

Hurewicz tests: separating and reducing analytic sets
the conscious way

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Abstract

In this thesis we develop the theory of topological Hurewicz test pairs. The central idea of this method is to construct special Polish topologies in such a way that simply structured sets, e.g. sets of low Borel class, are either “almost empty” or of second category in the corresponding special topology. Through this approach, which can be considered as a generalization of the Baire Category Theorem for higher Borel classes, we are able to give a quantitative aspect, in Baire category sense, to certain results related to Hurewicz test sets. As an application of the theory, we prove some results related to transfinite convergence of Borel functions, to the problem of finding simple generators of analytic ideals and to the problem of constructing Hurewicz test sets for generalized separation of analytic sets.

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Chapter 1

Classical results, motivating problems

If an open set is nonempty it is of second category. This is probably the simplest formulation of the Baire Category Theorem. The purpose of our thesis is to show that this theorem can be extended to the entire Borel hierarchy: for every $0 < \xi < \omega_1$ we define a fine topology such that if a Σ_ξ^0 set is just “a bit bigger than nonempty” then it is huge, i.e. it is of second category in our fine topology. And we present some typical Baire Category Theorem-like applications of this result: the intersection of countably many “fairly dense” Σ_ξ^0 sets is residual in the fine topology, hence it is nonempty.

We remark that there is a quite classical extension of the dichotomy expressed by the Baire Category Theorem for open sets. More than a half century ago Witold Hurewicz proved the following theorem, also called Hurewicz-dichotomy, about sets failing to be F_σ .

Theorem 1.1. (*W. Hurewicz*) *Let X be a Polish space and $A \subseteq X$ be an analytic set. If A is not F_σ , then there is a continuous injection of the Cantor set into X , $\varphi: 2^\omega \rightarrow X$ such that $2^\omega \setminus \varphi^{-1}(A)$ is countable dense in 2^ω ; that is $\varphi(2^\omega) \cap A$ is a relatively closed subset of A homeomorphic to the irrationals.*

If we add the well-known fact that any subset N of the Cantor set 2^ω homeomorphic to the irrationals is never F_σ we gave all the reasons why the pair $(2^\omega, N)$ is called the *Hurewicz test* for F_σ sets.

Theorem 1.1 has been strengthened in many successive steps. In some sense the most general existence theorem for Hurewicz tests is the following (see [7]).

Theorem 1.2. (*A. Louveau, J. Saint Raymond*) *Let $3 \leq \xi < \omega_1$ and (X, τ) be a Polish space. If $P_\xi \subseteq 2^\omega$ is $\Pi_\xi^0(\tau_{2^\omega})$ but not $\Sigma_\xi^0(\tau_{2^\omega})$ and $A_0, A_1 \subseteq X$ is any pair of disjoint analytic sets, then either A_0 can be separated from A_1 by a $\Sigma_\xi^0(\tau)$ set or there is a continuous one-to-one map $\varphi: (2^\omega, \tau_{2^\omega}) \rightarrow X$ with $\varphi(P_\xi) \subseteq A_0$ and $\varphi(2^\omega \setminus P_\xi) \subseteq A_1$.*

The same conclusion holds for $\xi = 2$ if $P_2 \subseteq 2^\omega$ is the complement of a dense countable set.

Thus we may call a $\Pi_\xi^0(\tau_{2^\omega})$ but not $\Sigma_\xi^0(\tau_{2^\omega})$ set P_ξ a *Hurewicz test* for $\Sigma_\xi^0(\tau_{2^\omega})$ sets, in the sense that either a set $A \subseteq X$ is $\Sigma_\xi^0(\tau_{2^\omega})$, a fact which can be witnessed by a description of the construction of A from simpler sets, or A is not $\Sigma_\xi^0(\tau_{2^\omega})$, a property which cannot be verified by checking some decomposition, but by Theorem 1.2 it can be witnessed by a continuous injection $\varphi: 2^\omega \rightarrow X$ satisfying $\varphi^{-1}(A) = P_\xi$.

Theorem 1.2 turned out to be the ideal starting point of our investigations. It allows us to isolate a simple reason, namely the image of the Hurewicz test $\varphi(P_\xi)$ in $\varphi(2^\omega)$ why the analytic set A_0 cannot be separated from the other analytic set A_1 with *one single* Σ_ξ^0 set. But the aspect of dichotomy which was present in the Baire Category Theorem for open sets is lacking, in particular Theorem 1.2 says nothing about how “crude” a separation attempt is if one insists on Σ_ξ^0 sets, or in other words, how well our A_0 can be approximated by Σ_ξ^0 sets laying outside A_1 . It was the problem of \mathcal{I} -convergent functions, raised by T. Natkaniec in [17], which shed light on this failure. We start with the definition.

Definition 1.3. Let λ be a cardinal, (X, τ) and (Y, d) be Polish spaces. Consider an ideal \mathcal{I} on λ . We say that a sequence of functions $f_\alpha: X \rightarrow Y$ ($\alpha < \lambda$) \mathcal{I} -converges to the function $f: X \rightarrow Y$, in notation $f_\alpha \rightarrow_{\mathcal{I}} f$, if

$$\{\alpha < \lambda: f_\alpha(x) \neq f(x)\} \in \mathcal{I}$$

for every $x \in X$.

The question of T. Natkaniec is whether \mathcal{I} -convergence keeps the Baire classes or not, that is under which condition on λ and \mathcal{I} it is true that if f_α is Baire- ξ ($\alpha < \omega$) and $f_\alpha \rightarrow_{\mathcal{I}} f$ the limit function f must be also Baire- ξ . Since the Baire class of a function is in close relation with the Borel class of its sublevel set it is not surprising that the problem of \mathcal{I} -convergent functions is equivalent to finding some quantitative analogue of Theorem 1.2. It was the first motivation for our investigations.

In some sense a particular case of the problem of T. Natkaniec is the problem of generalized separation. It is well known that analytic sets have the generalized separation property, that is for every sequence $(A_i)_{i < \omega}$ of analytic sets with $\bigcap_{i < \omega} A_i = \emptyset$ there is a sequence $(B_i)_{i < \omega}$ of Borel sets satisfying $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = \emptyset$. But of what complexity must B_i ($i < \omega$) be? That is, if ξ is fixed, when can each B_i ($i < \omega$) be chosen in Π_ξ^0 ? This can be regarded as an approximation problem where the analytic sets A_i ($i < \omega$) play the role of constraint. One can ask whether there is a theorem providing Hurewicz tests for generalized separation as Theorem 1.2 does for ordinary separation.

The last problem we mention here has been posed by A. Miller. S. Solecki in [21] proved the following result.

Theorem 1.4. (*S. Solecki*) *Given a family of closed sets \mathcal{I} and an analytic set A in some Polish space X , either A can be covered by the union of countably many members of \mathcal{I} or A contains a nonempty Π_2^0 set G with the property that $F \cap G$ is meager in G for every $F \in \mathcal{I}$.*

Observe that this result has the feature of testing: if A can be covered by the union of countably many members of \mathcal{I} this is witnessed by the countable collection of closed sets in \mathcal{I} realizing the covering, while the contrary, which is a priori unwitnessable, can be verified by a good Π_2^0 subset of A and Baire category. It is natural to ask whether this holds for every Borel class (see [15]).

Question 1.5. (*A. Miller*) *For $2 \leq \xi < \omega_1$ let \mathcal{I} be a σ -ideal which is generated by its Π_ξ^0 members. Is it true that for every analytic set $A \subseteq X$, either $A \in \mathcal{I}$ or there is a $\Pi_{\xi+1}^0$ set $B \subseteq A$ such that $B \notin \mathcal{I}$?*

These problems can be treated within the theory of *topological Hurewicz test pairs*, a concept we aim to develop in this thesis. Informally, it can be summarized as “sets of low Borel class are so simply structured comparing to sets of high Borel class that even Baire category can distinguish them in a suitable topology”. As an illustration, the precise topologized version of Theorem 1.2 is the following result (see [10], Theorem 4 on page 159).

Theorem 1.6. *Let (X, τ) be a Polish space. For every $2 \leq \xi < \omega_1$, there exist a $\Pi_\xi^0(\tau_{2^\omega})$ set $P_\xi \subseteq 2^\omega$ and a Polish topology τ_{P_ξ} on 2^ω which is finer than τ_{2^ω} such that P_ξ is nowhere dense and closed in the topology τ_{P_ξ} , and if a Borel set $A \subseteq X$ is*

1. *in $\Sigma_\xi^0(\tau)$, then whenever for a continuous one-to-one mapping $\varphi: (2^\omega, \tau_{2^\omega}) \rightarrow (X, \tau)$ we have that $\varphi^{-1}(A) \cap P_\xi$ is of second category in P_ξ in the relative topology $\tau_{P_\xi}|_{P_\xi}$ then $\varphi^{-1}(A) \subseteq 2^\omega$ is of second category in the topology τ_{P_ξ} ;*
2. *not in $\Sigma_\xi^0(\tau)$, then there is a continuous one-to-one mapping $\varphi: (2^\omega, \tau_{2^\omega}) \rightarrow (X, \tau)$ such that $\varphi(P_\xi) \subseteq A$ and $\varphi^{-1}(A) \subseteq 2^\omega$ is of first category in the topology τ_{P_ξ} .*

Moreover, if $\lambda < 2^{\aleph_0}$ is a cardinal and in our model the union of a λ number of meager sets is meager in Polish spaces the first statement holds for every (not necessarily Borel) set A which can be obtained as a union of a λ number of $\Sigma_\xi^0(\tau)$ sets.

This result extends Theorem 1.2 by providing the quantitative aspect we have been looking for: the Baire category in the topology τ_{P_ξ} . We remark that S. Solecki proved a theorem where Baire category was used to distinguish Σ_ξ^0 sets from a given Π_ξ^0 set (see [22], Theorem 2.2). In his approach the topology τ_P remains hidden.

In the next chapters we will build up the theory of topological Hurewicz test pairs. The basic definitions and notations in Chapter 2 are followed by an introductory chapter,

Chapter 3, where we make the reader familiar with the techniques and concepts in the $\xi = 3$ case. In particular we show that the answer to a special form of the question of A. Miller is consistently negative. In Chapter 4 we continue with the general $0 < \xi < \omega_1$ case, we prove Theorem 1.6 in an extended form and we construct Π_ξ^0 coverings for Σ_ξ^0 generated ideals. Given that Theorem 1.2 can essentially be extended to the difference hierarchy, we perform a partial extension for Theorem 1.6 in Chapter 5. Then in Chapter 6 we show that indeed, \mathcal{I} -convergence keeps the Baire class of functions. In Chapter 7 we construct a test allowing the determination of the minimal complexity of sets involved in a generalized separation or reduction, and we conclude the thesis with Chapter 8 where we formulate some problems which hopefully will be as motivating as the ones presented here.¹

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¹The interested reader is referred to [11], [12], [13] and [14] for an improved treatment of this material.

Chapter 2

Definitions and notation

2.1 Sequences and trees

Our terminology and notation follow [4]. Let (C, τ_C) denote the Polish space 2^ω with its usual product topology. For two finite sequences $s, t \in \omega^{<\omega}$, we write $s \subseteq t$ ($s \subset t$, resp.) if t is an extension (a proper extension, resp.) of s . The length of s is denoted by $|s|$. If $s = (s_0 s_1 \dots s_{n-1})$ and $i < \omega$, then $s \frown i$ stands for the sequence $(s_0 s_1 \dots s_{n-1} i)$.

If $T \subseteq \omega^{<\omega}$ is a subtree and $s \in \omega^{<\omega}$ then $T_s = \{t \in \omega^{<\omega} : s \frown t \in T\}$. The terminal nodes of T are denoted by $\mathfrak{T}(T)$.

Let ξ, ϑ_i ($i < \omega$) be ordinals. We write $\vartheta_i \rightarrow \xi$ if ξ is successor and $\vartheta_i + 1 = \xi$ ($i < \omega$) or if ξ is limit, $\vartheta_i \leq \vartheta_j$ ($i \leq j < \omega$) and $\sup_{i < \omega} \vartheta_i = \xi$.

For every ordinal $\xi < \omega_1$ we fix once and for all a sequence $(\vartheta_i)_{i < \omega}$ such that $\vartheta_i \rightarrow \xi$. To avoid complicated notations, we do not indicate the dependence of the sequence on ξ , it will be always clear which pair of ordinal and sequence is considered.

The order topology on an ordinal ξ is denoted by \mathfrak{o}_ξ .

2.2 Refining topologies, basic open sets

In this note we will notoriously refine Polish topologies by turning countably many closed sets into open sets. We do this as described in [4], that is the open sets of the ancient topology together with their portion on the members of our collection of closed sets serve as a subbase of the new, finer topology. We will use that the topology obtained in this way is also Polish.

Definition 2.1. Let (X, τ) be a Polish space, $\mathcal{P} = \{P_i : i < \omega\}$ be a countable collection of $\Pi_1^0(\tau)$ sets. Then $\tau[\mathcal{P}]$ denotes that Polish topology refining τ where each P_i ($i < \omega$) is turned successively into an open set.

It is easy to see that the resulting finer topology $\tau[\mathcal{P}]$ is independent from the enumeration of \mathcal{P} . This will be clear shortly when we fix a base of $\tau[\mathcal{P}]$. We also use the notation $\tau[\mathcal{P}]$ when the countable collection of not necessarily $\Pi_1^0(\tau)$ sets \mathcal{P} can be enumerated on such a way that P_n is $\Pi_1^0(\tau[\{P_i : i < n\}])$.

Definition 2.2. If τ_n ($n < \omega$) is a Polish topology on some base set X then $\bigvee_{n < \omega} \tau_n$ denotes the coarsest topology on X which refines each τ_n ($n < \omega$).

The resulting topology is also Polish and we will shortly fix a countable base for it. Before doing this we need a precise notion of basic open sets in our spaces.

Definition 2.3. Let (X_i, τ_i) ($i \in I$) be Polish spaces; if a basis \mathcal{G}_i is fixed in the spaces (X_i, τ_i) ($i \in I$), which are meant to be the basic open sets in (X_i, τ_i) , then the *basic open sets* of $(\prod_{i \in I} X_i, \prod_{i \in I} \tau_i)$ are the open sets of the form

$$\prod_{i \in J} G_i \times \prod_{i \in I \setminus J} X_i,$$

where $J \subseteq I$ is finite and $G_i \in \mathcal{G}_i$ for every $i \in J$.

If the basic open sets \mathcal{G} are fixed in the Polish space (X, τ) and $\tau[\mathcal{P}]$ makes sense for a countable collection \mathcal{P} of subsets of X , then *the basic open sets* of $\tau[\mathcal{P}]$ are of the form $G \cap F_0 \cap \dots \cap F_{n-1}$ or G with $G \in \mathcal{G}$, $F_i \in \mathcal{P}$ ($i < n$); a basic $\tau[\mathcal{P}]$ -open set is said to be *proper* if it is not τ -open.

If the basic open sets \mathcal{G}_n are fixed for the Polish topologies τ_n ($n < \omega$) then the *basic open sets* for $\bigvee_{n < \omega} \tau_n$ are the sets of the form $\bigcap_{i < m} G_{n_i}$ where $G_{n_i} \in \mathcal{G}_{n_i}$ ($m < \omega$, $n_i < \omega$ ($i < \omega$)).

Observe that the basic open sets defined on this way form a basis of $\prod_{i \in I} \tau_i$, $\tau[\mathcal{P}]$ and $\bigvee_{n < \omega} \tau_n$, respectively. From now on whenever a Polish space (X, τ) appears we assume that a countable basis comprised of basic τ -open sets is fixed; and this is done with respect to the convention of Definition 2.3 if applicable. We take X to be basic τ -open. Basic open sets are assumed to be regular open. In zero dimensional spaces we assume that our basic τ -open sets are $\Pi_1^0(\tau)$; note that our procedure of refinement results a zero dimensional space from a zero dimensional one with $\Pi_1^0(\tau[\mathcal{P}])$ basic $\tau[\mathcal{P}]$ -open sets. We denote ambiguously our special collection of basic τ -open sets also by τ .

The interior (closure, resp.) of a set $A \subseteq (X, \tau)$ is denoted by $\text{int}_\tau(A)$ ($\text{cl}_\tau(A)$, resp.). We will never have to fix a special compatible metric on our Polish spaces but we will condition on the diameter of sets. In this case diam_τ denotes the diameter in an arbitrary fixed metric generating τ . We assume that $\text{diam}_\tau(X) \leq 1$.

We recall that a $\Pi_2^0(\tau)$ subset G of the Polish space (X, τ) is itself a Polish space with the restricted topology τ_G (see e.g. [4], (3.11) Theorem). In particular, the notions related to category in the topology τ make sense relative to G .

We will have to return to the topologies on the coordinates in product spaces. If $(X, \sigma), (Y, \tau)$ are arbitrary topological spaces and $(\mathcal{X}, \mathcal{S}) = (X \times Y, \sigma \times \tau)$, then we define $\text{Pr}_X(\mathcal{S}) = \sigma$. The projection of product sets in product spaces is defined analogously. If $G_X \subseteq X$ and $G_Y \subseteq Y$, we say that the set of product form $G = G_X \times G_Y \subseteq \mathcal{X}$ is *nontrivial on the X coordinate* if $G_X \neq X$.

2.3 Hierarchies, separation, reduction

As usual, $\Pi_\xi^0(\tau)$ ($\Sigma_\xi^0(\tau)$ resp.) ($0 < \xi < \omega$) stands for the ξ^{th} multiplicative (additive resp.) Borel class in the Polish space (X, τ) , starting with $\Pi_1^0(\tau) =$ closed sets, $\Sigma_1^0(\tau) =$ open sets. The dual class $\check{\Gamma}$ of a class $\Gamma \subseteq 2^X$ is defined by $\check{\Gamma} = \{A \subseteq X : X \setminus A \in \Gamma\}$.

A set is called *proper* $\Pi_\xi^0(\tau)$ if it is $\Pi_\xi^0(\tau)$ but not $\Sigma_\xi^0(\tau)$ ($0 < \xi < \omega_1$).

We derive the *difference hierarchy* as follows. Every ordinal η can be uniquely written as $\alpha + n$ where α is limit and $n < \omega$. We call η even (odd, resp.) if n is even (odd, resp.).

Definition 2.4. Let $0 < \vartheta, \xi < \omega_1$ and let $(A_\eta)_{\eta < \vartheta}$ be a sequence of subsets of a set X , such that $A_\eta \subseteq A_{\eta'}$ ($\eta \leq \eta' < \vartheta$). Then $D_\vartheta((A_\eta)_{\eta < \vartheta}) \subseteq X$ is defined by

$$x \in D_\vartheta((A_\eta)_{\eta < \vartheta}) \iff x \in \bigcup_{\eta < \vartheta} A_\eta \text{ and the least } \eta < \vartheta \text{ with } x \in A_\eta$$

has parity opposite to that of ϑ .

With this operation in the Polish space (X, τ) we set

$$D_\vartheta(\Sigma_\xi^0(\tau)) = \{D_\vartheta((A_\eta)_{\eta < \vartheta}) : A_\eta \in \Sigma_\xi^0(\tau), A_\eta \subseteq A_{\eta'} (\eta \leq \eta' < \vartheta)\}.$$

For fixed $0 < \xi < \omega_1$, a set is called *proper* $D_\vartheta(\Sigma_\xi^0(\tau))$ if it is $D_\vartheta(\Sigma_\xi^0(\tau))$ but not $\check{D}_\vartheta(\Sigma_\xi^0(\tau))$ ($0 < \vartheta < \omega_1$).

Next we give two definition related to ideals.

Definition 2.5. Let $0 < \eta < \omega_1$. If (X, τ) is a Polish space and $P \subseteq X$, $\mathcal{S}_\eta^0(P)$ ($\mathcal{P}_\eta^0(P)$, resp.) denotes the collection of $\Sigma_\eta^0(\tau)$ ($\Pi_\eta^0(\tau)$, resp.) subsets of P .

Definition 2.6. Let \mathcal{I} be a σ -ideal and $\mathcal{F} \subseteq \mathcal{I}$. We say that \mathcal{I} is *generated by* \mathcal{F} if for every $G \in \mathcal{I}$ there exists $F_i \in \mathcal{F}$ ($i < \omega$) such that $G \subseteq \bigcup_{i < \omega} F_i$. We say that \mathcal{I} is *strongly generated by* \mathcal{F} if for every $G \in \mathcal{I}$ there is an $F \in \mathcal{F}$ such that $G \subseteq F$.

We recall some definitions related to separation.

Definition 2.7. Let X be an arbitrary set. For $A_0, A_1 \subseteq X$ we say that $G \subseteq X$ *separates* A_0 from A_1 if we have $A_0 \subseteq G \subseteq X \setminus A_1$.

A class $\Gamma \subseteq 2^X$ has the *separation property* if for every disjoint $A_0, A_1 \in \Gamma$ there is a set $G \in \Gamma$ such that $X \setminus G \in \Gamma$ and G separates A_0 from A_1 . A class $\Gamma \subseteq 2^X$ has the *generalized separation property* if for every sequence $(A_i)_{i < \omega} \subseteq \Gamma$ with $\bigcap_{i < \omega} A_i = \emptyset$ there is a sequence $(G_i)_{i < \omega} \subseteq \Gamma$ such that $(X \setminus G_i)_{i < \omega} \subseteq \Gamma$, $A_i \subseteq G_i$ ($i < \omega$) and $\bigcap_{i < \omega} G_i = \emptyset$.

A class $\Gamma \subseteq 2^X$ has the *reduction property* if for every $A, A' \in \Gamma$ there are $B, B' \in \Gamma$ such that $B \subseteq A$, $B' \subseteq A'$, $A \cup A' = B \cup B'$ and $B \cap B' = \emptyset$.

Our $\Gamma \subseteq 2^X$ has the *generalized reduction property* if for every sequence $(A_i)_{i < \omega} \subseteq \Gamma$ there is a sequence $(G_i)_{i < \omega} \subseteq \Gamma$ such that $G_i \subseteq A_i$, $G_i \cap G_j = \emptyset$ ($i, j < \omega$, $i \neq j$) and $\bigcup_{i < \omega} G_i = \bigcup_{i < \omega} A_i$.

Chapter 3

The $\xi = 3$ case

As we announced in the introduction, we start our work by constructing topological Hurewicz test pairs for $\xi = 3$ and then we prove the corresponding theorems for Π_2^0 generated ideals. From now on (X, τ) denotes a Polish space.

3.1 The Π_3^0 set of Lusin

We will investigate the following object.

Definition 3.1. Let $P \subseteq X$. We call the pair $\{P, \tau_P\}$ a $\Pi_3^0(\tau)$ *topological Hurewicz test pair in X* if

1. P is a $\Pi_3^0(\tau)$ set;
2. τ_P is a Polish topology refining τ ;
3. P is a nowhere τ_P -dense $\Pi_1^0(\tau_P)$ set;
4. for every $\Pi_2^0(\tau)$ set $A \subseteq X$ and basic τ_P -open set G with $G \cap P \neq \emptyset$, if $A \cap P$ is $\tau_P|_P$ -residual in $G \cap P$ then A is τ_P -residual in G .

As we shall see later, there exist $\Pi_3^0(\tau)$ topological Hurewicz test pairs in uncountable Polish spaces. We present a construction providing such sets in a sufficiently large variety for our purposes. First we define the topologies corresponding to P .

Definition 3.2. Consider a $\Pi_3^0(\tau)$ set P and fix a presentation

$$P = \bigcap_{i < \omega} P^i, \quad P^i = \bigcup_{j < \omega} P_j^i$$

with $\Pi_1^0(\tau)$ sets $P_j^i \subseteq X$ ($i, j < \omega$). Set

$$\mathcal{P}_n = \{P_j^i : i < n, j < \omega\} \quad (n \leq \omega), \quad \mathcal{P} = \{X \setminus P^i : i < \omega\}.$$

We define $\tau_P^< = \tau[\mathcal{P}_\omega]$ and $\tau_P = \tau_P^<[\mathcal{P}]$.

Claim 3.3. *We have the following.*

1. P is $\Pi_2^0(\tau_P^<)$ and $\Pi_1^0(\tau_P)$.
2. If G is basic τ_P -open and $G \cap P \neq \emptyset$ then G is in fact basic $\tau_P^<$ -open.
3. The topologies $\tau_P|_P$ and $\tau_P^<|_P$ coincide.
4. If G is basic $\tau_P^<$ -open and $G \cap X \setminus P^i \neq \emptyset$ ($n \leq i < \omega$) then G is basic $\tau[\mathcal{P}_n]$ -open.
5. If G is a $\tau[\mathcal{P}_n]$ -open set then the topologies

$$\tau_P|_{G \cap (\bigcap_{i < n} P^i) \setminus \bigcup_{n \leq j < \omega} P^j} \text{ and } \tau[\mathcal{P}_n]|_{G \cap (\bigcap_{i < n} P^i) \setminus \bigcup_{n \leq j < \omega} P^j}$$

coincide.

Proof. The first statement is obvious. Since $\tau_P = \tau_P^<[\mathcal{P}]$, proper basic τ_P -open sets do not intersect P , which shows 2. From this we immediately get 3.

By definition, if a set is basic $\tau_P^<$ -open then it is basic $\tau[\mathcal{P}_n]$ -open for some $n < \omega$. But a basic $\tau[\mathcal{P}_{i+1}]$ -open but not $\tau[\mathcal{P}_i]$ -open set G' satisfies $G' \subseteq P^i$ ($i < \omega$), so 4 follows.

For 5, it is enough to consider the case when $G \cap (\bigcap_{i < n} P^i) \setminus \bigcup_{n \leq j < \omega} P^j \neq \emptyset$. Let H be a nonempty basic τ_P -open set which intersects $G \cap (\bigcap_{i < n} P^i) \setminus \bigcup_{n \leq j < \omega} P^j$, say $H = H_0 \cap \bigcap_{k < l} X \setminus P^{m_k}$ where H_0 is basic $\tau_P^<$ -open, $l < \omega$ and $m_k < \omega$ ($k < l$). Then we have $n \leq m_k$ ($k < l$), so $G \cap (\bigcap_{i < n} P^i) \setminus \bigcup_{n \leq j < \omega} P^j \subseteq \bigcap_{k < l} X \setminus P^{m_k}$. Since $H_0 \cap X \setminus P^j \neq \emptyset$ ($n \leq j < \omega$), H_0 is basic $\tau[\mathcal{P}_n]$ -open by 4. Thus

$$H \cap G \cap \left(\bigcap_{i < n} P^i \right) \setminus \bigcup_{n \leq j < \omega} P^j = H_0 \cap G \cap \left(\bigcap_{i < n} P^i \right) \setminus \bigcup_{n \leq j < \omega} P^j$$

where H_0 is basic $\tau[\mathcal{P}_n]$ -open, so the statement follows and the proof is complete. ■

Using topologies of Definition 3.2 we can formulate a sufficient condition for that $\{P, \tau_P\}$ is a topological Hurewicz test pair. Observe that the requirements concern mainly the presentation of P instead of P itself; in the next chapter we will discuss the advantages and inconveniences of this.

Theorem 3.4. *With the notation of Definition 3.2, suppose that $P^j \subseteq P^i$ ($i \leq j < \omega$) and P^i is $\tau[\mathcal{P}_i]$ -dense and $\tau[\mathcal{P}_i]$ -meager in X ($i < \omega$); then*

1. P is $\tau_P^<$ -residual;
2. $\{P, \tau_P\}$ is a $\Pi_3^0(\tau)$ topological Hurewicz test pair.

Proof. For 1, it is enough to show that P^i is $\tau_P^<$ -dense ($i < \omega$); being $\tau_P^<$ -open this implies that each P^i and hence P is $\tau_P^<$ -residual. We have that P^i is $\tau[\mathcal{P}_i]$ -dense and contains P^j ($i \leq j < \omega$). So if G is a basic $\tau_P^<$ -open set then either G is $\tau[\mathcal{P}_i]$ -open and so $P^i \cap G \neq \emptyset$, or G is not $\tau[\mathcal{P}_i]$ -open hence $G \subseteq P^i$. This proves the statement.

For 2 we check the conditions of Definition 3.1: 1 holds by the choice of P . By Definition 3.2, the topology τ_P is Polish and refines τ , which proves 2.

For 3, by Claim 3.3.1 P is $\Pi_1^0(\tau_P)$ so it remains to show that P does not contain any nonempty basic τ_P -open set. Suppose that $G \subseteq P$ and G is nonempty basic τ_P -open. Then by Claim 3.3.2, G is basic $\tau_P^<$ -open hence basic $\tau[\mathcal{P}_n]$ -open for some $n < \omega$. But P^n is $\tau[\mathcal{P}_n]$ -meager, so $G \not\subseteq P^n$ which contradicts $G \subseteq P \subseteq P^n$.

For 4, let $A \subseteq X$ be $\Pi_2^0(\tau)$, G be a basic τ_P -open set with $G \cap P \neq \emptyset$ and suppose that $A \cap P$ is $\tau_P|_P$ -residual in $G \cap P$. Observe that since A is $\Pi_2^0(\tau)$, ($\Pi_2^0(\tau[\mathcal{P}_n])$ ($n \leq \omega$), $\Pi_2^0(\tau_P)$, resp.), the notions “meager” and “nowhere dense” coincide for A in the topology τ , ($\tau[\mathcal{P}_n]$ ($n \leq \omega$), τ_P , resp.).

