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MORE COFINAL TYPES OF DEFINABLE DIRECTED ORDERS

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> *"And thick and fast they came at last,*
> *And more, and more, and more – "*
>
> [4, Poem p. 80]

Abstract. We study the cofinal diversity of analytic p-ideals and analytic relative σ-ideals of compact sets. We prove that the σ-ideal of compact meager sets is not cofinally simpler than the asymptotic density zero ideal; this concludes the study of the cofinal types of classical analytic ideals. We obtain this result by using a Ramsey-type partition calculus, which allows us to capture the relevant combinatorial properties of our ideals.

We also show that the structure $(\mathcal{P}(\omega), \subseteq^\star)$ embeds into the family of F_σ p-ideals on ω and into the family of monotone σ-ideals of compact sets, partially ordered by Tukey reducibility. To this end, we introduce a general construction scheme for lower semicontinuous submeasures, and we carry out a detailed analysis of the combinatorial properties of monotone σ-ideals of compact sets.

1. Cofinal types

What is the complexity of a partial order? Obviously, the answer depends on our purpose for using a given partial order. If the only reason, which is often the case, for using a partially ordered set $(P, \leq)$ is to keep track of whether certain subsets of $(P, \leq)$ are bounded from above or not then for every *cofinal* subset C of $(P, \leq)$, i.e. such that $\forall p \in P\ \exists c \in C\ (p \leq c)$, $(P, \leq)$ and $(C, \leq)$ should have the same complexity. So one way to define the complexity of partial orders is to introduce their *cofinal types*. In the sequel we do not write out the partial order $\leq$ when it is not of special importance.

Following J. W. Tukey [34, Chapter 2.2], two partially ordered sets P, Q are called *cofinally similar* if there is a partially ordered set R with cofinal sets $C_P, C_Q \subseteq R$ such that P is order isomorphic to C_P and Q is order isomorphic to C_Q. Cofinal similarity turns out to be an equivalence relation, and the equivalence classes are called *cofinal types*.

In [34, Chapter 2.3], J. W. Tukey also introduced the following notion of comparison of cofinal types. Let P and Q be partially ordered sets. We say that P is *Tukey reducible to* Q, $P \leq_T Q$ in notation, if there is a function $g\colon Q \to P$ such that for every cofinal set $C \subseteq Q$, $g[C] \subseteq P$ is cofinal in P. Such a g is called a *convergent map*. The following characterization of Tukey reducibility is also available (see e.g. [26]).

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Proposition 1.1. *Let P, Q be partially ordered sets. Then $P \leq_T Q$ if and only if there is a function $f\colon P \to Q$ such that for every unbounded from above set $A \subseteq P$, $f[A] \subseteq Q$ is unbounded from above in Q.*

The function f of Proposition 1.1, witnessing Tukey reducibility, is called a *Tukey map.* For *directed* partially ordered sets, i.e. such that $\forall p, q \in P\ \exists r \in P\ (p, q \leq r)$, Tukey reducibility is compatible with cofinal similarity in the sense that if P, Q are directed partially ordered sets then $P \leq_T Q$ and $Q \leq_T P$ imply that P, Q are cofinally similar. In [32, Theorem 2 p. 712], S. Todorčević obtained that every partially ordered set P can be written as a union of some of its directed subsets, which are determined by the structure of antichains in P in a natural way; in particular, their number depends on the cardinalities of the antichains. In this sense, restricting our study to directed partial orders is affordable.

Tukey reducibility turns out to be the perfect tool for the comparison of cofinal types. The existence of a Tukey map between two directed partial orders relates many of their structural properties; e.g. it is easy to see, using the definition of Tukey reducibility and Proposition 1.1, that $P \leq_T Q$ implies $\mathrm{cof}(P) \leq \mathrm{cof}(Q)$ and $\mathrm{add}(Q) \leq \mathrm{add}(P)$. Moreover, recent research on Tukey reductions discovered that many previously known results about partial orders can be phrased and proved in a unified way using Tukey reductions (see [3], [9], [10] and [33]); e.g. all inequalities of the Cichoń diagram follow from Tukey reductions. Therefore a detailed description of the cofinal types of directed partial orders would be of great importance.

