U\in\operatorname{lev}_{2n+1}(\tau),\ u\in\mathfrak{U}_{n+1}\}.$$

This completes the inductive step of the definition of τ. Observe that by requiring the members of $\mathfrak{U}_n$ to be pairwise disjoint, for every $n<\omega$ and $u\in\mathfrak{U}_n$ there is a unique $U\in\operatorname{lev}_{2n+1}(\tau)$ such that $U(2n)=u$.

Let $U_n=\bigcup\mathfrak{U}_n$ $(n<\omega)$, we show that $\mathbb{U}=(U_n)_{n<\omega}$ is a non-empty open tower and $[\mathbb{U}]\subseteq A$. It is obvious that $U_0\neq\emptyset$. Let $n<\omega$ be arbitrary and let $u\subseteq U_n$ be an arbitrary non-empty open set; we show $(\operatorname{Pr}_{\prod_{k\leq n}X_k}(U_{n+1}))\cap u\neq\emptyset$. By passing to a subset we can assume $u\subseteq u'$ for some $u'\in\mathfrak{U}_n$. Then there is a unique $U\in\operatorname{lev}_{2n+1}(\tau)$ such that $U(2n)=u'$. By the definition of $\mathcal{G}_\omega$, there is a $v\in\mathcal{U}_\omega$ such that $U^\frown v\in\mathcal{G}_\omega$ and $v\subseteq u$. Since σ is a strategy of player I, there is a $w\in\mathcal{U}_\omega$ such that $U^\frown(v,w)\in\sigma$, in particular $\operatorname{Pr}_{\prod_{k\leq n}X_k}(w)\subseteq v$ hence $\operatorname{Pr}_{\prod_{k\leq n}X_k}(w)\subseteq u$. By the maximality of $\mathfrak{U}_{n+1}$, there is a $w'\in\mathfrak{U}_{n+1}$ such that $w\cap w'\neq\emptyset$. This shows $(\operatorname{Pr}_{\prod_{k\leq n}X_k}(U_{n+1}))\cap u\neq\emptyset$. Since $u\subseteq U_n$ was arbitrary, we obtained $U_n\setminus\operatorname{Pr}_{\prod_{k\leq n}X_k}(U_{n+1})$ is nowhere dense in $\prod_{k\leq n}X_k$. Since $\operatorname{Pr}_{\prod_{k\leq n}X_k}(U_{n+1})\subseteq U_n$ follows immediately from the definition, we proved $U_n\Delta\operatorname{Pr}_{\prod_{k\leq n}X_k}(U_{n+1})$

is meager in $\prod_{k\leq n} X_k$ $(n<\omega)$. Thus $\mathbb{U}$ is a non-empty open tower. We also obtained that τ is a pruned tree. Since $\tau\subseteq\sigma$, we have $W(\tau)\subseteq W(\sigma)\subseteq A$, so it remains to see $[\mathbb{U}]\subseteq W(\tau)$.

Let $(x_k)_{k<\omega}\in[\mathbb{U}]$ be arbitrary. For every $n<\omega$, by $\mathfrak{U}_n$ being pairwise disjoint, there is a unique $v_{2n}\in\mathfrak{U}_n$ such that $(x_k)_{k\leq n}\in v_{2n}$. By the definition of τ, $v_{2n}\in\mathfrak{U}_n$ means that there is a $V_n\in\operatorname{lev}_{2n+1}(\tau)$ with $V_n(2n)=v_{2n}$. But then $(x_k)_{k\leq i}\in V_n(2i)$, which implies $V_n(2i)=v_{2i}$ $(i<n)$. Thus there is a unique $V\in[\tau]$ such that $V(2n)=v_{2n}$ $(n<\omega)$. This shows $(x_k)_{k<\omega}\in W(\tau)$, which completes the proof of 1.

To see 2, let $\sigma\in\Sigma(II)$ be such that $W(\sigma)\subseteq A$. As in the proof of 1, we can define maximal pairwise disjoint families $\mathfrak{U}_n$ of open subsets of $\prod_{k\leq n} X_k$ and a pruned tree $\tau\subseteq\sigma$ such that for every $n<\omega$ and $u\in\mathfrak{U}_n$ there is a unique $U\in\operatorname{lev}_{2n+2}(\tau)$ satisfying $U(2n+1)=u$. Set $U_n=\bigcup\mathfrak{U}_n$ $(n<\omega)$. Since σ is a strategy of player II, the maximality of $\mathfrak{U}_n$ implies U_n is dense in $\prod_{k\leq n} X_k$. Hence $\mathbb{U}=(U_n)_{n<\omega}$ is a dense open tower, and as in the proof of 1, we have $[\mathbb{U}]\subseteq W(\tau)\subseteq W(\sigma)\subseteq A$.

To see 3, let $\sigma\in\Sigma^\star(II)$ be such that $W(\sigma)\cap F=\emptyset$. For every $y\in\omega^{<\omega}$ we say $U\in\mathcal{G}_\omega$ is *compatible with* σ,y if $|U|=2|y|$ and $(U(2i),y(i),U(2i+1))_{i<|y|}\in\sigma$.

For every $y\in\omega^{<\omega}\setminus\{\emptyset\}$ we construct a tree $\tau_y\subseteq\mathcal{G}_\omega$ of height $2|y|$ such that

(i) every $U\in\operatorname{lev}_{2|y|}(\tau_y)$ is compatible with σ,y;
(ii) $\{U(2|y|-1)\colon U\in\operatorname{lev}_{2|y|}(\tau_y)\}$ is a family of pairwise disjoint open sets and for every $U,V\in\operatorname{lev}_{2|y|}(\tau_y)$, $U(2|y|-2)\neq V(2|y|-2)$ implies $U(2|y|-1)\cap V(2|y|-1)=\emptyset$.
(iii) $y\sqsubseteq y'$ implies τ_y is the restriction of $\tau_{y'}$ to sequences of length $\leq 2|y|$;
(iv) τ_y is maximal with these properties.

Set $\tau_\emptyset=\emptyset$. Let $y\in\omega^{<\omega}\setminus\{\emptyset\}$ be arbitrary, set $y^-=y|_{|y|-1}$ and suppose that τ_{y^-} is already defined. Let

$$\mathfrak{U}_y\subseteq\{u\in\mathcal{U}_\omega\colon \exists U\in\tau_{y^-}\ \exists v\in\mathcal{U}_\omega\ (U^\frown(v,u)\text{ is compatible with }\sigma,y)\}$$

be a maximal family of pairwise disjoint open sets. For every $u\in\mathfrak{U}_y$ fix one $v(u)\in\mathcal{U}_\omega$ such that $U^\frown(v(u),u)$ is compatible with σ,y and set

$$\operatorname{lev}_{2|y|}(\tau_y)=\{U^\frown(v(u),u)\in\mathcal{G}_\omega\colon U\in\operatorname{lev}_{2|y|-2}(\tau_{y^-}),\ u\in\mathfrak{U}_y\}.$$

This completes the inductive step of the definition of τ_y $(y\in\omega^{<\omega}\setminus\{\emptyset\})$.

It is obvious from the definition that (i-iv) hold. As in the proof of statement 2, for every $y\in\omega^{<\omega}\setminus\{\emptyset\}$ the maximality of $\mathfrak{U}_y$ implies $U_y=\bigcup\mathfrak{U}_y$ is dense in $\prod_{k<|y|}X_k$. Thus with $M_n=\bigcup\{(\prod_{k<|y|}X_k)\setminus U_y\colon y\in\omega^{n+1}\}$, $M=\bigcup_{n<\omega}(M_n\times\prod_{n<k<\omega}X_k)$ satisfies $M\in\mathbb{M}$. It remains to show that $\operatorname{Pr}_{\prod_{k<\omega}X_k}(F)\subseteq M$.

Let $((x_k)_{k<\omega},(y_k)_{k<\omega})\in(\prod_{k<\omega}X_k)\times\omega^\omega$ be arbitrary and suppose $(x_k)_{k<\omega}\notin M$. Then in particular, $(x_k)_{k\leq n}\in U_{(y_k)_{k\leq n}}$ $(n<\omega)$ so by (ii), there is a unique $U\in[\mathcal{G}_\omega]$ such that $U|_{2n+2}\in\tau_{(y_k)_{k\leq n}}$ $(n<\omega)$. Then $(U(2k),y_k,U(k+1))_{k<\omega}\in[\sigma]$, so $((x_k)_{k<\omega},(y_k)_{k<\omega})\in W(\sigma)$ which implies $((x_k)_{k<\omega},(y_k)_{k<\omega})\notin F$. This completes the proof. □

Proof of Theorem 3.4. For 1, 1a ⇔ 1b follows from Proposition 3.9.1 and Proposition 3.10.1, while 1b ⇔ 1c is Corollary 3.8.1. For 2, 2a ⇔ 2b follows from Proposition 3.9.2 and Proposition 3.10.2, and 2b ⇔ 2c is Corollary 3.8.2. □

Proof of Theorem 3.6. For 1, if $F\in\mathcal{W}^\star(I)$ then by omitting the y_is, player I gets a strategy in $\mathcal{G}_\omega$ showing $\operatorname{Pr}_{\prod_{k<\omega}X_k}(F)\in\mathcal{W}(I)$. Statement 2 is Proposition 3.10.3. □

The following is an immediate corollary of Theorem 3.4 and Theorem 3.6.

Corollary 3.11. *With the notation of Definition 3.2 and Definition 3.5, let* $A \subseteq \prod_{k<\omega} X_k$ *and* $F \subseteq (\prod_{k<\omega} X_k) \times \omega^\omega$ *be arbitrary.*

(1) *If the game* $\mathcal{G}_\omega(A)$ *is determined then either* $(\prod_{k<\omega} X_k) \setminus A \in \mathbb{M}$ *or there is a non-empty open tower* $\mathbb{U}$ *such that* $A \cap [\mathbb{U}] = \emptyset$*.*
(2) *If the game* $\mathcal{G}^\star_\omega(F)$ *is determined then either* $\mathrm{Pr}_{\prod_{k<\omega} X_k}(F) \in \mathbb{M}$ *or there is a non-empty open tower* $\mathbb{U}$ *such that* $[\mathbb{U}] \subseteq \mathrm{Pr}_{\prod_{k<\omega} X_k}(F)$*.*

The following is to be compared to [21, Proposition 3.29 p. 31] and [29, Section 5.1.3 p. 231]. In Proposition 4.8 we will show that the complexity assumptions in this result are consistently optimal.

Corollary 3.12. *With the notation of Definition 3.3,*

(1) *if* $A \subseteq \prod_{k<\omega} X_k$ *is an analytic set then either* $A \in \mathbb{M}$ *or there is a non-empty open tower* $\mathbb{U}$ *such that* $[\mathbb{U}] \subseteq A$*.*
(2) *if* $A \subseteq \prod_{k<\omega} X_k$ *is a co-analytic set then either* $(\prod_{k<\omega} X_k) \setminus A \in \mathbb{M}$ *or there is a non-empty open tower* $\mathbb{U}$ *such that* $A \cap [\mathbb{U}] = \emptyset$*.*

Proof. For 1, let $F \subseteq (\prod_{k<\omega} X_k) \times \omega^\omega$ be a closed set such that $\mathrm{Pr}_{\prod_{k<\omega} X_k}(F) = A$. Then the game $\mathcal{G}^\star(F)$ is closed hence determined, i.e. the statement follows from Corollary 3.11.2. Statement 2 follows from 1 by taking complements. □

We note that Corollary 3.12.1 has an alternative proof based on Borel determinacy, as follows. Since the σ-ideal $\mathbb{M}$ is generated by closed sets, by [23, Theorem 1 p. 1023] either $A \in \mathbb{M}$ or there is a G_δ set $G \subseteq A$ such that $G \notin \mathbb{M}$. The game $\mathcal{G}_\omega((\prod_{k<\omega} X_k) \setminus G)$ is determined, so we conclude that there is a non-empty open tower $\mathbb{U}$ such that $[\mathbb{U}] \subseteq G \subseteq A$. However, we believe that by avoiding Borel determinacy and by presenting the unfolded game $\mathcal{G}^\star$ we give a better insight to the fusion game $\mathcal{G}_\omega$.

3.2. **Perfect Set Property games.**

For simplicity, in this section we assume X is a non-empty perfect Polish space and we fix a countable base $\mathcal{U}$ in X which consists of non-empty open sets. We study the following games.

Definition 3.13. Let $0 < N < \omega$ be fixed. For every $A \subseteq X^N$, $\mathcal{P}_N(A)$ denotes the *N-dimensional Perfect Set Property game with payoff set A*, in which two players play

$$\begin{array}{llllll} I: & ((U_k(0,0),U_k(0,1)))_{k<N} & & \dots & ((U_k(n,0),U_k(n,1)))_{k<N} & & \dots \\ II: & & (i_k(0))_{k<N} & \dots & & (i_k(n))_{k<N} & \dots \end{array}$$

where for every $k < N$, $l < \omega$ and $j < 2$ we have $i_k(l) \in \{0,1\}$, $U_k(l,j) \in \mathcal{U}$, $\mathrm{diam}_X(U_k(l,j)) < 2^{-l}$, $U_k(l,0) \cap U_k(l,1) = \emptyset$ and $U_k(l+1,j) \subseteq U_k(l, i_k(l))$. Player I wins the game if and only if for every $k < N$, $\bigcap_{l<\omega} U_k(l, i_k(l))$ is a singleton and $(\bigcap_{l<\omega} U_k(l, i_k(l)))_{k<N} \in A$.

Similarly, for every $A \subseteq X^\omega$, $\mathcal{P}_\omega(A)$ denotes the *fusion game of the Perfect Set Property games with payoff set* A, in which two players play

$$\begin{array}{llll} I\colon (U_0(0,0),U_0(0,1)) & & ((U_0(1,0),U_0(1,1)),(U_1(0,0),U_1(0,1))) & \dots \\ II\colon & i_0(0) & & (i_0(1),i_1(0)) \quad \dots \end{array}$$

$$\begin{array}{lll} \dots & ((U_k(n-k,0),U_k(n-k,1)))_{k\le n} & \dots \\ \dots & & (i_k(n-k))_{k\le n} \quad \dots \end{array}$$

where for every $k,l<\omega$ and $j<2$ we have $i_k(l)\in\{0,1\}$, $U_k(l,j)\in\mathcal{U}$, $\operatorname{diam}_X(U_k(l,j))<2^{-(k+l)}$, $U_k(l,0)\cap U_k(l,1)=\emptyset$ and $U_k(l+1,j)\subseteq U_k(l,i_k(l))$. Player I wins the game if and only if for every $k<\omega$, $\bigcap_{l<\omega}U_k(l,i_k(l))$ is a singleton and $(\bigcap_{l<\omega}U_k(l,i_k(l)))_{k<\omega}\in A$.