By Claim 3.3.2, G is actually $\tau_P^<$ -open. Since by 1, P is $\tau_P^<$ -residual in X and by Claim 3.3.3 the topologies $\tau_P|_P$ and $\tau_P^<|_P$ coincide, we have that A is $\tau_P^<$ -residual in G . Suppose that A is not τ_P -residual in G ; that is we have a nonempty basic τ_P -open set $G' \subseteq G$ such that $A \cap G'$ is τ_P -meager thus τ_P -nowhere dense in G' . So by passing to a nonempty basic τ_P -open subset we can assume that $A \cap G' = \emptyset$. Since $P^j \subseteq P^i$ ($i \leq j < \omega$), we have $G' = G'' \cap (X \setminus P^n)$ for some $n < \omega$ where $G'' \subseteq G$ is basic $\tau_P^<$ -open. Since $G'' \cap (X \setminus P^n) \neq \emptyset$, from Claim 3.3.4 we get that G'' is basic $\tau[\mathcal{P}_n]$ -open. Now $G'' \subseteq G$ implies that A is $\tau_P^<$ -residual hence $\tau_P^<$ -dense in G'' . Since $\tau[\mathcal{P}_n]$ is coarser than $\tau_P^<$, A is $\tau[\mathcal{P}_n]$ -dense so $\tau[\mathcal{P}_n]$ -residual in G'' . Thus A and $X \setminus P^n$ are both $\tau[\mathcal{P}_n]$ -residual in G'' which yields $A \cap G'' \cap (X \setminus P^n) \neq \emptyset$, a contradiction which completes the proof. ■

The following theorem describes important additional properties of the construction in Theorem 3.4, and in fact the first four statements can be considered as a reformulation. The fifth statement points out an obvious fact.

Corollary 3.5. *Let $\{P, \tau_P\}$ be a $\Pi_3^0(\tau)$ topological Hurewicz test pair as in Theorem 3.4. Then the following hold.*

1. *If $A \subseteq X$ is $\Pi_2^0(\tau)$ and $\tau_P^<$ -residual then A is τ_P -residual.*
2. *If $A \subseteq X$ is $\Sigma_3^0(\tau)$ and of $\tau_P^<$ -second category then A is of τ_P -second category.*
3. *If $A \subseteq X$ is $\Sigma_2^0(\tau)$ and of τ_P -second category then A is of $\tau_P^<$ -second category.*
4. *If $A \subseteq X$ is $\Pi_3^0(\tau)$ and τ_P -residual then A is $\tau_P^<$ -residual.*
5. *P is a proper $\Pi_3^0(\tau)$ set.*

Proof. Let A be $\Pi_2^0(\tau)$ and $\tau_P^<$ -residual. By Theorem 3.4.1, P is a $\tau_P^<$ -residual $\Pi_2^0(\tau_P^<)$ set so $A \cap P \neq \emptyset$ and $A \cap P$ is $\tau_P^<|_P$ -residual in P . By Claim 3.3.3 the topologies $\tau_P^<|_P$ and

$\tau_P|_P$ coincide, so we have that $A \cap P$ is $\tau_P|_P$ -residual in P . By Theorem 3.4.2, Definition 3.1.4 applies with $G = X$ and we conclude that A is τ_P -residual in X , as required.

For 2, let $A = \bigcup_{i < \omega} A_i$ with $\Pi_2^0(\tau)$ set A_i ($i < \omega$). As in the proof of 1, if A is of $\tau_P^<$ -second category then there exists an $i < \omega$ such that $A_i \cap P$ is $\tau_P^<|_P$ -residual in $G \cap P$ for some basic $\tau_P^<$ -open set G with $G \cap P \neq \emptyset$. Since by Claim 3.3.3 the topologies $\tau_P^<|_P$ and $\tau_P|_P$ coincide, we have that $A_i \cap P$ is $\tau_P|_P$ -residual in $G \cap P$. Thus G is a basic τ_P -open set with $G \cap P \neq \emptyset$, by Theorem 3.4.2, Definition 3.1.4 applies and gives that A_i is τ_P -residual in G . Thus A is of τ_P -second category, as required.

Statements 3 and 4 follow from 1 and 2 by taking complements.

Finally suppose that P is $\Sigma_3^0(\tau)$. Since P is $\tau_P^<$ -residual by Theorem 3.4.1, from 2 we get that P is of τ_P -second category. But P is τ_P -nowhere dense by Definition 3.1.3. This contradiction completes the proof. ■

We prove here another, a bit more technical lemma but of the same flavor as Corollary 3.5.

Lemma 3.6. *Let $\{P, \tau_P\}$ be a $\Pi_3^0(\tau)$ topological Hurewicz test pair as in Theorem 3.4. Let $A \subseteq X$ be a τ_P -meager $\Pi_2^0(\tau)$ set. If G is a basic $\tau[\mathcal{P}_n]$ -open but not $\tau[\mathcal{P}_{n-1}]$ -open set for some $n < \omega$, put*

$$Z_G^{X,P}(A) = cl_{\tau[\mathcal{P}_n]}(A \cap G).$$

Then $Z_G^{X,P}(A)$ is $\tau[\mathcal{P}_n]$ -meager in G .

Proof. Suppose that $Z_G^{X,P}(A)$ is of $\tau[\mathcal{P}_n]$ -second category in G . Now the $\Sigma_2^0(\tau)$ set P^n is $\tau[\mathcal{P}_n]$ -meager in G , so we have that A is of $\tau[\mathcal{P}_n]$ -second category in $G \setminus P^n$. Since G is not $\tau[\mathcal{P}_{n-1}]$ -open and $P^j \subseteq P^i$ ($i \leq j < \omega$), we have that $G \setminus P^n = G \cap (\bigcap_{i < n} P^i) \setminus \bigcup_{n \leq j < \omega} P^j$. Thus from Claim 3.3.5 we obtain that the topologies $\tau_P|_{G \setminus P^n}$ and $\tau[\mathcal{P}_n]|_{G \setminus P^n}$ coincide. So A is of τ_P -second category in the τ_P -open $G \setminus P^n$, a contradiction. This completes the proof. ■

Theorem 1.2 says in particular that in (C, τ_C) from the point of view of Hurewicz tests each proper $\Pi_3^0(\tau_C)$ set is the same, up to passing to a perfect subset of C . Precisely we have the following.

Corollary 3.7. *Let $P, P' \subseteq C$ be proper $\Pi_3^0(\tau_C)$ sets. Then there are continuous one-to-one maps $\varphi, \varphi': C \rightarrow C$ such that $\varphi^{-1}(P) = P'$ and $\varphi'^{-1}(P') = P$.*

Theorem 3.4 shows that a fairly big family of $\Pi_3^0(\tau_C)$ sets P form a topological Hurewicz test with the natural refinement τ_P of τ but in consequence the reasoning is at least notationally complicated. So in view of Corollary 3.7 we can fix one special $\Pi_3^0(\tau_C)$ set for which the topologies $\tau_P^<$ and τ_P become simple without losing too much generality. Since the method we apply had already been used by Lusin to build (probably the first) proper $\Pi_3^0(\tau_C)$ set, we denote it by P_L (see also [9]). We carry out the construction in an arbitrary perfect Polish space (X, τ) .

For every finite sequence $s \in \omega^{<\omega}$, fix a nonempty perfect set $P_s \subseteq X$ with the following properties:

- (3.1) $P_\emptyset = X$;
- (3.2) $P_{s \frown i} \cap P_{s \frown j} = \emptyset$ ($s \in \omega^{<\omega}$, $i < j < \omega$);
- (3.3) $P_s \subseteq P_t$ and P_s is $\tau|_{P_t}$ -nowhere dense in $(P_t, \tau|_{P_t})$ ($t \subset s \in \omega^{<\omega}$);
- (3.4) $\bigcup_{i < \omega} P_{s \frown i}$ is $\tau|_{P_s}$ -dense in $(P_s, \tau|_{P_s})$ ($s \in \omega^{<\omega}$).

To have $P_{s \frown i} \subseteq P_s$ ($i < \omega$), one simply has to take a countable dense subset $D_s = \{d_1, d_2, \dots\} \subseteq P_s$ and cover successively every d_i with a perfect set $P_{s \frown i}$ which is nowhere dense in $(P_s, \tau|_{P_s})$ and disjoint from $P_{s \frown j}$ ($j < i$). Then (3.1-3.4) obviously hold. Once this done, let

$$(3.5) \quad \mathcal{P}_n = \{P_s : s \in \omega^{<\omega}, |s| < n\} \quad (n \leq \omega),$$

$$P^n = \bigcup_{\substack{s \in \omega^{<\omega} \\ |s| = n}} P_s \quad (n < \omega), \quad P_L = \bigcap_{n < \omega} P^n.$$

Observe that this notation is in accordance with Definition 3.2. From now on if we take a P_L in X then we assume that the above construction has been carried out, such that (3.1-3.5) hold. Observe also that the conditions of Theorem 3.4 are satisfied: $P^j \subseteq P^i$ ($i \leq j < \omega$) by (3.3), P^i is $\tau[\mathcal{P}_i]$ -dense in X by (3.4) and $\tau[\mathcal{P}_i]$ -meager in X by (3.3). So $\{P_L, \tau_{P_L}\}$ is a topological Hurewicz test pair, in particular P_L is a proper $\Pi_3^0(\tau)$ set by Corollary 3.5.5. Finally we formulate a corollary of Lemma 3.6, which gets particularly simple for $\{P_L, \tau_{P_L}\}$. We fix a notation in advance.

Definition 3.8. For every $A \subseteq X$ set $Z_s^{X, P_L}(A) = cl_\tau(A \cap P_s)$ ($s \in \omega^{<\omega}$) and

$$Z^{X, P_L}(A) = P_L \cup \bigcup_{s \in \omega^{<\omega}} Z_s^{X, P_L}(A) \setminus P^{|s|+1}.$$

Corollary 3.9. The topologies $\tau|_{P_s}$ and $\tau[\mathcal{P}_{|s|+1}]|_{P_s}$ coincide ($s \in \omega^{<\omega}$). A basic $\tau_{P_L}^<$ -open set is of the form $G \cap P_s$ where G basic τ -open and $s \in \omega^{<\omega}$.

Let $A \subseteq X$ be a τ_{P_L} -meager $\Pi_2^0(\tau)$ set. Then

1. $Z_s^{X, P_L}(A) = Z_{P_s}^{X, P_L}(A)$ ($s \in \omega^{<\omega}$);
2. $Z_s^{X, P_L}(A)$ is $\tau|_{P_s}$ -nowhere dense in P_s ($s \in \omega^{<\omega}$);
3. $X \setminus Z^{X, P_L}(A)$ is $\tau_{P_L}^<$ -dense in X .

Proof. The coincidence of the topologies and the structure of basic $\tau_{P_L}^<$ -open sets follow from (3.2) and (3.3). This gives

$$Z_s^{X,P_L}(A) = \text{cl}_\tau(A \cap P_s) = \text{cl}_{\tau[\mathcal{P}_{|s|+1}]}(A \cap P_s) = Z_{P_s}^{X,P_L}(A) \quad (s \in \omega^{<\omega}),$$

which is 1.

By Lemma 3.6, $Z_{P_s}^{X,P_L}(A)$ is $\tau[\mathcal{P}_{|s|+1}]|_{P_s}$ -meager in P_s ($s \in \omega^{<\omega}$), so since “meager” and “nowhere dense” coincide for $\Pi_2^0(\tau)$ sets, 2 follows from 1.

For 3, let G be a basic $\tau_{P_L}^<$ -open set, say $G = G' \cap P_s$ where G' is basic τ -open and $s \in \omega^{<\omega}$. By (3.2) and (3.3) we have $P_s \cap Z^{X,P_L}(A) \subseteq Z_s^{X,P_L}(A) \cup P^{|s|+1}$, thus

$$G \cap X \setminus Z^{X,P_L}(A) = G' \cap P_s \setminus (P_s \cap Z^{X,P_L}(A)) \supseteq G' \cap P_s \setminus (Z_s^{X,P_L}(A) \cup P^{|s|+1}).$$

Since $Z_s^{X,P_L}(A)$ is $\tau|_{P_s}$ -nowhere dense in P_s by 2, and $P^{|s|+1}$ is $\tau|_{P_s}$ -meager in P_s by (3.3), $G' \cap P_s \setminus (Z_s^{X,P_L}(A) \cup P^{|s|+1}) \neq \emptyset$, which completes the proof. ■

3.2 Generating ideals

The next step is to show that if (X, τ) is a Polish space and $P \subseteq X$ is a Borel but not $\Sigma_3^0(\tau)$ set then the σ -ideal generated by the $\Sigma_3^0(\tau)$ subsets of P can be covered by a σ -ideal \mathcal{I} strongly generated by some $\Pi_3^0(\tau)$ subsets of P such that $P \notin \mathcal{I}$. This is surprising because as we will see later there are $\Sigma_3^0(\tau_C)$ sets contained by P_L which can be covered only by $\tau_{P_L}|_{P_L}$ -residual $\Pi_3^0(\tau_C)$ subsets of P_L . We think that this is the best covering result for Σ_3^0 -generated ideals which can be obtained in ZFC. We will return to this question in the last chapter.

Theorem 3.10. *Let (X, τ) be an uncountable Polish space and $P \subseteq X$ be a proper $\Pi_3^0(\tau)$ set. Then there is a mapping $\Phi: \mathcal{S}_3^0(P) \rightarrow \mathcal{P}_3^0(P)$ such that $A \subseteq \Phi(A)$ and*

$$P \setminus \bigcup_{i < \omega} \Phi(A^i) \neq \emptyset \quad (A, A^i \in \mathcal{S}_3^0(P) \quad (i < \omega)).$$

Proof. First we construct $\Phi = \Phi_L$ for $(X, \tau) = (C, \tau_C)$ and $P = P_L$. For every $A \in \mathcal{S}_3^0(P_L)$ fix a presentation $A = \bigcup_{j < \omega} A_j$ where A_j is $\Pi_2^0(\tau_C)$ ($j < \omega$). Set

$$\Phi_s(A) = \bigcup_{j < |s|} \text{cl}_{\tau_C}(A_j \cap P_s) \quad (s \in \omega^{<\omega}),$$

$$\Phi_n(A) = \bigcup_{|s|=n} \Phi_s(A) \quad (n < \omega), \quad \Phi_L(A) = \bigcap_{m < \omega} \bigcup_{m \leq n < \omega} \Phi_n(A).$$

It is clear that $\Phi_L(A)$ is $\Pi_3^0(\tau_C)$; $\Phi_n(A) \subseteq P^n \subseteq P^m$ ($m \leq n < \omega$) shows $\Phi_L(A) \subseteq P_L$. Since $A \subseteq P_L \subseteq P^n$ implies $A_j \subseteq \Phi_n(A)$ for $j < n < \omega$, we also have $A \subseteq \Phi_L(A)$. It

remains to show that if $A^i \in \mathcal{S}_3^0(P_L)$ with its fixed presentation $A^i = \bigcup_{j < \omega} A_j^i$ ($i < \omega$) then we can find a point in $P_L \setminus \bigcup_{i < \omega} \Phi_L(A^i)$. We do this by constructing inductively a sequence $s_n \in \omega^{<\omega}$, $|s_n| = n$ ($n < \omega$) and a basic τ_C -open set G_n ($n < \omega$) such that

$$(3.6) \quad s_n \subset s_{n+1} \quad (n < \omega);$$

$$(3.7) \quad G_n \cap P_{s_n} \neq \emptyset \quad (n < \omega);$$

$$(3.8) \quad \text{cl}_{\tau_C}(G_{n+1}) \subseteq G_n \quad (n < \omega);$$

$$(3.9) \quad G_{n+1} \cap P_{s_{n+1}} \subseteq C \setminus \bigcup_{i \leq n} \Phi_n(A^i) \quad (n < \omega).$$

Then by (3.6), (3.7) and (3.8) we have $\bigcap_{n < \omega} G_n \cap P_{s_n} \neq \emptyset$ and $\bigcap_{n < \omega} G_n \cap P_{s_n} \subseteq P_L$, so (3.9) gives

$$\bigcap_{n < \omega} G_n \cap P_{s_n} \subseteq P_L \setminus \bigcup_{i < \omega} \Phi_L(A^i).$$

It remains to make the construction. Set $s_0 = \emptyset$, $G_0 = C$. Suppose that s_n, G_n are already defined; we find our s_{n+1}, G_{n+1} . By (3.2), $\Phi_n(A^i) \cap P_{s_n} = \Phi_{s_n}(A^i)$ ($i < \omega$). First we obtain a basic τ_C -open set $G \subseteq G_n$ such that $G \cap P_{s_n} \neq \emptyset$ and $G \cap P_{s_n} \cap \Phi_{s_n}(A^i) = \emptyset$ ($i \leq n$). For this we show that $\Phi_{s_n}(A^i)$ ($i \leq n$) is $\tau_C|_{P_{s_n}}$ -nowhere dense in P_{s_n} .

We have $\Phi_{s_n}(A^i) = \bigcup_{j < |s_n|} \text{cl}_{\tau_C}(A_j^i \cap P_{s_n})$ ($i \leq n$). Since $A_j^i \subseteq P_L$, A_j^i is a τ_{P_L} -meager $\Pi_2^0(\tau_C)$ set ($i \leq n, j < |s_n|$). Hence by Corollary 3.9.2, $\text{cl}_{\tau_C}(A_j^i \cap P_{s_n})$ ($i \leq n, j < |s_n|$) is $\tau_C|_{P_{s_n}}$ -nowhere dense in P_{s_n} . So $\Phi_{s_n}(A^i)$ ($i \leq n$) is indeed $\tau_C|_{P_{s_n}}$ -nowhere dense in P_{s_n} .

We obtained that there is a basic τ_C -open set $G \subseteq G_n$ which satisfies $G \cap P_{s_n} \neq \emptyset$ and $G \cap P_{s_n} \subseteq C \setminus \bigcup_{i \leq n} \Phi_n(A^i)$. We can pass to a basic τ_C -open subset $G_{n+1} \subseteq \text{cl}_{\tau_C}(G) \subseteq G$ such that $G_{n+1} \cap P_{s_n} \neq \emptyset$; then we have (3.8). By (3.4) we can find an $s_n \subset s_{n+1} \in \omega^{<\omega}$, $|s_{n+1}| = n + 1$ with $P_{s_{n+1}} \cap G_{n+1} \neq \emptyset$, thus (3.6), (3.7) and (3.9) hold. This choice completes the inductive step and the proof of the special case.

If (X, τ) and P are arbitrary, by Theorem 1.2 we can take a continuous one-to-one map $\varphi: (C, \tau_C) \rightarrow (X, \tau)$ such that $\varphi^{-1}(P) = P_L$. For $A \in \mathcal{S}_3^0(P)$ let

$$\Phi(A) = (P \setminus \varphi(P_L)) \cup \varphi(\Phi_L(\varphi^{-1}(A))).$$

Since homeomorphism preserves the Borel class of sets, this definition makes sense and fulfills the requirements. ■

One may say that the construction in Theorem 3.10 is trivial if the $\Sigma_3^0(\tau_C)$ subsets of P_L (which are all $\tau_{P_L}|_{P_L}$ -meager by Corollary 3.5.2) could be covered by a $\tau_{P_L}|_{P_L}$ -meager $\Pi_3^0(\tau_C)$ subset of P_L . Then a category argument would give that

$$P_L \setminus \bigcup_{i < \omega} \Phi(A^i) \neq \emptyset \quad (A, A^i \in \mathcal{S}_3^0(P) \quad (i < \omega)).$$

However, this is not the case. The construction in the following claim has already been used by S. Solecki in [21] to prove Theorem 1.4.

Claim 3.11. *There is a $\Sigma_3^0(\tau_C)$ set $A \subseteq P_L$ such that if B is $\Pi_3^0(\tau_C)$ and $A \subseteq B \subseteq P_L$ then B is $\tau_{P_L}|_{P_L}$ -residual in P_L .*

Proof. In this proof $B(x, \varepsilon)$ denotes the τ_C -open ball centered at $x \in C$ with radius $\varepsilon > 0$. Fix a $v \in \omega^{<\omega} \setminus \{\emptyset\}$ and a basic τ_C -open set V satisfying $V \cap P_v \neq \emptyset$. We define an injection $\varphi: \omega^{<\omega} \rightarrow \omega^{<\omega}$, a map $j: \omega^{<\omega} \rightarrow \omega$ and basic τ_C -open sets $(U_t)_{t \in \omega^{<\omega}}$ with the following properties.

$$(3.10) \quad U_\emptyset = V, \quad \varphi(\emptyset) = v;$$

$$(3.11) \quad s \subseteq t \implies \varphi(s) \subseteq \varphi(t);$$

$$(3.12) \quad U_s \subseteq \text{cl}_{\tau_C}(U_s) \subseteq U_t \setminus P_{\varphi(t) \frown j(t)} \quad (s, t \in \omega^{<\omega}, t \subset s);$$

$$(3.13) \quad U_s \cap U_t = \emptyset \quad (s, t \in \omega^{<\omega}, |t| = |s|);$$

$$(3.14) \quad \text{diam}_{\tau_C}(U_s) \leq 2^{-|s|} \quad (s \in \omega^{<\omega});$$

$$(3.15) \quad U_s \cap P_{\varphi(s) \frown j(s)} \neq \emptyset \quad (s \in \omega^{<\omega});$$

$$(3.16) \quad \forall x \in U_s \cap P_{\varphi(s) \frown j(s)} \quad \forall \varepsilon > 0 \quad \exists i < \omega \quad (U_{s \frown i} \subseteq B(x, \varepsilon)) \quad (s \in \omega^{<\omega}).$$

Suppose that the construction is done. Let $A_{v,V} = \bigcap_{n < \omega} \bigcup_{s \in \omega^{<\omega}, |s|=n} U_s$. This set is $\Pi_2^0(\tau_C)$, we show that $A_{v,V} \subseteq P_L$. By (3.13) we have $A_{v,V} = \bigcup_{\sigma \in \omega^\omega} \bigcap_{n < \omega} U_{\sigma|_n}$. By (3.14), (3.15) and (3.3) we have

$$\bigcap_{n < \omega} U_{\sigma|_n} = \bigcap_{n < \omega} (U_{\sigma|_n} \cap P_{\varphi(\sigma|_n)}) \in P_L \quad (\sigma \in \omega^\omega)$$

so indeed $A_{v,V} \subseteq P_L$.

Next we show that $U_s \cap P_{\varphi(s) \frown j(s)} \subseteq \text{cl}_{\tau_C}(U_s \cap A_{v,V})$ ($s \in \omega^{<\omega}$). Let $x \in U_s \cap P_{\varphi(s) \frown j(s)}$. By (3.16) there is a sequence $(i_l)_{l < \omega} \subseteq \omega$ such that $U_{s \frown i_l} \subseteq B(x, 1/l)$. By (3.12) and (3.14) we have $A_{v,V} \cap U_s \neq \emptyset$ ($s \in \omega^{<\omega}$), in particular $A_{v,V} \cap U_{s \frown i_l} \neq \emptyset$ ($l < \omega$) so indeed $x \in \text{cl}_{\tau_C}(U_s \cap A_{v,V})$.

Finally we show that every $\Sigma_2^0(\tau_C)$ set $H \subseteq C$ for which $A_{v,V} \subseteq H$ is of $\tau_{P_L}^<$ -second category in $V \cap P_v$. Let $H = \bigcup_{i < \omega} H_i$ where H_i is $\Pi_1^0(\tau_C)$ ($i < \omega$). By the Baire Category Theorem in $A_{v,V}$, there is an $i < \omega$ such that $H_i \cap A_{v,V}$ is of $\tau_C|_{A_{v,V}}$ -second category in $A_{v,V}$, say H_i contains a basic $\tau_C|_{A_{v,V}}$ -open set $G \cap A_{v,V}$ where G is basic τ_C -open. By (3.14) there is an $s \in \omega^{<\omega}$ such that $U_s \subseteq G$. By (3.15) we have $U_s \cap P_{\varphi(s) \frown j(s)} \neq \emptyset$. As we have seen above,

$$U_s \cap P_{\varphi(s) \frown j(s)} \subseteq \text{cl}_{\tau_C}(U_s \cap A_{v,V}) \subseteq \text{cl}_{\tau_C}(G \cap A_{v,V}) \subseteq H_i.$$

Since $P_{\varphi(s) \frown j(s)} \subseteq P_v$ by (3.10) and (3.11), H_i and thus H is indeed of $\tau_{P_L}^<$ -second category in $V \cap P_v$.

To have A , fix an enumeration $\{(v_i, V_i) : i < \omega\}$ of the pairs (v, V) where $v \in \omega^{<\omega}$, V is basic τ_C -open and $V \cap P_v \neq \emptyset$. Set $A = \bigcup_{i < \omega} A_{v_i, V_i}$. If B is a $\Pi_3^0(\tau_C)$ set containing A , say

$B = \bigcap_{i < \omega} B_i$ with $B_i \in \Sigma_2^0(\tau_C)$ then as we have shown above, B_i ($i < \omega$) is of $\tau_{P_L}^<$ -second category in every nonempty $\tau_{P_L}^<$ -open set. So B_i ($i < \omega$) and hence B is $\tau_{P_L}^<$ -residual. Now if $B \subseteq P_L$ then B is $\tau_{P_L}^<|_{P_L}$ residual in P_L , by Claim 3.3.3 the topologies $\tau_{P_L}^<|_{P_L}$ and $\tau_{P_L}|_{P_L}$ coincide, so B is $\tau_{P_L}|_{P_L}$ -residual in P_L , as stated.

So it remains to make the construction; we do this recursively. Set $U_\emptyset = V$, $\varphi(\emptyset) = v$ and let $j(\emptyset) < \omega$ be such that $V \cap P_{v \frown j(\emptyset)} \neq \emptyset$; such a $j(\emptyset)$ exists by (3.4). So we have (3.10-3.15) for $s = \emptyset$. Suppose that we already have U_s , $\varphi(s)$ and $j(s)$ for $s \in \omega^{<\omega}$, $|s| \leq n$ such that (3.11-3.15) hold for $|s|, |t| \leq n$ and (3.16) holds for $|s| < n$. For every $s \in \omega^{<\omega}$ with $|s| = n$, using (3.3), we can fix a countable set $D_s = \{d_s(i) : i < \omega\} \subseteq U_s \cap P_{\varphi(s)} \setminus P_{\varphi(s) \frown j(s)}$ with the property that $\text{cl}_{\tau_C}(D) = D \cup \text{cl}_{\tau_C}(U_s \cap P_{\varphi(s) \frown j(s)})$. Then we can find a basic τ_C -open neighborhood $U_{s \frown i}$ of $d_s(i)$ such that

$$U_{s \frown i} \cap P_{\varphi(s)} \neq \emptyset, \text{cl}_{\tau_C}(U_{s \frown i}) \subseteq U_s \setminus P_{\varphi(s) \frown j(s)}, U_{s \frown i} \cap U_{s \frown j} = \emptyset \quad (i, j < \omega, i \neq j)$$

and $\text{diam}_{\tau_C}(U_{s \frown i}) \leq 2^{n+1+i}$ ($i < \omega$). Then (3.12-3.14) hold for $|s| \leq n+1$ and (3.16) holds for $|s| \leq n$. Define $\varphi(s) \subset \varphi(s \frown i)$ to have $P_{\varphi(s \frown i)} \cap U_{s \frown i} \neq \emptyset$ ($i < \omega$); this is possible by (3.4). Then (3.11) holds for $|s|, |t| \leq n+1$. Again by (3.4), we can have $j(s \frown i) < \omega$ such that $P_{\varphi(s \frown i) \frown j(s \frown i)} \cap U_{s \frown i} \neq \emptyset$ ($i < \omega$). Then also (3.15) holds for $|s| \leq n+1$. This completes the recursive step and the proof. ■

By assuming the Continuum Hypothesis we can cover the $\Sigma_3^0(\tau_C)$ subsets of P_L by $\Pi_2^0(\tau_C)$ sets such that P_L is not in the σ -ideal generated by the covering sets. Combining Theorem 3.4 with the previous example we see that in this case the covering $\Pi_2^0(\tau_C)$ sets cannot be the subsets of P_L : a $\Pi_2^0(\tau_C)$ subset of P_L should be $\tau_{P_L}|_{P_L}$ -meager in P_L while we have constructed a $\Sigma_3^0(\tau_C)$ subset of P_L which can be covered only by $\tau_{P_L}|_{P_L}$ -residual $\Pi_3^0(\tau_C)$ subsets of P_L . For this covering by $\Pi_2^0(\tau)$ sets, Theorem 1.2 is too ‘‘imprecise’’ in the sense that it does not care about the presentation of $\Pi_\xi^0(\tau)$ sets. We prove an alternative for $\xi = 3$.