If our only structural information about a directed partial order is the cardinality of its underlying set then the characterization of its cofinal type cannot be done in ZFC alone. In [32, Theorem 9 p. 718], S. Todorčević proved that it is consistent with ZFC that there are only five cofinal types of directed partial orders of cardinality $\leq \omega_1$; while by [32, Corollary 5 p. 714], under the Continuum Hypothesis there are 2^{ω_1} many different cofinal types of directed partial orders of cardinality ω_1. So toward a positive classifiaction result, it is necessary to restrict the study of cofinal diversity to special classes of directed partial orders, which are important for applications.

In [10], D. H. Fremlin obtained that the cofinal types of various partially ordered sets originating from a Maharam homogeneous Radon probability space (X, μ) are essentially the same and are determined by the Maharam type of (X, μ). In [9], he characterized the cofinal type of the family of compact subsets of a topological space, partially ordered by inclusion, in terms of its topological properties. An ongoing research project of N. Dobrinen, D. Raghavan and S. Todorčević aims to describe the cofinal diversity of ultrafilters (see [5] and [25]).

In the present paper we consider analytic p-ideals on ω and analytic relative σ-ideals of compact sets.

Definition 1.2. A family $\mathcal{I} \subseteq (\mathcal{P}(\omega), \subseteq)$ is an *analytic ideal* if it is closed under taking subsets and finite unions, and $\mathcal{I}$ is an analytic subset of $\mathcal{P}(\omega)$ endowed with the Cantor space topology. An ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a *p-ideal* if for every $\{H_n \colon n < \omega\} \subseteq \mathcal{I}$ there is an $H \in \mathcal{I}$ satisfying $|H_n \setminus H| < \omega$ $(n < \omega)$.

Definition 1.3. Let $(\mathcal{K}, \subseteq)$ denote the family of compact subsets of 2^ω endowed with the Vietoris topology, partially ordered by inclusion. Let $\mathcal{F} \subseteq \mathcal{K}$ be a closed set which is closed under taking finite unions. A family $\mathcal{I} \subseteq \mathcal{F}$ is a *σ-ideal of compact sets relatively to $\mathcal{F}$* if

(1) for every $K, L \in \mathcal{F}$, $L \subseteq K \in \mathcal{I}$ implies $L \in \mathcal{I}$;
(2) for every $\{K_n \colon n < \omega\} \subseteq \mathcal{I}$, $\bigcup_{n<\omega} K_n \in \mathcal{F}$ implies $\bigcup_{n<\omega} K_n \in \mathcal{I}$.

In the $\mathcal{F} = \mathcal{K}$ special case we call $\mathcal{I}$ a *σ-ideal of compact sets.*

Since being a p-ideal is the strongest σ-additivity-like property that a nontrivial ideal on ω can possess, analytic p-ideals play an important role in descriptive set theory, and in many other areas of mathematics, as well, e.g. Banach space theory, model theory, real analysis, etc. (see e.g. [7], [8], [13], [28], [35] and [36]). The theory of relative σ-ideals of compact sets is closely related to the theory of analytic p-ideals (see e.g. [27]); in addition, the descriptive set theory of σ-ideals of compact sets has important applications e.g. in harmonic analysis and definable forcing (see e.g. [15], [16] and [36]).