Notice that $\mathcal{P}_1$ is the usual Perfect Set Property game (see e.g. [11, Section 21.A p. 149]). The quasi-strategies of the players in the games $\mathcal{P}_N$ $(0<N<\omega)$ and $\mathcal{P}_\omega$ are defined analogously to Definition 3.2. The characterization of existence of winning quasi-strategies involves the following notions. Recall $N_s=\{\sigma\in 2^\omega\colon s\sqsubseteq\sigma\}$ $(s\in 2^{<\omega})$.

Definition 3.14. Let $0<N<\omega$ be fixed. We call $\mathcal{C}=((U_k(0),U_k(1)))_{k<N}$ an *N-cube* if for every $k<N$ and $j<2$ we have $U_k(l,j)\in\mathcal{U}$ and $U_k(0)\cap U_k(1)=\emptyset$. We define $[\mathcal{C}]=\prod_{k<N}(U_k(0)\cup U_k(1))$, and for every $t\in 2^N$, $\mathcal{C}(t)=\prod_{k<N}U_k(t(k))$. We set

$$\operatorname{diam}_X(\mathcal{C})=\max\{\operatorname{diam}_X(U_k(j))\colon k<N, j<2\}.$$

We say that $F\subseteq X^N$ is *N-cube free* if there is a $\delta>0$ such that for every N-cube $\mathcal{C}$ with $\operatorname{diam}_X(\mathcal{C})<\delta$ there is a $t\in 2^N$ with $F\cap\mathcal{C}(t)=\emptyset$. We set

$$\mathbb{F}_N=\Big\{F\subseteq X^N\colon \exists F_n\subseteq X^N\ (n<\omega)\ \Big(F_n \text{ is } N\text{-cube free } (n<\omega),\ F\subseteq\bigcup_{n<\omega}F_n\Big)\Big\},$$

$$\mathbb{F}_\omega=\Big\{F\subseteq X^\omega\colon \exists F_n\subseteq X^{n+1}\ (n<\omega)\\ \Big(F_n \text{ is } (n+1)\text{-cube free } (n<\omega),\ F\subseteq\bigcup_{n<\omega}\Big(F_n\times X^{\omega\setminus(n+1)}\Big)\Big)\Big\}.$$

A function $f\colon(2^\omega)^N\to X^N$ is *cube preserving* if for every $n<\omega$ and $s_k\in 2^n$ $(k<N)$ there is an N-cube $\mathcal{C}$ such that $\operatorname{diam}_X(\mathcal{C})<2^{-n}$ and

$$f\Big[\prod_{k<N}N_{s_k^\frown t(k)}\Big]=f\big[(2^\omega)^N\big]\cap\mathcal{C}(t)\ (t\in 2^N).$$

A function $f\colon(2^\omega)^\omega\to X^\omega$ is *cube preserving* if for every $n<\omega$ and $s_k\in 2^n$ $(k\le n)$ there is an $(n+1)$-cube $\mathcal{C}$ such that $\operatorname{diam}_X(\mathcal{C})<2^{-n}$ and

$$f\Big[\Big(\prod_{k\le n}N_{s_k^\frown t(k)}\Big)\times X^{\omega\setminus(n+1)}\Big]=f\big[(2^\omega)^\omega\big]\cap\big(\mathcal{C}(t)\times X^{\omega\setminus(n+1)}\big)\ (t\in 2^{n+1}).$$

Notice that $\mathbb{F}_1=[X]^{\le\omega}$. It is easy to see that every cube preserving function is continuous and injective. The statement corresponding the Lemma 3.7 is the following.

Lemma 3.15. *With the notation of Definition 3.14,*

(1) $F \subseteq X^N$ *is N-cube free if and only if* $cl_{X^N}(F)$ *is N-cube free.*
(2) $\mathbb{F}_N$ $(0 < N < \omega)$ *and* $\mathbb{F}_\omega$ *are σ-ideals generated by closed sets.*
(3) $\mathbb{F}_\omega = \Big\{F \subseteq X^\omega : \exists F_n \in \mathbb{F}_{n+1}\ (n < \omega)\ \Big(F \subseteq \bigcup_{n<\omega} \Big(F_n \times X^{\omega \setminus (n+1)}\Big)\Big)\Big\}.$

Proof. Since $\mathcal{C}(t)$ is open for every N-cube $\mathcal{C}$ and $t \in 2^N$, the first statement follows. Then by definition, 2 holds for $\mathbb{F}_N$ $(0 < N < \omega)$ and $\mathbb{F}_\omega$ is generated by closed sets. To see 3 and that $\mathbb{F}_\omega$ is a σ-ideal, observe that if $F \subseteq X^N$ is N-cube free then $F \times X^M$ is $N+M$ -cube free $(M < \omega)$. So the statement follows by decomposing and re-indexing, as in the proof of Lemma 3.7. □

Proposition 3.16. *With the notation of Definition 3.13 and Definition 3.14,*

(1) for every $A \subseteq X^N$*, player I has a winning quasi-strategy in* $\mathcal{P}_N(A)$ *if and only if there is a cube preserving function* $f\colon (2^\omega)^N \to X^N$ *such that* $f\big[(2^\omega)^N\big] \subseteq A$.
(2) for every $A \subseteq X^\omega$*, player I has a winning quasi-strategy in* $\mathcal{P}_\omega(A)$ *if and only if there is a cube preserving function* $f\colon (2^\omega)^\omega \to X^\omega$ *such that* $f\big[(2^\omega)^\omega\big] \subseteq A$.

Proof. For 1, suppose first player I has a winning quasi-strategy τ. By passing to a non-empty pruned subtree we can assume that τ is a strategy, i.e. for every $n < \omega$ and $s \in \mathrm{lev}_{2n}(\tau)$ there is a unique N-cube $\mathcal{C}$ with $s^\frown\mathcal{C} \in \tau$. For every $n < \omega$ and $s_k \in 2^n$ $(k < N)$ let $\mathcal{C}_{(s_k)_{k<N}}$ be an N-cube such that for every $\sigma_k \in 2^\omega$ $(k < N)$,

$$(\mathcal{C}_{(\sigma_k|_m)_{k<N}}, (\sigma_k|_{m+1})_{k<N})_{m \le n} \in \tau\ (n < \omega); \tag{3.1}$$

this assignment is possible and unique since τ is a strategy. Moreover, $\mathrm{diam}_X(\mathcal{C}_{(s_k)_{k<N}}) < 2^{-n}$ $((s_k)_{k<N} \in (2^n)^N,\ n < \omega)$. Define $f\colon (2^\omega)^N \to X^N$,

$$f((\sigma_k)_{k<N}) = \bigcap_{n<\omega} \mathcal{C}_{(\sigma_k|_n)_{k<N}}.$$

Then the cubes $\mathcal{C}_{(s_k)_{k<N}}$ $((s_k)_{k<N} \in (2^n)^N,\ n < \omega)$ witness that f is a cube preserving function. Since τ is a winning strategy of player I, $f\big[(2^\omega)^N\big] \subseteq A$ follows.

Suppose now there is a cube preserving function $f\colon (2^\omega)^N \to X^N$ such that $f\big[(2^\omega)^N\big] \subseteq A$, and let $\mathcal{C}_{(s_k)_{k<N}}$ $((s_k)_{k<N} \in (2^n)^N,\ n < \omega)$ be the witnessing cubes. By

$$f\big[(2^\omega)^N\big] \cap [\mathcal{C}_{(s_k)_{k<N}}] = f\Big[\prod_{k<N} N_{s_k}\Big] = f\big[(2^\omega)^N\big] \cap \mathcal{C}_{(s_k|_{n-1})_{k<N}}((s_k(n))_{k<N}),$$

we can assume

$$[\mathcal{C}_{(s_k)_{k<N}}] \subseteq \mathcal{C}_{(s_k|_{n-1})_{k<N}}((s_k(n))_{k<N})\ ((s_k)_{k<N} \in (2^n)^N,\ 0 < n < \omega).$$

Then the non-empty pruned tree τ defined by (3.1) is a winning quasi-strategy for player I.

Statement 2 follows by an analogous argument. □

Proposition 3.17. *With the notation of Definition 3.13 and Definition 3.14,*

(1) for every $A \subseteq X^N$*, player II has a winning quasi-strategy in* $\mathcal{P}_N(A)$ *if and only if* $A \in \mathbb{F}_N$.
(2) for every $A \subseteq X^\omega$*, player II has a winning quasi-strategy in* $\mathcal{P}_\omega(A)$ *if and only if* $A \in \mathbb{F}_\omega$.

Proof. For 1, suppose first player II has a winning quasi-strategy τ. Let $U_\emptyset = X^N$. For every $0 < n < \omega$ and $t \in \mathrm{lev}_{2n}(\tau)$, $\mathcal{C}_t = t(2n-2)$ in an N-cube so we can define $U_t = \mathcal{C}_t(t(2n-1))$. Set

$$F_t = U_t \setminus \bigcup\{\mathcal{C}(s) \colon \mathcal{C} \text{ is an } N\text{-cube},\ s \in 2^N,\ t^\frown\mathcal{C}^\frown s \in \tau\}\ (t \in \tau).$$

We show that F_t $(t \in \tau)$ are N-cube free. Fix $t \in \tau$ and let $\mathcal{C} = ((U_k(0), U_k(1)))_{k<N}$ be an arbitrary N-cube with $\mathrm{diam}_X(\mathcal{C}) < 2^{-|t|}$; we find $s \in 2^N$ such that $F_t \cap \mathcal{C}(s) = \emptyset$. Since $U_t = \prod_{k<N} V_k$ for some $V_k \in \mathcal{U}$ $(k < N)$, we have

$$[\mathcal{C}] \cap U_t = [((U_k(0) \cap V_k, U_k(1) \cap V_k))_{k<N}].$$

So since $F_t \subseteq U_t$, we can assume $[\mathcal{C}] \subseteq U_t$. Then $t^\frown\mathcal{C} \in \tau$, hence there is an $s \in 2^N$ with $t^\frown\mathcal{C}^\frown s \in \tau$. By definition, this implies $F_t \cap \mathcal{C}(s) = \emptyset$, as required.

Since τ is countable, it remains to show $A \subseteq \bigcup_{t\in\tau} F_t$. Suppose $(x_k)_{k<N} \in X^N \setminus \bigcup_{t\in\tau} F_t$. By induction on n, we define N-cubes $\mathcal{C}_n$ and $s_n \in 2^N$ $(n < \omega)$ such that $((\mathcal{C}_n, s_n))_{n<\omega} \in [\tau]$ and $(x_k)_{k<N} \in \bigcap_{n<\omega} \mathcal{C}_n(s_n)$. Since τ is a winning quasi-strategy of player II, this implies $(x_k)_{k<N} \notin A$, as stated.

Since $(x_k)_{k<N} \notin F_\emptyset$, there are $\mathcal{C}_0, s_0$ with ${\mathcal{C}_0}^\frown s_0 \in \tau$ such that $(x_k)_{k<N} \in \mathcal{C}_0(s_0)$. Let $n < \omega$ be arbitrary and suppose $\mathcal{C}_i, s_i$ $i \leq n$ are defined such that $((\mathcal{C}_i, s_i))_{i\leq n} \in \tau$ and $(x_k)_{k<N} \in \mathcal{C}_n(s_n)$. Since $(x_k)_{k<N} \in U_{((\mathcal{C}_i,s_i))_{i\leq n}} \setminus F_{((\mathcal{C}_i,s_i))_{i\leq n}}$, there are $\mathcal{C}_{n+1}, s_{n+1}$ with ${((\mathcal{C}_i, s_i))_{i\leq n}}^\frown \mathcal{C}_{n+1}{}^\frown s_{n+1} \in \tau$ and $(x_k)_{k<N} \in \mathcal{C}_{n+1}(s_{n+1})$. This completes the inductive step of the definition of $\mathcal{C}_n, s_n$ $(n < \omega)$, and the proof of 1.

Statement 2 follows by an analogous argument. □

Corollary 3.18. *With the notation of Definition 3.13 and Definition 3.14,*

(1) for every analytic set $A \subseteq X^N$, either $A \in \mathbb{F}_N$ or there is a cube preserving function $f\colon (2^\omega)^N \to X^N$ such that $f\left[(2^\omega)^N\right] \subseteq A$.
(2) for every analytic set $A \subseteq X^\omega$, either $A \in \mathbb{F}_\omega$ or there is a cube preserving function $f\colon (2^\omega)^\omega \to X^\omega$ such that $f\left[(2^\omega)^\omega\right] \subseteq A$.

Proof. To see 1, by Lemma 3.15.2, the σ-ideal $\mathbb{F}_N$ is generated by closed sets. So by [23, Theorem 1 p. 1023], either $A \in \mathbb{F}_N$ or there is a G_δ set $G \subseteq A$ such that $G \notin \mathbb{F}_N$. The game $\mathcal{P}_M(G)$ is determined, so the statement follows from Proposition 3.16.1 and Proposition 3.17.1.

Statement 2 follows by an analogous argument. □

4. Infinite dimensional perfect set theorems

Our infinite dimensional perfect set theorems are based on the following easy observation. Recall that for every set X and $\alpha \leq \omega$, $IS_\alpha(X) = \{(x_n)_{n<\alpha} \in X^\omega : x_n \neq x_m \ (n < m < \alpha)\}$.

Theorem 4.1. *Let X be a non-empty Choquet space such that X has no isolated points and there is a metric d on X whose balls are open in X. Let $A \subseteq X^\omega$ satisfy $X^\omega \setminus A \in \mathbb{M}$. Then there is a non-empty perfect set $P \subseteq X$ such that $IS_\omega(P) \subseteq A$.*

The proof of Theorem 4.1 uses the following version of Mycielski's Theorem (see e.g. [20, Theorem 1 p. 141] and [11, Exercise 19.5 p. 130]).

Theorem 4.2. *Let X be a non-empty Choquet space such that X has no isolated points and there is a metric d on X whose balls are open in X. Let $M_n \subseteq X^{n+1}$ $(n < \omega)$ be meager sets. Then there is a non-empty perfect set $P \subseteq X$ such that for every $n < \omega$ we have $IS_{n+1}(P) \cap M_n = \emptyset$.*

Proof of Theorem 4.1. Let $M_n \subseteq X^{n+1}$ $(n < \omega)$ be meager sets such that $X^\omega \setminus A \subseteq \bigcup_{n<\omega}(M_n \times X^{\omega\setminus(n+1)})$. By Theorem 4.2, there is a non-empty perfect set $P \subseteq X$ such that for every $n < \omega$ we have $IS_{n+1}(P) \cap M_n = \emptyset$. We show that P fulfills the requirements.

If $(x_i)_{i<\omega} \in X^\omega \setminus A$ then there is an $n < \omega$ such that $(x_i)_{i<\omega} \in M_n$. Then $(x_i)_{i\leq n} \notin IS_{n+1}(P)$ hence $(x_i)_{i<\omega} \notin IS_\omega(P)$, as required. □

Corollary 3.12.2 say that for co-analytic A, $X^\omega \setminus A \in \mathbb{M}$ holds if there is no non-empty open tower $\mathbb{U}$ such that $[\mathbb{U}] \cap A = \emptyset$. In the sequel we give various sufficient condition for this.