Lemma 3.12. *Let (X, τ) be a Polish space, $(P_s)_{s \in \omega^{<\omega}}$ be a family of $\Pi_1^0(\tau)$ subsets of X satisfying (3.1-3.4). Let \mathcal{P}_n, P^n ($n < \omega$) and P_L be as in (3.5). Let A be a fixed τ_{P_L} -meager $\Pi_2^0(\tau)$ set. If $x^* \in X \setminus Z^{X, P_L}(A)$ and $U \subseteq X$ is basic τ -open with $x^* \in U$ then there is a τ -compact set $F \subseteq X$ such that $x^* \in F \subseteq U \setminus A$ and $(F \cap P_s)_{s \in \omega^{<\omega}}$ satisfies (3.1-3.4) in the Polish space $(F, \tau|_F)$.*

Proof. Let $\mathcal{G} = \{G_n : n < \omega\}$ be an enumeration of the sets of the form $G \cap P_s$ ($s \in \omega^{<\omega}, G$ basic τ -open). Let $r \in \omega^{<\omega}$ be such that $x^* \in P_r \setminus P^{|r|+1}$. We define recursively a tree $T \subseteq \omega^{<\omega}$ and $\mathcal{T} = \{(V_s, X_s) : s \in T\}$ such that $T \cap \omega^n$ ($n < \omega$) is finite, $V_s \subseteq X$ ($s \in T$) is a basic τ -open set, $X_s = \{x_{s \frown i} : i < \omega, s \frown i \in T\} \subseteq X \setminus P_L$ ($s \in T$) is

a finite set, and with $V^n = \bigcup_{s \in T, |s|=n} V_s$, $X^n = \bigcup_{s \in T, |s|=n} X_s$ ($n < \omega$) the following hold:

$$(3.17) \quad x^* \in X^n \subseteq X^{n+1} \quad (n < \omega);$$

$$(3.18) \quad \text{cl}_\tau(V_{s \smallfrown i}) \subseteq V_s \subseteq U \quad (i < \omega, s, s \smallfrown i \in T);$$

$$(3.19) \quad V_s \cap V_t = \emptyset \quad (s, t \in T, |s| = |t|);$$

$$(3.20) \quad \text{diam}_\tau(V_s) < 2^{-|s|} \quad (s \in T);$$

$$(3.21) \quad V_{s \smallfrown i} \cap X_s = \{x_{s \smallfrown i}\} \quad (i < \omega, s \smallfrown i \in T);$$

$$(3.22) \quad \text{cl}_\tau(G_n) \cap X^n = \emptyset \text{ implies } G_n \cap V^{n+1} = \emptyset \quad (n < \omega);$$

$$(3.23) \quad \text{if } i \leq |r| \text{ is maximal such that } x_s \in P_{r|i} \text{ then } V_s \cap \text{cl}_\tau(P_{r|i} \cap A) = \emptyset \quad (s \in T);$$

$$(3.24) \quad \text{if } x_s \in P_t \text{ for some } s \in T, t \in \omega^{<\omega} \text{ then } X_s \cap V_s \cap (P_{t|m} \setminus P^{m+1}) \neq \emptyset \quad (m < |t|);$$

$$(3.25) \quad \text{if } x_s \in P_t \text{ for some } s \in T, t \in \omega^{<\omega} \text{ then } X_s \cap V_s \cap \bigcup_{i < \omega} P_{t \smallfrown i} \neq \emptyset.$$

Suppose that the construction is done, we show that

$$F = \bigcap_{n < \omega} V^n = \text{cl}_\tau\left(\bigcup_{n < \omega} X^n\right)$$

fulfills the requirements; here equality follows from (3.18), (3.20) and (3.21). By (3.18), (3.19) and (3.20), F is compact, (3.17) and (3.18) imply that $x^* \in F \subseteq U$.

We show that $F \cap A = \emptyset$. If $x \in F$ then by (3.19) and (3.20) there is a unique $\sigma \in \omega^\omega$ such that $x = \bigcap_{n < \omega} V_{\sigma|n}$. Let $i \leq |r|$ be maximal for which $x_{\sigma|n} \in P_{r|i}$ for infinitely many $n < \omega$. By (3.21) we have $\lim_{n \rightarrow \infty} x_{\sigma|n} = x$ so $x \in P_{r|i}$. Fix an $n < \omega$ such that i is maximal for which $x_{\sigma|n} \in P_{r|i}$. Then by (3.23), $V_{\sigma|n} \cap \text{cl}_\tau(P_{r|i} \cap A) = \emptyset$, in particular $x \in P_{r|i} \setminus A$, as stated.

Now we prove that (3.1-3.4) holds for $(F \cap P_s)_{s \in \omega^{<\omega}}$ in the Polish space $(F, \tau|_F)$. Since (3.1) and (3.2) are automatic we only have to check (3.3) and (3.4).

First we show that $F \cap P_s = \text{cl}_\tau(\bigcup_{n < \omega} X^n \cap P_s)$. It is clear that $F \cap P_s \supseteq \text{cl}_\tau(\bigcup_{n < \omega} X^n \cap P_s)$. For the reverse containment it is enough to show that whenever $G \cap F \cap P_s \neq \emptyset$ for some basic τ -open set G then $G \cap X^n \cap P_s \neq \emptyset$ for some $n < \omega$; so let $G \cap F \cap P_s \neq \emptyset$. By regularity we can find another basic τ -open set $G' \subseteq \text{cl}_\tau(G') \subseteq G$ such that $G' \cap F \cap P_s \neq \emptyset$. Let $G_n = G' \cap P_s$. Since $G_n \cap V^{n+1} \neq \emptyset$, by (3.22) we have $\text{cl}_\tau(G_n) \cap X^n \neq \emptyset$ hence $G \cap X^n \cap P_s \neq \emptyset$, which proves the statement.

For (3.3), let $s \subset t$ and suppose that $G \cap F \cap P_t \neq \emptyset$ for some basic τ -open set G . We have to show that $G \cap F \cap P_s \setminus P_t \neq \emptyset$. By the preceding, $x_u \in G \cap P_t$ for some $u \in T$, so by (3.17), (3.18), (3.20) and (3.21), $x_u = x_v \in V_v \subseteq G$ for some $u \subseteq v \in T$. Then by (3.24) for $m = |s| < |t|$ we get $X_v \cap V_v \cap (P_{t|m} \setminus P^{m+1}) \neq \emptyset$. Since $P_t \subseteq P^{m+1}$, $X_v \subseteq F$ and $V_v \subseteq G$, we have $G \cap F \cap (P_s \setminus P_t) \neq \emptyset$, as required.

For (3.4), suppose that $G \cap F \cap P_t \neq \emptyset$ for some basic τ -open set G and $t \in \omega^{<\omega}$. We have to find some $i < \omega$ such that $G \cap F \cap P_{t \smallfrown i} \neq \emptyset$. Let $x_u \in G \cap P_t$; as above, we have

a $u \subseteq v \in T$ with $x_u = x_v \in V_v \subseteq G$. Then by (3.25), $X_v \cap V_v \cap \bigcup_{i < \omega} P_{t \smallfrown i} \neq \emptyset$, so from $V_v \subseteq G$ and $X_v \subseteq F$ we have $G \cap F \cap \bigcup_{i < \omega} P_{t \smallfrown i} \neq \emptyset$ and we are done.

We do the construction recursively such that after the N^{th} step of the recursion (3.17-3.23) hold for $n \leq N$ while (3.24) and (3.25) hold for $x_s \in X^{N-1}$. Put $\emptyset \in T$, set $x_\emptyset = x^*$ and let $x^* \in V_\emptyset \subseteq U$ be a basic τ -open set such that $V_\emptyset \cap \text{cl}_\tau(P_r \cap A) = \emptyset$; this choice is possible since $x^* \notin Z^{X, P_L}(A)$, which means that $x^* \notin \text{cl}_\tau(P_r \cap A)$. So x_\emptyset, V_\emptyset meet the requirements.

Suppose that $T \cap \omega^N$ is defined, we have X^{N-1} and V^N such that (3.17-3.23) hold for $n \leq N$, (3.24) and (3.25) hold for $x_s \in X^{N-1}$. We extend T up to level $N+1$ and define X^N, V^{N+1} such that (3.17-3.23) hold for $n \leq N+1$, (3.24) and (3.25) hold for $x_s \in X^N$ as follows. Let $s \in T, |s| = N$ be arbitrary; since $x_s \notin P_L$, to satisfy (3.24) and (3.25) for x_s we have to take only finitely many points. We show that we can pick each of these points x in V_s such that

$$(3.26) \quad \text{if } i \leq |r| \text{ is maximal for which } x \in P_{r|i} \text{ then } x \notin \text{cl}_\tau(P_{r|i} \cap A).$$

Let $x_s \in P_t$ for some $t \in \omega^{<\omega}$. For (3.24) let $m < |t|$, and let $i \leq |r|$ be maximal such that $P_{t|m} \subseteq P_{r|i}$. We distinguish two cases. If $P_{t|m} = P_{r|i}$ then by Corollary 3.9.2, $\text{cl}_\tau(P_{t|m} \cap A)$ is $\tau|_{P_{t|m}}$ -nowhere dense in $P_{t|m}$ so by (3.3) we can pick a point in $V_s \cap P_{t|m} \setminus (P^{m+1} \cup \text{cl}_\tau(P_{t|m} \cap A))$. If $P_{t|m} \subset P_{r|i}$ then by (3.2) for every $y \in P_{t|m}$, i is the maximal for which $y \in P_{r|i}$. Since $x_s \in P_{t|m}$, we have $V_s \cap \text{cl}_\tau(P_{r|i} \cap A) = \emptyset$ by (3.23). By (3.3) we can pick a point x in $V_s \cap P_{t|m} \setminus P^{m+1}$, such that $x \notin \text{cl}_\tau(P_{r|i} \cap A)$ follows from $x \in V_s$.

For (3.25), if $x_s \in P^{|t|+1}$ then $x_s \in X_s$ shows (3.25). So suppose that $x_s \notin P^{|t|+1}$. By (3.4) we have a $j < \omega$ such that $V_s \cap P_{t \smallfrown j} \neq \emptyset$. By Corollary 3.9.2, $\text{cl}_\tau(P_{t \smallfrown j} \cap A)$ is $\tau|_{P_{t \smallfrown j}}$ -nowhere dense in $P_{t \smallfrown j}$ so we can pick a point x in $V_s \cap P_{t \smallfrown j} \setminus \text{cl}_\tau(P_{t \smallfrown j} \cap A)$. Let $i \leq |r|$ be maximal for which $x \in P_{r|i}$. If $r|i = t \smallfrown j$ then we have $x \notin \text{cl}_\tau(P_{r|i} \cap A)$. If $r|i \neq t \smallfrown j$ then $P_t \subseteq P_{r|i}$. We show that for x_s , as well, i is the maximal such that $x_s \in P_{r|i}$. If $P_t = P_{r|i}$ then this follows from $x \notin P^{|t|+1}$ while if $P_t \subset P_{r|i}$ then we are done by the maximality of i for x . Thus by (3.23), $V_s \cap \text{cl}_\tau(P_{r|i} \cap A) = \emptyset$ so $x \in V_s$ gives $x \notin \text{cl}_\tau(P_{r|i} \cap A)$.

Index x_s and the new points with $s \smallfrown i$ ($i < n_s$) for some $n_s < \omega$, put $s \smallfrown i \in T$ ($i < n_s$) and take pairwise disjoint basic τ -open sets $x_{s \smallfrown i} \in V_{s \smallfrown i} \subseteq \text{cl}_\tau(V_{s \smallfrown i}) \subseteq V_s$ such that (3.19), (3.20), (3.22) and (3.23) hold; observe that (3.23) can be satisfied by (3.26). Since (3.17), (3.18), (3.21), (3.24) and (3.25) hold, this completes the recursive step and the proof. ■

Instead of covering $\Sigma_3^0(\tau)$ sets by $\Pi_2^0(\tau)$ sets we define $\Sigma_2^0(\tau)$ sets avoiding them. The sequence of $\Sigma_2^0(\tau)$ sets constructed for this must be specially nested.

Definition 3.13. Let (X, τ) be a Polish space and consider P_L in X . Let $0 < \zeta < \omega_1$. Set $\mathcal{B}^0 = \{X\}$, while for $0 < \alpha < \zeta$, let $\mathcal{B}^\alpha = \{B_i^\alpha : i < \omega\}$ be a collection of pairwise disjoint $\Pi_1^0(\tau)$ subsets of X . Set $B^\alpha = \bigcup_{i < \omega} B_i^\alpha$ ($\alpha < \zeta$), $B_\zeta = \bigcap_{\alpha < \zeta} B^\alpha$. Let $\tau[\alpha] = \tau[\bigcup_{\theta < \alpha} \mathcal{B}^\theta]$, $\tau_{P_L}^\leq[\alpha] = \tau_{P_L}^\leq[\bigcup_{\theta < \alpha} \mathcal{B}^\theta]$ and $\tau_{P_L}[\alpha] = \tau_{P_L}[\bigcup_{\theta < \alpha} \mathcal{B}^\theta]$ ($\alpha \leq \zeta$). We say that $(\mathcal{B}^\alpha)_{\alpha < \zeta}$ is P_L -nested if

1. $B^\beta \subseteq B^\alpha$ ($\alpha \leq \beta < \zeta$);
2. B^α is $\tau_{P_L}^<[\alpha]$ -dense in X ($\alpha < \zeta$);
3. B_i^α is $\tau[\alpha]$ -compact ($\alpha < \zeta$, $i < \omega$);
4. $(P_s \cap B_\alpha)_{s \in \omega^{<\omega}}$ satisfies (3.1-3.4) in the Polish space $(B_\alpha, \tau[\alpha]|_{B_\alpha})$ for every $\alpha \leq \zeta$ if ζ is successor and for $\alpha < \zeta$ if ζ is limit.

Since a compact Polish topology on a base set has no nontrivial compact Polish refinement, Definition 3.13.3 says in particular that on B_i^α the topologies $\tau[\beta]$ ($\beta \leq \alpha$) coincide. In the sequel we use this property without further reference. Next we show that Definition 3.13.4 holds for limit $\alpha = \zeta$, as well.

Lemma 3.14. *Let $(\mathcal{B}^\alpha)_{\alpha < \zeta}$ be a P_L -nested sequence for some $\zeta < \omega_1$ and suppose that $A \subseteq B_\zeta$ is $\Pi_2^0(\tau[\zeta]|_{B_\zeta})$ and $\tau_{P_L}[\zeta]|_{B_\zeta}$ -meager. Then*

1. B_ζ is $\tau_{P_L}^<[\zeta]$ -residual in X ;
2. if ζ is a limit ordinal then $(P_s \cap B_\zeta)_{s \in \omega^{<\omega}}$ satisfies (3.1-3.4) in the Polish space $(B_\zeta, \tau[\zeta]|_{B_\zeta})$;
3. $B_\zeta \setminus Z^{B_\zeta, P_L \cap B_\zeta}(A)$ is $\tau_{P_L}^<[\zeta]|_{B_\zeta}$ -dense in B_ζ ;
4. $P_L \cap B_\zeta \neq \emptyset$.

Proof. By Definition 3.13.1 and 2, B^α ($\alpha < \zeta$) is $\tau_{P_L}^<[\alpha]$ -dense and $\tau_{P_L}^<[\zeta]$ -open ($\alpha < \zeta$). Since a $\tau_{P_L}^<[\zeta]$ -open but not $\tau_{P_L}^<[\alpha]$ -open set is contained in B^α ($\alpha < \zeta$), B^α ($\alpha < \zeta$) is also $\tau_{P_L}^<[\zeta]$ -dense. Hence B^α ($\alpha < \zeta$) and so B_ζ is $\tau_{P_L}^<[\zeta]$ -residual in X , which is 1.

For 2, we have (3.1) and (3.2) automatically. To have (3.3), let $t \subset s$ and suppose that $G \cap P_s \neq \emptyset$ for some basic $\tau[\zeta]|_{B_\zeta}$ -open set. That is, $G = G' \cap B_n^\alpha \cap B_\zeta$ for some basic τ -open set G' , $\alpha < \zeta$ and $n < \omega$. But (3.3) holds in B_n^α , in particular $U = B_n^\alpha \cap G' \cap P_t \setminus P_s \neq \emptyset$. Now U is a $\tau_{P_L}^<[\zeta]$ -open set, so given that B_ζ is $\tau_{P_L}^<[\zeta]$ -residual by 1, we have $U \cap B_\zeta \neq \emptyset$, as required.

To see (3.4) fix an $s \in \omega^{<\omega}$ and a basic $\tau[\zeta]|_{B_\zeta}$ -open set G with $G \cap P_s \neq \emptyset$. As before, we have that $G = G' \cap B_n^\alpha \cap B_\zeta$ for some basic τ -open set G' , $\alpha < \zeta$ and $n < \omega$. Now (3.4) holds in B_n^α hence $B_n^\alpha \cap G' \cap \bigcup_{i < \omega} P_{s \frown i} \neq \emptyset$, say $U = B_n^\alpha \cap G' \cap P_{s \frown i} \neq \emptyset$ for some $i < \omega$. Again, U is a $\tau_{P_L}^<[\zeta]$ -open set, so given that B_ζ is $\tau_{P_L}^<[\zeta]$ -residual by 1, we have $U \cap B_\zeta \neq \emptyset$, which proves the statement.

We have 3 since for ζ limit by 2, for ζ successor by Definition 3.13.4, Corollary 3.9.3 holds in the Polish space $(B_\zeta, \tau[\zeta]|_{B_\zeta})$.

Finally by 2 for ζ limit and by Definition 3.13.4 for ζ successor, Theorem 3.4.1 applies and we get that $P_L \cap B_\zeta$ is $\tau_{P_L}^<[\zeta]|_{B_\zeta}$ -residual in B_ζ ; in particular, $P_L \cap B_\zeta \neq \emptyset$. This shows 4 and completes the proof. ■

Now we have everything to prove the ideal generation theorem. We remark that CH is used only to assure that the cardinality of the family of $\Pi_2^0(\tau)$ sets is ω_1 .

Theorem 3.15. *Assume CH. Let (X, τ) be a Polish space and $P \subseteq X$ be a Borel not $\Sigma_3^0(\tau)$ set. Then there is a σ -ideal \mathcal{I} such that*

1. \mathcal{I} is strongly generated by its $\Pi_2^0(\tau)$ members;
2. $\mathcal{S}_3^0(P) \subseteq \mathcal{I}$;
3. $P \notin \mathcal{I}$.

Proof. First we prove the special case $(X, \tau) = (C, \tau_C)$, $P = P_L$. Using CH, let $\{A_\alpha : \alpha < \omega_1\}$ be an enumeration of $\mathcal{P}_2^0(P_L)$ such that $A_0 = \emptyset$. We shall construct a P_L -nested sequence $(\mathcal{B}^\alpha)_{\alpha < \omega_1}$ such that

$$(3.27) \quad A_\alpha \cap B^\alpha = \emptyset \quad (\alpha < \omega_1).$$

Once this done set

$$\mathcal{I}_{P_L} = \{G \subseteq C : \exists \alpha < \omega_1 (B_\alpha \cap G = \emptyset)\},$$

or, since $B_{\alpha+1} \subseteq B^\alpha \subseteq B_\alpha$ ($\alpha < \omega_1$) by Definition 3.13.1, equivalently

$$\mathcal{I}_{P_L} = \{G \subseteq C : \exists \alpha < \omega_1 (B^\alpha \cap G = \emptyset)\}.$$

By Definition 3.13, \mathcal{I}_{P_L} is a σ -ideal. Also by Definition 3.13.1, \mathcal{I}_{P_L} is strongly generated by its $\Pi_2^0(\tau_C)$ members. By (3.27) it contains $\mathcal{P}_2^0(P_L)$, hence $\mathcal{S}_3^0(P_L)$, as well. Finally Lemma 3.14.4 implies that $P_L \notin \mathcal{I}$, as required.

It remains to make the construction. We proceed by induction; to start with, set $\mathcal{B}^0 = \{B_0^0\}$ with $B_0^0 = C$. Suppose that \mathcal{B}^α is defined for $\alpha < \zeta$ such that the sequence $(\mathcal{B}^\alpha)_{\alpha < \zeta}$ is P_L -nested and (3.27) hold. By Definition 3.13.4 if ζ is a successor and by Lemma 3.14.2 if ζ is limit, $\{P_L \cap B_\zeta, \tau_{P_L}[\zeta]|_{B_\zeta}\}$ is a topological Hurewicz test pair; in particular, $P_L \cap B_\zeta$ is $\tau_{P_L}[\zeta]|_{B_\zeta}$ meager in B_ζ . Since $A_\zeta \cap B_\zeta \subseteq P_L \cap B_\zeta$ and $A_\zeta \cap B_\zeta$ is a $\tau_{P_L}[\zeta]|_{B_\zeta}$ -meager $\Pi_2^0(\tau[\zeta]|_{B_\zeta})$ set. So by Lemma 3.14.3 we can fix a countable set

$$Y = \{x_n : n < \omega\} \subseteq B_\zeta \setminus Z^{B_\zeta, P_L \cap B_\zeta}(A_\zeta \cap B_\zeta)$$

such that Y is $\tau_{P_L}^\zeta[\zeta]|_{B_\zeta}$ -dense in the Polish space $(B_\zeta, \tau_{P_L}^\zeta[\zeta]|_{B_\zeta})$. We define recursively B_n^ζ ($n < \omega$) such that $\bigcup_{i < n} B_i^\zeta \neq B_\zeta$ ($n < \omega$), $(\mathcal{B}^\alpha)_{\alpha < \zeta+1}$ is P_L -nested and (3.27) hold.

Suppose that we already have $B_i^\zeta \subseteq B_\zeta$ for $i < n$. Let $m < \omega$ be minimal with $x_m \notin \bigcup_{i < n} B_i^\zeta$ and take a basic $\tau_C[\zeta]|_{B_\zeta}$ -open set $U \subseteq B_\zeta$ such that $x = x_m \in U \subseteq B_\zeta \setminus \bigcup_{i < n} B_i^\zeta$ and $U \cup \bigcup_{i < n} B_i^\zeta \neq B_\zeta$. By Definition 3.13.4 if ζ is successor and by Lemma 3.14.2 if ζ is limit, we can apply Lemma 3.12 in the Polish space $(X, \tau) = (B_\zeta, \tau_C[\zeta]|_{B_\zeta})$ for $x^* = x$,

U , $(P_s \cap B_\zeta)_{s \in \omega^{<\omega}}$ and $A_\zeta \cap B_\zeta$. Let $B_n^\zeta \subseteq B_\zeta$ be the resulting $\tau_C[\zeta]|_{B_\zeta}$ -compact set. This defines recursively \mathcal{B}^ζ .

We have $B^\zeta \cap A_\zeta = \emptyset$ so we have to check the conditions of Definition 3.13; 1 and 3 are obvious while 2 follows from $Y \subseteq B^\zeta$ using Lemma 3.14.1. If ζ is limit, 4 follows for $\alpha = \zeta$ from Lemma 3.14.2. If ζ is a successor then we have 4 for $\alpha = \xi$ by the induction hypothesis. Now we check 4 for $\alpha = \xi + 1$; (3.1) and (3.2) are automatic. To have (3.3), let $t \subset s$ and suppose that $G \cap P_s \neq \emptyset$ for some basic $\tau_C[\zeta + 1]|_{B_{\zeta+1}}$ -open set G . Since B^ζ is $\tau_{P_L}^\zeta[\zeta + 1]|_{B_{\zeta+1}}$ -open and $\tau_{P_L}^\zeta[\zeta + 1]|_{B_{\zeta+1}}$ -dense, we can assume that G is of the form $G' \cap B_n^\zeta$ for some basic τ_C -open set G' and $n < \omega$. But (3.3) holds in B_n^ζ , in particular $B_n^\zeta \cap G' \cap P_t \setminus P_s \neq \emptyset$, as required.

To see (3.4) fix an $s \in \omega^{<\omega}$ and a basic $\tau_C[\zeta + 1]|_{B_{\zeta+1}}$ -open set G with $G \cap P_s \neq \emptyset$; as before, we can assume that $G = G' \cap B_n^\zeta$ for some basic τ_C -open set G' and $n < \omega$. Now (3.4) holds in B_n^ζ hence $B_n^\zeta \cap G' \cap \bigcup_{i < \omega} P_{s \sim i} \neq \emptyset$ which proves the statement. This completes the recursive step and the proof of the special case.

For an arbitrary uncountable Polish space (X, τ) and Borel not $\Sigma_3^0(\tau)$ set P take a continuous one-to-one map $\varphi: (C, \tau_C) \rightarrow (X, \tau)$ such that $\varphi^{-1}(P) = P_L$. Let

$$\mathcal{I} = \{G \subseteq X : \exists G' \in \mathcal{I}_{P_L}(G \cap \varphi(C) \subseteq \varphi(G'))\}.$$

Since homeomorphism preserves the Borel class of sets and $X \setminus \varphi(C)$ is τ -open hence $\Pi_2^0(\tau)$, this σ -ideal is strongly generated by its $\Pi_2^0(\tau)$ members. If $A \subseteq P$ is $\Sigma_3^0(\tau)$ then $\varphi^{-1}(A) \subseteq P_L$ is $\Sigma_3^0(\tau_C)$ so $\varphi^{-1}(A) \in \mathcal{I}_{P_L}$ and hence $A \in \mathcal{I}$. This shows $\mathcal{S}_3^0(P) \subseteq \mathcal{I}$. Since $P_L \notin \mathcal{I}_{P_L}$ we have $P \notin \mathcal{I}$, which completes the proof. ■

Let us turn back to Question 1.5. The analogous question for strong generation is the following.

Question 3.16. *Let (X, τ) be a Polish space and let $0 < \xi < \omega_1$. If a σ -ideal $\mathcal{I} \subseteq 2^X$ is strongly generated by its $\Pi_\xi^0(\tau)$ members and a Borel set $A \subseteq X$ is not in \mathcal{I} then does there exist a $\Sigma_{\xi+1}^0(\tau)$ set $B \subseteq A$ such that $B \notin \mathcal{I}$?*

If $\xi = 1$ then the answer is obviously affirmative, since if B is a countable $\tau|_A$ -dense subset of A then $B \notin \mathcal{I}$ is shown by $A \subseteq \text{cl}_\tau(B)$.

Theorem 3.15 shows that for $\xi = 2$ the answer is consistently negative. We think that Question 3.16 is independent. In particular, we believe that Theorem 3.10 gives the best constructible covering for the ideal $\mathcal{S}_3^0(P)$. We will come back to this problem in Chapter 8.

Chapter 4

Topological Hurewicz test pairs

This chapter is devoted to the extensions of the results of Chapter 4 we have for $4 \leq \xi < \omega_1$: the existence of Π_ξ^0 topological Hurewicz test pairs and the ideal generation theorem.

4.1 Distinguishing Borel classes

In this section we extend Theorem 3.4 to higher levels of the Borel hierarchy. In order to produce a sufficiently big family of test pairs we need a machinery which allows us to condition on the construction of a given Borel set from simpler sets. For this, we handle a $\Pi_\xi^0(\tau)$ set by coding its construction from closed sets in a tree. The following inductive definition makes this concrete.

Definition 4.1. Let $0 < \xi < \omega_1$ and $\vartheta_i \rightarrow \xi$. For $\xi = 1$, $[P, (P_\emptyset)]$ is called a $\Pi_1^0(\tau)$ set with presentation if $P = P_\emptyset$ is a $\Pi_1^0(\tau)$ set.

Suppose that the $\Pi_\vartheta^0(\tau)$ sets with presentation are defined for $\vartheta < \xi$. Then $[P, (P_t)_{t \in T}]$ is a $\Pi_\xi^0(\tau)$ set with presentation if $T \subseteq \omega^{<\omega}$ is a subtree such that $\{(i) : i < \omega\} \subseteq T$, $P = X \setminus \bigcup_{i < \omega} P_{(i)}$ and $[P_{(i)}, (P_{i \smallfrown t})_{t \in T_{(i)}}]$ is a $\Pi_{\vartheta_i}^0(\tau)$ set with presentation ($i < \omega$).

It is important to note that a $\Pi_\xi^0(\tau)$ sets with presentation is not necessarily a proper $\Pi_\xi^0(\tau)$ set. For example, it can easily be empty.

Next we define the test sets and the corresponding topologies. These test pairs are exactly what one may expect; in particular, the following definition for $\xi = 3$ is in accordance with Definition 3.1.

Definition 4.2. Let $0 < \xi < \omega_1$ and $P \subseteq X$. We call the pair $\{P, \tau_P\}$ a $\Pi_\xi^0(\tau)$ topological Hurewicz test pair in X if

1. P is a $\Pi_\xi^0(\tau)$ set;
2. τ_P is a Polish topology refining τ ;

3. P is a τ_P -nowhere dense $\Pi_1^0(\tau_P)$ set;
4. (a) $\xi = 1$: for every τ -open set $A \subseteq X$ and basic τ_P -open set G with $G \cap P \neq \emptyset$ if $A \cap P$ is $\tau_P|_P$ -residual in $G \cap P$ then A is τ_P -residual in a τ_P -open set $G' \subseteq G$ such that $G \cap P \subseteq \text{cl}_{\tau_P}(G' \cap P)$.
- (b) $1 < \xi$ is a successor ordinal: for every $\vartheta < \xi$, $\Pi_\vartheta^0(\tau)$ set $A \subseteq X$ and basic τ_P -open set G with $G \cap P \neq \emptyset$ if $A \cap P$ is $\tau_P|_P$ -residual in $G \cap P$ then A is τ_P -residual in G .
- (c) $1 < \xi$ is a limit ordinal: there is a τ_P -open set $\mathcal{H}_{X,P}(\vartheta)$ ($\vartheta < \xi$) such that $P \subseteq \mathcal{H}_{X,P}(\vartheta)$ ($\vartheta < \xi$), and for every $\vartheta < \xi$, $\Pi_\vartheta^0(\tau)$ set $A \subseteq X$ and basic τ_P -open set G with $G \cap P \neq \emptyset$ if $A \cap P$ is $\tau_P|_P$ -residual in $G \cap P$ then A is τ_P -residual in $G \cap \mathcal{H}_{X,P}(\vartheta)$.

We associate inductively the topologies $\tau_P^<$ and τ_P to $\Pi_\xi^0(\tau)$ sets with presentation ($0 < \xi < \omega_1$).

Definition 4.3. Consider a $\Pi_\xi^0(\tau)$ set with presentation $[P, (P_t)_{t \in T}]$. For $\xi = 1$ we define $\tau_P^< = \tau_P = \tau$. If $1 < \xi < \omega_1$ and τ_P is defined for $\Pi_\vartheta^0(\tau)$ sets with presentation for $\vartheta < \xi$, set $\mathcal{P} = \{P_{(n)} \cap \bigcap_{i < n} (X \setminus P_{(i)}) : n < \omega\}$. We define $\tau_P^< = \bigvee_{i < \omega} \tau_{P_{(i)}}$ and $\tau_P = \tau_P^<[\mathcal{P}]$.