The following picture summarizes our knowledge about the cofinal types of analytic p-ideals on ω and analytic relative σ-ideals of compact sets, including the results of the present paper. Here $\mathcal{I} \longrightarrow \mathcal{J}$ stands for $\mathcal{I} <_T \mathcal{J}$, and $\mathcal{I} \dashrightarrow\!\!\!\!\!\!\times\ \ \mathcal{J}$ denotes $\mathcal{I} \not\leq_T \mathcal{J}$. Alphabetical labels indicate the results which are contributions of the present paper.

$$\begin{tikzcd}
 & & (\mathcal{P}(\omega),\subseteq^\star) \arrow[r, hook, "(13)"] & \left\{\begin{matrix}\text{density-like}\\ F_{\sigma\delta}\ p\text{-ideals}\end{matrix}\right\} \arrow[r, "(8)"] & \{F_\sigma\ p\text{-ideals}\} & (\mathcal{P}(\omega),\subseteq^\star) \arrow[l, hook, "(c)"'] & \\
 & & & \Cup & \Cup & & \\
 & & & \mathcal{Z}_0 \arrow[r, "(5)"] \arrow[ddr, dashed, "\times" marking, "(b)"] & \mathcal{I}_{1/n} \arrow[drr, "(11)"] \arrow[ddr, dashed, "\times" marking, "(9)"] & & \\
1 \arrow[r, "(1)"] & \omega \arrow[r, "(2)"] & \omega^\omega \arrow[ur, "(3)"] \arrow[dr, "(4)"'] & & & & \mathcal{I}_{\max} \in \{\text{non-}p\ F_\sigma\ \text{ideals}\} \\
 & & & \mathcal{K}_{\mathcal{N}} \arrow[uu, "(6)"] \arrow[r, "(a)"'] & \mathcal{K}_{\mathcal{M}} \arrow[uu, "(7)"'] \arrow[uul, dashed, "\times" marking] \arrow[r, "(10)"'] & \mathfrak{J} \arrow[uul, dashed, "\times" marking] \arrow[ur, "(12)"'] & \\
 & & & & & & (\mathcal{P}(\omega),\subseteq^\star) \arrow[uu, hook', "(14)"] \arrow[dd, hook, "(d)"] \\
 & & & \Cap & \Cap & \Cap & \\
 & & & & \{G_\delta\ \sigma\text{-ideals of compact sets}\} & \subseteq & \left\{\begin{matrix}\text{relaitve }\sigma\text{-ideals}\\ \text{of compact sets}\end{matrix}\right\}
\end{tikzcd}$$

Tukey picture

Before introducing the notation, let us point out some fundamental facts. By a result of S. Solecki, every analytic p-ideal on ω is $F_{\sigma\delta}$ (see Theorem 2.2). Of course, a non-p ideal may be complete analytic. Similarly, by [16, Theorem 7 pp. 268] every analytic σ-ideal of compact sets is G_δ; but this does not hold for an arbitrary analytic relative σ-ideal of compact sets. These remarks explain the moderate Borel rank of the classes of ideals in the picture.

We introduce the notation and give the partial orders on the sets which appear above.

- 1 denotes the one element set with the trivial order;
- ω stands for the first infinite ordinal with its usual well-order;
- the order on ω^ω, the set of all functions from ω to ω, is the (not necessarily strict) dominance at every coordinate;
- all ideals on ω and relative σ-ideals of compact sets are partially ordered by inclusion;
- $\mathcal{N}$ denotes the σ-ideal of Lebesgue null subsets of 2^ω;
- $\mathcal{M}$ stands for the σ-ideal of meager subsets of 2^ω;
- $\mathcal{K}_{\mathcal{N}}$ denotes the σ-ideal of compact Lebesgue null subsets of 2^ω;
- $\mathcal{K}_{\mathcal{M}}$ stands for the σ-ideal of compact meager subsets of 2^ω;
- $\mathcal{Z}_0 = \{H \subseteq \omega\colon \lim_{n<\omega} |H \cap n|/n = 0\}$ is the *asymptotic density zero ideal*;
- $\mathcal{I}_{1/n} = \{H \subseteq \omega\colon \sum_{h\in H} 1/(h+1) < \infty\}$ is the *summable ideal*;

- $\mathfrak{I}$ is a G_δ σ-ideal of compact sets, which was constructed in [19] (see also [20]);
- $\mathcal{I}_{\max}$ is an F_σ ideal on ω which represents the maximal cofinal type $([2^{\aleph_0}]^{<\omega}, \subseteq)$ of directed partial orders of cardinality $\leq 2^{\aleph_0}$ (see e.g. [17, Proposition 3 p. 185] or [18, Proposition 5.5]).