4.1. **Largeness in category.** Recall that for every $S_n \subseteq X^{n+1}$ $(n < \omega)$ we have $[(S_n)_{n<\omega}] = \bigcap_{n<\omega}(S_n \times X^{\omega\setminus(n+1)})$. If $n < \omega$, $(x_i)_{i\leq n} \in X^{n+1}$ and $S \subseteq X^{n+2}$ then we set

$$[S]_{(x_i)_{i\leq n}} = \{x_{n+1} \in X : (x_i)_{i\leq n+1} \in S\}.$$

The most important additional property our topological spaces have to satisfy is the following.

Definition 4.3. A topological space X has the *Kuratowski-Ulam property* if for every $n < \omega$ and for every meager set $M \subseteq X^{n+2}$,

$$\{(x_i)_{i\leq n} \in X^{n+1} : [M]_{(x_i)_{i\leq n}} \text{ is non-meager in } X\}$$

is meager in X^{n+1}.

By [11, Theorem 8.41 p. 53], every second countable topological space has the Kuratowski-Ulam property. In particular, all of our results hold for Polish spaces and for the canonical refinement of Polish topologies turning a countable family of analytic sets into clopen sets (see e.g. [11, Theorem 25.18 p. 203]).

Our main technical notion is the following.

Definition 4.4. Let X be a topological space. We call $\mathbb{W} = (W_n)_{n<\omega}$ a *flag* if for every $n < \omega$, $W_n \subseteq X^{n+1}$ and $W_n = \mathrm{Pr}_{X^{n+1}}(W_{n+1})$. A flag $\mathbb{W}$ is *of second category everywhere* if $W_0 \subseteq X$ is of second category everywhere and for every $n < \omega$ and $(x_i)_{i\leq n} \in W_n$ we have $[W_{n+1}]_{(x_i)_{i\leq n}}$ is of second category everywhere in X.

If $\mathbb{W}$ is a flag and $\mathbb{U}$ is an open tower, then we say $\mathbb{W}$ *is co-meager in* $\mathbb{U}$ if $U_0 \setminus W_0$ is meager and for every $n < \omega$ and $(x_i)_{i\leq n} \in W_n \cap U_n$ we have $[U_{n+1} \setminus W_{n+1}]_{(x_i)_{i\leq n}}$ is meager.

Lemma 4.5. *Let X be a topological space and let $\mathbb{U}$ be a non-empty open tower. If X has the Kuratowski-Ulam property then there is a flag $\mathbb{W} = (W_n)_{n<\omega}$ which is co-meager in $\mathbb{U}$ and $W_n \subseteq U_n$ $(n < \omega)$. Moreover, if X is Polish then W_n $(n < \omega)$ can be taken G_δ.*

Proof. We define $T_n(i) \subseteq X^{n+1}$ $(n, i < \omega)$ by induction, as follows. Set $T_n(0) = U_n \Delta \mathrm{Pr}_{X^{n+1}}(U_{n+1})$ $(n < \omega)$. Let $i < \omega$ and suppose that $T_n(i)$ $(n < \omega)$ are defined. Then let

$$T_n(i+1) = \{(x_i)_{i\leq n} \in X^{n+1} : [T_{n+1}(i)]_{(x_i)_{i\leq n}} \text{ is non-meager}\} \ (n < \omega).$$

We show that $W_n = U_n \setminus \bigcup_{i<\omega} \bigcup_{k\leq n}(T_k(i) \times X^{n-k})$ $(n < \omega)$ fulfill the requirements.

It is obvious that $W_n \subseteq U_n$ $(n < \omega)$. Next we show that $U_0 \setminus W_0$ is meager and that for every $n < \omega$ and $(x_i)_{i\leq n} \in W_n$ we have $[U_{n+1} \setminus W_{n+1}]_{(x_i)_{i\leq n}}$ is meager. Since $\mathbb{U}$ is an open tower, $T_n(0)$ $(n < \omega)$ are meager. Using the Kuratowski-Ulam property of X, it is easy to see that for every $n, i < \omega$, $T_n(i) \subseteq X^{n+1}$ is meager. Hence $U_0 \setminus W_0$ is meager, as required.

Now let $n < \omega$ and let $(x_i)_{i\leq n} \in W_n$. By $(x_i)_{i\leq n} \notin T_n(j+1)$ $(j < \omega)$ we have $[T_{n+1}(j)]_{(x_i)_{i\leq n}}$ is meager $(j < \omega)$. So $[U_{n+1} \setminus W_{n+1}]_{(x_i)_{i\leq n}}$ is meager, as required.

To see that $\mathbb{W}$ is a flag, we have to show $W_n = \mathrm{Pr}_{X^{n+1}}(W_{n+1})$ $(n < \omega)$. Let first $n < \omega$ and $(x_i)_{i\leq n} \in W_n$ be arbitrary. By $(x_i)_{i\leq n} \notin T_n(0)$ we have $[U_{n+1}]_{(x_i)_{i\leq n}} \neq \emptyset$. As we have seen above, $[U_{n+1} \setminus W_{n+1}]_{(x_i)_{i\leq n}}$ is meager, in particular $[W_{n+1}]_{(x_i)_{i\leq n}} \neq \emptyset$, as required. While if $n < \omega$ and $(x_i)_{i\leq n+1} \in W_{n+1}$ is arbitrary, then by $(x_i)_{i\leq n+1} \in U_{n+1}$ and $(x_i)_{i\leq n+1} \notin \bigcup_{i<\omega} \bigcup_{k\leq n+1}(T_k(i) \times X^{n+1-k})$ we get $(x_i)_{i\leq n} \in U_n$ and $(x_i)_{i\leq n} \notin \bigcup_{i<\omega} \bigcup_{k\leq n}(T_k(i) \times X^{n-k})$, i.e. $(x_i)_{i\leq n} \in W_n$, as required.

Finally if X is Polish, then $T_n(0)$ $(n < \omega)$ are F_σ. By Montgomery's Theorem (see e.g. [11, Exercise 22.22 p. 174]), $T_n(i)$ $(n, i < \omega)$ are F_σ, so by definition, W_n $(n < \omega)$ are G_δ.■ □

Corollary 4.6. *Let X be a topological space with the Kuratowski-Ulam property, and let $\mathbb{U}$ be a non-empty open tower. Let $\mathbb{V}$ be flag which is of second category everywhere. Then $[\mathbb{U}] \cap [\mathbb{V}] \neq \emptyset$.*

Proof. By Lemma 4.5, there is a flag $\mathbb{W}$ which is co-meager in $\mathbb{U}$ and $W_n \subseteq U_n$ $(n < \omega)$. By induction on $n < \omega$, we define $x_n \in X$ $(n < \omega)$ such that $(x_i)_{i \leq n} \in W_{n+1} \cap V_{n+1}$ $(n < \omega)$. Then by $(x_n)_{n<\omega} \in [\mathbb{W}] \cap [\mathbb{V}]$ and $[\mathbb{W}] \subseteq [\mathbb{U}]$ the statement follows.

Since $U_0 \neq \emptyset$, $U_0 \setminus W_0$ is meager and V_0 is of second category everywhere, we have $W_0 \cap V_0 \neq \emptyset$. Let $x_0 \in W_0 \cap V_0$ be arbitrary.

Let $n < \omega$ and suppose that x_i $(i \leq n)$ are defined such that $(x_i)_{i \leq n} \in W_{n+1} \cap V_{n+1}$. Since $\mathbb{W}$ is a flag, we get $[W_{n+2}]_{(x_i)_{i \leq n}} \neq \emptyset$. By $W_{n+2} \subseteq U_{n+2}$ this implies $[U_{n+2}]_{(x_i)_{i \leq n}} \neq \emptyset$. Moreover $[U_{n+2} \setminus W_{n+2}]_{(x_i)_{i \leq n}}$ is meager , so since $[V_{n+2}]_{(x_i)_{i \leq n}}$ is of second category everywhere in X, we have $[V_{n+2}]_{(x_i)_{i \leq n}} \cap [W_{n+2}]_{(x_i)_{i \leq n}} \neq \emptyset$. Let $x_{n+1} \in [V_{n+2}]_{(x_i)_{i \leq n}} \cap [W_{n+2}]_{(x_i)_{i \leq n}}$ be arbitrary. This completes the inductive step of the construction and finishes the proof. □

Corollary 4.7. *Let X be a non-empty Choquet space such that X has no isolated points, X has the Kuratowski-Ulam property and there is a metric d on X whose balls are open in X. Let $A \subseteq X^\omega$.*

(1) Suppose $\mathcal{G}_\omega(A)$ is determined and there is a $H \subseteq X$ which is of second category everywhere and $IS_\omega(H) \subseteq A$. Then there exists a non-empty perfect set $P \subseteq X$ such that $IS_\omega(P) \subseteq A$.
(2) Suppose A is co-analytic and there is a $H \subseteq X$ which is non-meager and $IS_\omega(H) \subseteq A$. Then there exists a non-empty perfect set $P \subseteq X$ such that $IS_\omega(P) \subseteq A$.

Proof. To see 1, set $V_n = \{(x_i)_{i \leq n} \in H^{n+1} \colon x_i \neq x_j \ (i < j \leq n)\}$; then $\mathbb{V} = (V_n)_{n<\omega}$ is a flag which is of second category everywhere. Since $[\mathbb{V}] \subseteq A$, by Corollary 4.6 we have $A \cap [\mathbb{U}] \neq \emptyset$ for every non-empty open tower $\mathbb{U}$. Hence by Corollary 3.11.1, $X^\omega \setminus A \in \mathbb{M}$. So the statement follows from Theorem 4.1.

By Corollary 3.12.2, statement 2 reduces to 1 by passing to a non-empty open subset of X where H is of second category everywhere. □

As we pointed out above, Corollary 4.7 is applicable if X is Polish or X is obtained from a Polish space by turning a countable family of analytic sets into clopen sets the usual way. Even if largeness in Baire category is not preserved during such refinement of topologies, this observation shows that the game $\mathcal{G}_\omega(A)$ is informative for $A = C \cap \prod_{k<\omega} A_k$ where $C \subseteq X$ is co-analytic and $A_k \subseteq X$ $(k < \omega)$ are analytic.

Finally we show that the complexity assumptions in Corollary 3.12 are consistently optimal. The assumption of the following proposition holds e.g. in L (see e.g. [9, Corollary 25.28 p. 495]).

Proposition 4.8. *Assume there exists a $\mathbf{\Sigma}^1_2$ set $D \subseteq 2^\omega$ which does not have the Baire property. Then there exists a co-analytic set $A \subseteq (2^\omega)^\omega$ such that $A \notin \mathbb{M}$ but $[\mathbb{U}] \not\subseteq A$ for every non-empty open tower $\mathbb{U}$.*

Proof. By passing to a relative open subset we can assume D is non-meager and $2^\omega \setminus D$ is of second category everywhere. Let $C \subseteq 2^\omega \times 2^\omega$ be a co-analytic set such that projection of C to the first coordinate is D. Let

$$A = \{(x_i)_{i<\omega} \in (2^\omega)^\omega \colon (x_0, (x_i(0))_{0<i<\omega}) \in C\}.$$

We show that A fulfills the requirements.

To see that A is co-analytic, let $\varphi\colon \omega\times\omega\to\omega\times\omega$ be a bijection defined by $\varphi(0,i)=(0,i)$ $(i<\omega)$, $\varphi(i,0)=(1,i-1)$ $(0<i<\omega)$ and $\varphi\colon\{(i,j)\in\omega\times\omega\colon i\cdot j\neq 0\}\to(\omega\setminus\{0,1\})\times\omega$ being any bijection. Then the automorphism of $(2^\omega)^\omega$ induced by ϕ is a bijection between A and $C\times(2^\omega)^{\omega\setminus\{0,1\}}$, hence A is co-analytic.

Observe that the projection of A to the first coordinate is D. If $\mathbb{U}$ is a non-empty open tower then by Lemma 4.5, there exists a flag $\mathbb{W}=(W_n)_{n<\omega}$ which is co-meager in $\mathbb{U}$ and $W_n\subseteq U_n$ $(n<\omega)$; in particular $W_0\not\subseteq D$. Since the projection of $[\mathbb{W}]$ to the first coordinate is W_0, $[\mathbb{W}]\not\subseteq A$ hence $[\mathbb{U}]\not\subseteq A$, as required.

To see $A\notin\mathbb{M}$, let $M\in\mathbb{M}$ be arbitrary. By Corollary 3.8.2, there is a dense open tower $\mathbb{U}$ satisfying $[\mathbb{U}]\cap M=\emptyset$. So by Lemma 4.5, there exists a flag $\mathbb{W}=(W_n)_{n<\omega}$ which is co-meager in $\mathbb{U}$ and $W_n\subseteq U_n$ $(n<\omega)$, in particular W_0 is co-meager in 2^ω and $[\mathbb{W}]\cap M=\emptyset$.

Let $x_0\in W_0\cap D$ be arbitrary and let $x_1\in 2^\omega$ satisfy $(x_0,x_1)\in C$. Set

$$U_n=\prod_{i\leq n}\{x\in 2^\omega\colon x(0)=x_1(i)\}\ (n<\omega)$$

and $V_n=W_{n+1}\cap(\{x_0\}\times(2^\omega)^{n+1})$ $(n<\omega)$. Then $\mathbb{U}=(U_n)_{n<\omega}$ is a non-empty open tower and $\{x_0\}\times[\mathbb{U}]\subseteq A$; while $\mathbb{V}=(V_n)_{n<\omega}$ is a co-meager flag with $\{x_0\}\times[\mathbb{V}]\subseteq[\mathbb{W}]$. By Corollary 4.6 we have $[\mathbb{U}]\cap[\mathbb{V}]\neq\emptyset$, i.e. $A\cap[\mathbb{W}]\neq\emptyset$ and so $A\not\subseteq M$. This proves $A\notin\mathbb{M}$ and completes the proof. □

4.2. **Largeness in cardinality.** We show that in the iterated perfect set model and in Cohen extensions the existence of a homogeneous set of sufficiently large cardinality implies the existence of a non-empty perfect homogeneous set.

Theorem 4.9. *Let V be a model obtained from a model of the Continuum Hypothesis by adding ω_2 Sacks reals. Let $A\subseteq\mathbb{R}^\omega$ be a co-analytic set such that there is an A-homogeneous set of cardinality ω_2. Then there exists a non-empty perfect A-homogeneous set.*

Proof. Let $H\in[\mathbb{R}]^{\omega_2}$ be A-homogeneous. By [18, Theorem p. 581], there is a non-empty perfect set $X\subseteq\mathbb{R}$ such that $H\cap X$ is of second category everywhere in X. Then the statement follows from Corollary 4.7.2 applied to $A\cap X^\omega$ and $H\cap X$. □

For every cardinal μ, $C[\mu]=\{f\subseteq\mu\times\omega\to 2\colon |f|<\omega\}$ denotes the forcing for adding μ many Cohen reals. We will use the elementary properties of the Cohen forcing stated in [9, Lemma 26.4 p. 514], [14, Lemma 2.2 p. 250] and [14, Theorem 2.1 p. 252] without further reference.