Observe that P is disjoint to the members of \mathcal{P} and the sets in \mathcal{P} are pairwise disjoint. Next we prove an auxiliary claim on how P is related to the topologies $\tau_P^<$, τ_P . For its proof we will need the Kuratowski-Ulam Theorem in the following form (see [4], (8.41) Theorem).

Theorem 4.4. (*Kuratowski-Ulam*) Let (X, τ) and (Y, σ) be Polish spaces, let $G = G_X \times G_Y$ be a basic $\tau \times \sigma$ -open set in $X \times Y$ and consider a Borel set $A \subseteq X \times Y$. Set $A^y = \{x \in X : (x, y) \in A\}$. Then A is $\tau \times \sigma$ -residual in G if and only if

$$\{y \in G_Y : A^y \text{ is } \tau\text{-residual in } G_X\}$$

is σ -residual in G_Y .

Claim 4.5. With the notation of Definition 4.3, we have the following.

1. P is $\Pi_2^0(\tau_P^<)$ and $\Pi_1^0(\tau_P)$.
2. If G is basic τ_P -open and $G \cap P \neq \emptyset$ then G is in fact basic $\tau_P^<$ -open.
3. The topologies $\tau_P|_P$ and $\tau_P^<|_P$ coincide.
4. The topologies

$$\tau_P|_{P_{(n)} \setminus \bigcup_{i < n} P_{(i)}} \text{ and } \tau_P^<|_{P_{(n)} \setminus \bigcup_{i < n} P_{(i)}} \quad (n < \omega)$$

coincide.

5. If (Y, σ) is any nonempty Polish space and $\{P, \tau_P\}$ is a $\Pi_\xi^0(\tau)$ topological Hurewicz test pair in (X, τ) then $\{P \times Y, \tau_P \times \sigma\}$ is a $\Pi_\xi^0(\tau \times \sigma)$ topological Hurewicz test pair in $(X \times Y, \tau \times \sigma)$.

Proof. We prove the first statement by induction on ξ . For $\xi = 1$ the statement is obvious. Let now $1 < \xi < \omega_1$ and suppose that the statement holds for $\vartheta < \xi$. We have

$$(4.1) \quad P = X \setminus \bigcup_{n < \omega} P_{(n)} = X \setminus \bigcup_{n < \omega} \left(P_{(n)} \cap \bigcap_{i < n} (X \setminus P_{(i)}) \right),$$

where $P_{(n)}$ is τ_P^{\leq} closed ($n < \omega$) by the inductive hypothesis and $P_{(n)} \cap \bigcap_{i < n} (X \setminus P_{(i)})$ is τ_P -open ($n < \omega$) by definition, so 1 follows. By Definition 4.3, proper basic τ_P -open sets do not intersect P , which shows 2. This immediately implies 3.

Since the sets in \mathcal{P} are pairwise disjoint, if G is a proper basic τ_P -open set which intersects $P_{(n)} \setminus \bigcup_{i < n} P_{(i)}$ then $G = G' \cap P_{(n)} \setminus \bigcup_{i < n} P_{(i)}$ where G' is basic τ_P^{\leq} -open, so 4 holds.

For 5, let G be a basic $\tau_P \times \sigma$ -open set, say $G = G_X \times G_Y$ where G_X (G_Y , resp.) is basic τ_P -open in X (basic σ -open in Y , resp.). If $\xi = 1$, let A be a $\tau \times \sigma$ -open set such that $A \cap (P \times Y)$ is $\tau_P \times \sigma$ -residual in $(G_X \cap P) \times G_Y$. Let

$$G' = \bigcup \{H \subseteq X \times Y : H \text{ is basic } \tau_P \times \sigma\text{-open, } A \text{ is } \tau_P \times \sigma\text{-residual in } H\}.$$

Then A is $\tau_P \times \sigma$ -residual in G' so it remains show that $G \cap (P \times Y) \subseteq \text{cl}_{\tau_P \times \sigma}(G' \cap (P \times Y))$. Suppose that $K = K_X \times K_Y \subseteq G$ is a nonempty basic $\tau_P \times \sigma$ -open set such that $K \cap (P \times Y) \subseteq G \cap (P \times Y) \setminus \text{cl}_{\tau_P \times \sigma}(G' \cap (P \times Y))$. Then $A \cap (P \times Y)$ is $\tau_P \times \sigma$ -residual in $(K_X \cap P) \times K_Y$, so by Theorem 4.4,

$$W = \{y \in K_Y : A^y \cap P \text{ is } \tau_P|_P\text{-residual in } K_X \cap P\}$$

is σ -residual in K_Y . Since $\{P, \tau_P\}$ is a $\Pi_1^0(\tau)$ -topological Hurewicz test pair, by Definition 4.2.4a, for every $y \in W$ there is a τ_P -open set $K'_X(y) \subseteq K_X$ such that A^y is τ_P -residual in $K'_X(y)$ and $K_X \cap P \subseteq \text{cl}_\tau(K'_X(y) \cap P)$. Since (X, τ_P) has countable base, $K'_X(y)$ contains the same basic τ_P -open set for σ -residually many y 's, that is there is a basic τ_P -open set $K'_X \subseteq K_X$ and a basic σ -open set $K'_Y \subseteq K_Y$ such that

$$\{y \in K'_Y : A^y \text{ is } \tau_P\text{-residual in } K'_X\}$$

is σ -residual in K'_Y . Then by Theorem 4.4, A is $\tau_P \times \sigma$ -residual in $K' = K'_X \times K'_Y$, that is $K' \subseteq G'$, a contradiction.

Let now $1 < \xi < \omega_1$ be a successor ordinal, $\vartheta < \xi$ and A be a $\Pi_\vartheta^0(\tau \times \sigma)$ set such that $A \cap (P \times Y)$ is $\tau_P \times \sigma$ -residual in $(G_X \cap P) \times G_Y$. We show that A is $\tau_P \times \sigma$ -residual in G . By Theorem 4.4,

$$W = \{y \in G_Y : A^y \cap P \text{ is } \tau_P|_P\text{-residual in } G_X \cap P\}$$

is σ -residual in G_Y . Since $\{P, \tau_P\}$ is a $\Pi_\xi^0(\tau)$ -topological Hurewicz test pair, by Definition 4.2.4b, A^y ($y \in W$) is τ_P -residual in G_X . Then again by Theorem 4.4, A is $\tau_P \times \sigma$ -residual in G , as stated.

Let now $1 < \xi < \omega_1$ be a limit ordinal. We show that $\mathcal{H}_{X \times Y, P \times Y}(\vartheta) = \mathcal{H}_{X, P}(\vartheta) \times Y$ fulfills the requirements. Let $\vartheta < \xi$ and A be a $\Pi_\vartheta^0(\tau \times \sigma)$ set such that $A \cap (P \times Y)$ is $\tau_P \times \sigma$ -residual in $(G_X \cap P) \times G_Y$. By Theorem 4.4,

$$W = \{y \in G_Y : A^y \cap P \text{ is } \tau_P|_P\text{-residual in } G_X \cap P\}$$

is σ -residual in G_Y . Since $\{P, \tau_P\}$ is a $\Pi_\xi^0(\tau)$ -topological Hurewicz test pair, by Definition 4.2.4c, A^y ($y \in W$) is τ_P -residual in $G_X \cap \mathcal{H}_{X, P}(\vartheta)$. Then again by Theorem 4.4, A is $\tau_P \times \sigma$ -residual in $G \cap \mathcal{H}_{X \times Y, P \times Y}(\vartheta) = (G_X \cap \mathcal{H}_{X, P}(\vartheta)) \times G_Y$, as stated. This completes the proof. ■

The following claim describes the behavior of a topological test pair with respect to $\Sigma_\xi^0(\tau)$ sets.

Claim 4.6. *Let $0 < \xi < \omega_1$ and let $\{P, \tau_P\}$ be a $\Pi_\xi^0(\tau)$ topological Hurewicz test pair. If for a $\Sigma_\xi^0(\tau)$ set W and τ_P -open set G with $G \cap P \neq \emptyset$ we have that $W \cap P$ is $\tau_P|_P$ -residual in $G \cap P$, then W is τ_P -residual in a τ_P -open set H satisfying that $G \cap P \subseteq \text{cl}_{\tau_P}(H \cap P)$.*

Proof. For $\xi = 1$ the statement follows from the definition. Let now $1 < \xi < \omega_1$ and write $W = \bigcup_{i < \omega} Q_i$, where Q_i is $\Pi_{\vartheta_i}^0(\tau)$ and $\vartheta_i \rightarrow \xi$. If $W \cap G \cap P$ is $\tau_P|_P$ -residual in $G \cap P$ then let H_i denote the maximal τ_P -open set in which Q_i is τ_P -residual ($i < \omega$). By Definition 4.2.4, the τ_P -open set $H = \bigcup_{i < \omega} H_i$ meets every τ_P -open set intersecting $G \cap P$, which proves the statement. ■

In the following theorem we give a method allowing to build up inductively a topological Hurewicz test pair from simpler test sets .

Theorem 4.7. *Let $0 < \xi < \omega_1$, $\vartheta_i \rightarrow \xi$ and let $[P, (P_t)_{t \in T}]$ be a nonempty $\Pi_\xi^0(\tau)$ set with presentation. If $\xi = 1$ and P is τ -nowhere dense then $\{P, \tau_P\}$ is a topological Hurewicz test pair.*

For $1 < \xi < \omega_1$ suppose that $\bigcup_{i < \omega} P_{(i)}$ is $\tau_P^<$ -dense and $\{P_{(i)}, \tau_P^<\}$ is a $\Pi_{\vartheta_i}^0(\tau)$ topological Hurewicz test pair ($i < \omega$). Then

1. P is $\tau_P^<$ -residual;
2. $\{P, \tau_P\}$ is a $\Pi_\xi^0(\tau)$ topological Hurewicz test pair.

Proof. If $\xi = 1$ then $G' = A \cap G$ does the job. Let now $1 < \xi < \omega_1$. Since $\{P_{(n)}, \tau_P^<\}$ ($n < \omega$) is a $\Pi_{\vartheta_n}^0(\tau)$ topological Hurewicz test pair, $P_{(n)}$ ($n < \omega$) is $\tau_P^<$ -nowhere dense so 1 follows from (4.1).

For 2 we have to check the conditions of Definition 4.2; 1 holds by the choice of P , 2 follows from Definition 4.3.

For 3, by Claim 4.5.1 it remains to show that P does not contain any nonempty basic τ_P -open set. Suppose that $G \subseteq P$ and G is nonempty basic τ_P -open. Then by Claim 4.5.2, G is basic $\tau_P^<$ -open, we have that $\bigcup_{i<\omega} P_{(i)}$ is $\tau_P^<$ -dense hence $P_{(n)} \cap G \neq \emptyset$ for some $n < \omega$, a contradiction.

Let now $\vartheta < \xi$, $A \subseteq X$ be $\Pi_{\vartheta}^0(\tau)$, G be a basic τ_P -open set with $G \cap P \neq \emptyset$ and suppose that $A \cap P$ is $\tau_P|_P$ -residual in $G \cap P$. By Claim 4.5.2, G is actually $\tau_P^<$ -open while by 1 and Claim 4.5.3, A is $\tau_P^<$ -residual in G .

Set $G' = G$ if ξ is a successor. If ξ is limit then let $I < \omega$ be minimal such that $\vartheta \leq \vartheta_I$, set $\mathcal{H}_{X,P}(\vartheta) = \bigcap_{i<I} X \setminus P_{(i)}$ and $G' = G \cap \mathcal{H}_{X,P}(\vartheta) = G \setminus \bigcup_{i<I} P_{(i)}$. We have $P \subseteq \mathcal{H}_{X,P}(\vartheta)$ by Definition 4.1. It remains to show that A is τ_P -residual in G' . Note that G' is $\tau_P^<$ -open and that A is $\tau_P^<$ -residual in G' .

Suppose that A is not τ_P -residual in G' ; that is we have a nonempty basic τ_P -open set $\tilde{G} \subseteq G'$ such that $A \cap \tilde{G}$ is τ_P -meager in \tilde{G} . By passing to a nonempty basic τ_P -open subset we can assume that

$$\tilde{G} = G_0 \cap P_{(n)} \cap \bigcap_{i<n} X \setminus P_{(i)} = G_0 \cap P_{(n)} \setminus \bigcup_{i<n} P_{(i)}$$

where G_0 is basic $\tau_P^<$ -open and $n < \omega$. Note that if ξ is limit then $I \leq n$ by the choice of G' . So we can assume $G_0 \cap \bigcap_{i<n} X \setminus P_{(i)} \subseteq G'$.

We obtained that the $\Sigma_{\vartheta}^0(\tau)$ set $X \setminus A$ is $\tau_P|_{P_{(n)} \setminus \bigcup_{i<n} P_{(i)}}$ -residual in the τ_P -open set $P_{(n)} \cap G_0 \cap \bigcap_{i<\omega} X \setminus P_{(i)}$. Thus by Claim 4.5.4, $X \setminus A$ is $\tau_P^<|_{P_{(n)} \setminus \bigcup_{i<n} P_{(i)}}$ -residual in the $\tau_P^<|_{P_{(n)}}$ -open set $P_{(n)} \cap G_0 \cap \bigcap_{i<n} X \setminus P_{(i)}$. Since $\vartheta \leq \vartheta_n$, we can apply Claim 4.6 for the $\Sigma_{\vartheta_n}^0(\tau)$ set $\underline{W} = X \setminus A$, the $\Pi_{\vartheta_n}^0(\tau)$ topological Hurewicz test pair $\{P_{(n)}, \tau_P^<\}$ and the $\tau_P^<$ -open set $\underline{G} = G_0 \cap \bigcap_{i<n} X \setminus P_{(i)}$ satisfying $\underline{G} \cap P_{(n)} \neq \emptyset$. We get that $X \setminus A$ is $\tau_P^<$ -residual in a $\tau_P^<$ -open set H such that $\underline{G} \cap P_{(n)} \subseteq \text{cl}_{\tau_P^<}(H \cap P_{(n)})$, in particular $H' = H \cap \underline{G} \neq \emptyset$ and $H' \subseteq G'$. Thus both A and $X \setminus A$ are $\tau_P^<$ -residual in the nonempty $\tau_P^<$ -open set H , a contradiction. This completes the proof. ■

Just as for the $\xi = 3$ case, the conditions of Theorem 4.7 concern the presentation of the $\Pi_{\xi}^0(\tau)$ set P instead of P itself. This handicap seems to be inevitable. First, because up to our knowledge there are no results providing some method to build up $\Sigma_{\xi}^0(\tau)$ sets from simpler sets, there is not even a canonical decomposition of $\Sigma_2^0(\tau)$ sets to $\Pi_1^0(\tau)$ sets. It is easy to see that by taking a wrong presentation the topology τ_P becomes wrong either, that is we cannot just condition our test set to be proper $\Pi_{\xi}^0(\tau)$, a suitably chosen, not natural presentation must be involved. Second, because the only way to build up a proper $\Pi_{\xi}^0(\tau)$ set for $\xi > 4$ is to use induction, so in view of our first reason one could hardly imagine Theorem 4.7 without some inductive condition on the presentation. Even if well explained, this handicap remains painful and this is responsible for most of the complication we have to face later. We close this section with the usual corollaries.

Corollary 4.8. *For a $0 < \xi < \omega_1$, let $\{P, \tau_P\}$ be a $\Pi_\xi^0(\tau)$ topological Hurewicz test pair as in Theorem 4.7. Let G be a basic τ_P -open set with $G \cap P \neq \emptyset$, or equivalently let G be a nonempty basic τ_P^\prec -open. Then the following hold.*

1. *If ξ is a successor, $\vartheta < \xi$ and $A \subseteq X$ is $\Pi_\vartheta^0(\tau)$ and τ_P^\prec -residual in G then A is τ_P -residual in G .*
2. *If $A \subseteq X$ is $\Sigma_\xi^0(\tau)$ and of τ_P^\prec -second category in G then A is of τ_P -second category in G .*
3. *If ξ is a successor, $\vartheta < \xi$ and $A \subseteq X$ is $\Sigma_\vartheta^0(\tau)$ and of τ_P -second category in G then A is of τ_P^\prec -second category in G .*
4. *If $A \subseteq X$ is $\Pi_\xi^0(\tau)$ and τ_P -residual in G then A is τ_P^\prec -residual in G .*
5. *P is a proper $\Pi_\xi^0(\tau)$ set.*

Proof. Let A be $\Pi_\vartheta^0(\tau)$ and τ_P^\prec -residual in G . Since P is a τ_P^\prec -residual $\Pi_2^0(\tau_P^\prec)$ set, $A \cap P$ is $\tau_P^\prec|_P$ -residual in $G \cap P$. Since by Claim 4.5.3 the topologies $\tau_P^\prec|_P$ and $\tau_P|_P$ coincide, $A \cap P$ is $\tau_P|_P$ -residual in $G \cap P$. So Definition 4.2.4b applies and we conclude that A is τ_P -residual in G , which proves 1.

For 2, let $A = \bigcup_{i < \omega} A_i$ with $\Pi_{\vartheta_i}^0(\tau)$ set A_i ($i < \omega$) where $\vartheta_i \rightarrow \xi$. Since P is a τ_P^\prec -residual $\Pi_2^0(\tau_P^\prec)$ set, if A is of τ_P^\prec -second category in G then for an $i < \omega$, $A_i \cap P$ is $\tau_P^\prec|_P$ -residual in $G' \cap P$ for some basic τ_P^\prec -open set $G' \subseteq G$. Since by Claim 4.5.3 the topologies $\tau_P^\prec|_P$ and $\tau_P|_P$ coincide, we have that $A_i \cap P$ is $\tau_P|_P$ -residual in the basic τ_P -open set G' with $G' \cap P \neq \emptyset$ so by Definition 4.2.4, A_i is τ_P -residual in some nonempty τ_P -open set $G'' \subseteq G'$ thus A is of τ_P -second category in G , as required.

Statements 3 and 4 follow from 1 and 2 by taking complements.

For 5, suppose that P is $\Sigma_\xi^0(\tau)$. By Theorem 4.7.1, P is τ_P^\prec -residual in X so by Corollary 4.8.2, P is of τ_P -second category in X . But by Definition 4.2.3, P is τ_P -nowhere dense, a contradiction. This completes the proof. ■

There is an asymmetry in our approach to topological Hurewicz test sets: the test set is of some multiplicative class and the sets tested are of the dual additive class. The reason for this is that Σ_ξ^0 is closed under taking countable union while Π_ξ^0 is not. However, there is a testing theorem like Theorem 4.7 for special Σ_ξ^0 sets (see e.g. Theorem 5.2 or Corollary 5.3) but the statement of this theorem cannot go beyond Corollary 4.8. So we do not work for that.

4.2 Intersection criteria

It turns out that the conditions of Theorem 4.7 are combinatorial, they are the same as requiring that countably many intersections are nonempty. We give these intersections.

Our purpose is to show that if Theorem 4.7 proves that a set P is a topological Hurewicz test then P remains a test set if the initial topology τ of the Polish space is changed.

Definition 4.9. Let $0 < \xi < \omega_1$ and $[P, (P_t)_{t \in T}]$ be a $\Pi_\xi^0(\tau)$ set with presentation. If $\xi = 1$, set $\mathcal{C}_1(X, \tau, P) = \{(X \setminus P, G) : G \in \tau\}$. If $\mathcal{C}_\vartheta(X', \tau', P')$ is defined for every $\vartheta < \xi$, Polish space (X', τ') and $\Pi_\vartheta^0(\tau')$ set with presentation P' then let $\vartheta_i \rightarrow \xi$ and set

$$\mathcal{C}_\xi(X, \tau, P) = \{(X \setminus P, G) : G \in \tau_P^\leq\} \cup \bigcup_{i < \omega} \mathcal{C}_{\vartheta_i} \left(X, \bigvee_{j < \omega, j \neq i} \tau_{P_{(j)}}, P_{(i)} \right).$$

We say that $[P, (P_t)_{t \in T}]$ satisfies \mathfrak{C}_ξ in (X, τ) if

$$\forall (C, G) \in \mathcal{C}_\xi(X, \tau, P) \quad (G \neq \emptyset \implies C \cap G \neq \emptyset).$$

Claim 4.10. Let $0 < \xi < \omega_1$. If a $\Pi_\xi^0(\tau)$ set with presentation $[P, (P_t)_{t \in T}]$ satisfies \mathfrak{C}_ξ in (X, τ) then $\{P, \tau_P\}$ satisfies the conditions of Theorem 4.7, so in particular $\{P, \tau_P\}$ is a $\Pi_\xi^0(\tau)$ topological Hurewicz test pair.

Proof. We prove the statement by induction on ξ . For $\xi = 1$, \mathfrak{C}_1 means that P is τ -nowhere dense in X , that is $\{P, \tau_P\}$ is a $\Pi_1^0(\tau)$ topological Hurewicz test pair by Theorem 4.7. Suppose now that the statement holds for $\vartheta < \xi$ and let $\vartheta_i \rightarrow \xi$. By \mathfrak{C}_ξ , $X \setminus P = \bigcup_{i < \omega} P_{(i)}$ is τ_P^\leq -dense in X and $[P_{(i)}, (P_{i \setminus t})_{t \in T_{(i)}}]$ satisfies $\mathfrak{C}_{\vartheta_i}$ in the Polish space $(X, \bigvee_{j < \omega, j \neq i} \tau_{P_{(j)}})$. We have

$$\tau_P^\leq = \bigvee_{j < \omega} \tau_{P_{(j)}} = \left(\bigvee_{j < \omega, j \neq i} \tau_{P_{(j)}} \right)_{P_{(i)}},$$

so by the induction hypothesis $\{P_{(i)}, \tau_P^\leq\}$ is a $\Pi_{\vartheta_i}^0(\tau)$ topological Hurewicz test pair. Thus the conditions of Theorem 4.7 are satisfied, Theorem 4.7.2 can be applied and we conclude that $\{P, \tau_P\}$ is a $\Pi_\xi^0(\tau)$ topological Hurewicz test pair. ■

We need that if P lives in a product space but it is nontrivial only on one coordinate then \mathfrak{C}_ξ also conditions only on one coordinate.

Claim 4.11. Let $0 < \xi < \omega_1$ and let $[P, (P_t)_{t \in T}]$ be a $\Pi_\xi^0(\tau)$ set with presentation in the Polish space (X, τ) . Let (Y, σ) be a Polish space and set $Q = P \times Y$, $Q_t = P_t \times Y$ ($t \in T$). Then $[Q, (Q_t)_{t \in T}]$ is a $\Pi_\xi^0(\tau \times \sigma)$ set with presentation and for every $(C, G) \in \mathcal{C}_\xi(X \times Y, \tau \times \sigma, Q)$, C is nontrivial only on the X coordinate.

Proof. The statement easily follows by induction on ξ . ■

The next claim gives that \mathfrak{C}_ξ remains true if the initial topology gets coarser.

Claim 4.12. *Let τ' be a Polish topology on X refining τ . If for some $0 < \xi < \omega_1$ a $\Pi_\xi^0(\tau)$ set with presentation $[P, (P_t)_{t \in T}]$ satisfies \mathfrak{C}_ξ in (X, τ') then it satisfies \mathfrak{C}_ξ in (X, τ) as well.*

Proof. We prove by induction on ξ that $\mathcal{C}_\xi(X, \tau, P) \subseteq \mathcal{C}_\xi(X, \tau', P)$ ($0 < \xi < \omega_1$). From this the statement follows.

For $\xi = 1$ we have $\tau \subseteq \tau'$ and so $\mathcal{C}_1(X, \tau, P) \subseteq \mathcal{C}_1(X, \tau', P)$. Suppose now that the statement holds for $\vartheta < \xi$ and let $\vartheta_i \rightarrow \xi$. Since $\tau \subseteq \tau'$ we also have $\tau_P^\prec \subseteq \tau'_{P^\prec}$, $\tau_{P(i)} \subseteq \tau'_{P(i)}$ ($i < \omega$) so

$$\{(X \setminus P, G) : G \in \tau_P^\prec\} \subseteq \{(X \setminus P, G) : G \in \tau'_{P^\prec}\}$$

and by the induction hypothesis,

$$\mathcal{C}_{\vartheta_i} \left(X, \bigvee_{j < \omega, j \neq i} \tau_{P(j)}, P(i) \right) \subseteq \mathcal{C}_{\vartheta_i} \left(X, \bigvee_{j < \omega, j \neq i} \tau'_{P(j)}, P(i) \right).$$

This proves $\mathcal{C}_\xi(X, \tau, P) \subseteq \mathcal{C}_\xi(X, \tau', P)$ and completes the proof. ■

From now on in this section we work to prove the main lemma of the extension of Theorem 3.10. The technique of the proof is to exploit the low class Hurewicz test sets appearing in the construction of a $\Pi_\xi^0(\tau)$ test set. For this we need some more topologies.

Definition 4.13. Let $0 < \xi < \omega_1$ and let $[P, (P_t)_{t \in T}]$ be a $\Pi_\xi^0(\tau)$ set with presentation which satisfies \mathfrak{C}_ξ in (X, τ) . We define the topologies

$$\tau_P(n) = \bigvee_{i \leq n} \tau_{P(i)} \vee \bigvee_{n < i < \omega} \tau_{P(i)}^\prec \quad (n < \omega).$$

Corollary 4.14. *Let $0 < \xi < \omega_1$, $\vartheta_i \rightarrow \xi$ and let $[P, (P_t)_{t \in T}]$ be a $\Pi_\xi^0(\tau)$ set with presentation which satisfies \mathfrak{C}_ξ in (X, τ) . With the notation of Definition 4.13, $\{P(n), \tau_P(n)\}$ satisfies the conditions of Theorem 4.7, thus it is a $\Pi_{\vartheta_n}^0(\tau)$ topological Hurewicz test pair. We have*

$$\tau_P(0)^\prec = \bigvee_{i < \omega} \tau_{P(i)}^\prec$$

and $\tau_P(n+1)^\prec = \tau_P(n)$ ($n < \omega$).

Proof. Since $[P, (P_t)_{t \in T}]$ satisfies \mathfrak{C}_ξ , $[P(n), (P_{n \smallfrown t})_{t \in T(n)}]$ satisfies $\mathfrak{C}_{\vartheta_n}$ in the Polish space $(X, \bigvee_{j < \omega, j \neq i} \tau_{P(j)})$. Then by Claim 4.12, $[P(n), (P_{n \smallfrown t})_{t \in T(n)}]$ satisfies $\mathfrak{C}_{\vartheta_n}$ in the Polish space

$$\left(X, \bigvee_{i < n} \tau_{P(i)} \vee \bigvee_{n < i < \omega} \tau_{P(i)}^\prec \right).$$

We have

$$\tau_P(n) = \tau_{P(n)} \vee \bigvee_{i < n} \tau_{P(i)} \vee \bigvee_{n < i < \omega} \tau_{P(i)}^\prec$$

so by Claim 4.10, $\{P_{(n)}, \tau_P(n)\}$ satisfies the conditions of Theorem 4.7, so it is a $\Pi_{\vartheta_n}^0(\tau)$ topological Hurewicz test pair. Similarly,

$$\tau_P(n+1)^{<} = \tau_{P_{(n+1)}}^{<} \vee \bigvee_{i < n+1} \tau_{P_{(i)}} \vee \bigvee_{n+1 < i < \omega} \tau_{P_{(i)}}^{<} = \tau_P(n) \quad (n < \omega)$$

and

$$\tau_P(0)^{<} = \tau_{P_{(0)}}^{<} \vee \bigvee_{0 < i < \omega} \tau_{P_{(i)}}^{<} = \bigvee_{i < \omega} \tau_{P_{(i)}}^{<},$$

which completes the proof. ■

The next lemma is an application of our newly found Hurewicz test sets.

Lemma 4.15. *Let $2 < \xi < \omega_1$ be such that we have $\xi = \xi' + 1$ where $\xi' = \xi'' + 1$ is a successor. Let $[P, (P_t)_{t \in T}]$ be a $\Pi_{\xi}^0(\tau)$ set with presentation which satisfies \mathfrak{C}_{ξ} . Fix an $n < \omega$. If A is a $\Sigma_{\xi''}^0(\tau)$ set and A is $\tau_P(n)$ -meager in a $\tau_P(n)$ -open set G then A is also $\tau_P(m)$ -meager in G ($n \leq m < \omega$).*

Proof. If G is $\tau_P(n)$ -open then it is $\tau_P(m)$ -open ($n \leq m < \omega$) so it is enough to prove that A is $\tau_P(n+1)$ -meager in G ; from this the statement follows by induction.

By Corollary 4.14 we have that $\{P_{(n+1)}, \tau_P(n+1)\}$ is a $\Pi_{\xi'}^0(\tau)$ topological Hurewicz test pair and $\tau_P(n+1)^{<} = \tau_P(n)$. So by Corollary 4.8.3 our A cannot be of $\tau_P(n+1)$ -second category in G . This completes the proof. ■

The crucial point in the proof of Theorem 3.10 was the fact the a $\Pi_2^0(\tau)$ set of first category is nowhere dense whatever is the underlying Polish topology. This is not true for $\Pi_{\vartheta}^0(\tau)$ sets when $3 \leq \vartheta < \omega_1$ so in order to avoid them we have to reduce complicated sets to $\Pi_2^0(\tau)$ sets. This is the motivation of the following concept.