The Tukey picture summarizes the following results. The nontrivial parts of (1)-(7) can be found in [10, Proposition 3K p. 208], [10, Proposition p. 211], [10, Theorem 3B p. 198], [10, Proposition p. 211] and [17, Theorem 7 p. 187].

The ideal $\mathcal{I}_{1/n}$ is Tukey maximal among all analytic p-ideals (see [17, Theorem 5 p. 181]). We remark that $\mathcal{I}_{1/n}$ is ω-Tukey equivalent with $\mathcal{N}$ (see [10, Theorem 2F p. 191] for the details), so roughly speaking, $\mathcal{I}_{1/n}$ and $\mathcal{N}$ have the same cofinal types. We remark that by [10, Theorem 3B (b) p. 198], $\mathcal{M}$ and $\mathcal{K}_\mathcal{M}$ are also ω-Tukey equivalent, so $\mathcal{M}$ and $\mathcal{K}_\mathcal{M}$ also have essentially the same cofinal types.

As a substantial improvement of earlier results, it was shown in [30, Corollary 6.4 p. 1895] that if an analytic p-ideal $\mathcal{I}$ is Tukey reducible to a relative σ-ideal of compact sets then $\mathcal{I} \leq_T \omega^\omega$. This result, together with (6) and (7), motivate the arrangement that puts analytic p-ideals above relative σ-ideals of compact sets. Arrow (8) refers to [30, Corollary 6.9 p. 1899], which states that if an F_σ p-ideal $\mathcal{I}$ is Tukey reducible to a density-like ideal then $\mathcal{I} \leq_T \omega$.

In [19] we constructed a G_δ σ-ideal of compact sets $\mathfrak{I}$ which turned out to be strictly above $\mathcal{K}_\mathcal{M}$ and incomparable with $\mathcal{I}_{1/n}$ (see [20, Theorem 3.1]). In [24, Theorem 3.1] a similar G_δ σ-ideal of compact sets was shown to be strictly above $\mathcal{K}_\mathcal{M}$. Arrows (9) and (10) refer to these results. Arrows (11) and (12) stand for the obvious fact that $\mathcal{I}_{1/n}$ and $\mathfrak{I}$ are strictly below $\mathcal{I}_{\max}$.

Arrows (13) and (14) summarize cofinal diversity results. In [17, Theorem 6 p. 183] it was obtained that the set of all subsets of ω partially ordered by essential inclusion can be embedded into the set of density-like $F_{\sigma\delta}$ p-ideals partially ordered by Tukey reducibility. In [18, Proposition 5.4.] we showed that $(\mathcal{P}(\omega), \subseteq^\star)$ embeds into the family of non-p F_σ ideals on ω partially ordered by $\leq_T$.

In the present paper we prove the following results.

Theorem 1.4. *We have* $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{Z}_0$.

Theorem 1.5. *The structure* $(\mathcal{P}(\omega), \subseteq^\star)$ *embeds into the family of* F_σ *p-ideals on* ω *partially ordered by* $\leq_T$.

Theorem 1.6. *The structure* $(\mathcal{P}(\omega), \subseteq^\star)$ *embeds into the family of relative* σ*-ideals of compact sets Tukey reducible to* $\mathcal{K}_\mathcal{M}$*, partially ordered by* $\leq_T$.