Theorem 4.10. *In V, let $\kappa=2^{\aleph_0}$ and let $\kappa<\lambda\leq\mu$ be arbitrary cardinals. In $V^{C[\mu]}$, let $A\subseteq(2^\omega)^\omega$ be a co-analytic set such that there is an A-homogeneous set of cardinality λ. Then there exists a non-empty perfect A-homogeneous set.*

Proof. In $V^{C[\mu]}$, let $R=\{r_\alpha\colon\alpha<\mu\}$ be the μ many Cohen reals added to V. First suppose $H\in[R]^\lambda$, i.e. that there is an $I\in[\mu]^\lambda$ such that $H=\{r_\alpha\colon\alpha\in I\}$.

In $V^{C[\mu]}$, let $\mathcal{U}=\{U\subseteq 2^\omega\colon U\text{ open}, |U\cap H|<\lambda\}$. Set $U=\bigcup\mathcal{U}$. Then there is an $I_U\in[\mu]^{\leq\omega}$ such that $U\in V[\{r_\alpha\colon\alpha\in I_U\}]$. Since $\{r_\alpha\colon\alpha\notin I_U\}$ are Cohen reals over $V[\{r_\alpha\colon\alpha\in I_U\}]$, U cannot be dense in 2^ω. So we can find a non-empty open set $O\subseteq 2^\omega\setminus U$.

We show that in $V^{C[\mu]}$, every non-meager G_δ set $G\subseteq O$ satisfies $|G\cap H|=\lambda$. Given such a G, let $W\subseteq O$ be a non-empty open set such that G is dense in W. Let $I_{G,W}\in[\mu]^{\leq\omega}$ be such that $G,W\in V[\{r_\alpha\colon\alpha\in I_{G,W}\}]$. Since $W\subseteq O$, by the definition of O we have $|H\cap W|=\lambda$. The set $H_{G,W}=\{r_\alpha\colon\alpha\in I_{G,W}\}$ is countable. So $H\cap W\setminus H_{G,W}$ has cardinality λ, and each member of this set is a Cohen real over $V[\{r_\alpha\colon\alpha\in I_{G,W}\}]$. Thus $H\cap W\setminus H_{G,W}\subseteq G$, so the statement follows.

By Theorem 4.1, in order to conclude the existence of a non-empty perfect A-homogeneous set, it is enough to show that $O^\omega \setminus A \in \mathbb{M}$. By Corollary 3.12.2, this follows if we show that $A \cap [\mathbb{U}] \neq \emptyset$ for every non-empty open tower $\mathbb{U} = (U_n)_{n<\omega}$ with $U_n \subseteq O^{n+1}$ $(n < \omega)$. So let $\mathbb{U}$ be such an open tower. By Lemma 4.5, there is a flag $\mathbb{W} = (W_n)_{n<\omega}$ which is co-meager in $\mathbb{U}$ and W_n $(n < \omega)$ are G_δ. The proof will be complete if we show $A \cap [\mathbb{W}] \neq \emptyset$.

Let $I_{\mathbb{W}} \in [\mu]^{\leq\omega}$ be such that $\mathbb{W} \in V[\{r_\alpha : \alpha \in I_{\mathbb{W}}\}]$. We define a sequence $(h_n)_{n<\omega} \in IS_\omega(H) \cap [\mathbb{W}]$ by induction, as follows. Since $W_0 \subseteq O$ is a non-meager G_δ set, we have $|H \cap W_0| = \lambda$. Let $h_0 \in H \cap W_0$ be arbitrary.

Let $n < \omega$ be arbitrary and suppose that h_i $(i \leq n)$ are defined such that $(h_i)_{i\leq n} \in IS_{n+1}(H) \cap W_n$. By definition, $[W_{n+1}]_{(h_i)_{i\leq n}} \subseteq O$ is a non-meager G_δ set. So $|H \cap [W_{n+1}]_{(h_i)_{i\leq n}}| = \lambda$, thus we can pick $h_{n+1} \in H \cap [W_{n+1}]_{(h_i)_{i\leq n}} \setminus \{h_i : i \leq n\}$. This completes the inductive step of the definition of $(h_n)_{n<\omega}$ and completes the proof of the special $H \in [R]^\lambda$ case.

In the general case, let $H \in [2^\omega]^\lambda$ be A-homogeneous. For every $h \in H$ fix an $I_h \in [\mu]^{\leq\omega}$ such that $h \in V[\{r_\alpha : \alpha \in I_h\}]$. By the standard Δ-system argument and by extending V, we can assume $I_h \cap I_{h'} = \emptyset$ $(h, h' \in H, h \neq h')$. By passing to a subset of H, we can assume in addition that $\mathrm{tp}(I_h) = \mathrm{tp}(I_{h'}) = \eta < \omega_1$ $(h, h' \in H)$.

For every $h \in H$, let $f_h : (2^\omega)^\eta \to 2^\omega$ be a Borel function in V such that $f_h((r_\alpha)_{\alpha\in I_h}) = h$. By passing again to a subset of H, we can assume $f_h = f_{h'} = f$ $(h, h' \in H)$. Let $\mathfrak{X} \subseteq (2^\omega)^\eta$ be a co-meager G_δ set such that $f|_{\mathfrak{X}}$ is continuous. Observe that for every $h \in H$, $(r_\alpha)_{\alpha\in I_h}$ is a Cohen real in $(2^\omega)^\eta$, so in particular $(r_\alpha)_{\alpha\in I_h} \in \mathfrak{X}$ $(h \in H)$.

Set $\mathfrak{A} = \{(\mathfrak{x}_k)_{k<\omega} \in \mathfrak{X}^\omega : (f(\mathfrak{x}_k))_{k<\omega} \in A \cap IS_\omega(2^\omega)\}$. As we observed above, $\mathfrak{H} = \{(r_\alpha)_{\alpha\in I_h} : h \in H\} \subseteq \mathfrak{X}$ is an $\mathfrak{A}$-homogeneous set of Cohen reals of cardinality λ. Since $\mathfrak{A}$ is co-analytic, by the special case of Theorem 4.10 proved above, there exists a non-empty perfect $\mathfrak{A}$-homogeneous set $\mathfrak{P} \subseteq \mathfrak{X}$.

Set $P = f[\mathfrak{P}]$. By definition, if $(\mathfrak{p}_k)_{k<\omega} \in IS_\omega(\mathfrak{P})$ then $(f(\mathfrak{p}_k))_{k<\omega} \in IS_\omega(2^\omega)$. Hence f is injective on $\mathfrak{P}$, i.e. by the continuity of f, P is a non-empty perfect set. Similarly, for every $(\mathfrak{p}_k)_{k<\omega} \in IS_\omega(\mathfrak{P})$ we have $(f(\mathfrak{p}_k))_{k<\omega} \in A$, so P is A-homogeneous. This completes the proof. □

Proof of Theorem 1.3. Let $1 < \alpha < \omega_1$ be an ordinal. By [22, Claim 3.9 p. 39], it is consistent with ZFC that there exists an F_σ set $C \subseteq [2^\omega]^2$ such that there exists a C-homogeneous set of cardinality $\aleph_\alpha$ but there is no non-empty perfect C-homogeneous set. Then by Theorem 1.1, there exists a symmetric open set $U \subseteq (2^\omega)^\omega$ such that there exists a U-homogeneous set of cardinality $\aleph_\alpha$ but there is no non-empty perfect U-homogeneous set.

On the other hand, by starting from a model with $2^{\aleph_0} = \aleph_1$ and by adding $\aleph_{\alpha+1}$ many Cohen reals, we get a model in which $2^{\aleph_0} = \aleph_{\alpha+1}$ and Theorem 4.10 implies the statement. □

5. An application: definability of Tukey maps

In this section a set is called *bounded* if it is bounded from above. Similarly, *directed* means upward directed. Let $(P, \leq)$ and $(Q, \leq)$ be directed partial orders. We say that $(P, \leq)$ is *Tukey reducible to* $(Q, \leq)$, $(P, \leq) \leq_T (Q, \leq)$ in notation, if there is a function $f : P \to Q$ such that for every unbounded set $A \subseteq P$, $f[A] \subseteq Q$ is unbounded. Such an f is called a *Tukey map*. If $(P, \leq) \leq_T (Q, \leq)$ and $(Q, \leq) \leq_T (P, \leq)$ then $(P, \leq)$ and $(Q, \leq)$ are called *Tukey equivalent*, $(P, \leq) \equiv_T (Q, \leq)$ in notation. In the sequel we do not write out the partial order when it is obvious from the context.

An equivalent definition of Tukey reducibility is that $P \leq_T Q$ if and only if there is a function $g : Q \to P$ such that for every cofinal set $A \subseteq Q$, $g[A] \subseteq P$ is cofinal. This characterization indicates that Tukey reductions provide information about the cofinal types of directed partial orders, and explains why Tukey equivalence classes are also called cofinal types.

Tukey reducibility turns out to be the right tool for the comparison of cofinal types. Not only the existence of a Tukey reduction between two directed partial orders relates many of their structural properties, e.g. it is easy to see, using the equivalent definitions given above, that $P \leq_T Q$ implies $\mathrm{add}(Q) \leq \mathrm{add}(P)$ and $\mathrm{cof}(P) \leq \mathrm{cof}(Q)$, but in addition, Tukey reductions account for many known inequalities between cardinal invariants, e.g. all inequalities in the Cichoń diagram can be witnessed by Tukey maps (see e.g. [5] and [1]). So a natural question arises: how many cofinal types of directed partial orders are there?

The following result indicates that such a general endeavor has to face independence. As usual, for every cardinal κ, $[\kappa]^{<\omega}$ denotes the set of finite subsets of κ partially ordered by inclusion. We remark that $[\kappa]^{<\omega}$ is the maximal cofinal type of directed partial orders of cardinality $\leq \kappa$ (see e.g. [28, Theorem 5.1 p. 13]).

Theorem 5.1. *(*[26, Theorem 9 p. 718]*)*

1. *If the Continuum Hypothesis holds then there are 2^{ω_1} many different cofinal types of directed partial orders of cardinality ω_1.*
2. *It is consistent with ZFC that $\{1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega}\}$ are the only cofinal types of directed orders of cardinality $\leq \omega_1$.*

Therefore it is reasonable to restrict the cofinal diversity problem to classes of directed orders which carry additional structures (see e.g. [5], [6]). One possible restriction is to assume definability properties. Accordingly, in the present section we study analytic ideals on ω, i.e. such families $\mathcal{I} \subseteq \mathcal{P}(\omega)$ which form an ideal under the partial order $\subseteq$, and which are analytic subsets of $\mathcal{P}(\omega)$, endowed with the Cantor space topology. In Section 5.4 we will examine how restrictive this assumption is (see Proposition 5.25 and Proposition 5.26).

Recall that in the definition of Tukey reducibility the reducing functions are *not* required to possess any regularity properties. However, Tukey maps are not unique; e.g. if f is a Tukey map and $f \leq f'$ pointwise then f' is also a Tukey map. So it is reasonable to ask whether a Tukey reduction between analytic ideals can be witnessed by "nice" Tukey maps (see [5, Problem 3N (c) p. 212], [16, Question 2 p. 193] and [27, Question 6.69] for analogous problems).

Problem 5.2. *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be analytic ideals satisfying $\mathcal{I} \leq_T \mathcal{J}$. Is there then a "definable" Tukey map $f\colon \mathcal{I} \to \mathcal{J}$?*

Depending on $\mathcal{I}$ and $\mathcal{J}$, "definable" may mean continuous, Borel measurable, Souslin measurable (i.e. measurable with respect to the σ-algebra generated by analytic sets), Baire measurable, Lebesgue measurable, etc. An affirmative answer to this problem could allow the use of descriptive set theoretic methods for the study of an originally non-definable object.

Surprisingly, Problem 5.2 has an affirmative answer for many analytic ideals. In [24] the notion of *basic* directed partial orders was introduced and the following result was proved.

Theorem 5.3. *(*[24, Theorem 5.3 p. 1890]*) Let P, Q be basic directed partial orders satisfying $P \leq_T Q$. Then there is a Souslin measurable Tukey map $f\colon P \to Q$.*

We will recall the definition of basic directed partial orders in Section 5.4 (see Definition 5.27). Here we only mention that every analytic P-ideal on ω is basic. However, there are many analytic ideals on ω which are not basic in any topology; we will call such ideals *non-basic*. In [16, Section 7 p. 190] a sequence of non-basic Borel ideals was constructed which is strictly decreasing in the Tukey hierarchy. In Section 5.4 we will prove the following.

Proposition 5.4. *The structure $(\mathcal{P}(\omega), \subseteq^\star)$ embeds into the family of non-basic F_σ ideals on ω partially ordered by $\leq_T$.*

As we pointed out above, $[2^{\aleph_0}]^{<\omega}$ is the maximal Tukey type among directed partial orders of cardinality $\leq 2^{\aleph_0}$, so in particular among analytic ideals. As we will recall in Section 5.4, this maximal cofinal type admits a representation as an F_σ ideal on ω.

Proposition 5.5. *(*[16, Proposition 3 p. 185]*) There exists an F_σ ideal $\mathcal{I}_{\max} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{I}_{\max} \equiv_T [2^{\aleph_0}]^{<\omega}$.*

As we will see in Section 5.4, $\mathcal{I}_{\max}$ is not basic; in particular, Theorem 5.3 does not apply to its Tukey reductions. Therefore the following special case of Problem 5.2 is of particular interest (see e.g. [27, Question 6.69]).

Problem 5.6. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an analytic ideal satisfying $\mathcal{I}_{\max} \leq_T \mathcal{I}$. Is there then a "definable" Tukey map $f: \mathcal{I}_{\max} \to \mathcal{I}$?*

The purpose of this section is to show that even Problem 5.6 is independent of ZFC.

Theorem 5.7. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an arbitrary analytic ideal.*

(1) Let V be a model obtained from a model of the Continuum Hypothesis by adding ω_2 Sacks reals. Then in V, if $\mathcal{I}_{\max} \leq_T \mathcal{I}$ then there is a continuous Tukey map $f: \mathcal{I}_{\max} \to \mathcal{I}$.

(2) If the Continuum Hypothesis holds then there is an analytic ideal $\mathfrak{J} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{I}_{\max} \leq_T \mathfrak{J}$, but if $f: \mathcal{I}_{\max} \to \mathfrak{J}$ is a Tukey map then $f[\mathcal{I}_{\max}]$ has no non-empty perfect subsets. In particular, a Tukey map $f: \mathcal{I}_{\max} \to \mathfrak{J}$ cannot be Lebesgue measurable or have the Baire property.

As a corollary, in Section 5.2 we obtain that it is consistent with ZFC that $\mathcal{I}_{\max}$ has the primality property (see Definition 5.13 and Corollary 5.15).