Definition 4.16. Let $T \subseteq \omega^{<\omega}$ be a tree. We say that a subtree $T' \subseteq T$ is *even-complete* if

$$t \in T', |t| \text{ odd} \implies \{t \smallfrown i : i < \omega\} \cap T' = \{t\} \ \& \ \{t \smallfrown i : i < \omega\} \cap T \neq \emptyset \ \& \ \{t \smallfrown i : i < \omega\} \cap T \subseteq T',$$

and T' is maximal with respect to this property.

If $T' \subseteq T$ is an even complete subtree, $t \in T' \setminus \mathfrak{T}(T')$ with $|t|$ even then $t^{+T'} \in T'$ denotes the unique extension of t with $|t^{+T'}| = |t| + 1$.

Lemma 4.17. *For some $0 < \xi < \omega_1$ let $[P, (P_t)_{t \in T}]$ be a $\Pi_{\xi}^0(\tau)$ set with presentation. If $T' \subseteq T$ is an even-complete subtree and $x \notin P_t$ ($t \in \mathfrak{T}(T')$) then $x \notin P$.*

Proof. We prove the statement by induction on ξ . For $\xi = 1$ and $\xi = 2$ the only even-complete subtree of T is $T' = \{\emptyset\}$ that is $x \notin P_{\emptyset} = P$, as stated. Suppose now that

$3 \leq \xi$ and the statement holds for $\vartheta < \xi$. By maximality we have $T' \neq \emptyset$, hence there is a unique $i < \omega$ such that $(i) \in T'$. But $T'_{i \smallfrown j}$ is an even-complete subtree of $T_{i \smallfrown j}$ ($j < \omega$) so by the induction hypothesis $x \notin P_{i \smallfrown j}$ ($j < \omega$). That is $x \in P_{(i)}$ and so $x \notin P$, which completes the proof. ■

Now we can prove the main result of the section.

Claim 4.18. *Let $1 < \xi < \omega_1$ be such that we have $\xi = \xi' + 1$ where $\xi' = \xi'' + 1$ is a successor. Let $[P, (P_t)_{t \in T}]$ be a $\Pi_\xi^0(\tau)$ set with presentation which satisfies \mathfrak{C}_ξ . If A^n is a $\Pi_{\xi'}^0(\tau)$ set such that $A^n \cap P_{(n)} = \emptyset$ ($n < \omega$) then $P \setminus \bigcup_{n < \omega} A^n \neq \emptyset$.*

Proof. Fix a presentation $[A^n, (A_t^n)_{t \in T^n}]$ ($n < \omega$). Take maps $\eta_1: \omega \rightarrow \omega$ and $\eta_2: \omega \rightarrow \omega^{<\omega}$ such that

$$\eta = (\eta_1, \eta_2): \omega \rightarrow \bigcup_{n < \omega} \{n\} \times T^n$$

is a bijection satisfying

$$(4.2) \quad t, t' \in T^n, t \subseteq t' \implies \eta^{-1}(n, t) \leq \eta^{-1}(n, t') \quad (n < \omega)$$

and

$$(4.3) \quad \eta_1(n) \leq \max\{0, n - 1\} \quad (n < \omega).$$

We construct inductively a basic $\tau_P(n)$ -open set G_n ($n < \omega$) and an even-complete subtree $F^n \subseteq T^n$ ($n < \omega$) such that

$$(4.4) \quad \text{cl}_{\tau_P(n)}(G_{n+1}) \subseteq G_n \quad (n < \omega);$$

$$(4.5) \quad G_n \cap P_{(n)} = \emptyset \quad (n < \omega);$$

$$(4.6) \quad G_n \cap A^n \text{ is } \tau_P(n)\text{-meager} \quad (n < \omega);$$

$$(4.7) \quad \text{if } \eta_2(n) \in F^{\eta_1(n)} \setminus \mathfrak{T}(F^{\eta_1(n)}) \text{ and } |\eta_2(n)| \text{ is even, then}$$

$$G_n \cap A_{\eta_2(n) + F^{\eta_1(n)} \smallfrown i}^{\eta_1(n)} \text{ is } \tau_P(n)\text{-meager} \quad (n, i < \omega);$$

$$(4.8) \quad \text{if } \eta_2(n) \in \mathfrak{T}(F^{\eta_1(n)}) \text{ then } G_n \subseteq X \setminus A_{\eta_2(n)}^{\eta_1(n)} \quad (n < \omega).$$

Since the topology $\tau_P^{<}$ is finer than $\tau_P(n)$ ($n < \omega$), $\text{cl}_{\tau_P(n)}(G_{n+1}) \subseteq G_n$ implies that $\text{cl}_{\tau_P^{<}}(G_{n+1}) \subseteq G_n$ ($n < \omega$), so we have that $\bigcap_{n < \omega} G_n \neq \emptyset$ by (4.4) and $\bigcap_{n < \omega} G_n \subseteq P$ by (4.5). By Lemma 4.17, (4.8) gives

$$\bigcap_{n < \omega} G_n \subseteq P \setminus \bigcup_{n < \omega} A^n.$$

In the following construction we will successively grow the trees F^n , so a node can be terminal after some intermediate step but not in the final tree. To be concrete, we will

grow F^n at \emptyset in the n^{th} step or never, while if $s \in F^n$ for $s \in T^n \setminus \{\emptyset\}$ with $\eta(m) = (n, s)$ then we will grow F^n at s in the m^{th} step or never. We will declare when a tree does not grow any more from a node, such that this node remains terminal.

For $n = 0$, by (4.2) and (4.3) we have $\eta(0) = (0, \emptyset)$. To find our G_0 observe that $X \setminus A^0$ is a $\Sigma_{\xi'}^0(\tau)$ set containing $P_{(0)}$. By Corollary 4.14, $\{P_{(0)}, \tau(0)\}$ satisfies the conditions of Theorem 4.7, in particular it is a $\Pi_{\xi'}^0(\tau)$ topological Hurewicz test pair. By Theorem 4.7.1, $P_{(0)}$ and hence $X \setminus A^0$ is $\tau_P(0)^{<}$ -residual, by Corollary 4.8.2, $X \setminus A^0$ is of $\tau_P(0)$ -second category, that is A^0 is $\tau_P(0)$ -meager in some nonempty basic $\tau_P(0)$ -open set G .

If A^0 is $\Pi_{\vartheta}^0(\tau)$ with $3 \leq \vartheta < \omega_1$ then we have $A^0 = \bigcap_{i < \omega} X \setminus A_{(i)}^0$, so for some $k_0 < \omega$ and nonempty basic $\tau_P(0)$ -open set $G' \subseteq G$ we have that $X \setminus A_{(k_0)}^0$ is $\tau_P(0)$ -meager in G' . Put $k_0 \widehat{\ } i \in F^0$ ($i < \omega$). Then (4.7) holds for $n = 0$ and (4.8) does not apply. Since $P_{(0)}$ is $\tau_P(0)$ -nowhere dense we can pass to some basic $\tau_P(0)$ -open subset $G_0 \subseteq G'$ such that $G_0 \cap P_{(0)} = \emptyset$; so (4.5-4.8) are satisfied for $n = 0$.

Else we have that A^0 is $\Pi_1^0(\tau)$ or $\Pi_2^0(\tau)$, that is A^0 is $\tau_P(0)$ -nowhere dense in G . In this case choose $G_0 \subseteq G \setminus (P_{(0)} \cup A^0)$, put $\emptyset \in F^0$ and F^0 does not grow any more, that is $\eta_2(0) = \emptyset \in \mathfrak{T}(F^0)$. Now (4.7) does not apply; so we again have (4.5-4.8) for $n = 0$.

Suppose that G_n ($n < N$) is already defined such that (4.5-4.8) hold for $n < N$ and (4.4) holds for $n < N - 1$; we find our G_N . Again, $X \setminus A^N$ is a $\Sigma_{\xi'}^0(\tau)$ set containing $P_{(N)}$. By Corollary 4.14, $\{P_{(N)}, \tau(N)\}$ satisfies the conditions of Theorem 4.7, it is a $\Pi_{\xi'}^0(\tau)$ topological Hurewicz test pair, $\tau_P(N)^{<} = \tau_P(N - 1)$, so G_{N-1} is $\tau_P(N)^{<}$ -open. By Theorem 4.7.1, $P_{(N)}$ and hence $X \setminus A^N$ is $\tau_P(N)^{<}$ -residual in G_{N-1} , so by Corollary 4.8.2, $X \setminus A^N$ is of $\tau_P(N)$ -second category in G_{N-1} , that is A^N is $\tau_P(N)$ -meager in some nonempty basic $\tau_P(N)$ -open set $G \subseteq G_{N-1}$.

If A^N is not a $\Pi_1^0(\tau)$ or a $\Pi_2^0(\tau)$ set with presentation then we have $A^N = \bigcap_{i < \omega} X \setminus A_{(i)}^N$. So for some $k_N < \omega$ and nonempty basic $\tau_P(N)$ -open set $G' \subseteq G$ we have that $X \setminus A_{(k_N)}^N$ is $\tau_P(N)$ -meager on G' . If A^N is a $\Pi_1^0(\tau)$ or a $\Pi_2^0(\tau)$ set with presentation then set $G' = G$.

Since $P_{(N)}$ is $\tau_P(N)$ -nowhere dense, we can pass to some basic $\tau_P(N)$ -open subset $G'' \subseteq G'$ such that $G'' \cap P_{(N)} = \emptyset$ and $\text{cl}_{\tau_P(N)}(G'') \subseteq G_{N-1}$. We put $k_N \widehat{\ } i \in F^N$, ($i < \omega$), then (4.4-4.6) hold for every basic $\tau_P(N)$ -open set $G_N \subseteq G''$.

If $\eta_2(N) \notin \mathfrak{T}(F^{\eta_1(N)})$ then set $G_N = G''$. If $|\eta_2(N)|$ is odd then neither (4.7) nor (4.8) apply so the inductive step is complete. If $|\eta_2(N)|$ is even then (4.8) does not apply so it remains to show (4.7). If $\eta_2(N) = \emptyset$ then after the $\eta_1(N)^{\text{th}}$ step of the construction we had that $A^{\eta_1(N)}$ is neither a $\Pi_1^0(\tau)$ nor a $\Pi_2^0(\tau)$ set with presentation, $2 < \xi$ and $G_{\eta_1(N)} \cap A_{\emptyset + F^{\eta_1(N)} \widehat{\ } i}^{\eta_1(N)}$ is $\tau_P(\eta_1(N))$ -meager ($i < \omega$). By (4.3) we have that $\eta_1(N) < N$. Since $A_{\emptyset + F^{\eta_1(N)} \widehat{\ } i}^{\eta_1(N)}$ is $\Sigma_{\xi''}^0(\tau)$ ($i < \omega$), Lemma 4.15 gives that $A_{\emptyset + F^{\eta_1(N)} \widehat{\ } i}^{\eta_1(N)}$ is $\tau_P(N)$ -meager in $G_{\eta_1(N)}$ hence also in G_N , as required.

If $\eta_2(N) \notin \mathfrak{T}(F^{\eta_1(N)})$, $|\eta_2(N)|$ is even but $\eta_2(N) \neq \emptyset$ then we show that $\eta_2(N)$ will never be a node of $F^{\eta_1(N)}$. Let u be the terminal node of $F^{\eta_1(N)}$ on the branch of $\eta_2(N)$ in $T^{\eta_1(N)}$. By (4.2) there is an $m < N$ such that $\eta_1(m) = \eta_1(N)$ and $\eta_2(m) = u$. After the

m^{th} step of the construction u remained a terminal node of $F^{\eta_1(N)}$, that is $F^{\eta_1(N)}$ never grows from u so $\eta_2(N)$ will never be a node of $F^{\eta_1(N)}$. So again neither (4.7) nor (4.8) apply and the inductive step is complete.

If $\eta_2(N) \in \mathfrak{T}(F^{\eta_1(N)})$ then we do the following. Let $m < \omega$ be such that $\eta_1(m) = \eta_1(N)$ and $\eta_2(m) = \eta_2(N)|_{|\eta_2(N)|-2}$. By (4.2) we have $m < N$. Since $\eta_2(N) \neq \emptyset$, $A^{\eta_1(N)}$ is neither a $\Pi_1^0(\tau)$ nor a $\Pi_2^0(\tau)$ set with presentation and $2 < \xi$. Also, $A_{\eta_2(N)}^{\eta_1(N)}$ is a $\Sigma_{\xi''}^0(\tau)$ set. We had (4.7) in the m^{th} step of the construction so by Lemma 4.15, $G'' \subseteq G_m$ implies that $A_{\eta_2(N)}^{\eta_1(N)}$ is $\tau_P(N)$ -meager in G'' . If $A_{\eta_2(N)}^{\eta_1(N)}$ is a $\Pi_1^0(\tau)$ or a $\Pi_2^0(\tau)$ set then it is actually $\tau_P(N)$ -nowhere dense in G'' so we can find a nonempty basic $\tau_P(N)$ -open set $G_N \subseteq G'' \setminus A_{\eta_2(N)}^{\eta_1(N)}$. We do not grow $F^{\eta_1(N)}$ from the node $\eta_2(N)$, so (4.7) does not apply and (4.8) holds.

If $A_{\eta_2(N)}^{\eta_1(N)}$ is $\Pi_{\vartheta}^0(\tau)$ for some $3 \leq \vartheta < \xi'$ then since $A_{\eta_2(N)}^{\eta_1(N)} = \bigcap_{i < \omega} X \setminus A_{\eta_2(N) \smallfrown i}^{\eta_1(N)}$ for some $l_N < \omega$ and nonempty basic $\tau_P(N)$ -open set $G_N \subseteq G''$ we have that $X \setminus A_{\eta_2(N) \smallfrown l_N}^{\eta_1(N)}$ is $\tau_P(N)$ -meager on G_N . We put $\eta_2(N) \smallfrown l_N \smallfrown i \in F^{\eta_1(N)}$, ($i < \omega$), then (4.7) holds and (4.8) does not apply. This completes the recursive step and the proof. ■

4.3 A moment of being concrete

In order to proceed we need to show at least one concrete $\Pi_{\xi}^0(\tau)$ topological Hurewicz test pair. We do this as was done in [15].

Definition 4.19. We set $(C_1, \tau_{C_1}) = (C, \tau_C)$, $P_1 = \{x \in C_1 : \forall m \in \omega (x(m) = 1)\}$, $T_1 = \{\emptyset\}$, $P_{\emptyset}^1 = P_1$ and $\mathfrak{r}_1 = \omega$. Suppose that the spaces $(C_{\vartheta}, \tau_{C_{\vartheta}})$, the $\Pi_{\vartheta}^0(\tau_{C_{\vartheta}})$ sets with presentation $[P_{\vartheta}, (P_t^{\vartheta})_{t \in T_{\vartheta}}]$ and the ordinal \mathfrak{r}_{ϑ} are defined for every $\vartheta < \xi$. Then let

$$C_{\xi} = \prod_{i < \omega} C_{\vartheta_i}, \quad \tau_{C_{\xi}} = \prod_{i < \omega} \tau_{C_{\vartheta_i}},$$

$$(4.9) \quad P_{\xi} = \{x \in C_{\xi} : \forall i < \omega (x(i, \cdot) \in C_{\vartheta_i} \setminus P_{\vartheta_i})\},$$

$$(4.10) \quad T_{\xi} = \{n \smallfrown t : t \in T_{\vartheta_n}, n < \omega\},$$

$$(4.11) \quad P_{n \smallfrown t}^{\xi} = \prod_{i < n} C_{\vartheta_i} \times P_t^{\vartheta_n} \times \prod_{n < i < \omega} C_{\vartheta_i} \quad (t \in T_{\vartheta_n}, n < \omega)$$

and

$$\mathfrak{r}_{\xi} = \sum_{i < \omega} \mathfrak{r}_{\vartheta_i}.$$

Claim 4.20. *Let $0 < \xi < \omega_1$ and $\vartheta_i \rightarrow \xi$. We have $C_\xi = 2^{\mathfrak{r}_\xi}$. The $\Pi_\xi^0(\tau)$ set with presentation $[P_\xi, (P_t^\xi)_{t \in T_\xi}]$ satisfies \mathfrak{C}_ξ ; in particular $\{P_\xi, \tau_{P_\xi}\}$ is a $\Pi_\xi^0(\tau)$ topological Hurewicz test pair. We have $\tau_{P_\xi}^< = \prod_{i < \omega} \tau_{P_{\vartheta_i}}$ and $\tau_{P_\xi} = \tau_{P_\xi}^<[\{U_{\xi,n} : n < \omega\}]$ ($1 < \xi < \omega_1$) where*

$$(4.12) \quad \begin{aligned} U_{\xi,n} &= \prod_{i < n} (C_{\vartheta_i} \setminus P_{\vartheta_i}) \times P_{\vartheta_n} \times \prod_{n < i < \omega} C_{\vartheta_i} \subseteq \\ &\subseteq \prod_{i < n} C_{\vartheta_i} \times C_{\xi_n} \times \prod_{n < i < \omega} C_{\vartheta_i} = C_\xi \quad (1 < \xi < \omega_1, n < \omega). \end{aligned}$$

Proof. We prove the statements by induction on ξ . For $\xi = 1$, $C = 2^\omega$, P_1 is a single point so it is nowhere dense, as stated. Let now $1 < \xi < \omega_1$ and suppose that the statements are true for $\vartheta < \xi$. Then

$$C_\xi = \prod_{i < \omega} C_{\vartheta_i} = \prod_{i < \omega} 2^{\mathfrak{r}_{\vartheta_i}} = 2^{\sum_{i < \omega} \mathfrak{r}_{\vartheta_i}} = 2^{\mathfrak{r}_\xi}.$$

By definition,

$$\tau_{P_\xi}^< = \bigvee_{i < \omega} \tau_{P_{(i)}^\xi} = \prod_{i < \omega} \tau_{P_{\vartheta_i}},$$

as stated.

Let now $(C, G) \in \mathcal{C}_\xi(C_\xi, \tau_{C_\xi}, P_\xi)$. If $C = X \setminus P$ and $G \in \tau_{P_\xi}^<$ is nonempty then G is nontrivial only on finitely many coordinates so it intersects $X \setminus P = \bigcup_{i < \omega} P_{(i)}^\xi$. If $(C, G) \in \mathcal{C}_{\vartheta_i}(C_\xi, \bigvee_{j < \omega, j \neq i} \tau_{P_{(j)}^\xi}, P_{(i)}^\xi)$ then by Claim 4.11, C is nontrivial only on the i^{th} coordinate and $G = \prod_{j < \omega} G_j$ where $G_j = C_{\vartheta_j}$ except for finitely many j 's, G_j is basic $\tau_{P_{(j)}^\xi}$ -open ($j \in \omega \setminus \{i\}$) while G_i is basic $\tau_{C_{\vartheta_i}}$ -open. Since $[P_{\vartheta_i}, (P_t)_{t \in T_{\vartheta_i}}]$ satisfies $\mathfrak{C}_{\vartheta_i}$ by the induction hypothesis, we have $\text{Pr}_i(C) \cap \text{Pr}_i(G) \neq \emptyset$, which implies $C \cap G \neq \emptyset$. So $[P_\xi, (P_t^\xi)_{t \in T_\xi}]$ indeed satisfies \mathfrak{C}_ξ .

Finally we have $P_{(n)}^\xi \cap \bigcap_{i < n} (C_\xi \setminus P_{(i)}^\xi) = U_{\xi,n}$, so by definition $\tau_{P_\xi} = \tau_{P_\xi}^<[\{U_{\xi,n} : n < \omega\}]$. This completes the proof. ■

We point out a property of this construction.

Lemma 4.21. *Let $G \subseteq C_\xi$ be a nonempty basic τ_{P_ξ} -open set. Then $\text{int}_{\tau_{P_\xi}^<}(\text{cl}_{\tau_{P_\xi}^<}(G)) \neq \emptyset$.*

Proof. If G is basic $\tau_{P_\xi}^<$ -open then by regularity we have $\text{int}_{\tau_{P_\xi}^<}(\text{cl}_{\tau_{P_\xi}^<}(G)) = G \neq \emptyset$. So let G be proper basic τ_{P_ξ} -open, say $G = G' \cap U_{\xi,n}$ where G' is $\tau_{P_\xi}^<$ -open and $n < \omega$. Since $U_{\xi,n}$ is $\tau_{P_\xi}^<$ -dense in $\prod_{i < n} C_{\vartheta_i} \times P_{\vartheta_n} \times \prod_{n < i < \omega} C_{\vartheta_i}$, we have

$$G' \cap \left(\prod_{i < n} C_{\vartheta_i} \times P_{\vartheta_n} \times \prod_{n < i < \omega} C_{\vartheta_i} \right) \subseteq \text{int}_{\tau_{P_\xi}^<}(\text{cl}_{\tau_{P_\xi}^<}(G))$$

so $\text{int}_{\tau_{P_\xi}^\leq}(\text{cl}_{\tau_{P_3}^\leq}(G)) \neq \emptyset$, as stated. ■

We close this section with the proof of Theorem 1.6.

Proof of Theorem 1.6. First we prove 1. and 2. in the $\xi = 2$ case while this is exceptional in Theorem 1.2. Then we show 1. and 2. for $3 \leq \xi < \omega_1$ and finally we treat the extension for every $2 \leq \xi < \omega_1$.

So let $\xi = 2$. We set

$$U'_{2,N} = \{x \in 2^\omega : \forall n \geq N (x(n) = 0)\} \quad (N < \omega),$$

$$T'_2 = \{(i) : i < \omega\}, \quad P^{2'}_{(i)} = U'_{2,i} \quad (i < \omega), \quad P'_2 = 2^\omega \setminus \left(\bigcup_{N < \omega} U'_{2,N} \right).$$

Then the topology $\tau_{P'_2}$ is the refinement of τ_{C_1} by turning each point of the finite sets $U'_{2,N}$ ($N < \omega$) into an open set. Clearly, P'_2 is the complement of a dense countable subset in $(2^\omega, \tau_{C_1})$, so in particular P'_2 is $\Pi_2^0(\tau_{C_1})$ and τ_{C_1} -residual. As a complement of the dense $\tau_{P'_2}$ -open set $\bigcup_{N < \omega} U'_{2,N}$, we also have that P'_2 is $\tau_{P'_2}$ -nowhere dense.

Let $A \subseteq X$ be $\Sigma_2^0(\tau)$ and take a continuous one-to-one mapping $\varphi: (2^\omega, \tau_{C_1}) \rightarrow (X, \tau)$ such that $\varphi^{-1}(A) \cap P'_2$ is of $\tau_{P'_2}|_{P'_2}$ -second category in P'_2 . Then $\varphi^{-1}(A) \subseteq (2^\omega, \tau_{C_1})$ is $\Sigma_2^0(\tau_{C_1})$ and $\varphi^{-1}(A) \cap P'_2$ is of $\tau_{P'_2}|_{P'_2}$ -second category; thus $\varphi^{-1}(A)$ is of τ_{C_1} -second category, as well. Since a $\Sigma_2^0(\tau_{C_1})$ set in $(2^\omega, \tau_{C_1})$ is of second category only if its interior is nonempty, $\varphi^{-1}(A)$ contains a nonempty τ_{C_1} -open set so $\varphi^{-1}(A) \cap U'_{2,N} \neq \emptyset$ for some $N < \omega$. Then $\varphi^{-1}(A)$, having nonempty interior, is of $\tau_{P'_2}$ -second category.

If A is not $\Sigma_2^0(\tau)$, we apply Theorem 1.2 for $A_0 = A$, $A_1 = X \setminus A$. These sets cannot be separated by a $\Sigma_\xi^0(\tau)$ -set, so since P'_2 is the complement of a countable dense subset of $(2^\omega, \tau_{C_1})$, there is a continuous one-to-one mapping $\varphi: 2^\omega \rightarrow X$ with $\varphi(P'_2) \subseteq A$, $\varphi(2^\omega \setminus P'_2) \subseteq X \setminus A$. So as we have seen above, $\varphi^{-1}(A) = P'_2$ is indeed $\tau_{P'_2}$ -meager.

We turn to the $\xi \geq 3$ case. The Polish space (C_ξ, τ_{C_ξ}) is obviously homeomorphic to (C_1, τ_{C_1}) (see e.g. [4], Theorem 7.4 on page 35). We show that $\{P_\xi, \tau_{P_\xi}\}$ fulfills the requirements for every $3 \leq \xi < \omega_1$.

Let $A \subseteq X$ be $\Sigma_\xi^0(\tau)$ for some $\xi < \omega_1$ and take a continuous one-to-one mapping $\varphi: C_\xi \rightarrow X$ such that $\varphi^{-1}(A) \cap P_\xi$ is of $\tau_{P_\xi}|_{P_\xi}$ -second category in P_ξ . Then $\varphi^{-1}(A) \subseteq C_\xi$ is $\Sigma_\xi^0(\tau_{C_\xi})$ and $\varphi^{-1}(A) \cap P_\xi$ is $\tau_{P_\xi}|_{P_\xi}$ -residual in $G \cap P_\xi$ for some basic open set G . So according to Claim 4.6, $\varphi^{-1}(A)$ is of τ_{P_ξ} -second category, as required.

Suppose now that A is not $\Sigma_\xi^0(\tau)$. We apply Theorem 1.2 for $A_0 = A$, $A_1 = X \setminus A$. These sets cannot be separated by a $\Sigma_\xi^0(\tau)$ -set, so since P_ξ is $\Pi_\xi^0(\tau_{C_\xi})$ but not $\Sigma_\xi^0(\tau_{C_\xi})$, there is a continuous one-to-one mapping $\varphi: C_\xi \rightarrow X$ with $\varphi(P_\xi) \subseteq A$, $\varphi(2^\omega \setminus P_\xi) \subseteq X \setminus A$. So by Definition 4.2.3, $\varphi^{-1}(A) = P_\xi$ is indeed τ_{P_ξ} -meager.

Finally suppose that for some cardinal $\lambda < 2^{\aleph_0}$, in our model the the union of a λ number of meager sets is meager in Polish spaces. Let A_i ($i < \lambda$) be $\Sigma_\xi^0(\tau)$ and $A = \bigcup_{i < \lambda} A_i$ be Borel. Since $\varphi^{-1}(A) \cap P_\xi$ is of $\tau_{P_\xi}|_{P_\xi}$ -second category in P_ξ , by our

assumption $\varphi^{-1}(A_i) \cap P_\xi$ is also of second category in $(P_\xi, \tau_{P_\xi}|_{P_\xi})$ for some $i < \lambda$. So by the first statement, $\varphi^{-1}(A_i) \subseteq \varphi^{-1}(A)$ is of τ_{P_ξ} -second category. This finishes the proof. ■

4.4 Constructible coverings

In this section we give the extension of Theorem 3.10.

Theorem 4.22. *Let $1 < \xi < \omega_1$ be such that $\xi = \xi' + 1$ where ξ' is a successor. Let (X, τ) be an uncountable Polish space and $P \subseteq X$ be a proper $\Pi_\xi^0(\tau)$ set. Then there is a mapping $\Phi: \mathcal{S}_\xi^0(P) \rightarrow \mathcal{P}_\xi^0(P)$ such that $B \subseteq \Phi(B)$ and*

$$P \setminus \bigcup_{i < \omega} \Phi(B^i) \neq \emptyset \quad (B, B^i \in \mathcal{S}_\xi^0(P), i < \omega).$$

Proof. Fix our ξ . First we construct $\Phi = \Phi_\xi$ for $(X, \tau) = (C_\xi, \tau_{C_\xi})$ and $P = P_\xi$. For every $B \in \mathcal{S}_\xi^0(P_\xi)$ fix a decomposition $B = \bigcup_{j < \omega} B_j$ where B_j is $\Pi_{\xi'}^0(\tau_{C_\xi})$ ($j < \omega$). Since the class $\Pi_{\xi'}^0(\tau_{C_\xi})$ has the separation property (see e.g. [4], (22.16) Theorem; and if $\xi = 2$ note that C_2 is zero dimensional) we can take a sequence $(\Delta_n(B))_{n < \omega} \subseteq \Delta_{\xi'}^0(\tau_{C_\xi})$ such that

$$(4.13) \quad \bigcup_{i \leq n} B_i \subseteq \Delta_n(B) \subseteq C_\xi \setminus \bigcup_{i \leq n} P_{(i)}^\xi \quad (n < \omega).$$

Set

$$\Phi_\xi(B) = \bigcap_{m < \omega} \bigcup_{m \leq n < \omega} \Delta_n(B).$$

It is clear that $\Phi_\xi(B)$ is $\Pi_\xi^0(\tau_{C_\xi})$ and (4.13) implies $B \subseteq \Phi_\xi(B) \subseteq P_\xi$. It remains to show that if $B^i \in \mathcal{S}_\xi^0(P_\xi)$ with its fixed decomposition $B^i = \bigcup_{j < \omega} B_j^i$ ($i < \omega$) then we can find a point in $P \setminus \bigcup_{i < \omega} \Phi_\xi(B^i)$.

We apply Claim 4.18 for $A^n = \bigcup_{i \leq n} \Delta_n(B^i)$ ($n < \omega$). We obtain that $P \setminus \bigcup_{n < \omega} A^n \neq \emptyset$. Since

$$\Phi_\xi(B^i) \subseteq \bigcup_{i \leq n < \omega} \Delta_n(B^i) \subseteq \bigcup_{n < \omega} A^n,$$

we have $P \setminus \bigcup_{i < \omega} \Phi_\xi(B^i) \neq \emptyset$, which completes the proof of the special case.

If (X, τ) , P are arbitrary, by Theorem 1.2 we can take a continuous one-to-one map $\varphi: (C_\xi, \tau_{C_\xi}) \rightarrow (X, \tau)$ such that $\varphi^{-1}(P) = P_\xi$. For $B \in \mathcal{S}_\xi^0(P)$ let

$$\Phi(B) = (P \setminus \varphi(P_\xi)) \cup \varphi(\Phi_\xi(\varphi^{-1}(B))).$$

Since homeomorphism preserves the Borel class of sets this definition makes sense and fulfills the requirements. ■

With a more careful approach one could prove Theorem 4.22 for $\xi = \xi' + 1$ where ξ' is a limit ordinal. We think that this is the maximum one can do in ZFC, that is Theorem 4.22 is consistently false when ξ is a limit ordinal. It seems that proving Theorem 4.22 for such a ξ is equivalent with an affirmative answer to the question of A. Miller for Π_ξ^0 generated ideals, the consistency of which is not established at the moment. We return to this problem in Chapter 8.

Chapter 5

Testing the difference hierarchy

In this chapter we show that for $0 < \xi \leq 3$ a proper $D_\eta(\Sigma_\xi^0)$ set cannot be well approximated by a $D_{\eta'}(\Sigma_\xi^0)$ set for $\eta' < \eta$. As usual in this paper, approximation is measured by Baire category in a suitable topology. The interesting feature of this is that such a testing is possible even if $D_\eta(\Sigma_\xi^0)$ is not closed under taking countable union; this is why the result we obtain is weaker than what we have got used to for the Borel hierarchy.

The reasons for restricting ξ are twofold. The more important one is that argument which follows is not applicable for $\xi = \eta = \omega$. The less important, aesthetic reason is that for $\xi \leq 3$ all the ideas which are new comparing to the previous chapters appear without serious technicalities. We will discuss possible generalizations after the proof of Claim 5.5 and in Chapter 8.

We start with the definition of the relevant spaces. From now on in this section let $0 < \xi \leq 3$ and $0 < \eta < \omega_1$.

Definition 5.1. Take sequences $(C_\xi(\alpha), \tau_{C_\xi(\alpha)})$ and $\{P_\xi(\alpha), \tau_{P_\xi(\alpha)}\}$ ($\alpha < \omega_1$) of copies of the Polish space (C_ξ, τ_{C_ξ}) and the $\Pi_\xi^0(\tau_{C_\xi})$ topological Hurewicz test pair $\{P_\xi, \tau_{P_\xi}\}$ defined in Definition 4.19. Let

$$C_{\xi,\eta} = \prod_{\alpha < \eta} C_\xi(\alpha), \quad \tau_{C_{\xi,\eta}} = \prod_{\alpha < \eta} \tau_{C_\xi(\alpha)} \quad (0 < \eta < \omega_1).$$

The difference operator will act on

$$U_{\xi,\eta}(\alpha) = \bigcup_{\beta \leq \alpha} \left(\prod_{\gamma < \beta} C_\xi(\gamma) \times (C_\xi(\beta) \setminus P_\xi(\beta)) \times \prod_{\beta < \gamma < \eta} C_\xi(\gamma) \right) \quad (\alpha < \eta < \omega_1).$$

We set

$$V_{\xi,\eta}(\alpha) = \prod_{\gamma < \alpha} P_\xi(\gamma) \times (C_\xi(\alpha) \setminus P_\xi(\alpha)) \times \prod_{\alpha < \gamma < \eta} C_\xi(\gamma) \quad (\alpha < \eta < \omega_1),$$

$$V_{\xi,\eta}(\eta) = \prod_{\gamma < \eta} P_{\xi}(\gamma),$$

$$W_{\xi,\eta}(\alpha) = \prod_{\gamma < \alpha} P_{\xi}(\gamma) \times \prod_{\alpha \leq \gamma < \eta} C_{\xi}(\gamma) \quad (\alpha \leq \eta < \omega_1).$$

Now for every $0 < \eta < \omega_1$ let

$$P_{\xi,\eta} = D_{\eta} \left((U_{\xi,\eta}(\alpha))_{\alpha < \eta} \right) = \bigcup \{V_{\xi,\eta}(\alpha) : \alpha \leq \eta, \alpha \text{ is odd} \leftrightarrow \eta \text{ is even}\},$$

$$Q_{\xi,\eta} = C_{\xi,\eta} \setminus P_{\xi,\eta} = \bigcup \{V_{\xi,\eta}(\alpha) : \alpha \leq \eta, \alpha \text{ is odd} \leftrightarrow \eta \text{ is odd}\}$$

and define the topologies

$$\tau_{\xi,\eta}^< = \prod_{\alpha < \eta} \tau_{P_{\xi}(\alpha)}^<, \quad \tau_{\xi,\eta}^<(\alpha) = \prod_{\gamma < \alpha} \tau_{P_{\xi}(\gamma)}^< \times \prod_{\alpha \leq \gamma < \eta} \tau_{P_{\xi}(\gamma)} \quad (\alpha \leq \eta < \omega_1),$$

$$\tau_{\xi,\eta} = \tau_{\xi,\eta}^<(0) [\{W_{\xi,\eta}(\alpha) : \alpha \leq \eta\}].$$

It is clear that $U_{\xi,\eta}(\alpha)$ is $\Sigma_{\xi}^0(\tau_{C_{\xi,\eta}})$ ($\alpha < \eta < \omega_1$). Observe that $P_{\xi,\eta}, Q_{\xi,\eta}$ ($0 < \eta < \omega_1$) are $\tau_{\xi,\eta}$ -open sets and that the sets $V_{\xi,\eta}(\alpha)$ ($\alpha \leq \eta$) are pairwise disjoint. Notice that $W_{\xi,\eta}(\eta) = V_{\xi,\eta}(\eta)$ and $\tau_{\xi,\eta}^< = \tau_{\xi,\eta}^<(\eta)$ ($0 < \eta < \omega_1$). The testing theorem can be stated as follows.

Theorem 5.2. *Let $0 < \xi \leq 3$ and $0 < \eta < \omega_1$ be fixed and consider a $D_{\eta}(\Sigma_{\xi}^0(\tau_{C_{\xi,\eta}}))$ set $A \subseteq C_{\xi,\eta}$. Let G be $\tau_{\xi,\eta}^<$ -open; and if $\xi = 1$ suppose that $W_{1,\eta}(\eta) \in G$ in addition. Then if $A \cap Q_{\xi,\eta}$ is $\tau_{\xi,\eta}$ -residual in $G \cap Q_{\xi,\eta}$ then $A \cap P_{\xi,\eta}$ is of $\tau_{\xi,\eta}$ -second category in $G \cap P_{\xi,\eta}$.*

We formulate a corollary of this theorem.

Corollary 5.3. *Let $0 < \xi \leq 3$ and $0 < \eta < \omega_1$ be fixed.*

1. *If $A \subseteq C_{\xi,\eta}$ is a $D_{\eta}(\Sigma_{\xi}^0(\tau_{C_{\xi,\eta}}))$ set and $Q_{\xi,\eta} \subseteq A$ then $A \cap P_{\xi,\eta}$ is of $\tau_{\xi,\eta}$ -second category.*
2. *If $\check{A} \subseteq C_{\xi,\eta}$ is a $\check{D}_{\eta}(\Sigma_{\xi}^0(\tau_{C_{\xi,\eta}}))$ set and $P_{\xi,\eta} \subseteq \check{A}$ then $\check{A} \cap Q_{\xi,\eta}$ is of $\tau_{\xi,\eta}$ -second category.*

Proof. The first statement is the special $G = C_{\xi,\eta}$ case of Theorem 5.2, while the second follows from Theorem 5.2 applied to $A = C_{\xi,\eta} \setminus \check{A}$ and $G = C_{\xi,\eta}$. ■

Corollary 5.3 has already the feature of dichotomy we are looking for: it allows to derive from the information that a set A is simply structured the conclusion that either it does not contain $Q_{\xi,\eta}$ or it is big in category in $C_{\xi,\eta} \setminus Q_{\xi,\eta}$, and vice versa.

The proof of Theorem 5.2 is based on Claim 5.5 stating that sets like $W_{\xi,\eta}(\eta)$ behave as topological Hurewicz test sets with the topology $\tau_{\xi,\eta}$. Before stating and proving it we need the usual lemma on the coincidence of topologies.

Lemma 5.4. *Let $0 < \eta < \omega_1$ and $\alpha \leq \eta$ be fixed. Then*

1. *for $\xi = 1$, if G is $\tau_{1,\eta}^<$ -open and $W_{1,\eta}(\eta) \in G$ then $G \cap V_{1,\eta}(\gamma) \neq \emptyset$ ($\gamma \leq \eta$);*
2. *for $\xi = 2, 3$, $V_{\xi,\eta}(\alpha)$ is a $\tau_{\xi,\eta}^<$ -dense set ($\alpha \leq \gamma \leq \eta$);*
3. *for $\xi = 2, 3$, $W_{\xi,\eta}(\alpha)$ is a $\tau_{\xi,\eta}^<$ -dense $\Pi_2^0(\tau_{\xi,\eta}^<)$ set ($\alpha \leq \gamma \leq \eta$);*
4. *if G is basic $\tau_{\xi,\eta}^<$ (0)-open and $G \cap V_{\xi,\eta}(\alpha) \neq \emptyset$ then G is basic $\tau_{\xi,\eta}^<$ (γ)-open ($\gamma \leq \alpha$);*
5. *if G is basic $\tau_{\xi,\eta}$ -open and $G \cap V_{\xi,\eta}(\alpha) \neq \emptyset$ then there is a basic $\tau_{\xi,\eta}^<$ (α)-open set G' such that $G \cap V_{\xi,\eta}(\alpha) = G' \cap V_{\xi,\eta}(\alpha)$;*
6. *the topologies $\tau_{\xi,\eta}|_{V_{\xi,\eta}(\alpha)}$ and $\tau_{\xi,\eta}^<$ (γ)| $_{V_{\xi,\eta}(\alpha)}$ ($\gamma \leq \alpha$) coincide;*
7. *if G is basic $\tau_{\xi,\eta}^<$ (0)-open and $G \cap W_{\xi,\eta}(\alpha) \neq \emptyset$ then G is basic $\tau_{\xi,\eta}^<$ (γ)-open ($\gamma \leq \alpha$);*
8. *if G is basic $\tau_{\xi,\eta}$ -open and $G \cap W_{\xi,\eta}(\eta) \neq \emptyset$ then there is a basic $\tau_{\xi,\eta}^<$ -open set G' such that $G' \cap W_{\xi,\eta}(\eta) = G \cap W_{\xi,\eta}(\eta)$;*
9. *the topologies $\tau_{\xi,\eta}|_{W_{\xi,\eta}(\eta)}$ and $\tau_{\xi,\eta}^<$ (γ)| $_{W_{\xi,\eta}(\eta)}$ ($\gamma \leq \eta$) coincide.*

Proof. Since $\tau_{1,\eta}^< = \tau_{C_{1,\eta}}$, 1 is obvious. By Definition 4.19, both $P_\xi(\beta)$ and $C_\xi(\beta) \setminus P_\xi(\beta)$ are $\tau_{P_\xi(\beta)}^<$ -dense ($\beta < \alpha$) sets, so 2 follows. By Theorem 4.7.1 and Claim 4.5.1, $P_\xi(\beta)$ is a $\tau_{P_\xi(\beta)}^<$ -residual $\Pi_2^0(\tau_{P_\xi(\beta)}^<)$ set ($\beta < \alpha$), so we have 3.

For 4, let G be basic $\tau_{\xi,\eta}^<$ (0)-open with $G \cap V_{\xi,\eta}(\alpha) \neq \emptyset$. If for some $\gamma < \alpha$, $\text{Pr}_{C_\gamma}(G)$ is proper basic $\tau_{P_\xi(\gamma)}^<$ -open then $\text{Pr}_{C_\gamma}(G) \cap P_\xi(\gamma) = \emptyset$ hence $G \cap V_{\xi,\eta}(\alpha) = \emptyset$, which is not the case. So $\text{Pr}_{C_\gamma}(G)$ is $\tau_{P_\xi(\gamma)}^<$ -open ($\gamma < \alpha$), thus 4 holds.

For 5, let G be basic $\tau_{\xi,\eta}$ -open, say $G = G' \cap W_{\xi,\eta}(\beta)$ where G' is basic $\tau_{\xi,\eta}^<$ (0)-open and $\beta \leq \eta$, such that $G \cap V_{\xi,\eta}(\alpha) \neq \emptyset$. Since $V_{\xi,\eta}(\alpha) \subseteq W_{\xi,\eta}(\gamma)$ ($\gamma \leq \alpha$) and $V_{\xi,\eta}(\alpha) \cap W_{\xi,\eta}(\gamma) = \emptyset$ ($\alpha < \gamma \leq \eta$), we have $G \cap V_{\xi,\eta}(\alpha) = G' \cap V_{\xi,\eta}(\alpha) \neq \emptyset$. Thus by 4, G' is basic $\tau_{\xi,\eta}^<$ (α)-open so 5 holds.

Since $\tau_{\xi,\eta}^<$ (γ) ($\gamma \leq \alpha$) is finer than $\tau_{\xi,\eta}^<$ (α) and coarser than $\tau_{\xi,\eta}$, 5 immediately gives 6.

For 7, observe that $W_{\xi,\eta}(\alpha) = \bigcup_{\alpha \leq \beta \leq \eta} V_{\xi,\eta}(\beta)$. Thus $G \cap V_{\xi,\eta}(\beta) \neq \emptyset$ for some $\alpha \leq \beta \leq \eta$. So by 4, G is basic $\tau_{\xi,\eta}^<$ (γ)-open ($\gamma \leq \beta$), as required.

For 8, let G be a basic $\tau_{\xi,\eta}$ -open set, say $G = G' \cap W_{\xi,\eta}(\beta)$ where G' is basic $\tau_{\xi,\eta}^<$ (0)-open and $\beta \leq \eta$, and suppose that $G \cap W_{\xi,\eta}(\eta) \neq \emptyset$. From $W_{\xi,\eta}(\eta) \subseteq W_{\xi,\eta}(\gamma)$ ($\gamma \leq \eta$) we get $G' \cap W_{\xi,\eta}(\eta) = G \cap W_{\xi,\eta}(\eta) \neq \emptyset$ thus from 7 we have that G' is $\tau_{\xi,\eta}^<$ -open, as stated.

Since $\tau_{\xi,\eta}^<$ (γ) ($\gamma \leq \eta$) is finer than $\tau_{\xi,\eta}^<$ and coarser than $\tau_{\xi,\eta}$, 9 immediately follows from 8, so the proof is complete. ■

Claim 5.5. *Let $0 < \eta < \omega_1$, (Y, σ) be an arbitrary nonempty Polish space and $G \subseteq C_{\xi,\eta} \times Y$ be a nonempty $\tau_{\xi,\eta}^<$ $\times \sigma$ -open set, such that if $\xi = 1$ then $G \cap (W_{1,\eta}(\eta) \times Y) \neq \emptyset$ in*

addition. If $A \subseteq C_{\xi,\eta} \times Y$ is $\Sigma_{\xi}^0(\tau_{C_{\xi,\eta}} \times \sigma)$ and $A \cap (W_{\xi,\eta}(\eta) \times Y)$ is of $(\tau_{\xi,\eta} \times \sigma) |_{W_{\xi,\eta}(\eta) \times Y}$ -second category in $G \cap (W_{\xi,\eta}(\eta) \times Y)$ then there is a nonempty basic $\tau_{\xi,\eta}^{\leq} \times \sigma$ -open set $G_0 \subseteq G$ such that A is $\tau_{\xi,\eta} \times \sigma$ -residual in G_0 and $G_0 \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$.

Proof. Before starting the proof, observe that for $\xi = 2, 3$ by Lemma 5.4.3, $G \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$ for every nonempty $\tau_{\xi,\eta}^{\leq} \times \sigma$ -open set G .

For $\xi = 1$ we have that $\tau_{1,\eta}^{\leq} = \tau_{C_{1,\eta}}$ and that A is $\tau_{C_{1,\eta}} \times \sigma$ -open so every basic $\tau_{C_{1,\eta}} \times \sigma$ -open set G_0 can be chosen which satisfies $G_0 \subseteq G \cap A$ and $W_{1,\eta}(\eta) \in \text{Pr}_{C_{1,\eta}}(G_0)$.

Let now $\xi = 2$ or $\xi = 3$. Since A is $\Sigma_{\xi}^0(\tau_{C_{\xi,\eta}} \times \sigma)$, there is a $\Pi_{\xi-1}^0(\tau_{C_{\xi,\eta}} \times \sigma)$ set $B \subseteq A$ and a nonempty basic $\tau_{\xi,\eta} \times \sigma$ -open set $G^* \subseteq G$ such that $G^* \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$ and B is $(\tau_{\xi,\eta} \times \sigma) |_{W_{\xi,\eta}(\eta) \times Y}$ -residual in $G^* \cap (W_{\xi,\eta}(\eta) \times Y)$. By Lemma 5.4.8 there is a basic $\tau_{\xi,\eta}^{\leq} \times \sigma$ -open set G_0 for which $G_0 \cap (W_{\xi,\eta}(\eta) \times Y) = G^* \cap (W_{\xi,\eta}(\eta) \times Y)$. Then by Lemma 5.4.9, B is $(\tau_{\xi,\eta}^{\leq} \times \sigma) |_{W_{\xi,\eta}(\eta) \times Y}$ -residual in $G_0 \cap (W_{\xi,\eta}(\eta) \times Y)$. We show that B is $\tau_{\xi,\eta} \times \sigma$ -residual in G_0 ; then by $B \subseteq A$, G_0 fulfills the requirements.

For $\xi = 2$ we have $\tau_{2,\eta}^{\leq} = \tau_{2,\eta}^{\leq}(\eta) = \tau_{C_{2,\eta}}$. So by Lemma 5.4.9,

$$(\tau_{2,\eta} \times \sigma) |_{W_{2,\eta}(\eta) \times Y} = (\tau_{2,\eta}^{\leq}(\eta) \times \sigma) |_{W_{2,\eta}(\eta) \times Y} = (\tau_{C_{2,\eta}} \times \sigma) |_{W_{2,\eta}(\eta) \times Y}.$$

Hence B is $(\tau_{C_{2,\eta}} \times \sigma) |_{W_{2,\eta}(\eta) \times Y}$ -residual in $G_0 \cap (W_{2,\eta}(\eta) \times Y)$. By Lemma 5.4.3 we have that $W_{2,\eta}(\eta) \times Y$ is $\tau_{C_{2,\eta}} \times \sigma$ -residual in $X \times Y$ so the $\Pi_1^0(\tau_{C_{2,\eta}} \times \sigma)$ set B contains G_0 . This proves the $\xi = 2$ case.

For $\xi = 3$ suppose that B is not $\tau_{3,\eta} \times \sigma$ -residual in G_0 . Since $C_{3,\eta} \setminus B$ is $\Sigma_2^0(\tau_{C_{3,\eta}} \times \sigma)$, there is some nonempty basic $\tau_{3,\eta} \times \sigma$ -open set $G' \subseteq G_0$ and $\Pi_1^0(\tau_{C_{3,\eta}} \times \sigma)$ set $F \subseteq C_{3,\eta} \setminus B$ such that F is $\tau_{3,\eta} \times \sigma$ -residual in G' , that is $G' \subseteq F$ which gives $\text{cl}_{\tau_{3,\eta}^{\leq} \times \sigma}(G') \subseteq F$. We can assume that G' is not $\tau_{3,\eta}^{\leq}(0)$ -open, so $G' = (W_{3,\eta}(\alpha) \times Y) \cap G''$ for some $\alpha \leq \eta$ where G'' is basic $\tau_{3,\eta}^{\leq}(0) \times \sigma$ -open; thus by Lemma 5.4.7, G'' is $\tau_{3,\eta}^{\leq}(\alpha) \times \sigma$ -open. Then for some finite set $I \subseteq \eta$, $\text{Pr}_{C_3(\gamma)}(G'')$ is basic $\tau_{P_3(\gamma)}^{\leq}$ -open ($\gamma \in I \cap \alpha$), $\text{Pr}_{C_3(\gamma)}(G'')$ is basic $\tau_{P_3(\gamma)}$ -open ($\gamma \in I \setminus \alpha$) and $\text{Pr}_{C_3(\gamma)}(G'') = C_3(\gamma)$ ($\gamma \in \eta \setminus I$). By Lemma 5.4.3 we have

$$(5.1) \quad G'' \subseteq \text{cl}_{\tau_{3,\eta}^{\leq}(\alpha) \times \sigma}(G') \subseteq \text{cl}_{\tau_{3,\eta}^{\leq} \times \sigma}(G').$$

By Lemma 4.21,

$$\text{int}_{\tau_{P_3(\gamma)}^{\leq}}(\text{cl}_{\tau_{P_3(\gamma)}^{\leq}}(\text{Pr}_{C_3(\gamma)}(G''))) \neq \emptyset \quad (\gamma \in I).$$

So by the regularity of $\text{Pr}_Y(G'')$ in Y ,

$$\text{int}_{\tau_{3,\eta}^{\leq} \times \sigma}(\text{cl}_{\tau_{3,\eta}^{\leq} \times \sigma}(G'')) \neq \emptyset$$

which by (5.1) gives

$$H = \text{int}_{\tau_{3,\eta}^{\leq} \times \sigma}(\text{cl}_{\tau_{3,\eta}^{\leq} \times \sigma}(G')) \neq \emptyset.$$

Since G_0 is regular $\tau_{3,\eta}^{\leq} \times \sigma$ -open,

$$H \subseteq \text{int}_{\tau_{3,\eta}^{\leq} \times \sigma}(\text{cl}_{\tau_{3,\eta}^{\leq} \times \sigma}(G_0)) = G_0,$$

and from $\text{cl}_{\tau_{3,\eta}^{\times} \times \sigma}(G') \subseteq F$ we have $H \subseteq F$. Since B is $(\tau_{3,\eta}^{\times} \times \sigma)|_{W_{3,\eta}(\eta) \times Y}$ -residual in $G_0 \cap (W_{3,\eta}(\eta) \times Y)$ and by Lemma 5.4.3, $W_{3,\eta}(\eta)$ is $\tau_{3,\eta}^{\times}$ -residual in $C_{3,\eta}$, B is $\tau_{3,\eta}^{\times} \times \sigma$ -residual in G_0 . But $H \subseteq G_0 \setminus B$, a contradiction. ■

In our present approach the validity of this claim limits our capacities in testing the difference hierarchy. It is fairly simple to see that the proof of the lemma can be repeated for $3 < \xi < \omega$, $0 < \eta < \omega_1$; it is an imitation of the proof of Theorem 4.7 for the $\Pi_{\xi}^0(\tau_{C_{\xi,\eta}})$ set $W_{\xi,\eta}$. Also, if $\eta < \omega$ then the statement of the lemma follows for every $0 < \xi < \omega_1$ from the fact that $\{P_{\xi}, \tau_{P_{\xi}}\}$ is a topological Hurewicz test pair, using Kuratowski-Ulam Theorem as in the proof of Claim 4.5.5. The handicap of this approach is that Claim 5.5 is false for $\xi = \eta = \omega$. This failure is the main reason why we restrict our attention to $\xi \leq 3$.

In the proof of Theorem 5.2 the product structure of Definition 5.1 must be exploited. So we prove it in the following more general form. When Y is a single point, we get back Theorem 5.2.

Theorem 5.6. *Let $0 < \xi \leq 3$ and $0 < \eta < \omega_1$ be fixed. Let (Y, σ) be an arbitrary nonempty Polish space and consider a $D_{\eta}(\Sigma_{\xi}^0(\tau_{C_{\xi,\eta}} \times \sigma))$ set $A \subseteq C_{\xi,\eta} \times Y$. Let $G \subseteq C_{\xi,\eta} \times Y$ be $\tau_{\xi,\eta}^{\times} \times \sigma$ -open, and for $\xi = 1$ suppose that $G \cap (W_{1,\eta}(\eta) \times Y) \neq \emptyset$ in addition. Then if $A \cap (Q_{\xi,\eta} \times Y)$ is $\tau_{\xi,\eta} \times \sigma$ -residual in $G \cap (Q_{\xi,\eta} \times Y)$ then $A \cap (P_{\xi,\eta} \times Y)$ is of $\tau_{\xi,\eta} \times \sigma$ -second category in $G \cap (P_{\xi,\eta} \times Y)$.*

Proof. We prove the statement by induction on η . Let first $\eta = 1$, then A is a $\Sigma_{\xi}^0(\tau_{C_{\xi,1}} \times \sigma)$. By Claim 4.20, Claim 4.10 and Claim 4.5.5, $\{P_{\xi} \times Y, \tau_{P_{\xi}} \times \sigma\}$ is a topological Hurewicz test pair in $C_{\xi,1} \times Y$, so the statement follows from Claim 4.6.

Suppose now that $1 < \eta < \omega_1$ and that the statement holds for $\eta' < \eta$. Let $A = D_{\eta}((A_{\alpha})_{\alpha < \eta})$ with $\Sigma_{\xi}^0(\tau_{C_{\xi,\eta}} \times \sigma)$ sets A_{α} ($\alpha < \eta$) satisfying $A_{\beta} \subseteq A_{\alpha}$ ($\beta \leq \alpha < \eta$). We have that $W_{\xi,\eta}(\eta) \times Y \subseteq Q_{\xi,\eta} \times Y$ is $\tau_{\xi,\eta} \times \sigma$ -open. By assumption for $\xi = 1$ and by Lemma 5.4.3 for $\xi = 2, 3$, $G \cap W_{\xi,\eta}(\eta) \times Y \neq \emptyset$ so A is of $\tau_{\xi,\eta} \times \sigma$ -second category in $G \cap W_{\xi,\eta}(\eta) \times Y$. Thus there is a minimal α such that the parity of α and η are different and for some basic $\tau_{\xi,\eta} \times \sigma$ -open set $G^* \subseteq G$, A_{α} of $\tau_{\xi,\eta} \times \sigma$ -second category in the nonempty $G^* \cap (W_{\xi,\eta}(\eta) \times Y)$. Then by Lemma 5.4.8 there is a $\tau_{\xi,\eta}^{\times} \times \sigma$ -open set G' such that $G^* \cap (W_{\xi,\eta}(\eta) \times Y) = G' \cap (W_{\xi,\eta}(\eta) \times Y) \neq \emptyset$.

We apply Claim 5.5 for A_{α} and G' ; we obtain that for some nonempty basic $\tau_{\xi,\eta}^{\times} \times \sigma$ -open set $G_0 \subseteq G'$, A_{α} is $\tau_{\xi,\eta} \times \sigma$ -residual in G_0 , and $G \cap (W_{1,\eta}(\eta) \times Y) \neq \emptyset$. So in particular, A_{α} is $\tau_{\xi,\eta} \times \sigma$ -residual in $G_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$, which is nonempty by Lemma 5.4.1 if $\xi = 1$ and by Lemma 5.4.2 if $\xi = 2, 3$. But the parity of α and η differ, so $V_{\xi,\eta}(\alpha) \times Y \subseteq P_{\xi,\eta} \times Y$. That is if $A_{\alpha} \setminus \bigcup_{\beta < \alpha} A_{\beta}$ is also of $\tau_{\xi,\eta} \times \sigma$ -second category in $G_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$ then we are done.

Suppose that this is not the case. Then there is a $\beta < \alpha$ and a basic $\tau_{\xi,\eta} \times \sigma$ -open set $G_0^* \subseteq G_0$ such that $G_0^* \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$ and A_{β} is $\tau_{\xi,\eta} \times \sigma$ -residual in $G_0^* \cap (V_{\xi,\eta}(\alpha) \times Y)$, and the parity of β and α differ, that is parity of β and η coincide. By Lemma 5.4.5,

there is a basic $\tau_{\xi,\eta}^<(\alpha) \times \sigma$ -open set G'_0 such that $G_0^* \cap (V_{\xi,\eta}(\alpha) \times Y) = G'_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$. Since G_0 is basic $\tau_{\xi,\eta}^< \times \sigma$ -open, we can and do assume that $G'_0 \subseteq G_0$. Then by Lemma 5.4.6, A_β is $\tau_{\xi,\eta}^<(\alpha) \times \sigma$ -residual in $G'_0 \cap (V_{\xi,\eta}(\alpha) \times Y)$. By passing to a subset if necessary, we assume that $\text{Pr}_{C_\xi(\alpha)}(G'_0)$ is proper $\tau_{P_\xi}(\alpha)$ -open, that is

$$(5.2) \quad \text{Pr}_{C_\xi(\alpha)}(G'_0) \subseteq C_\xi(\alpha) \setminus P_\xi(\alpha).$$

Set $\tilde{\eta} = \alpha$,

$$\tilde{Y} = \left(\prod_{\alpha \leq \gamma < \eta} C_\xi(\gamma) \right) \times Y, \quad \tilde{\sigma} = \left(\prod_{\alpha \leq \gamma < \eta} \tau_{P_\xi(\gamma)} \right) \times \sigma$$

and $\tilde{G} = G'_0$. With this setting, using (5.2),

$$(5.3) \quad \tilde{G} \cap \left(W_{\xi,\tilde{\eta}}(\tilde{\eta}) \times \tilde{Y} \right) = G'_0 \cap (V_{\xi,\eta}(\alpha) \times Y) \neq \emptyset$$

and by Lemma 5.4.9,

$$(5.4) \quad \begin{aligned} (\tau_{\xi,\tilde{\eta}} \times \tilde{\sigma})|_{\tilde{G} \cap (W_{\xi,\tilde{\eta}}(\tilde{\eta}) \times \tilde{Y})} &= (\tau_{\xi,\tilde{\eta}}^< \times \tilde{\sigma})|_{\tilde{G} \cap (W_{\xi,\tilde{\eta}}(\tilde{\eta}) \times \tilde{Y})} = \\ &= (\tau_{\xi,\tilde{\eta}}^< \times \tilde{\sigma})|_{G'_0 \cap (V_{\xi,\eta}(\alpha) \times Y)} = (\tau_{\xi,\eta}^<(\alpha) \times \sigma)|_{G'_0 \cap (V_{\xi,\eta}(\alpha) \times Y)}. \end{aligned}$$

We apply Claim 5.5 in $C_{\xi,\tilde{\eta}} \times \tilde{Y}$ for A_β , which by (5.3) and (5.4) is $(\tau_{\xi,\tilde{\eta}} \times \tilde{\sigma})|_{W_{\xi,\tilde{\eta}}(\tilde{\eta}) \times \tilde{Y}}$ -residual in the nonempty $\tilde{G} \cap \left(W_{\xi,\tilde{\eta}}(\tilde{\eta}) \times \tilde{Y} \right)$. We get that for some nonempty basic $\tau_{\xi,\tilde{\eta}}^< \times \tilde{\sigma}$ -open set $\tilde{G}_0 \subseteq \tilde{G}$, A_β is $\tau_{\xi,\tilde{\eta}} \times \tilde{\sigma}$ -residual in \tilde{G}_0 and $\tilde{G}_0 \cap \left(W_{\xi,\tilde{\eta}}(\tilde{\eta}) \times \tilde{Y} \right) \neq \emptyset$. In particular, $\tilde{G}_0 \cap \left(V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y} \right) \neq \emptyset$ ($\gamma \leq \tilde{\eta}$) by Lemma 5.4.1 if $\xi = 1$ and by Lemma 5.4.2 if $\xi = 2, 3$, and A_β is $\tau_{\xi,\tilde{\eta}} \times \tilde{\sigma}$ -residual in $\tilde{G}_0 \cap \left(V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y} \right)$ ($\gamma \leq \tilde{\eta}$).