Presently, we do not know about any other special cases of Problem 5.2 where the same independence phenomenon appears. What makes Tukey reductions of $\mathcal{I}_{\max}$ particularly easy to describe is the following simple characterization.

Definition 5.8. Let $(P, \leq)$ be a directed partial order. A set $H \subseteq P$ is called *strongly unbounded* if every $A \in [H]^\omega$ is unbounded in $(P, \leq)$.

Proposition 5.9. *(*[16, Section 1 p. 174]*) Let $(P, \leq)$ be a directed partial order and let κ be an arbitrary cardinal. Then $[\kappa]^{<\omega} \leq_T P$ if and only if there exists a strongly unbounded set $H \in [P]^\kappa$.*

5.1. **A consistent positive answer to Problem 5.6.** We will need the following simple observation.

Lemma 5.10. *Let P and Q be directed partial orders and let $f: P \to Q$ be a Tukey map. If $H \subseteq P$ is strongly unbounded then $f|_H$ is finite-to-one and $f[H] \subseteq Q$ is also strongly unbounded.*

Proof. By definition, every $A \in [H]^\omega$ is unbounded. Since f is a Tukey map, $f[A]$ cannot be a singleton, i.e. f is finite-to-one. The second statement immediately follows from the definition. □

The following implies Theorem 5.7.1, and also shows that the conclusion of Theorem 5.7.1 holds for every projective ideal under suitable large cardinal assumptions.

Theorem 5.11. *Let $\mathbf{\Gamma}$ be a projective pointclass such that every $I \in \mathbf{\Gamma}(\mathcal{P}(\omega))$ has the Baire property and for every $A \in \mathbf{\Gamma}(\mathcal{P}(\omega)^\omega)$, the game $\mathcal{G}_\omega(A)$ is determined. Let $\kappa \leq 2^{\aleph_0}$ be a cardinal such that* *for every $H \in [\mathcal{P}(\omega)]^\kappa$ there exists a non-empty perfect set $Q \subseteq \mathcal{P}(\omega)$ such that $H \cap Q$ is of second category everywhere in Q. Let $I \subseteq \mathcal{P}(\omega)$ be an ideal with $\mathcal{P}(\omega) \setminus I \in \mathbf{\Gamma}(\mathcal{P}(\omega))$ such that $[\kappa]^{<\omega} \leq_T I$. Then there is a continuous Tukey reduction from $[2^\omega]^{<\omega}$ to I.*

Proof. Let $f \colon [\kappa]^{<\omega} \to I$ be a Tukey map and let $H = f[\kappa]$. By Lemma 5.10, f is finite-to-one, hence H has cardinality κ. So there exists a non-empty perfect set $Q \subseteq \mathcal{P}(\omega)$ such that $H \cap Q$ is of second category everywhere in Q. Since $I \cap Q$ has the Baire property in Q, by $H \cap Q \subseteq I \cap Q$ there is a co-meager G_δ set $X \subseteq I \cap Q \subseteq \mathcal{P}(\omega)$ satisfying $H \cap X$ is of second category everywhere in X.

Set $A = \left\{(x_i)_{i<\omega} \in X^\omega \colon \bigcup_{i<\omega} x_i \notin I\right\}$. Since the union function from $\mathcal{P}(\omega)^\omega$ to $\mathcal{P}(\omega)$ is Borel and $\mathcal{P}(\omega) \setminus I \in \mathbf{\Gamma}(\mathcal{P}(\omega))$, we have $A \in \mathbf{\Gamma}(X^\omega)$. Then the game $\mathcal{G}_\omega(A)$ is determined and $IS_\omega(H \cap X) \subseteq A$, so by Corollary 4.7.1 there is a non-empty perfect set $P \subseteq X$ such that $IS_\omega(P) \subseteq A$. Then any continuous bijection $\tilde{f} \colon 2^\omega \to P$ is a continuous Tukey map from $[2^\omega]^{<\omega}$ to I. □

Proof of Theorem 5.7.1. Recall that $\mathcal{I}_{\max} \equiv_T [2^\omega]^{<\omega}$. By [18, Theorem p. 581], in the iterated perfect set model for every $H \in [2^\omega]^{2^{\aleph_0}}$ there exists a non-empty perfect set $Q \subseteq \mathcal{P}(\omega)$ such that $H \cap Q$ is of second category everywhere in Q. So by Corollary 3.12.2, the statement follows from Theorem 5.11 with $\mathbf{\Gamma} = \mathbf{\Pi}^1_1$ and $\kappa = 2^{\aleph_0} = \omega_2$. □

5.2. **Consistent primality of $\mathcal{I}_{\max}$.** It is easy to see that the least upper bound in the Tukey order of two ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ is their direct sum, or disjoint union, defined as follows.

Definition 5.12. Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be arbitrary ideals. Let $E = \{2n \colon n < \omega\}$. Then $\mathcal{I} \oplus \mathcal{J} \subseteq \mathcal{P}(\omega)$ is the ideal defined by

$$A \in \mathcal{I} \oplus \mathcal{J} \Leftrightarrow \{n < \omega \colon 2n \in A \cap E\} \in \mathcal{I} \text{ and } \{n < \omega \colon 2n + 1 \in A \setminus E\} \in \mathcal{J}.$$

For a complete description of the cofinal types of analytic ideals, ideals having the following primality property are of particular importance.

Definition 5.13. We say that an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ has the *primality property* if for every ideals $\mathcal{J}, \mathcal{K} \subseteq \mathcal{P}(\omega)$, $\mathcal{I} \leq_T \mathcal{J} \oplus \mathcal{K}$ implies $\mathcal{I} \leq_T \mathcal{J}$ or $\mathcal{I} \leq_T \mathcal{K}$.

It is reasonable to ask which ideals have the primality property, and especially whether $\mathcal{I}_{\max}$ has the primality property (see e.g. [27, Question 6.68]). Note that by [26, Theorem 6 p. 715] the primality of $\mathcal{I}_{\max}$ fails among non-definable ideals. On the other hand, by [27, Theorem 6.71], the primality of $\mathcal{I}_{\max}$ holds for Souslin measurable Tukey reductions, as follows.

Theorem 5.14. *(*[27, Theorem 6.71]*) Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be analytic ideals such that there exists a Souslin-measurable Tukey reduction of $\mathcal{I}_{\max}$ to $\mathcal{I} \oplus \mathcal{J}$. Then either $\mathcal{I}_{\max} \leq_T \mathcal{I}$ or $\mathcal{I}_{\max} \leq_T \mathcal{J}$.*

So by Theorem 5.7.1, we get the following.

Theorem 5.15. *Let V be a model obtained from a model of the Continuum Hypothesis by adding ω_2 Sacks reals. In V, let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be analytic ideals such that $\mathcal{I}_{\max} \leq_T \mathcal{I} \oplus \mathcal{J}$. Then either $\mathcal{I}_{\max} \leq_T \mathcal{I}$ or $\mathcal{I}_{\max} \leq_T \mathcal{J}$.*

Proof. By Theorem 5.7.1, there exists a continuous Tukey reduction of $\mathcal{I}_{\max}$ to $\mathcal{I} \oplus \mathcal{J}$. So the statement follows from Theorem 5.14. □

5.3. **A consistent negative answer to Problem 5.6.** In this section we construct the ideal $\mathfrak{J}$ of Theorem 5.7.2. To this end, the main technical step is to observe that it is enough to construct an ideal of compact sets with the same properties.

Definition 5.16. Let $\mathcal{K}(2^\omega)$ denote the space of compact subsets of the Cantor set endowed with the Vietoris topology. A family $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ is called an *ideal of compact sets* if $\mathcal{I}$ is closed under taking finite unions, and $\mathcal{I}$ is closed downward, i.e. for every $K, L \in \mathcal{K}(2^\omega)$, $K \subseteq L \in \mathcal{I}$ implies $K \in \mathcal{I}$.

Throughout this section we use the following notation.

Definition 5.17. For every $s \in 2^{<\omega}$, set $N_s = \{x \in 2^\omega : s \sqsubseteq x\}$. Let $\Omega = 2^{<\omega}$. We define $\Phi \colon \mathcal{K}(2^\omega) \to \mathcal{P}(\Omega)$,

$$\Phi(K) = \{s \in \Omega : N_s \cap K \neq \emptyset\}.$$

For every $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, set $\mathcal{A}^\downarrow = \{B \in \mathcal{P}(\Omega) : \exists A \in \mathcal{A}\ (B \subseteq A)\}$. For every $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ we define $\mathfrak{I} = (\Phi[\mathcal{I}])^\downarrow$.

Lemma 5.18. *With the notation of Definition 5.17, we have the following.*

1. *Φ is continuous, and for every $K, L \in \mathcal{K}(2^\omega)$, $K \subseteq L \Leftrightarrow \Phi(K) \subseteq \Phi(L)$. In particular, $\Phi \colon \mathcal{K}(2^\omega) \to \Phi[\mathcal{K}(2^\omega)]$ is a homeomorphism.*
2. *If $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ is an ideal then $\mathfrak{I} \subseteq \mathcal{P}(\Omega)$ is also an ideal.*
3. *If $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is analytic then $\mathcal{A}^\downarrow \subseteq \mathcal{P}(\Omega)$ is also analytic.*
4. *If $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ is an analytic ideal then $\mathfrak{I} \subseteq \mathcal{P}(\Omega)$ is also an analytic ideal.*
5. *3 and 4 also hold if we replace "analytic" with "F_σ".*

Proof. It is obvious that for every $K, L \in \mathcal{K}(2^\omega)$, $K \subseteq L \Leftrightarrow \Phi(K) \subseteq \Phi(L)$. In particular, Φ is injective. For every $s \in \Omega$, $\Phi^{-1}(\{A \in \mathcal{P}(\Omega) : s \in A\}) = \{K \in \mathcal{K}(2^\omega) : K \cap N_s \neq \emptyset\}$ is clopen in the Vietoris topology, thus Φ is continuous. Since $\mathcal{K}(2^\omega)$ is compact, Φ is a homeomorphism.

For 2, it is obvious that $\mathfrak{I}$ is closed downward. To see that it is closed under taking unions, pick $A_i \in \mathfrak{I}$ $(i < 2)$. Then there are $K_i \in \mathcal{I}$ $(i < 2)$ such that $A_i \subseteq \Phi(K_i)$ $(i < 2)$. By 1, $A_0 \cup A_1 \subseteq \Phi(K_0) \cup \Phi(K_1) = \Phi(K_0 \cup K_1)$, so the statement follows.

To see 3, observe that $\{(A, B) \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) : B \subseteq A\}$ is a closed relation, and $\mathcal{A}^\downarrow$ is the projection of $\{(A, B) \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) : B \subseteq A \in \mathcal{A}\}$ to the second coordinate. Hence $\mathcal{A}^\downarrow$ is analytic if $\mathcal{A}$ is analytic; and by $\mathcal{P}(\Omega)$ being compact, $\mathcal{A}^\downarrow$ is F_σ if $\mathcal{A}$ is F_σ.

Finally, 4 and 5 follow from 1, 2 and 3. □

The ideals $\mathcal{I}$ and $\mathfrak{I}$ are cofinally similar, as well.

Lemma 5.19. *Let $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ be an analytic ideal. Then $\mathcal{I} \equiv_T \mathfrak{I}$; moreover,*

1. *$\Phi \colon \mathcal{I} \to \mathfrak{I}$ and $\Phi^{-1} \colon \Phi[\mathcal{I}] \to \mathcal{I}$ are continuous Tukey maps;*
2. *there is a Tukey reduction $\Phi^{-1} \circ f \colon \mathfrak{I} \to \mathcal{I}$ where $f \colon \mathfrak{I} \to \mathfrak{I}$ is a Souslin measurable Tukey map such that $H \subseteq f(H) \in \Phi[\mathcal{I}]$ $(H \in \mathfrak{I})$.*

Proof. To see 1, let $\mathcal{H} \subseteq \mathcal{I}$ be an arbitrary set. If $\Phi[\mathcal{H}] \subseteq \mathfrak{I}$ is bounded then there is an $L \in \mathcal{I}$ such that $\Phi(K) \subseteq \Phi(L)$ $(K \in \mathcal{H})$. By Lemma 5.18.1, this implies $K \subseteq L$ $(K \in \mathcal{H})$, i.e. $\mathcal{H}$ is also bounded. Hence Φ is a Tukey map, which is continuous by Lemma 5.18.1. The argument for Φ^{-1} is identical.

For 2, consider the set $P = \{(A, B) \in \mathfrak{I} \times \Phi[\mathcal{I}] : A \subseteq B\}$. Then $P \subseteq \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$ is analytic, so by the Jankov-von Neumann Uniformization Theorem (see e.g. [11, Theorem 18.1 p. 120]) P has a Souslin measurable uniformizing function $f \colon \mathfrak{I} \to \Phi[\mathcal{I}]$. Also, $H \subseteq f(H) \in \Phi[\mathcal{I}]$ $(H \in \mathfrak{I})$ holds.

Since f pointwise dominates the identity function, it is a Tukey map. Then by 1, $\Phi^{-1} \circ f \colon \mathfrak{I} \to \mathcal{I}$ is also a Tukey map, which completes the proof. □

Corollary 5.20. *Let $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ be an analytic ideal.*

(1) For every cardinal κ we have $[\kappa]^{<\omega} \leq_T \mathcal{I} \Leftrightarrow [\kappa]^{<\omega} \leq_T \mathfrak{I}$.
(2) There exists a non-empty perfect strongly unbounded subset of $\mathcal{I}$ if and only if there exists a non-empty perfect strongly unbounded subset of $\mathfrak{I}$.

Proof. Statement 1 follows from Lemma 5.19. To see 2, suppose first that $P \subseteq \mathcal{I}$ is a non-empty perfect strongly unbounded set. By Lemma 5.18.1, Φ is continuous and injective. So $\Phi[P]$ is also non-empty and perfect. By Lemma 5.19.1, Φ is a Tukey map, i.e. by Lemma 5.10, $\Phi[P]$ is strongly unbounded.

Finally let $P \subseteq \mathfrak{I}$ be a non-empty perfect strongly unbounded set. Let f be the function of Lemma 5.19.2. Then $f|_P$ is Souslin measurable, hence it has the Baire property, so by passing to a non-empty perfect subset of P we can assume that $f|_P$ is continuous (see e.g. [11, Theorem 8.38 p. 52]).

By Lemma 5.10 and Lemma 5.19.2, $f|_P$ is finite-to one. Thus $f[P] \subseteq \Phi[\mathcal{I}]$ is an uncountable compact strongly unbounded set, in particular it has a non-empty perfect subset Q, which is also strongly unbounded. So by Lemma 5.19.1 and Lemma 5.10, $\Phi^{-1}(Q) \subseteq \mathcal{I}$ is a non-empty perfect strongly unbounded set. □

Our main theorem is the following.

Theorem 5.21. *There is an analytic ideal $\mathcal{J} \subseteq \mathcal{K}(2^\omega)$ with the following properties.*
(1) $[\omega_1]^{<\omega} \leq_T \mathcal{J}$.
(2) $[\omega_2]^{<\omega} \not\leq_T \mathcal{J}$.
(3) There is no non-empty perfect strongly unbounded subset of $\mathcal{J}$.

The following corollary answers [16, Conjecture 2 p. 194] in the negative.

Corollary 5.22. *There is an analytic ideal $\mathfrak{J} \subseteq \mathcal{P}(\omega)$ with the following properties.*
(1) $[\omega_1]^{<\omega} \leq_T \mathfrak{J}$.
(2) $[\omega_2]^{<\omega} \not\leq_T \mathfrak{J}$.
(3) There is no non-empty perfect strongly unbounded subset of $\mathfrak{J}$.

Proof. Let $\mathcal{J}$ be the ideal of Theorem 5.21 and set $\mathfrak{J} = (\Phi[\mathcal{J}])^{\downarrow}$. By Lemma 5.18.4, $\mathfrak{J}$ is an analytic ideal. By Corollary 5.20, 1, 2 and 3 follows from Theorem 5.21. □

Also, Theorem 5.7.2 easily follows.

Proof of Theorem 5.7.2. Let $\mathfrak{J}$ be the ideal of Corollary 5.22. Assume the Continuum Hypothesis holds. Then $\mathcal{I}_{\max} \equiv_T [\omega_1]^{<\omega} \leq_T \mathfrak{J}$ by Corollary 5.22.1.

Let $f : \mathcal{I}_{\max} \to \mathfrak{J}$ be an arbitrary Tukey map. By Lemma 5.10, $f[\mathcal{I}_{\max}] \subseteq \mathfrak{J}$ is strongly unbounded, so every subset of $f[\mathcal{I}_{\max}]$ is strongly unbounded, as well. Hence by Corollary 5.22.3, $f[\mathcal{I}_{\max}]$ has no non-empty perfect subsets.

If f is either Lebesgue measurable or has the Baire property then there is an uncountable Borel set $B \subseteq \mathcal{I}_{\max}$ such that $f|_B$ is continuous. By Lemma 5.10, $f[B] \subseteq \mathfrak{J}$ is an uncountable strongly unbounded analytic set, in particular it has a non-empty perfect strongly unbounded subset. This contradicts Corollary 5.22.3. □

It remains to prove Theorem 5.21. Recall the definition of the Cantor-Bendixson derivative (see e.g. [11, Definition 6.10 p. 33]): for every $K \in \mathcal{K}(2^\omega)$, set $K' = \{x \in K : x \in \mathrm{cl}_{2^\omega}(K \setminus \{x\})\}$. We define, by induction on $n < \omega$, the iterated derivatives: $K^{(0)} = K$, $K^{(n+1)} = (K^{(n)})'$ $(n < \omega)$. Also recall that there exists an analytic equivalence relation $E \subseteq 2^\omega \times 2^\omega$ such that E has exactly ω_1 many equivalence classes (see e.g. [9, Exercise 32.3 p. 625]).

Definition 5.23. Set $\mathcal{R} = \{K \in \mathcal{K}(2^\omega) \colon \exists n < \omega \ (K^{(n)} = \emptyset)\}$. Let $E \subseteq 2^\omega \times 2^\omega$ be an analytic equivalence relation with exactly ω_1 many equivalence classes. We define

$$\mathcal{L} = \{K \in \mathcal{K}(2^\omega) \colon K \in \mathcal{R},\ \forall x, y \in K\ ((x, y) \in E)\}. \tag{5.1}$$

It is known that $\mathcal{R}$ is a Borel subset of $\mathcal{K}(2^\omega)$ (see [3] and [4] for its exact Borel class). However, due to the universal quantification in the definition of $\mathcal{L}$, the following is not straightforward.

Proposition 5.24. *$\mathcal{L} \subseteq \mathcal{K}(2^\omega)$ is analytic.*

Proof. For every $n < \omega$, let $\vartheta_n \colon \mathcal{K}(2^\omega)^n \to \mathcal{K}(2^\omega)$ denote the union function, i.e. for every $(K_i)_{i<n} \in \mathcal{K}(2^\omega)^n$, $\vartheta_n((K_i)_{i<n}) = \bigcup_{i<n} K_i$. It is clear that ϑ_n $(n < \omega)$ are continuous.

Recall $N_s = \{\sigma \in 2^\omega \colon s \sqsubseteq \sigma\}$ $(s \in 2^{<\omega})$. We define $\mathcal{L}_n, \mathcal{L}_n^+ \subseteq \mathcal{K}(2^\omega)$ by induction on n, as follows. Set $\mathcal{L}_0 = [2^\omega]^{\le 1}$. If $n < \omega$ and $\mathcal{L}_n$ is already defined, set

$$\mathcal{L}_n^+ = \{\emptyset\} \cup \bigcup_{m<\omega} \vartheta_m[\{(L_i)_{i<m} \colon \forall i < m\ \exists u_i \in L_i\ ((u_i, u_j) \in E\ (i, j < m))\}] \tag{5.2}$$

and

$$\mathcal{L}_{n+1} = \mathcal{L}_0 \cup \{L \in \mathcal{K}(2^\omega) \colon \exists u, v \in L\ (u \ne v,\ (u, v) \in E,\ L \setminus N_{u|_m} \in \mathcal{L}_n^+\ (m < \omega))\}. \tag{5.3}$$

It follows from the definition that $\mathcal{L}_n, \mathcal{L}_n^+ \subseteq \mathcal{K}(2^\omega)$ $(n < \omega)$ are analytic sets and $\mathcal{L}_n \subseteq \mathcal{L}_n^+$ $(n < \omega)$. So the proof will be complete if we show $\mathcal{L} = \bigcup_{n<\omega} \mathcal{L}_n^+$.

To see $\bigcup_{n<\omega} \mathcal{L}_n^+ \subseteq \mathcal{L}$, we show by induction on n that for every $L \in \mathcal{L}_n^+$ we have $L^{(n+1)} = \emptyset$ and $(x, y) \in E$ $(x, y \in L)$. The statement is obvious for $n = 0$; so let $n < \omega$ be arbitrary and suppose that the statement holds for $\mathcal{L}_n^+$. Let $L \in \mathcal{L}_{n+1}^+$ be arbitrary. Suppose first $L \in \mathcal{L}_{n+1}$, and let $u, v \in L$ be as in (5.3). By the inductive hypothesis, $L^{(n)} \subseteq \{u\}$ and $(x, y) \in E$ $(x, y \in L \setminus \{u\})$. Thus $L^{(n+1)} = \emptyset$, and $(u, v) \in E$, $u \ne v$ imply $(x, y) \in E$ $(x, y \in L)$.

In the general case, for some $m < \omega$, let $L = \bigcup_{i<m} L_i$ where $L_i \in \mathcal{L}_{n+1}$ $(i < m)$. Since $L_i^{(n+1)} = \emptyset$ $(i < m)$, we have $L^{(n+1)} = \emptyset$. Let $u_i \in L_i$ $(i < m)$ be as in (5.2). Since $(x, y) \in E$ $(x, y \in L_i, i < m)$ and $(u_i, u_j) \in E$ $(i, j < m)$, we get $(x, y) \in E$ $(x, y \in L)$, as required.

To see $\mathcal{L} \subseteq \bigcup_{n<\omega} \mathcal{L}_n^+$, we show by induction on n that for every $L \in \mathcal{L}$, $L^{(n+1)} = \emptyset$ implies $L \in \mathcal{L}_n^+$. For $n = 0$, $L' = \emptyset$ holds if and only if L is finite; and then $L \in \mathcal{L}_0^+$. So let $n < \omega$ and suppose that the statement holds for n. Let $L \in \mathcal{L}$ be arbitrary satisfying $L^{(n+2)} = \emptyset$. If $L \in \mathcal{L}_0$ then $L \in \mathcal{L}_{n+1}$ follows. If $L \notin \mathcal{L}_0$ and $L^{(n+1)} = \emptyset$ then let $u, v \in L$ be arbitrary with $u \ne v$. By $L \in \mathcal{L}$, we have $(u, v) \in E$. By $L^{(n+1)} = \emptyset$ and by the inductive assumption, $L \setminus N_{u|_m} \in \mathcal{L}_n^+$ $(m < \omega)$, so $L \in \mathcal{L}_{n+1}$.

If $L^{(n+1)}$ is a singleton then set $\{u\} = L^{(n+1)}$ and let $v \in L \setminus \{u\}$ be arbitrary. As in the previous case, by the inductive assumption, $L \setminus N_{u|_m} \in \mathcal{L}_n^+$ $(m < \omega)$, so $L \in \mathcal{L}_{n+1}$.

Finally if $|L^{(n+1)}| > 1$ then it is a finite set, say $L^{(n+1)} = \{u_i \colon i < m\}$ for some $m < \omega$. Let $L_i \in \mathcal{K}(2^\omega)$ $(i < m)$ be arbitrary satisfying $u_i \in L_i \subseteq L \setminus \{u_j \colon j < m,\ j \ne i\}$ $(i < m)$ and $L = \bigcup_{i<m} L_i$. Since $L_i^{(n+1)} = \{u_i\}$ $(i < m)$, we have $L_i \in \mathcal{L}_{n+1}$ $(i < m)$. By $L \in \mathcal{L}$, $(u_i, u_j) \in E$ $(i, j < m)$ hence $L \in \mathcal{L}_{n+1}^+$, as stated. □

Proof of Theorem 5.21. Recall that for every $n < \omega$, $\vartheta_n \colon \mathcal{K}(2^\omega)^n \to \mathcal{K}(2^\omega)$ denotes the union function. With the set $\mathcal{L}$ defined in (5.1), set

$$\mathcal{J} = \bigcup_{n<\omega} \vartheta_n[\mathcal{L}^n].$$

We show that $\mathcal{J}$ fulfills the requirements.

Since $\mathcal{L}$ is closed downward and $\mathcal{J}$ is closed under taking finite unions, $\mathcal{J}$ is an ideal of compact sets. The functions ϑ_n $(n < \omega)$ are continuous hence by Proposition 5.24, $\mathcal{J}$ is analytic. It is immediate from the definition that for every $K \in \mathcal{K}(2^\omega)$, $K \in \mathcal{J}$ if and only if

(i) K intersects only finitely many equivalence classes of E;
(ii) for every equivalence class $C \subseteq 2^\omega$ of E, $K \cap C \in \mathcal{K}(2^\omega)$ and $(K \cap C)^{(n)} = \emptyset$ for some $n < \omega$.

Let $H \in [2^\omega]^{\omega_1}$ be an arbitrary set such that $(x, y) \notin E$ $(x, y \in H)$ and set $\mathcal{H} = [H]^1$. By (i), $\mathcal{H} \subseteq \mathcal{J}$ is strongly unbounded, i.e. by Proposition 5.9, 1 holds.

To see 2, by Proposition 5.9 we have to show that if $\mathcal{H} \in [\mathcal{J}]^{\omega_2}$ then $\mathcal{H}$ is not strongly unbounded. So let $\mathcal{H} \in [\mathcal{J}]^{\omega_2}$ be arbitrary. Let $\{E_\alpha : \alpha < \omega_1\}$ be an enumeration of the equivalence classes of E. By (i) and (ii), there are $n < \omega$, $A \in [\omega_1]^{<\omega}$ and $\mathcal{H}' \in [\mathcal{H}]^{\omega_2}$ such that for every $K \in \mathcal{H}'$,

$$K \cap E_\alpha \neq \emptyset \Leftrightarrow \alpha \in A, \text{ and } (K \cap E_\alpha)^{(n)} = \emptyset \ (\alpha \in A).$$

Consider the space $\mathcal{X} = \prod_{\alpha \in A} (\mathcal{J} \cap \mathcal{K}(E_\alpha))$ and the set

$$X = \left\{ (K_\alpha)_{\alpha \in A} : K_\alpha \in \mathcal{J} \cap \mathcal{K}(E_\alpha), \bigcup_{\alpha \in A} K_\alpha \in \mathcal{H}' \right\}.$$

Then $\mathcal{X}$ is a separable metric space and $X \subseteq \mathcal{X}$ is an uncountable set. Since $\mathcal{X}$ is Lindelöf, there is a $(K_\alpha)_{\alpha \in A} \in \mathcal{X}$ and an injective sequence $(K_\alpha(m))_{\alpha \in A} \in X$ $(m < \omega)$ such that $\lim_{m<\omega} K_\alpha(m) = K_\alpha$ $(\alpha \in A)$. Set $H(m) = \bigcup_{\alpha \in A} K_\alpha(m)$ $(m < \omega)$, $H = \bigcup_{\alpha \in A} K_\alpha$ and $L = H \cup \bigcup_{n<\omega} H(m)$. Observe that $H \in \mathcal{J}$, in particular there is a $d < \omega$ such that $H^{(d)} = \emptyset$. Also, $H(m) \in \mathcal{H}'$ implies $H(m)^{(n)} = \emptyset$ $(m < \omega)$.

We have $L \cap E_\alpha \neq \emptyset \Leftrightarrow \alpha \in A$ $(\alpha < \omega_1)$ and $L \cap E_\alpha \in \mathcal{K}(2^\omega)$ $(\alpha \in A)$. Moreover, $L^{(n)} \subseteq H$ hence $L^{(n+d)} = \emptyset$. By (i) and (ii) this implies $L \in \mathcal{J}$. Since $((K_\alpha(m))_{\alpha \in A})_{m<\omega}$ is injective, $\{H(m) : m < \omega\} \subseteq \mathcal{H}$ is an infinite set. By $H(m) \subseteq L$ $(m < \omega)$, $\mathcal{H}$ is not strongly unbounded, as stated.

Finally observe that by Shoenfield's Absoluteness (see e.g. [9, Theorem 25.20 p. 490]), a non-empty perfect strongly unbounded subset of $\mathcal{J}$ would remain strongly unbounded in any forcing extension. Hence 3 follows from 2. □

5.4. **Miscellanea.** In the previous sections we restricted our investigations to analytic ideals $\mathcal{I} \subseteq (\mathcal{P}(\omega), \subseteq)$. Unfortunately, there are simple directed partial orders which are not cofinally similar to any such ideal. The following may be considered as a negative answer to [16, Question 1 p. 193].

Proposition 5.25. *Let* 1 *denote the one element set with the trivial order, let* ω *denote the first infinite ordinal with its usual well-order, and let* $(\omega^\omega, \leq^\star)$ *be the set of all functions from* ω *to* ω*, partially ordered by eventual dominance.*

(1) If $\mathcal{I} \subseteq (\mathcal{P}(\omega), \subseteq)$ *is an arbitrary directed set then* $\mathcal{I} \leq_T 1$ *or* $\omega \leq_T \mathcal{I}$.
(2) The directed partial order $(\omega^\omega, \leq^\star)$ *satisfies* $\leq^\star \subseteq \omega^\omega \times \omega^\omega$ *is* F_σ *but it is incomparable with* ω *in the Tukey order. In particular,* $(\omega^\omega, \leq^\star) \not\equiv_T \mathcal{I}$ *for every directed set* $\mathcal{I} \subseteq (\mathcal{P}(\omega), \subseteq)$.