Now $\tilde{G}_0 \subseteq G$ is $\tau_{\xi,\eta} \times \sigma$ -open and $V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y}$ is also $\tau_{\xi,\eta} \times \sigma$ -open ($\gamma \leq \tilde{\eta}$). So A is $\tau_{\xi,\eta} \times \sigma$ -residual in $\tilde{G}_0 \cap \left(V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y} \right)$ for every $\gamma \leq \tilde{\eta}$ with parity different from the parity of $\tilde{\eta}$. By (5.2) and $\tilde{G}_0 \subseteq G'_0$,

$$\text{Pr}_{C_\xi(\alpha)}(\tilde{G}_0) \subseteq C_\xi(\alpha) \setminus P_\xi(\alpha),$$

so we have $\tilde{G}_0 \cap (V_{\xi,\eta}(\gamma) \times Y) = \tilde{G}_0 \cap \left(V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y} \right) \neq \emptyset$ ($\gamma \leq \tilde{\eta}$). Then by Lemma 5.4.6,

$$(5.5) \quad \begin{aligned} (\tau_{\xi,\eta} \times \sigma)|_{\tilde{G}_0 \cap (V_{\xi,\eta}(\gamma) \times Y)} &= (\tau_{\xi,\eta}^<(0) \times \sigma)|_{\tilde{G}_0 \cap (V_{\xi,\eta}(\gamma) \times Y)} = \\ &= (\tau_{\xi,\tilde{\eta}}^<(0) \times \tilde{\sigma})|_{\tilde{G}_0 \cap (V_{\xi,\eta}(\gamma) \times Y)} = \\ &= (\tau_{\xi,\tilde{\eta}}^<(0) \times \tilde{\sigma})|_{\tilde{G}_0 \cap (V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y})} = (\tau_{\xi,\tilde{\eta}} \times \tilde{\sigma})|_{\tilde{G}_0 \cap (V_{\xi,\tilde{\eta}}(\gamma) \times \tilde{Y})} \quad (\gamma \leq \tilde{\eta}). \end{aligned}$$

We get that A is $\tau_{\xi, \tilde{\eta}} \times \tilde{\sigma}$ -residual in $\tilde{G}_0 \cap \left(V_{\xi, \tilde{\eta}}(\gamma) \times \tilde{Y} \right)$ for every $\gamma \leq \tilde{\eta}$ with parity different from the parity of $\tilde{\eta}$. Since A_β is also $\tau_{\xi, \tilde{\eta}} \times \tilde{\sigma}$ -residual in $\tilde{G}_0 \cap \left(V_{\xi, \tilde{\eta}}(\gamma) \times \tilde{Y} \right)$ ($\gamma \leq \tilde{\eta}$), this is possible only if $D_\beta \left((A_\gamma)_{\gamma < \beta} \right)$ is $\tau_{\xi, \tilde{\eta}} \times \tilde{\sigma}$ -residual in $\tilde{G}_0 \cap \left(V_{\xi, \tilde{\eta}}(\gamma) \times \tilde{Y} \right)$ for every $\gamma \leq \tilde{\eta}$ with parity different from the parity of $\tilde{\eta}$.

Set $\underline{\eta} = \beta$,

$$\underline{Y} = \left(\prod_{\beta \leq \gamma < \alpha} C_\xi(\gamma) \right) \times \tilde{Y}, \quad \underline{\sigma} = \left(\prod_{\beta \leq \gamma < \alpha} \tau_{P_\xi(\gamma)} \right) \times \tilde{\sigma},$$

$\underline{A} = D_{\underline{\eta}} \left((A_\gamma)_{\gamma < \underline{\eta}} \right)$ and $\underline{G} = \tilde{G}_0 \cap H \neq \emptyset$ where H is nontrivial only on the β^{th} coordinate and $\text{Pr}_{C_\xi(\beta)}(H)$ is proper basic $\tau_{P_\xi(\beta)}$ -open, i.e. $\text{Pr}_{C_\xi(\beta)}(H) \subseteq C_\xi(\beta) \setminus P_\xi(\beta)$. Since \tilde{G}_0 is basic $\tau_{\xi, \tilde{\eta}} \times \tilde{\sigma}$ -open, it is $\tau_{\xi, \tilde{\eta}} \times \tilde{\sigma}$ -open and so \underline{G} is basic $\tau_{\xi, \underline{\eta}} \times \underline{\sigma}$ -open. As above, we have

$$\underline{G}_0 \cap \left(V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y} \right) = \underline{G}_0 \cap \left(V_{\xi, \tilde{\eta}}(\gamma) \times \tilde{Y} \right) \neq \emptyset \quad (\gamma \leq \underline{\eta})$$

so by Lemma 5.4.6,

$$\begin{aligned} (5.6) \quad \left(\tau_{\xi, \underline{\eta}} \times \underline{\sigma} \right) |_{\underline{G}_0 \cap (V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y})} &= \left(\tau_{\xi, \underline{\eta}}^<(0) \times \underline{\sigma} \right) |_{\underline{G}_0 \cap (V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y})} = \\ &= \left(\tau_{\xi, \tilde{\eta}}^<(0) \times \tilde{\sigma} \right) |_{\underline{G}_0 \cap (V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y})} = \\ &= \left(\tau_{\xi, \tilde{\eta}}^<(0) \times \tilde{\sigma} \right) |_{\underline{G}_0 \cap (V_{\xi, \tilde{\eta}}(\gamma) \times \tilde{Y})} = \left(\tau_{\xi, \tilde{\eta}} \times \tilde{\sigma} \right) |_{\underline{G}_0 \cap (V_{\xi, \tilde{\eta}}(\gamma) \times \tilde{Y})} \quad (\gamma \leq \underline{\eta}). \end{aligned}$$

Since $\underline{A} = D_{\underline{\eta}} \left((A_\gamma)_{\gamma < \underline{\eta}} \right)$ is $\tau_{\xi, \tilde{\eta}} \times \tilde{\sigma}$ -residual in $\tilde{G}_0 \cap \left(V_{\xi, \tilde{\eta}}(\gamma) \times \tilde{Y} \right)$, we get that \underline{A} is $\tau_{\xi, \underline{\eta}} \times \underline{\sigma}$ -residual in $\underline{G} \cap \left(V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y} \right)$ for every $\gamma \leq \underline{\eta}$ with parity equal to the parity of $\underline{\eta}$. That is, \underline{A} is $\tau_{\xi, \underline{\eta}} \times \underline{\sigma}$ -residual in $\underline{G} \cap \left(Q_{\xi, \underline{\eta}} \times \underline{Y} \right)$. So by the induction hypothesis \underline{A} is of $\tau_{\xi, \underline{\eta}} \times \underline{\sigma}$ -second category in $\underline{G} \cap \left(P_{\xi, \underline{\eta}} \times \underline{Y} \right)$. Since $\underline{A} = D_\beta \left((A_\gamma)_{\gamma < \beta} \right) \subseteq A$, this means that A is of $\tau_{\xi, \underline{\eta}} \times \underline{\sigma}$ -second category in $P_{\xi, \underline{\eta}} \times \underline{Y}$. We have

$$V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y} = V_{\xi, \underline{\eta}}(\gamma) \times Y \quad (\gamma < \underline{\eta})$$

so by Lemma 5.4.6,

$$\begin{aligned} (5.7) \quad \left(\tau_{\xi, \underline{\eta}} \times \underline{\sigma} \right) |_{V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y}} &= \left(\tau_{\xi, \underline{\eta}}^<(0) \times \underline{\sigma} \right) |_{V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y}} = \\ &= \left(\tau_{\xi, \underline{\eta}}^<(0) \times \sigma \right) |_{V_{\xi, \underline{\eta}}(\gamma) \times \underline{Y}} = \\ &= \left(\tau_{\xi, \underline{\eta}}^<(0) \times \sigma \right) |_{V_{\xi, \underline{\eta}}(\gamma) \times Y} = \left(\tau_{\xi, \underline{\eta}} \times \sigma \right) |_{V_{\xi, \underline{\eta}}(\gamma) \times Y} \quad (\gamma < \underline{\eta}). \end{aligned}$$

Since β and η have the same parity,

$$P_{\xi, \eta} \times \underline{Y} = P_{\xi, \beta} \times \prod_{\beta \leq \gamma < \eta} C_{\xi}(\gamma) \times Y = \bigcup \{V_{\xi, \eta}(\alpha) : \alpha < \beta, \alpha \text{ is odd} \leftrightarrow \beta \text{ is even}\} \subseteq P_{\xi, \eta} \times Y.$$

So A is of $\tau_{\xi, \eta} \times \sigma$ -second category in $P_{\xi, \eta} \times Y$, which completes the proof. ■

Possible further extensions of testing theorems will be discussed in Chapter 8.

Chapter 6

\mathcal{I} -convergent functions

It is a fact of life that the class of continuous real functions is not closed under pointwise convergence: instead, we obtain a realization of the Baire-1 functions. On the other hand, it is an easy exercise that the pointwise limit of a sequence of continuous functions with length ω_1 is necessarily continuous.

This problem and other properties of the pointwise convergence of transfinite sequences of real functions has been first considered by W. Sierpiński [20]. In particular, he studied which class of functions will be closed under such convergencies. Since most of the classes, for example the class of Baire- ξ functions for $\xi \geq 2$, are not, T. Natkaniec [17] introduced a stronger notion of pointwise convergence. We recall the precise setting in the following definition.

Definition 6.1. Let λ be a cardinal, (X, τ) be a Polish space, (Y, d) be a separable metric space and consider an ideal \mathcal{I} on λ . We say that a sequence of functions $f_\alpha: X \rightarrow Y$ ($\alpha < \lambda$) \mathcal{I} -converges to the function $f: X \rightarrow Y$, in notation $f_\alpha \rightarrow_{\mathcal{I}} f$, if

$$\{\alpha < \lambda: f_\alpha(x) \neq f(x)\} \in \mathcal{I}$$

for every $x \in X$.

Similarly, we write $f_\alpha \rightarrow_{\frac{d}{\mathcal{I}}} f$ if for every $\varepsilon > 0$ and $x \in X$ we have

$$\{\alpha < \lambda: d(f(x), f_\alpha(x)) > \varepsilon\} \in \mathcal{I}.$$

In case of the ordinary ω_1 convergence, as used in [5] and [20], we have $\lambda = \omega_1$ and $\mathcal{I} = [\omega_1]^{\leq \omega}$, that is the ideal of countable subsets of ω_1 . However, our motivating theorem, answering Problem 1 in [17] on page 490, is related to the particular case, when the ideal contains the finite subsets of ω_1 , that is $\mathcal{I}_< = [\omega_1]^{< \omega}$.

Theorem 6.2. Let (X, τ) be a Polish space, (Y, d) be a separable metric space, and for a fixed $\xi < \omega_1$ consider a family $f_\alpha: X \rightarrow Y$ ($\alpha < \omega_1$) of Baire- ξ functions. If $f: X \rightarrow Y$ is such that $f_\alpha \rightarrow_{\frac{d}{\mathcal{I}_<}} f$, then f is Baire- ξ .

We note here that the original question asked by T. Natkaniec referred to $\mathcal{I}_{<}$ -convergence. However, it is easy to see that $\mathcal{I}_{<}$ -convergence implies $\frac{d}{\mathcal{I}_{<}}$ -convergence, so the result above is formally stronger than the required. The sufficiency of $\frac{d}{\mathcal{I}_{<}}$ -convergence was pointed out to the author by Petr Holický.

As W. Sierpiński showed ([20], Theorem 1 on page 133 and Theorem 2 on page 137), for the class of continuous and Baire-1 functions Theorem 6.2 holds also for $\mathcal{I} = [\omega_1]^{\leq \omega}$ instead of $\mathcal{I}_{<}$. On the other hand, it is independent for every $2 \leq \xi < \omega_1$ whether there is a $[\omega_1]^{\leq \omega}$ -convergent sequence of Baire- ξ functions whose limit function is Borel but not Baire- ξ (observe that the $\frac{d}{\mathcal{I}_{<}}$ -convergence implies the $[\omega_1]^{\leq \omega}$ -convergence). The first part of the following theorem has already been proved by W. Sierpiński ([20], Section 6, pages 139 and 140) and further discussed by P. Komjáth ([5], Theorem 3, page 499). Its second part, related to Problem 3 in [17] on page 490, is a simple analogue of Theorem 6.2.

Theorem 6.3. *Let (X, τ) be a Polish space, (Y, d) be a separable metric space.*

1. *(W. Sierpiński, P. Komjáth) Assuming CH, there exists an $[\omega_1]^{\leq \omega}$ -convergent sequence of real Baire-2 functions whose limit function is not Borel.*
2. *Let $\lambda < 2^{\aleph_0}$ be an infinite cardinal with $\text{cf}(\lambda) > \omega$ and set $\mathcal{J} = [\lambda]^{< \lambda}$. For a fixed $\xi < \omega_1$, consider a family $f_\alpha: X \rightarrow Y$ ($\alpha < \lambda$) of Baire- ξ functions and a Borel function $f: X \rightarrow Y$. If $f_\alpha \xrightarrow{\mathcal{J}} f$ and in our model the union of λ meager sets is meager in Polish spaces, then f is necessarily Baire- ξ .*

The assumption on the additivity of meager sets holds under $MA(\lambda)$ (see e.g. [3], Theorem 1.2 on page 505 or [16], Theorem on page 170). The convergence of transfinite sequences of Baire-2 functions of length ω_2 has also been investigated by P. Komjáth (see [5], Theorem 4 and Theorem 5 on page 500). It is consistent (with $2^{\aleph_0} = \omega_2$ and $MA(\omega_1)$) that every real function can be obtained as such a limit. It is also consistent, under more complicated assumptions, that the limit function is necessarily Baire-2. The case when the underlying space X is not necessarily Polish but merely metric has been considered in [2] and [19]. Transfinite convergence of derivatives is examined in [1] while in [18] transfinite convergence of Baire- ξ functions is treated under various set theoretic assumptions.

In order to establish the connection between function classes and sublevel sets we will use the following classical result (see e.g. [4], Chapter II, Theorem 24.3 on page 190).

Theorem 6.4. *Let (X, τ) be a Polish space, (Y, d) be a separable metric space. Then for every $1 \leq \xi < \omega_1$, a function $f: X \rightarrow Y$ is Baire- ξ if and only if $f^{-1}(U) \subseteq X$ is $\Sigma_{\xi+1}^0(\tau)$ for every open set $U \subseteq Y$.*

In the metric space (Y, d) , the open ball centered at $x \in Y$ with radius ρ is denoted by $B_d(x, \rho)$. After these preparations, Theorem 6.2 and Theorem 6.3.2 are simple corollaries of Theorem 1.6.

Proof of Theorem 6.2. By Theorem 1 on page 133 and Theorem 2 on page 137 of [20], the statement holds for $\xi \leq 1$. So let $2 \leq \xi < \omega_1$ be fixed and suppose that $f_\alpha \xrightarrow[\mathcal{I}_<]{d} f$ for a family $f_\alpha: X \rightarrow Y$ ($\alpha < \omega_1$) of Baire- ξ functions.

Suppose that f is not Baire- ξ . As the pointwise limit of the functions $\{f_\alpha: \alpha < \omega\}$, f is clearly Borel, so by Theorem 6.4, there is an open ball $B_d(x, \rho) \subseteq Y$ such that $f^{-1}(B_d(x, \rho))$ is Borel but not $\Sigma_{\xi+1}^0(\tau)$. Set

$$H(\varepsilon) = f^{-1}(B_d(x, \rho - \varepsilon)), \quad H_\alpha(\varepsilon) = f_\alpha^{-1}(B_d(x, \rho - \varepsilon))$$

for every $\alpha < \omega_1$ and $0 < \varepsilon < \rho$. Note that by Theorem 6.4, $H_\alpha(\varepsilon)$ is in $\Sigma_{\xi+1}^0(\tau)$ for every $\alpha < \omega_1$ and $0 < \varepsilon < \rho$.

Since $H(0)$ is not $\Sigma_{\xi+1}^0(\tau)$, by Theorem 1.6.2 there is a continuous one-to-one map $\varphi: (2^\omega, \tau_{C_1}) \rightarrow (X, \tau)$ such that

(i) $\varphi(P_{\xi+1}) \subseteq H(0)$, and

(ii) $\varphi^{-1}(H(0)) \subseteq 2^\omega$ is of first category in the topology $\tau_{P_{\xi+1}}$.

By (i), there is an $\varepsilon_0 > 0$ such that $\varphi^{-1}(H(\varepsilon_0)) \cap P_{\xi+1}$ is of $\tau_{P_{\xi+1}}|_{P_{\xi+1}}$ -second category. Let $J_1(\varepsilon)$ denote that set of those indices $\alpha < \omega_1$ for which $\varphi^{-1}(H_\alpha(\varepsilon))$ is of $\tau_{P_{\xi+1}}$ -second category.

We prove that $\omega_1 \setminus J_1(\varepsilon)$ is finite for every $\varepsilon < \varepsilon_0$. Suppose that this is not true and take a countably infinite set $J'(\varepsilon) \subseteq \omega_1 \setminus J_1(\varepsilon)$. By the definition of $\xrightarrow[\mathcal{I}_<]{d}$ -convergence, $\varepsilon < \varepsilon_0$ implies that

$$H(\varepsilon_0) \subseteq H'(\varepsilon) := \bigcup_{\alpha \in J'(\varepsilon)} H_\alpha(\varepsilon),$$

so we have that $\varphi^{-1}(H'(\varepsilon)) \cap P_{\xi+1}$ is of $\tau_{P_{\xi+1}}|_{P_{\xi+1}}$ -second category in $P_{\xi+1}$; that is, since $H'(\varepsilon)$ is $\Sigma_{\xi+1}^0(\tau)$, by Theorem 1.6.1 $\varphi^{-1}(H'(\varepsilon))$ is of $\tau_{P_{\xi+1}}$ -second category. This is a contradiction, since by the definition of $J_1(\varepsilon)$, $\varphi^{-1}(H'(\varepsilon))$ is $\tau_{P_{\xi+1}}$ -meager.

So $J_1(\varepsilon)$ is of cardinality ω_1 for every $\varepsilon < \varepsilon_0$. In particular, given that $(2^\omega, \tau_{P_{\xi+1}})$ has countable base, there is a $\tau_{P_{\xi+1}}$ -open set $U \subseteq 2^\omega$ such that for a countably infinite set of indices $J'' \subseteq J_1(\varepsilon_0/2)$ we have that $\varphi^{-1}(H_\alpha(\varepsilon_0/2))$ is $\tau_{P_{\xi+1}}$ -residual in U whenever $\alpha \in J''$. Hence for

$$H'' = \bigcap_{\alpha \in J''} H_\alpha(\varepsilon_0/2),$$

$\varphi^{-1}(H'')$ is also $\tau_{P_{\xi+1}}$ -residual in U , so by (ii) we can find a point $x_0 \in H'' \setminus H(0)$. Thus f_α ($\alpha < \omega_1$) is not $\xrightarrow[\mathcal{I}_<]{d}$ -convergent since

$$J'' \subseteq \left\{ \alpha < \omega_1 : d(f(x_0), f_\alpha(x_0)) > \frac{\varepsilon_0}{2} \right\}$$

is infinite; a contradiction. The proof is complete. ■

Proof of Theorem 6.3.2. Again, for $\xi \leq 1$ the statement follows from the proofs of Theorem 1 on page 133 and Theorem 2 on page 137 of [20] ; so let $\xi \geq 2$. Now f is Borel by assumption; and the proof is the same as for Theorem 6.2, until the definition of J_1 . Now we show that $\text{card}(\lambda \setminus J_1(\varepsilon)) < \lambda$ for every $\varepsilon < \varepsilon_0$.

Suppose that this is not true and take a set $J'(\varepsilon) \subseteq \lambda \setminus J_1(\varepsilon)$ of cardinality λ . By the definition of $\overset{d}{\mathcal{J}}$ -convergence, $\varepsilon < \varepsilon_0$ implies that

$$H(\varepsilon_0) \subseteq H'(\varepsilon) := \bigcup_{\alpha \in J'(\varepsilon)} H_\alpha(\varepsilon),$$

so we have that $\varphi^{-1}(H'(\varepsilon)) \cap P_{\xi+1}$ is of $\tau_{P_{\xi+1}|P_{\xi+1}}$ -second category in $P_{\xi+1}$; that is, by the extension of Theorem 1.6.1, since $H'(\varepsilon)$ is the union of the λ number of $\Sigma_{\xi+1}^0(\tau)$ sets $H_\alpha(\varepsilon)$ ($\alpha \in J'(\varepsilon)$), $\varphi^{-1}(H'(\varepsilon))$ is of $\tau_{P_{\xi+1}}$ -second category. Now this contradicts the assumption that the union of a λ number of meager sets is meager in $(2^\omega, \tau_{P_{\xi+1}})$, since by the definition of $J_1(\varepsilon)$, $\varphi^{-1}(H_\alpha(\varepsilon))$ ($\alpha \in J'(\varepsilon)$) is $\tau_{P_{\xi+1}}$ -meager.

We continue as above; $J_1(\varepsilon)$ is of cardinality λ for every $\varepsilon < \varepsilon_0$. In particular, given that $\text{cf}(\lambda) > \omega$ and $(2^\omega, \tau_{P_{\xi+1}})$ has countable base, there is a $\tau_{P_{\xi+1}}$ -open set $U \subseteq 2^\omega$ such that for a set of indices $J'' \subseteq J_1(\varepsilon_0/2)$ of cardinality λ we have that $\varphi^{-1}(H_\alpha(\varepsilon_0/2))$ is $\tau_{P_{\xi+1}}$ -residual in U whenever $\alpha \in J''$. Since in our model the intersection of a λ number of $\tau_{P_{\xi+1}}$ -residual sets is again $\tau_{P_{\xi+1}}$ -residual, for

$$H'' = \bigcap_{\alpha \in J''} H_\alpha(\varepsilon_0/2),$$

$\varphi^{-1}(H'')$ is also $\tau_{P_{\xi+1}}$ -residual in U , so by (ii) we can find a point $x_0 \in H'' \setminus H(0)$. Again, this contradicts the $\overset{d}{\mathcal{J}}$ -convergence. The proof is complete. ■

Chapter 7

Generalized separation and reduction

It is well known that a pair of disjoint analytic sets can be separated by a Borel set. If we need to estimate the Borel class of this separating set then Theorem 1.2 is very useful: we only need to test the pair of analytic sets via injections of 2^ω . Since analytic sets also have the generalized separation property, that is for every sequence $(A_i)_{i < \omega}$ of analytic sets with $\bigcap_{i < \omega} A_i = \emptyset$ there is a sequence $(B_i)_{i < \omega}$ of Borel sets such that $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = \emptyset$, one can request a test for the complexity of the B_i 's. We cannot refuse such a demand so we present test sets corresponding to this problem. If topologization seems to be artificial for Theorem 1.2, here the topological feature of our Hurewicz test sets will be essential, namely for the proof that testing generalized separation is possible; this application is the main motivation of this section.

The concepts are similar that of [7]. On the other hand we do not follow the spirit of [7], that is we do not define closed games, neither will be reducing maps be one-to-one. We will have to refer to Borel determinacy, so in particular this proof is not effective. We also restrict our attention to sequences of Borel sets; all these simplification to make the argument as short as possible for having a nice application of topological test sets.

First we examine the problem of generalized separation; generalized reduction will follow by taking complements.

7.1 Witnessing generalized separation

We aim to prove the following theorem.

Theorem 7.1. *For every $0 < \xi < \omega_1$, there exists a Polish space (D_ξ, τ_{D_ξ}) homeomorphic to (C, τ_C) and $\Delta_{\xi+1}^0(\tau_{D_\xi})$ sets $R_\xi(n) \subseteq D_\xi$ ($n < \omega$) such that the following hold.*

1. *If $B_i \subseteq D_\xi$ is a $\Pi_\xi^0(\tau_{D_\xi})$ set and $R_\xi(i) \subseteq B_i$ ($i < \omega$) then $\bigcap_{i < \omega} B_i \neq \emptyset$.*

2. If (X, τ) is a Polish space and $(A_i)_{i < \omega}$ is a sequence of Borel sets in X such that $\bigcap_{i < \omega} A_i = \emptyset$ then
- (a) either there is a sequence $(B_i)_{i < \omega}$ of $\Pi_\xi^0(\tau)$ sets such that $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = \emptyset$,
 - (b) or there is a continuous map $\varphi: D_\xi \rightarrow X$ such that $\varphi(R_\xi(i)) \subseteq A_i$ ($i < \omega$).

Observe that if 2b holds then the map φ indeed shows that 2a fails: if $(B_i)_{i < \omega}$ a sequence of $\Pi_\xi^0(\tau)$ sets such that $A_i \subseteq B_i \subseteq X$ ($i < \omega$) then $\bigcap_{i < \omega} \varphi^{-1}(B_i) \neq \emptyset$ follows from 1 using $R_\xi(i) \subseteq \varphi^{-1}(B_i)$ ($i < \omega$), so $\bigcap_{i < \omega} B_i \neq \emptyset$. So this theorem is of Hurewicz type.

Before starting the construction we feel obliged to clarify why the testing is set up only for Π_ξ^0 sets and not for Σ_ξ^0 sets. We have three reasons for this, the first is a corollary of Theorem 7.1.

Corollary 7.2. *If in Theorem 7.1 case 2a holds then the sequence $(B_i)_{i < \omega}$ can be chosen such that $B_i \in \Delta_\xi^0(\tau)$ ($i < \omega$).*

Proof. Since the class Π_ξ^0 has the generalized separation property in Polish spaces (see e.g. [4], (22.16) Theorem on page 172), the statement follows. ■

That is generalized reduction with Π_ξ^0 sets is stronger than generalized reduction with Σ_ξ^0 sets. The following result says that in a particular case Theorem 1.2 gives an answer to the problem for Σ_ξ^0 sets .

Claim 7.3. *Let $A_0, A_1 \subseteq X$ be disjoint analytic sets. Then there are disjoint $\Sigma_\xi^0(\tau)$ sets $B_0, B_1 \subseteq X$ such that $A_i \subseteq B_i$ ($i = 0, 1$) if and only if A_0 can be separated from A_1 by a $\Sigma_\xi^0(\tau)$ set and vice versa, A_1 can be separated from A_0 by a $\Sigma_\xi^0(\tau)$ set.*

Proof. If we have the disjoint $\Sigma_\xi^0(\tau)$ sets $B_0, B_1 \subseteq X$ such that $A_i \subseteq B_i$ ($i = 0, 1$) then the separation is obviously possible in both directions. Suppose now that a $\Sigma_\xi^0(\tau)$ set B'_i separates A_i from A_{1-i} ($i = 0, 1$). The class Σ_ξ^0 has the reduction property in Polish spaces (see again [4], (22.16) Theorem on page 172), thus we have disjoint $\Sigma_\xi^0(\tau)$ sets $B_i \subseteq B'_i$ ($i = 0, 1$) such that $B_0 \cup B_1 = B'_0 \cup B'_1$. Since $B'_i \cap A_{1-i} = \emptyset$ ($i = 0, 1$), we have $B_i \cap A_{1-i} = \emptyset$ ($i = 0, 1$). But $A_0 \cup A_1 \subseteq B'_0 \cup B'_1 = B_0 \cup B_1$, so we get $A_i \subseteq B_i$ ($i = 0, 1$). The proof is complete. ■

The next claim is simply trivial.