Proof. To see 1, let $S = \bigcup I$. If $S \in I$ then $I \leq_T 1$. If $S \notin I$, let $(S_n)_{n<\omega} \subseteq I$ be an increasing sequence satisfying $S = \bigcup_{n<\omega} S_n$; such a sequence exists since I is directed. Then $f : \omega \to I$, $f(n) = S_n$ $(n < \omega)$ is a Tukey map.

For 2, $\omega \not\leq_T (\omega^\omega, \leq^\star)$ follows from the fact that every countable subset of $(\omega^\omega, \leq^\star)$ is bounded. To see $(\omega^\omega, \leq^\star) \not\leq_T \omega$ let $f : (\omega^\omega, \leq^\star) \to \omega$ be an arbitrary function. Then there is an $n < \omega$ such that $f^{-1}(\{n\}) \subseteq \omega^\omega$ is non-meager. Since for every $\varphi \in \omega^\omega$, $\{\psi \in \omega^\omega : \psi \leq^\star \varphi\}$ is meager, such a set cannot be bounded. Hence f is not a Tukey map. The second statement follows from 1. □

On the other hand, closed directed partial orders on analytic spaces admit representations as analytic ideals on ω.

Proposition 5.26. *Let $(I,\leq)$ be a topological space endowed with a directed partial order such that I is a continuous image of a Polish space and $\leq\subseteq I\times I$ is closed. Then there is an analytic ideal $\mathfrak{I}\subseteq(\mathcal{P}(\omega),\subseteq)$ such that $(I,\leq)\equiv_T\mathfrak{I}$.*

Proof. Let $G\subseteq 2^\omega$ be an arbitrary set homeomorphic to ω^ω, and let $f\colon G\to I$ be a continuous surjection (see e.g. [11, Theorem 7.9 p. 38]). For every $a\in I$, set $L_a=\{b\in I\colon b\leq a\}$. Note that $L_a\subseteq I$ $(a\in I)$ are closed. We show that

$$\mathcal{I}=\{K\in\mathcal{K}(2^\omega)\colon \exists a\in I\ (K\subseteq \mathrm{cl}_{2^\omega}(f^{-1}(L_a))\}$$

is an analytic ideal of compact subsets of 2^ω such that $(I,\leq)\equiv_T\mathcal{I}$. Then, with the notation of Definition 5.17, by Lemma 5.19, $\mathfrak{I}=(\Phi[\mathcal{I}])^\downarrow$ fulfills the requirements.

It is obvious that $\mathcal{I}$ is closed downward. Since I is directed, $\mathcal{I}$ is closed under taking finite unions. Next we show that $\mathcal{I}$ is analytic. Let $\mathcal{F}(G)$ denote the set of closed subsets of G endowed with the Effros Borel structure (see e.g. [11, Section 12.C p. 75]); recall that a sub-basis of the corresponding topology consists of set of the form $\{F\in\mathcal{F}(G)\colon F\cap U\neq\emptyset\}$ where U varies over the open subsets of G. First we show that

$$R=\{(F,a)\in\mathcal{F}(G)\times I\colon F\subseteq f^{-1}(L_a)\}$$

is a closed set. Let $(F,a)\in\mathcal{F}(G)\times I$, $(F_n,a_n)\in R$ $(n<\omega)$ be such that $\lim_{n<\omega}(F_n,a_n)=(F,a)$. Let $x\in F$ be arbitrary; then by $F_n\subseteq f^{-1}(L_{a_n})$ $(n<\omega)$ there are $x_n\in f^{-1}(L_{a_n})$ $(n<\omega)$ such that $\lim_{n<\omega}x_n=x$. Since f is continuous, $f(x)=\lim_{n<\omega}f(x_n)$. By $\leq\subseteq I\times I$ being closed, $f(x_n)\leq a_n$ $(n<\omega)$ implies $f(x)\leq a$. Thus $x\in f^{-1}(L_a)$. Since $x\in F$ was arbitrary, $F\subseteq f^{-1}(L_a)$ thus $(F,a)\in R$ follows.

Observe that $\{(K,F)\in\mathcal{K}(2^\omega)\times\mathcal{F}(G)\colon K\subseteq\mathrm{cl}_{2^\omega}(F)\}$ is a Borel set (see e.g. the proof of [11, Theorem 12.6 p. 75]). Hence

$$S=\{(K,F,a)\in\mathcal{K}(2^\omega)\times\mathcal{F}(G)\times I\colon K\subseteq\mathrm{cl}_{2^\omega}(F),\ F\subseteq f^{-1}(L_a)\}$$

is also Borel. It is obvious that

$$T=\{(K,a)\in\mathcal{K}(2^\omega)\times I\colon K\subseteq\mathrm{cl}_{2^\omega}(f^{-1}(L_a))\}=\mathrm{Pr}_{\mathcal{K}(2^\omega)\times I}(S),$$

so T is an analytic set. Since I is analytic, $\mathcal{I}=\mathrm{Pr}_{\mathcal{K}(2^\omega)}(T)$ is analytic, as well.

We show that $g\colon I\to\mathcal{I}$, $g(a)=\mathrm{cl}_{2^\omega}(f^{-1}(L_a))$ $(a\in I)$ is a Tukey map. Let $A\subseteq I$ be arbitrary. If $g[A]\subseteq\mathcal{I}$ is bounded then there is a $b\in I$ satisfying $\mathrm{cl}_{2^\omega}(f^{-1}(L_a))\subseteq\mathrm{cl}_{2^\omega}(f^{-1}(L_b))$ $(a\in A)$. Then $f^{-1}(a)\subseteq\mathrm{cl}_{2^\omega}(f^{-1}(L_b))\cap G=f^{-1}(L_b)$ $(a\in A)$, so $a\leq b$ $(a\in A)$. Thus g maps unbounded sets into unbounded sets, as stated.

Finally let $h\colon\mathcal{I}\to I$ be any function satisfying $K\subseteq\mathrm{cl}_{2^\omega}(f^{-1}(L_{h(K)}))$ $(K\in\mathcal{I})$; we show that h is a Tukey map. Let $\mathcal{A}\subseteq\mathcal{I}$ be arbitrary. If $h[\mathcal{A}]\subseteq I$ is bounded then there is a $b\in I$ satisfying $h(K)\leq b$ hence $L_{h(K)}\subseteq L_b$ $(K\in\mathcal{A})$. Then $K\subseteq\mathrm{cl}_{2^\omega}(f^{-1}(L_b))$ $(K\in\mathcal{A})$; i.e. h maps unbounded sets into unbounded sets, as required. □

Next we present a simple proof of Proposition 5.5.

Proof of Proposition 5.5. Observe that for every $n<\omega$, $[2^\omega]^{\leq n}\subseteq\mathcal{K}(2^\omega)$ is closed. Hence $[2^\omega]^{<\omega}\subseteq\mathcal{K}(2^\omega)$ is an F_σ ideal of compact sets. Then, with the notation of Definition 5.17, by Lemma 5.18.5 and Lemma 5.19, $\mathcal{I}_{\max}=(\Phi[[2^\omega]^{<\omega}])^\downarrow$ fulfills the requirements. □

As we mentioned above, $\mathcal{I}_{\max}$ is not basic. To see this, let us recall the definition of basic directed partial orders. Recall that a set is called *bounded* if it is bounded from above.

Definition 5.27. ([24, Definition p. 1881]) Let $(D, \leq)$ be a separable metric space endowed with a partial order. Then $(D, \leq)$ is *basic* if

(1) each pair of elements of D has the least upper bound with respect to $\leq$ and the binary operation of least upper bound from $D \times D$ to D is continuous;
(2) each bounded sequence has a converging subsequence;
(3) each converging sequence has a bounded subsequence.

Since every infinite subset of $\mathcal{I}_{\max}$ is unbounded, in a topology making $\mathcal{I}_{\max}$ satisfy Definition 5.27.3 there are no injective convergent sequences. The only such metric topology is the discrete topology, which is not separable in this case.

We close this section with the proof of Proposition 5.4. The construction originates from [16, Theorem 6 p. 183], and an analogous construction was used in [17] to show that the structure $(\mathcal{P}(\omega), \subseteq^\star)$ embeds into the family of F_σ ideals on ω partially ordered by *Borel* reduction. Our main improvement compared to [16] and [17], which makes possible to omit definability assumptions, is that our proof for non-reducibility is purely combinatorial.

We define sequences $(b_j)_{j<\omega}$ and $(m_j)_{j<\omega}$ by induction on j, as follows. Set $m_0 = 0$; if $j < \omega$ and m_j is already defined, set $b_j = 2^{j \cdot m_j}$ and $m_{j+1} = m_j + b_j$.

Let $I_j = [m_j, m_{j+1})$ $(j < \omega)$. Let log stand for logarithm of base 2. For every $S \in [\omega]^\omega$ and $N < \omega$, let $I_S(N) = \bigcup_{j \in S \cap N} I_j$ and $I_S = \bigcup_{j \in S} I_j$. For every $j < \omega$ and $x \subseteq \omega$,

$$\|x\|_j = \frac{\log(|x \cap I_j| + 1)}{m_j + 1}, \ \ \|x\| = \sup_{j<\omega} \|x\|_j.$$

We define $\mathcal{F}_S = \{x \subseteq I_S \colon \sup_{j<\omega} \|x\|_j < \infty\}$. For every $N < \omega$, let $\mathcal{F}_S(N) = \{x \in \mathcal{F}_S \colon \|x\| \leq N\}$. We will use the property that for arbitrary $n, j < \omega$ and $(x_i)_{i<n} \subseteq \mathcal{P}(\omega)$,

$$\left\| \bigcup_{i<n} x_i \right\|_j \leq \sup_{i<n} \|x_i\|_j + \frac{\log(n)}{m_j + 1}. \tag{5.4}$$

Proposition 5.4 is an immediate corollary of the following two statements.

Proposition 5.28. *For every $S \in [\omega]^\omega$, $\mathcal{F}_S \subseteq \mathcal{P}(\omega)$ is an F_σ ideal which is not basic in any topology.*

Proposition 5.29. *For every $S, T \in [\omega]^\omega$, $\mathcal{F}_S \leq_T \mathcal{F}_T$ if and only if $S \subseteq^\star T$.*

Proof of Proposition 5.28. It is obvious that $\mathcal{F}_S(N) \subseteq \mathcal{P}(\omega)$ $(N < \omega)$ are closed sets, so $\mathcal{F}_S$ is an F_σ ideal. Suppose there is a topology on $\mathcal{F}_S$ which makes it basic. For every $N < \omega$, let $(x_i(N))_{i<\omega} \subseteq [\omega]^{<\omega}$ be a sequence such that

(1) for every $i < \omega$ we have $x_i(N) \subseteq I_S$, $N - 1 \leq \|x_i(N)\| \leq N$;
(2) for every $j < \omega$ we have $|\{i < \omega \colon x_i(N) \cap I_j \neq \emptyset\}| \leq 1$.

Set $X_i(N) = \bigcup_{i \leq k < \omega} x_k(N)$ $(i, N < \omega)$. Then by (2), $X_i(N) \subseteq I_S$ and $\|X_i(N)\| \leq N$, i.e. $X_i(N) \in \mathcal{F}_S$ $(i, N < \omega)$.

Fix $N < \omega$. The sequence $(x_k(N))_{k<\omega}$ is bounded by $X_0(N)$, so by Definition 5.27.2, it has a convergent subsequence $(x_k(N))_{k \in I_N}$ for some $I_N \in [\omega]^\omega$.

By [24, Lemma 3.1 p. 1882], for every $i < \omega$, $X_i^+(N) = \{y \in \mathcal{F}_S \colon y \subseteq X_i(N)\}$ is compact. Since the convergent sequence $(x_k(N))_{k \in I_N}$ is eventually a subsequence of $X_i^+(N)$ $(i < \omega)$, $\lim_{k \in I_N} x_k(N) \in \bigcap_{i<\omega} X_i^+(N) = \{\emptyset\}$. Thus $\lim_{i \in I_N} x_i(N) = \emptyset$ $(N < \omega)$.

Since a basic topology is metric, by an easy diagonalization argument we can find a function $\varphi \colon \omega \to \omega$ such that $\varphi(N) \in I_N$ $(N < \omega)$ and $\lim_{N<\omega} x_{\varphi(N)}(N) = \emptyset$. Thus $(x_{\varphi(N)}(N))_{N<\omega}$ is a convergent sequence, which by (1) has no bounded subsequence. This contradicts Definition 5.27.3. □

The non-reduction part of Proposition 5.29 is based on the following. For every $t \in 2^{\leq \omega}$ we set $[t = 1] = \{n < \omega \colon t(n) = 1\}$. Recall $N_s = \{x \in \mathcal{P}(\omega) \colon x \cap |s| = [s = 1]\}$ $(s \in 2^{<\omega})$.

Lemma 5.30. *Let $S, T \in [\omega]^\omega$ satisfy $S \cap T = \emptyset$. Let $A \subseteq \mathcal{F}_S(1)$, $n \in S$ and $s \in 2^{m_n}$ be such that $N_s \cap \mathcal{F}_S(1) \neq \emptyset$ and A is of second category everywhere in $N_s \cap \mathcal{F}_S(1)$ in the relative topology of $\mathcal{F}_S(1)$. Let $N < \omega$, and let $f \colon A \to \mathcal{F}_T(N)$ be an arbitrary function. Then there is a $t \in 2^{m_n}$, such that $[t = 1] \subseteq T$, $\|t\| \leq N$, $B = \{x \in A \colon f(x)|_{m_n} = t\} \subseteq \mathcal{F}_S(1)$ is of second category and $\|\bigcup B\|_n \geq n - 2$.*

Proof. Let $\mathcal{T} = \{t \in 2^{m_n} \colon [t = 1] \subseteq T, \|t\| \leq N\}$. For every $t \in \mathcal{T}$, set $B_t = \{x \in A \colon f(x)|_{m_n} = t\}$. If there is a $t \in \mathcal{T}$ such that B_t is of second category in $\mathcal{F}_S(1)$ and $\|\bigcup B_t\|_n \geq n - 2$ then we are done.

So suppose no such $t \in \mathcal{T}$ exists. Set

$$C = \bigcup \{B_t \colon t \in \mathcal{T},\ B_t \text{ is of second category in } \mathcal{F}_S(1)\}.$$

Then by (5.4), using $|\mathcal{T}| \leq 2^{m_n}$,

$$\|\bigcup C\|_n \leq \sup\nolimits_{t \in \mathcal{T}} \|\bigcup B_t\| + \frac{\log(2^{m_n})}{m_n + 1} < n - 2 + \frac{\log(2^{m_n})}{m_n + 1} < n - 1.$$

However, since A is of second category everywhere in $N_s \cap \mathcal{F}_S(1)$, we get $I_n \subseteq \bigcup A = \bigcup C$. Thus $\|\bigcup C\|_n = \log(2^{n \cdot m_n} + 1)/(m_n + 1) \geq n - 1$. This contradiction completes the proof. □

Proof of Proposition 5.29. Let first $S, T \in [\omega]^\omega$ satisfy $S \subseteq^\star T$. Set $f \colon \mathcal{F}_S \to \mathcal{F}_T$, $f(x) = x \cap I_T$ $(x \in \mathcal{F}_S)$. Is is easy to verify that f is a Tukey map.