Claim 7.4. *Let $0 < \xi < \omega_1$ be fixed and let $\vartheta_i \rightarrow \xi$. Let $(A_i)_{i < \omega}$ be a decreasing sequence of analytic sets in X such that $\bigcap_{i < \omega} A_i = \emptyset$. If there is a sequence $(B_i)_{i < \omega}$ of $\Pi_\xi^0(\tau)$ sets such that $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = \emptyset$ then there is a sequence $(B'_i)_{i < \omega}$ satisfying $A_i \subseteq B'_i$ ($i < \omega$) and $\bigcap_{i < \omega} B'_i = \emptyset$ such that B'_i is $\Sigma_{\vartheta_i}^0(\tau)$ ($i < \omega$).*

Proof. Write $B_i = \bigcap_{j < \omega} B_i(j)$ where $B_i(j)$ is $\Sigma_{\vartheta_j}^0(\tau)$ ($j < \omega$) and $B_i(j) \subseteq B_i(k)$ ($k \leq j < \omega, i < \omega$). Set $B'_i = \bigcap_{j \leq i} B_j(i)$, then B'_i is $\Sigma_{\vartheta_i}^0(\tau)$ ($i < \omega$). Since $A_i \subseteq A_j \subseteq B_j \subseteq B_j(i)$ ($j \leq i < \omega$) we have $A_i \subseteq B'_i$ ($i < \omega$). We also have $\bigcap_{i < \omega} B'_i = \bigcap_{i,j < \omega} B_j(i) = \bigcap_{i < \omega} B_i = \emptyset$, which completes the proof. ■

Claim 7.3 and Claim 7.4 handle the two extreme situations in generalized separation: when we have only two sets which are disjoint or we have infinitely many sets which are decreasing. The cases in-between can be handled with similar tricks but the separation conditions become numerous and complicated, so we omit them and turn to the construction of test sets for generalized separation by Π_ξ^0 sets.

Definition 7.5. For $\xi = 1$ let $(D_1, \tau_{D_1}) = (\omega + 1, \mathbf{o}_{\omega+1})$. Let $\lambda_1: \omega \times \omega \rightarrow \omega$ be the bijection defined by

$$\lambda_1(n, i) = \frac{1}{2}(i + n + 1)(i + n) + n \quad (n, i < \omega)$$

and set $R_1(n) = \{\lambda_1(n, i) : i < \omega\}$ ($n < \omega$) and $R_1 = \{\omega\}$.

For $1 < \xi < \omega_1$, let $(C_\xi(i), \tau_{C_\xi(i)})$, $\{P_\xi(i), \tau_{P_\xi(i)}\}$ ($i < \omega$) be copies of the space (C_ξ, τ_{C_ξ}) and Π_ξ^0 Hurewicz test pair $\{P_\xi, \tau_{P_\xi}\}$ defined in Definition 4.19. Set

$$\begin{aligned} D_\xi &= \prod_{i < \omega} C_\xi(i), \quad \tau_{D_\xi} = \prod_{i < \omega} \tau_{C_\xi(i)}, \\ \hat{P}_\xi(n) &= \prod_{i < n} C_\xi(i) \times P_\xi(n) \times \prod_{n < i < \omega} C_\xi(i), \\ \tau_{\hat{P}_\xi(n)} &= \prod_{i < n} \tau_{P_\xi(i)}^< \times \tau_{P_\xi(n)} \times \prod_{n < i < \omega} \tau_{P_\xi(i)}^< \end{aligned}$$

and

$$R_\xi(n) = \left(D_\xi \setminus \hat{P}_\xi(n) \right) \cap \bigcap_{m < \omega, m \neq n} \hat{P}_\xi(m) = \prod_{i < n} P_\xi(i) \times (C_\xi(n) \setminus P_\xi(n)) \times \prod_{n < i < \omega} P_\xi(n),$$

$$R_\xi = \bigcap_{i < \omega} \hat{P}_\xi(i) = \prod_{i < \omega} P_\xi(i), \quad \tau_{R_\xi} = \prod_{i < \omega} \tau_{P_\xi(i)}^<$$

We state a trivial lemma on the Borel class of the preceding sets.

Lemma 7.6. *For every $1 < \xi < \omega_1$, $\hat{P}_\xi(n)$ is a $\Pi_\xi^0(\tau_{D_\xi})$ set and $R_\xi(n)$ is a $\Delta_{\xi+1}^0(\tau_{D_\xi})$ set ($n < \omega$).*

Next we show that the sets $R_\xi(n)$ do the job for Theorem 7.1.

Claim 7.7. *Let $0 < \xi < \omega_1$ be fixed and suppose that $B_i \subseteq D_\xi$ is a $\Pi_\xi^0(\tau_{D_\xi})$ set and $R_\xi(i) \subseteq B_i$ ($i < \omega$). Then $R_\xi \cap \bigcap_{i < \omega} B_i \neq \emptyset$.*

Proof. Consider first $\xi = 1$. If B_i is $\Pi_1^0(\tau_{D_1})$ then $\omega \in \text{cl}_{\tau_{D_1}}(R_1(i)) \subseteq B_i$ ($i < \omega$) so indeed $\omega \in R_1 \cap \bigcap_{i < \omega} B_i$.

Let now $1 < \xi < \omega_1$. By Claim 4.5.5, $\{\hat{P}_\xi(n), \tau_{\hat{P}_\xi(n)}\}$ ($n < \omega$) is a topological Hurewicz test pair in the Polish space

$$\left(D_\xi, \prod_{i < n} \tau_{P_\xi^<(i)} \times \tau_{C_\xi(n)} \times \prod_{n < i < \omega} \tau_{P_\xi^<(i)} \right).$$

Moreover, also by Claim 4.5.5, Theorem 4.7 holds for $\hat{P}_\xi(n)$ with the presentation inherited from the presentation of P_ξ defined in (4.9-4.11) of Definition 4.19. We also have $\tau_{\hat{P}_\xi(n)}^< = \tau_{R_\xi}^<$ ($n < \omega$). So by Theorem 4.7.1, $\hat{P}_\xi(i)$ ($i < \omega$) hence R_ξ is $\tau_{R_\xi}^<$ residual in D_ξ . We show that B_i ($i < \omega$) is also $\tau_{R_\xi}^<$ -residual in D_ξ , this will complete the proof.

Fix some $i < \omega$. Since $D_\xi \setminus \hat{P}_\xi(i)$ is $\tau_{\hat{P}_\xi(i)}$ -residual and $\hat{P}_\xi(j)$ is also $\tau_{\hat{P}_\xi(i)}$ -residual ($j < \omega$, $j \neq i$), $R_\xi(i)$ and thus B_i are also $\tau_{\hat{P}_\xi(i)}$ -residual in D_ξ ($i < \omega$). Since B_i is $\Pi_\xi^0(\tau_{D_\xi})$, it is also

$$\Pi_\xi^0 \left(\prod_{i < n} \tau_{P_\xi^<(i)} \times \tau_{C_\xi(n)} \times \prod_{n < i < \omega} \tau_{P_\xi^<(i)} \right).$$

Thus Corollary 4.8.4 can be applied and gives that B_i is $\tau_{\hat{P}_\xi(i)}^<$ -residual, that is $\tau_{R_\xi}^<$ -residual in D_ξ , as stated. ■

The next task is to find the appropriate image of $R_\xi(n)$ ($n < \omega$) if generalized separation by Π_ξ^0 sets is not possible. Fix a $1 < \xi < \omega_1$. Let $\lambda_\xi: \omega \rightarrow \mathfrak{r}_\xi^\omega$ be a bijection; using $D_\xi = (2^{\mathfrak{r}_\xi})^\omega$ by Claim 4.20, let $\Lambda_\xi: C \rightarrow D_\xi$ denote the corresponding homeomorphism. We define the Borel game \mathfrak{G}_ξ , which is the heart of the proof of Theorem 7.1.

Let $A_i, G, H \subseteq D_\xi$ be Borel sets such that $A_i \subseteq G$ ($i < \omega$). In the game $\mathfrak{G}_\xi((A_i)_{i < \omega}, G, H)$ players I and II play

$$\begin{array}{rcc} \text{I} & \alpha(0) & \alpha(1) & \dots \\ \text{II} & \beta(0) & \beta(1) & \dots \end{array}$$

where $\alpha(i), \beta(i) \in \{0, 1\}$ ($i < \omega$), and II wins if and only if

$$\Lambda_\xi(\beta) \in G \text{ and } \begin{cases} \Lambda_\xi(\alpha) \in R_\xi(i) & \implies \Lambda_\xi(\beta) \in A_i \ (i < \omega), \\ \Lambda_\xi(\alpha) \in R_\xi & \implies \Lambda_\xi(\beta) \notin H; \end{cases}$$

else I wins.

We associate maps to the strategies of I.

Definition 7.8. Fix a $1 < \xi < \omega_1$ and let σ be a strategy of I in \mathfrak{G}_ξ . Then $\rho_\sigma: D_\xi \rightarrow D_\xi$ denotes the function mapping to a $y \in D_\xi$ the unique $x \in D_\xi$ such that the run $\{\Lambda_\xi^{-1}(x), \Lambda_\xi^{-1}(y)\}$ is according to σ .

Note that ρ_σ is continuous. In the sequel we will use this property in the form that ρ_σ^{-1} keeps the Borel class of sets.

By the following claim the determinacy of the game \mathfrak{G}_ξ indeed proves Theorem 7.1.

Claim 7.9. *For every $1 < \xi < \omega_1$, in the game \mathfrak{G}_ξ a winning strategy*

1. *for player I gives a sequence $(B_i)_{i < \omega}$ of $\Pi_\xi^0(\tau_{D_\xi})$ sets such that $A_i \subseteq B_i$ ($i < \omega$) and $(G \setminus H) \cap \bigcap_{i < \omega} B_i = \emptyset$;*
2. *for player II gives a continuous map $\varphi: D_\xi \rightarrow G$ such that $\varphi(R_\xi) \cap H = \emptyset$ and $\varphi(R_\xi(i)) \subseteq A_i$ ($i < \omega$).*

Proof. Consider first a winning strategy σ for I. Let $B_i = \rho_\sigma^{-1}(\hat{P}_\xi(i))$ ($i < \omega$). By Lemma 7.6, B_i is $\Pi_\xi^0(\tau_{D_\xi})$ ($i < \omega$). So it remains to show that $A_i \subseteq B_i$ and that $(G \setminus H) \cap \bigcap_{i < \omega} B_i = \emptyset$.

Suppose that for some $i < \omega$ we have a $y \in A_i \setminus B_i$, that is $y \in A_i \setminus \rho_\sigma^{-1}(\hat{P}_\xi(i))$. Since $A_i \setminus \rho_\sigma^{-1}(\hat{P}_\xi(i)) = A_i \cap \rho_\sigma^{-1}(D_\xi \setminus \hat{P}_\xi(i))$, there is some $x \in D_\xi \setminus \hat{P}_\xi(i)$ such that the run $\{\Lambda_\xi^{-1}(x), \Lambda_\xi^{-1}(y)\}$ in \mathfrak{G}_ξ is according to σ . We have

$$D_\xi \setminus \hat{P}_\xi(i) = R_\xi(i) \cup \bigcup_{j < \omega, j \neq i} \left(D_\xi \setminus \left(\hat{P}_\xi(j) \cup \hat{P}_\xi(i) \right) \right).$$

But if $x \in R_\xi(i)$ then II wins since $y \in A_i \subseteq G$ while if $x \notin R_\xi(i)$ and

$$x \in \bigcup_{j < \omega, j \neq i} \left(D_\xi \setminus \left(\hat{P}_\xi(j) \cup \hat{P}_\xi(i) \right) \right)$$

then $x \notin R_\xi(j)$ ($j < \omega$) and $x \notin R_\xi$ so II wins already by $y \in G$. This contradicts the definition of σ and proves that $A_i \subseteq B_i$ ($i < \omega$).

Suppose now that $y \in (G \setminus H) \cap \bigcap_{i < \omega} B_i$, that is $y \in G \setminus H$ and

$$(7.1) \quad y \in \rho_\sigma^{-1}(\hat{P}_\xi(i)) \quad (i < \omega).$$

Let $x \in D_\xi$ be arbitrary such that the run $\{\Lambda_\xi^{-1}(x), \Lambda_\xi^{-1}(y)\}$ in \mathfrak{G}_ξ is according to σ . Then by (7.1), $x \in \bigcap_{i < \omega} \hat{P}_\xi(i)$ so $x \notin R_\xi(j)$ ($j < \omega$). Thus II wins by $y \in G \setminus H$, which again contradicts the choice of σ and proves $(G \setminus H) \cap \bigcap_{i < \omega} B_i = \emptyset$.

Let now σ be a winning strategy of II. Let $\varphi: D_\xi \rightarrow D_\xi$ map to an $x \in D_\xi$ the unique $y \in D_\xi$ such that the run $\{\Lambda_\xi^{-1}(x), \Lambda_\xi^{-1}(y)\}$ in \mathfrak{G}_ξ is according to σ . This map is clearly

continuous. We have $\varphi(D_\xi) \subseteq G$, $\varphi(R_\xi) \cap H = \emptyset$ and if $i < \omega$ and $x \in R_\xi(i)$ then $\varphi(x) \in A_i$ since the strategy σ is winning for II. This completes the proof. ■

From this Theorem 7.1 will follow by standard arguments (see [7], Corollary 2 and Theorem 3); that we give in the remaining part of this section. We have to extend the results to arbitrary Polish spaces; for this we need the following result (see e.g. [7], Theorem on page 455).

Theorem 7.10. (Saint Raymond) *Let (E, τ_E) , (F, τ_F) be compact metrizable spaces and (Q, τ_Q) be a Polish space. If $h: E \rightarrow F$ is a continuous surjection and $f: E \rightarrow Q$ is a first class function then h has a first class section (i.e. there is a Baire-1 function $s: F \rightarrow E$ such that $h(s(y)) = y$ ($y \in F$)) such that $f \circ s: F \rightarrow Q$ is also a first class function.*

This theorem is applicable for a class of sets Γ if this Γ can be built up from Σ_2^0 sets on some canonical way.

Definition 7.11. Let $D \subseteq 2^\omega$ and let $(C(n))_{n < \omega}$ be a sequence of arbitrary subsets of some base set X . The the Hausdorff operation associated to D acts on the sequence $(C(n))_{n < \omega}$ as

$$D((C(n))_{n < \omega}) = \{x \in X : \{n < \omega : x \in C(n)\} \in D\}.$$

A class Γ is called a Σ_2^0 generated Hausdorff class if there is a basis $D \subseteq 2^\omega$ such that in every Polish space (X, τ) the members of Γ are the sets of the form $D((C(n))_{n < \omega})$ where $(C(n))_{n < \omega} \subseteq \Sigma_2^0(\tau)$.

We recall that each class Σ_ξ^0 ($1 < \xi < \omega_1$) is a Σ_2^0 generated Hausdorff class (see e.g. [4], (23.5) Exercise on page 180.) The following corollary is a variant of the so-called transfer lemma (for a version, see e.g. [7], Corollary 2 on page 455). It will allow us to extend our results to arbitrary compact spaces.

Corollary 7.12. *Let (E, τ_E) , (F, τ_F) be compact metrizable spaces, $h: E \rightarrow F$ be a continuous surjection and Γ be a Σ_2^0 generated Hausdorff class. Suppose that $(A_i)_{i < \omega}$ is a sequence of subsets of F , $T \subseteq F$ and there is a sequence $(H_i)_{i < \omega} \subseteq \Gamma$ of subsets of E satisfying $h^{-1}(A_i) \cap H_i = \emptyset$ ($i < \omega$) and $\bigcup_{i < \omega} H_i = h^{-1}(T)$. Then there is a sequence $(B_i)_{i < \omega} \subseteq \Gamma$ of subsets of F satisfying $A_i \cap B_i = \emptyset$ ($i < \omega$) and $\bigcup_{i < \omega} B_i = T$.*

Proof. Let $H_i = D((C_i(n))_{n < \omega})$ where D is a basis for Γ and $C_i(n)$ is $\Sigma_2^0(\tau_E)$ ($i, n < \omega$). Let $f_{i,n}: E \rightarrow [0, 1]$ be a first class function such that $C_i(n) = \{x \in E : f_{i,n}(x) > 0\}$ ($i, n < \omega$). Set $f: E \rightarrow [0, 1]^{\omega \times \omega}$, $f = \prod_{i,n < \omega} f_{i,n}$. From Theorem 7.10 for h and f we get a first class section s of h such that $f \circ s$ is also of first class, that is $V_i(n) = \{y \in F : f_{i,n}(s(y)) > 0\}$ is a $\Sigma_2^0(\tau_F)$ set. Thus for $B_i = D((V_i(n))_{n < \omega})$ we have $B_i \in \Gamma$ ($i < \omega$). It is clear that $s^{-1}(H_i) = B_i$ ($i < \omega$). Since $s^{-1}(h^{-1}(T)) = T$ and $\bigcup_{i < \omega} H_i = h^{-1}(T)$, we obtain $\bigcup_{i < \omega} B_i = T$. From $s^{-1}(H)_i = B_i$ we have $s(B_i) \subseteq H_i$, which gives $B_i = h(s(B_i)) \subseteq h(H_i)$

($i < \omega$). Since $H_i \cap h^{-1}(A_i) = \emptyset$ implies $h(H_i) \cap A_i = \emptyset$ ($i < \omega$), we have $B_i \cap A_i = \emptyset$ ($i < \omega$) which completes the proof. ■

It remains to finish with the proof of Theorem 7.1. We prove a slightly stronger extension; Theorem 7.1 is the $\bigcap_{i < \omega} A_i = \emptyset$ case of Theorem 7.13 together with Claim 7.7.

Theorem 7.13. *Let $0 < \xi < \omega_1$. In the Polish space (X, τ) let $(A_i)_{i < \omega}$ be a sequence of Borel sets. Then*

1. *either there is a sequence $(B_i)_{i < \omega}$ of $\Pi_\xi^0(\tau)$ set such that $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = \bigcap_{i < \omega} A_i$,*
2. *or there is a continuous map $\varphi: D_\xi \rightarrow X$ such that $\varphi(R_\xi) \cap \bigcap_{i < \omega} A_i = \emptyset$ and $\varphi(R_\xi(i)) \subseteq A_i$ ($i < \omega$).*

Proof. First we show that 2 excludes 1. If a map φ satisfies the conditions of 2 then by Claim 7.7, $R_\xi \cap \bigcap_{i < \omega} \varphi^{-1}(B_i) \neq \emptyset$. This implies $\varphi(R_\xi) \cap \bigcap_{i < \omega} B_i \neq \emptyset$ so since $\varphi(R_\xi) \subseteq X \setminus \bigcap_{i < \omega} A_i$, we have $\bigcap_{i < \omega} B_i \neq \bigcap_{i < \omega} A_i$.

To return to the proof of the theorem, set $H = \bigcap_{i < \omega} A_i$, and let first $\xi = 1$. If $\bigcap_{i < \omega} \text{cl}_\tau(A_i) = H$ then we have 1. Else let $x_\omega \in \bigcap_{i < \omega} \text{cl}_\tau(A_i) \setminus H$. Since $x_\omega \in \text{cl}_\tau(A_i)$ ($i < \omega$), we have

$$A_i \cap B_\tau(x_\omega, 1/n) \neq \emptyset \quad (i, n < \omega)$$

where $B_\tau(x_\omega, 1/n)$ is the open ball around x_ω with radius $1/n$ ($n < \omega$). So we can pick points $x_{\lambda_1(n,i)} \in A_i \cap B_\tau(x_\omega, 1/n)$ ($i, n < \omega$). Let $\varphi: D_1 \rightarrow X$ be defined by $\varphi(i) = x_i$ ($i \leq \omega$). This map is clearly continuous, $\varphi(R_1(i)) \subseteq A_i$ ($i < \omega$) and $\varphi(R_1) \not\subseteq H$, as required.

Let now $1 < \xi < \omega_1$ and let $(A_i)_{i < \omega}$ be a sequence of Borel sets in (X, τ) such that $\bigcap_{i < \omega} A_i = H$. Let $(\hat{X}, \hat{\tau})$ be a Polish compactification of (X, τ) (see e.g. [4], (4.14) Theorem); then $X \subseteq \hat{X}$ is a $\Pi_2^0(\hat{\tau})$ set. Take a continuous surjection $h: D_\xi \rightarrow \hat{X}$ and consider the game

$$\mathfrak{G}_\xi((h^{-1}(A_i))_{i < \omega}, h^{-1}(X), h^{-1}(H)).$$

This game is clearly Borel so by Borel determinacy either player I or player II has a winning strategy.

If player I has a winning strategy then by Claim 7.9.1 we have a sequence $(U_i)_{i < \omega}$ of $\Pi_\xi^0(\tau_{D_\xi})$ sets such that $h^{-1}(A_i) \subseteq U_i$ ($i < \omega$) and

$$(h^{-1}(X) \setminus h^{-1}(H)) \cap \bigcap_{i < \omega} U_i = \emptyset.$$

That is by

$$h^{-1}(H) \subseteq h^{-1}(A_i) \subseteq h^{-1}(X) \quad (i < \omega)$$

we have

$$(7.2) \quad h^{-1}(X) \cap \bigcap_{i < \omega} U_i = h^{-1}(H).$$

Let $(E, \tau_E) = (D_\xi, \tau_{D_\xi})$, $(F, \tau_F) = (\hat{X}, \hat{\tau})$, $\Gamma = \Sigma_\xi^0(\tau_{D_\xi})$, $T = \hat{X} \setminus H$ and $H_i = D_\xi \setminus (h^{-1}(X) \cap U_i)$ ($i < \omega$). Since $h^{-1}(A_i) \subseteq h^{-1}(X) \cap U_i$ implies $h^{-1}(A_i) \cap H_i = \emptyset$ ($i < \omega$) and

$$\bigcup_{i < \omega} H_i = D_\xi \setminus \left(h^{-1}(X) \cap \bigcap_{i < \omega} U_i \right) = D_\xi \setminus h^{-1}(H) = h^{-1}(T)$$

by (7.2), we can apply Corollary 7.12 to get a sequence $(V_i)_{i < \omega}$ of $\Sigma_\xi^0(\hat{\tau})$ sets such that $A_i \cap V_i = \emptyset$ and $\bigcup_{i < \omega} V_i = \hat{X} \setminus H$. Then $B_i = X \setminus V_i$ is $\Pi_\xi^0(\tau)$, $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = H$, as required.

If Player II has a winning strategy then we have a continuous map $\psi: D_\xi \rightarrow h^{-1}(X)$ such that $\psi(R_\xi) \cap h^{-1}(H) = \emptyset$ and $\psi(R_\xi(i)) \subseteq h^{-1}(A_i)$ ($i < \omega$). Then for $\varphi = h \circ \psi$, $\varphi: D_\xi \rightarrow X$ is also continuous and satisfies $\varphi(R_\xi) \cap H = \emptyset$, $\varphi(R_\xi(i)) \subseteq A_i$ ($i < \omega$). This completes proof. ■

7.2 Generalized reduction

Generalized reduction follows from Theorem 7.13 by taking complements.

Theorem 7.14. *Let $0 < \xi < \omega_1$. In the Polish space (X, τ) let $(A_i)_{i < \omega}$ be a sequence of Borel sets. Then*

1. *either there is a sequence $(B_i)_{i < \omega}$ of $\Sigma_\xi^0(\tau)$ set such that $B_i \subseteq A_i$, $B_i \cap B_j = \emptyset$ ($i, j < \omega$, $i \neq j$) and $\bigcup_{i < \omega} B_i = \bigcup_{i < \omega} A_i$,*
2. *or there is a continuous map $\varphi: D_\xi \rightarrow X$ such that $\varphi(R_\xi) \subseteq \bigcup_{i < \omega} A_i$ and $\varphi(R_\xi(i)) \cap A_i = \emptyset$. ($i < \omega$).*

Proof. By repeating the argument in the proof of Theorem 7.13 one easily obtains that by Claim 7.7, 2 excludes 1. Set $A'_i = X \setminus A_i$ ($i < \omega$). By Theorem 7.13, either there is a sequence $(B'_i)_{i < \omega}$ of $\Pi_\xi^0(\tau)$ set such that $A'_i \subseteq B'_i$ and $\bigcap_{i < \omega} A'_i = \bigcap_{i < \omega} B'_i$ or there is a continuous map $\varphi: D_\xi \rightarrow X$ such that $\varphi(R_\xi) \cap \bigcap_{i < \omega} A'_i = \emptyset$ and $\varphi(R_\xi(i)) \subseteq A'_i$ ($i < \omega$).

In the first case let $B''_i = X \setminus B'_i$ ($i < \omega$). The class Σ_ξ^0 has the generalized reduction property so there is a $\Sigma_\xi^0(\tau)$ set $B_i \subseteq B''_i$ ($i < \omega$) such that $B_i \cap B_j = \emptyset$ ($i, j < \omega$, $i \neq j$) and $\bigcup_{i < \omega} B_i = \bigcup_{i < \omega} B''_i = \bigcup_{i < \omega} A_i$; which is 1.

In the second case we have $\varphi(R_\xi(i)) \cap A_i = \emptyset$ ($i < \omega$) and $\varphi(R_\xi) \subseteq X \setminus \bigcap_{i < \omega} A'_i = \bigcup_{i < \omega} A_i$; which is 2. So the proof is complete. ■

For reduction by Π_ξ^0 sets the same remarks apply as for separation by Σ_ξ^0 sets (see Corollary 7.2, Claim 7.3, Claim 7.4 and their discussion).

Chapter 8

Concluding remarks

We constructed Π_ξ^0 topological Hurewicz test pairs for every $0 < \xi < \omega_1$ and we proved the applications announced in the introduction. To conclude this thesis we would like to present some problems which were left open implicitly or explicitly in the preceding chapters.

Problem 8.1. *Extend Theorem 4.7 or Theorem 5.2 to the entire Borel-Wadge hierarchy.*

The aesthetic motivation for this is the fact that Theorem 1.2 has its extension to all the Borel-Wadge degrees (see [6] and [8]). The real motivation is that it is not clear at all how one could get rid of the rigid structure of the sets P_ξ ($0 < \xi < \omega_1$), which were practically our only examples of Hurewicz test sets. The reason why we did not prove Theorem 5.2 for $3 < \xi$ was the failure of Claim 5.5 for $\xi = \eta = \omega$. But the reason for this failure is that we cannot allow too much flexibility in the structure of our topological Hurewicz test sets so the proofs get lost in technicalities. An approach which is able to handle the entire Borel-Wadge hierarchy will hopefully be free of these superfluous complications.

Next let us recall the theorem of S. Solecki mentioned in the introduction ([21], Theorem 1 on page 1023).

Theorem 8.2. *(S. Solecki) Let \mathcal{I} be a family of closed subsets of a Polish space (X, τ) . Let $A \subseteq X$ be a $\Sigma_1^1(\tau)$ set. Then either A can be covered by a countable union of members of \mathcal{I} or there is a $\Pi_2^0(\tau)$ set $G \subseteq A$ such that $F \cap G$ relatively meager in G for every $F \in \mathcal{I}$.*

Less formally this result says that if a σ -ideal \mathcal{I} is generated by its Π_1^0 members the problem whether an analytic set belongs to \mathcal{I} or not depends only on sets of low complexity, namely on Π_2^0 sets. This led to the formulation of Question 1.5 and Question 3.16. It is important to note that these questions are already refuted by the following unpublished result of A. Kechris and M. Zelený.

Theorem 8.3. (*A. Kechris-M. Zelený*) *Assume $V = L$. Then there is an analytic set $A \subseteq C$ and a σ -ideal \mathcal{I} strongly generated by its $\Pi_2^0(\tau_C)$ members such that \mathcal{I} contains every Borel subset of A .*

However, in view of Theorem 3.15 it seems to be interesting to look for an alternative counterexample. Mostly because in the proof of Theorem 3.15, CH was used for having only ω_1 many sets to handle, so this result is more “descriptive” than “set theory”.

Problem 8.4. *Prove (without $V = L$) for every $2 \leq \xi < \omega_1$ that the answer to Question 1.5 and/or to Question 3.16 is consistently negative.*

We know that using the approach of nested sequences applied in the proof of Theorem 3.15 it can be proved that Question 1.5 has a negative answer under CH ; this will be published elsewhere. The reason why we could not include it in this thesis is the same as above: the proof gets extremely technical due to the fact that the structural conditions on P_4 are so restrictive that this set cannot be used. We admit that we used P_L instead of P_3 for the same reason. And we also admit that this approach seems to be completely inadequate to handle the $4 \leq \xi < \omega_1$ case. It seems that Problem 8.4 is in close relation also to the question whether Theorem 4.22 holds for $\xi = \xi' + 1$ where ξ' is a limit ordinal.

Problem 8.5. *Prove for every $2 \leq \xi < \omega_1$ that the answer to Question 1.5 and/or to Question 3.16 is consistently positive.*

We have no results in this direction. It can easily happen that the conditions for a positive answer are much more natural and easier to handle. Since the proof of Theorem 8.2 is as elegant as numerous its corollaries are, it would be nice if an analogous statement was true for higher Borel classes, at least consistently.

Problem 8.6. *Make the arguments in Chapter 7 effective.*

It seems to require only endurance. But a lot more than effective generalized separation and reduction might be achieved. We could ask for a much closer analogy with the Baire Category Theorem and related techniques than what we have up till now through our topological Hurewicz test pairs. Namely for $1 < \xi < \omega_1$, we intend to have some machinery which isolates a “maximal” Σ_ξ^0 set inside a given Borel set which behaves like “taking the interior”. The best candidate for the framework of such a theory seems to be the effective machinery. It may allow to define topologies τ_P to a much larger family of Π_ξ^0 set in such a way that $\{P, \tau_P\}$ is a topological Hurewicz test pair. We may have an elegant, natural proof for Theorem 4.22 and for ideal generation theorems in general. And we may be in the position to answer Question 1.5 and Question 3.16: in the negative by repeating the relatively simple argument of the proof of Theorem 3.15 and in the positive by repeating the absolute simple argument of the proof of Theorem 8.2 with a properly chosen notion of an “effective closure”.

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