Let now $S, T \in [\omega]^\omega$ satisfy $S \not\subseteq^\star T$. As we have seen, $\mathcal{F}_{S \setminus T} \leq_T \mathcal{F}_S$. So it is enough to prove $\mathcal{F}_{S \setminus T} \not\leq_T \mathcal{F}_T$; that is, we can assume $S \cap T = \emptyset$.

Let $f \colon \mathcal{F}_S \to \mathcal{F}_T$ be an arbitrary function. We find $N < \omega$ and $n_i < \omega$, $s_i, t_i \in 2^{<\omega}$, $A_i \subseteq \mathcal{F}_S(1)$ $(i < \omega)$ such that

(1) $n_i \in S$, $n_i < n_{i+1}$, $s_i \in 2^{m_{n_i}}$ $(i < \omega)$;
(2) $t_i \in 2^{m_{n_i}}$, $\|t_i\| \leq N$, $t_i \sqsubseteq t_{i+1}$, $[t_i = 1] \subseteq T$ $(i < \omega)$;
(3) $N_{s_i} \cap \mathcal{F}_S(1) \neq \emptyset$ and A_i is of second category everywhere in $N_{s_i} \cap \mathcal{F}_S(1)$ $(i < \omega)$;
(4) for every $i < w$, $f[A_i] \subseteq \mathcal{F}_T(N)$, $\|\bigcup A_{i+1}\|_{n_i} \geq n_i - 2$ and $f(x)|_{m_{n_i}} = t_i$ $(x \in A_{i+1})$.

Suppose first the construction is done; we show that f is not a Tukey map. For every $i < \omega$, let $X_{i+1} \subseteq A_{i+1}$ be minimal such that $(\bigcup X_{i+1}) \cap I_{n_i} = (\bigcup A_{i+1}) \cap I_{n_i}$. Then $|X_i| \leq b_{n_i}$ $(i < \omega)$. By (4), $x = \bigcup_{i<\omega} \bigcup X_i$ satisfies $\|x\|_{n_i} \geq n_i - 2$ $(i < \omega)$, i.e. $\bigcup_{i<\omega} X_i$ is unbounded in $\mathcal{F}_S$.

We show that $\bigcup_{i<\omega}\{f(x)\colon x\in X_i\}\subseteq\mathcal{F}_T$ is bounded. Set $\tau=\bigcup_{i<\omega}t_i$; then by (2), $\tau\in\mathcal{F}_T$ and $\|\tau\|\le N$. For every $n\in T$, by (5.4), (4) and $n\notin S$,

$$\left\|\bigcup\bigcup_{i<\omega}\{f(x)\colon x\in X_i\}\right\|_n\le\left\|\tau\cup\bigcup\{f(x)\colon x\in X_i,\ n_i<n\}\right\|_n$$
$$\le N+\frac{\log(1+\sum\{b_{n_i}\colon n_i<n\}+1)}{m_n+1}\le N+\frac{\log(m_n+2)}{m_n+1}\le N+1,$$

so the statement follows.

It remains to perform the construction. We have $f[\mathcal{F}_S(1)]\subseteq\bigcup_{N<\omega}\mathcal{F}_T(N)$ so there are $N<\omega$ and $A_0\subseteq\mathcal{F}_S(1)$ such that $f[A_0]\subseteq\mathcal{F}_T(N)$ and A_0 is of second category in $\mathcal{F}_S(1)$. So we can find $n_0\in S$, $s_0\in 2^{n_0}$ such that $N_{s_0}\cap\mathcal{F}_S(1)\ne\emptyset$ and A_0 is of second category everywhere in $N_{s_0}\cap\mathcal{F}_S(1)$.

Let $i<\omega$ be arbitrary and suppose that n_i, s_i, and A_i have already been found. We apply Lemma 5.30 with S, T, N, $A=A_i$, $n=n_i$ and $s=s_i$. We get a $t_i\in 2^{m_{n_i}}$ such that $[t_i=1]\subseteq T$, $\|t_i\|\le N$, $A_{i+1}=\{x\in A_i\colon f(x)|_{m_{n_i}}=t_i\}\subseteq\mathcal{F}_S(1)$ is of second category and $\|\bigcup A_{i+1}\|_{n_i}\ge n_i-2$. It remains to choose $n_{i+1}\in S$, $n_{i+1}>n_i$, $s_{i+1}\in 2^{n_{i+1}}$ such that $N_{s_{i+1}}\cap\mathcal{F}_S(1)\ne\emptyset$ and A_{i+1} is of second category everywhere in $N_{s_{i+1}}\cap\mathcal{F}_S(1)$. Then (1), (3) and (4) hold, and for (2) it remains to show $t_{i-1}\sqsubseteq t_i$ if $i>0$. However, this follows from $A_{i+1}\subseteq A_i$ using the inductive hypothesis $f(x)|_{m_{n_{i-1}}}=t_{i-1}$ $(x\in A_i)$. So the proof is complete. □

6. Problems

In our present work we did not apply some well-understood methods for studying infinite dimensional perfect set theorems. We close this paper with a survey of possible further research directions, and state some related open problems.

6.1. **General symmetric topologies, analytic relations.** Apart from the usual product topology, there are several topologies on $\mathbb{R}^\omega$ which are symmetric, i.e. open sets remain open under arbitrary permutation of coordinates, and which are important in applications; e.g. the box topology or the topologies induced by the ℓ^p $(1\le p\le\infty)$ norms. There is no reason to believe that the product topology is the most appropriate for the formulation of optimal infinite dimensional perfect set theorems. It would be interesting to find weaker largeness assumptions in these finer topologies than our $X^\omega\setminus A\in\mathbb{M}$ in Theorem 4.1, which still imply the existence of a non-empty perfect A-homogeneous set. E.g. it is easy to construct a symmetric dense open set $A\subseteq\mathbb{R}^\omega$ such that there is no infinite A-homogeneous set; but in $\mathbb{R}^\omega$ endowed with the box topology, for every symmetric dense open set there is an infinite homogeneous set. Compare this with the remarks following Corollary 2.3.

Note also that ironically, all of our perfect set theorems hold for co-analytic relations, while the rank approach of [13] and [22] is able to handle analytic relations. Observe that to every construction using finite approximations, one can associate the tree of finite approximations ordered by end-extension, such that the ill-foundedness of this tree is equivalent with the existence of the limit object of the construction. Therefore the infinite dimensional counterpart of [13, Proposition 4.1 p. 151] is natural to formulate and easy to prove. The more involved task would be to study the existence of universal relations as in [13, Section 5], and to characterize the resulting rank as in [22]. It seems that these investigations can be carried out for any of the above mentioned refinements of the product topology, as well.

6.2. **More fusion games.** It seems informative to study fusions of other games of descriptive set theory, especially those of Separation games and Wadge games. We remark that the way we increased the dimension in our fusion scheme was completely arbitrary; different schemes characterize other notions of smallness.

We propose an explicit modification of $\mathcal{G}_\omega$ of Definition 3.2 which seems particularly interesting to study. Consider the game $\mathcal{G}^{\mathrm{m}}_\omega \subseteq \mathcal{G}_\omega$, where in addition, in the $(n+1)^{\mathrm{th}}$ move player I is required to play $(U_i(2(n+1-i)))_{i\leq n+1}$ such that $U_{n+1}(0) \subseteq \bigcup_{i\leq n} U_i(2(n-i)+1)$; else the game is unchanged. For player II this game is easier to win, still one can show that the existence of a winning strategy for player II in $\mathcal{G}^{\mathrm{m}}_\omega(A)$ implies the existence of a non-empty perfect A-homogeneous set.

Unfortunately we could not characterize the existence of a winning strategy for player I in $\mathcal{G}^{\mathrm{m}}_\omega(A)$. Nevertheless, we expect that fusion games modified in such ways can provide sharper results.

6.3. **Other Choquet topologies and forcings.** The Ellentuck topology and the density topologies are Choquet, in particular Theorem 4.1 can be applied to them. However, they fail the Kuratowski-Ulam property, which is crucial for Lemma 4.4 and so for all the results of Section 4.1 and Section 4.2. E.g. our methods does not allow us to prove the counterparts of Theorem 1.2 and Theorem 4.10, in which "meager" is replaced by "Lebesgue null" and "Cohen" is replaced by "random".

Problem 6.1. *Let $A \subseteq \mathbb{R}^\omega$ be a co-analytic set. Does the existence of an A-homogeneous set of positive outer Lebesgue measure imply the existence of a non-empty perfect set A-homogeneous set?*

Problem 6.2. *Let V be a model obtained from a model of the Continuum Hypothesis by adding ω_2 random reals. In V, let $A \subseteq \mathbb{R}^\omega$ be a co-analytic set. Does the existence of an A-homogeneous set of cardinality ω_2 imply the existence of a non-empty perfect set A-homogeneous set?*

Note that the finite dimensional analogue of Problem 6.1 holds for every Lebesgue measurable set A by the measure version of Mycielski's Theorem (see e.g. [27, Theorem 6.40]), while the finite dimensional analogue of Problem 6.2 holds for every analytic set A by [22, Fact 1.16 p. 23].

These problems may be related to the following.

Problem 6.3. *Let $A \subseteq 2^\omega \times 2^\omega$ be a co-meager set. Is it true in ZFC that there is an A-homogeneous set which is of second category everywhere?*

If e.g. $\mathrm{cof}(\mathcal{M}) = \mathrm{cov}(\mathcal{M})$, then by an easy transfinite argument, for every co-meager set $A \subseteq 2^\omega \times 2^\omega$ one can construct an A-homogeneous set which is of second category everywhere. Also note the counterpart of Problem 6.3 involving Lebesgue measure fails.

Theorem 6.4. ([2], [7]) *Let V be a model obtained from a model of the Continuum Hypothesis by adding ω_2 Cohen reals. Then in V, there is an F_σ set $A \subseteq 2^\omega \times 2^\omega$ of Lebesgue measure 1 such that there exists no A-homogeneous set of positive outer Lebesgue measure.*

The proof of Theorem 6.4 is based on the observation that in Cohen extensions, if for an F_σ set $A \subseteq 2^\omega \times 2^\omega$ there is an A-homogeneous set of positive outer Lebesgue measure then there is an A-homogeneous set of positive Lebesgue measure. Thus it is likely that Problem 6.1 has an affirmative answer in Cohen extensions. However, the proof of Theorem 6.4 does not have a straightforward modification valid for random extensions, so we could not obtain a negative answer to Problem 6.3.

Problem 6.3 was motivated by the question whether the largeness assumption in Theorem 1.2 is the natural analogue of the largeness assumption of Mycielski's Theorem. As we pointed out above, this is consistently true, since for sets $A \subseteq 2^\omega \times 2^\omega$ having the Baire property, being co-meager and having an A-homogeneous set which is of second category everywhere are consistently equivalent.

In Cohen extensions there are perfectly meager sets of cardinality $2^{\aleph_0}$, so the largeness assumption in Theorem 1.2 is consistently not optimal. It would be interesting to know whether the converse is also consistent; the following is also related to Problem 6.3.

Problem 6.5. *Is it consistent with ZFC that $H \subseteq \mathbb{R}$ is perfectly meager if and only if there is an open set $A \subseteq \mathbb{R}^\omega$ such that $IS_\omega(H) \subseteq A$ but there is no non-empty perfect A-homogeneous set?*

6.4. **Local results.** In [15] the following "local" infinite dimensional perfect set theorem was obtained (see also [27, Corollary 6.48]). We call a sequence $(x_n)_{n<\omega} \in \mathbb{R}^\omega$ *rapidly increasing* if $0 < x_{n+2} - x_{n+1} < x_{n+1} - x_n$ $(n < \omega)$. The set of rapidly increasing sequences is denoted by $\mathcal{R}$.

Theorem 6.6. *(*[15, Theorem 2 p. 275]*) For every finite Baire measurable coloring of $\mathcal{R}$ there is a perfect set $P \subseteq \mathbb{R}$ such that $P^\omega \cap \mathcal{R}$ is monochromatic.*

Ramsey-type theorems like Theorem 6.6 are very important, e.g. Theorem 5.14 is also based on Theorem 6.6. The most general results of this nature are known as the Halpern-Läuchli Theorems (see e.g. [10] or [27] and the references therein). It is easy to see that there is an open tower $\mathbb{U}$ such that $[\mathbb{U}] = \mathcal{R}$, in particular $\mathcal{R} \notin \mathbb{M}$. It would be very useful to explore the possible interplay between our approach and such local results.

6.5. **Cofinal types of analytic ideals.** A deeper analysis of the construction in the proof of Theorem 5.21 reveals that it uses that E of Definition 5.23 is an equivalence relation in an essential way. Thus this method cannot yield a Borel ideal satisfying the conditions of Theorem 5.21. On the other hand, the construction in the proof of Theorem 2.1 is the base step of the construction of an F_σ ideal satisfying the conditions of Theorem 5.21. Therefore we expect a positive answer to the following problems.

Problem 6.7. *Is there an F_σ ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that $[\omega_1]^{<\omega} \leq_T \mathcal{I}$ but $\mathcal{I}$ has no non-empty perfect strongly unbounded subsets?*

Problem 6.8. *(*[16, Conjecture 1 p. 194]*) Is there an F_σ ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that $[\omega_1]^{<\omega} \leq_T \mathcal{I}$ but $\omega^\omega \not\leq_T \mathcal{I}$?*

We also expect the consistency of the failure of the primality property for $\mathcal{I}_{\max}$.

Problem 6.9. *Is it consistent with ZFC that $\mathcal{I}_{\max}$ fails the primality property, i.e. there exist analytic ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{I}_{\max} \leq_T \mathcal{I} \oplus \mathcal{J}$ but $\mathcal{I}_{\max} \not\leq_T \mathcal{I}$ and $\mathcal{I}_{\max} \not\leq_T \mathcal{J}$?*

At the present stage of research, one could wonder whether for every analytic ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$, $[\omega_2]^{<\omega} \leq_T \mathcal{I}$ implies that $\mathcal{I}$ has a non-empty perfect strongly unbounded subset. However, we expect that the affirmative answer to Problem 6.7 will be based on a construction which is flexible enough to rule out such speculations.

Note also that presently the iterated perfect set model is our only example where the conclusion of Theorem 5.7.1 holds. Nevertheless, we think that a better understanding of infinite dimensional perfect set theorem vs. Lebesgue measure and random extensions will provide more models where such results hold. Especially because by [19], in the random real model every universal measure zero set has cardinality $\leq \omega_1$.

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Department of Mathematics, University of Toronto, 40 St. George St., Toronto, ON, Canada
E-mail address: matrait@renyi.hu