Theorem 1.4 establishes (a) and (b), since the $\mathcal{K}_\mathcal{N} \leq_T \mathcal{K}_\mathcal{M}$ part of (a) was proved in [10, Corollary 3E p. 202], and $\mathcal{Z}_0 \not\leq_T \mathcal{K}_\mathcal{M}$ follows from the above mentioned general non-reducibility result [30, Corollary 6.4 p. 1895]. The proof of Theorem 1.4 is of independent interest. We introduce partition properties which are inherited by cofinally finer directed partial orders; i.e. if $(T, \tau, \mathcal{S})$ is a partition property and $\mathcal{I}$, $\mathcal{J}$ are directed partial orders such that $\mathcal{I} \leq_T \mathcal{J}$ and $\mathcal{I} \to (T, \tau, \mathcal{S})$ then $\mathcal{J} \to (T, \tau, \mathcal{S})$ (see Definition 3.1 and Proposition 3.2). Thus a suitably chosen partition property can witness non-reducibility. We remark that $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{Z}_0$ was independently obtained in [29] by different methods.

It will be an easy observation that every Tukey non-reducibility can be witnessed by an appropriate partition property. The real benefit of this concept is that for each concrete non-reduction problem it allows us to capture those combinatorial characteristics of the involved partial orders which account for non-reducibility. This way, partition properties allow us to measure the difference between cofinal types: if a non-reduction is witnessed by a "simple" partition property then the cofinal types are "very different". E.g. the partition property behind $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{K}_\mathcal{N}$ will capture in a mathematically meaningful way why $\mathcal{K}_\mathcal{M}$ cannot be the ideal of negligible sets for any measure (see Section 3.2). In Section 3 we discuss the partition properties behind the non-reducibility results on the Tukey picture.

Theorem 1.5 accounts for (c). Before this result, $\mathcal{I}_{1/n}$ was the only known cofinal type of F_σ p-ideals (see [30, Question 5 p. 1909]). For its proof, in Section 4.2 we introduce a way to construct submeasures. In Section 4.3 we discuss the possibilities offered by our construction scheme; in particular, we relate it to the work in [2] and we show that Tsirelson submeasures can be constructed this way. In fact, we expect that this construction scheme together with the above mentioned partition properties make possible the complete description of the cofinal types of analytic p-ideals.

Theorem 1.6 gives (d). Again, before this result and (a) it was unknown whether there exist any cofinal types strictly between ω^ω and $\mathcal{K}_\mathcal{M}$. In [30, Section 7 p. 1899], S. Solecki and S. Todorčević introduced an operator D which maps analytic p-ideals to relative σ-ideals of compact sets. They showed that D preserves Tukey reducibility, and asked whether it keeps Tukey non-reducibility, as well. In Section 5 we show that D preserves Tukey non-reducibility for the cofinal types accounting for arrow (13). This answers [30, Question 4 p. 1909] in the affirmative.

It is important to point out a conceptual difference between out present work and the recent research in the area. A common feature of the proofs of the result we cited from [17], [30], [24] and [29] is the observation that Tukey reductions between the ideals in question can always be witnessed by a Tukey map which is continuous when restricted to an appropriate set. The proof of arrow (13) in [17] uses Blumberg's theorem. In [30] it is obtained that every Tukey reducibility between analytic p-ideals and appropriate relative σ-ideals of compact sets can be witnessed by a Souslin-measurable Tukey map, which becomes continuous when restricted to a co-meager set. In [24], this result is applied to $\mathcal{K}_\mathcal{M}$ endowed with its usual topology; while in [29], the same is applied to $\mathcal{K}_\mathcal{M}$ endowed with the Ochan topology. The non-reducibility results of [30] are also based on notions of calibration, which condition on the interplay between the order and the topology of the ideals.

In contrast, the proofs we present in this paper are purely combinatorial; in particular, we do not use special topologies or measurable Tukey maps. This allows to isolate the combinatorial characteristics which distinguish the different cofinal types, and we think that the proofs become more transparent, as well.

We note that there are many notions of reducibility for ideals on ω other than Tukey reducibility, e.g. the Katětov, the Katětov-Blass, the Rudin-Kreisler or the Rudin-Blass orderings, etc. (see e.g. [1] or [11]). In comparison with these, cofinal similarity yields a very rough classification. So Tukey non-equivalent ideals are very different objects. In Section 6 we recall several open problems related to the Tukey picture.

Finally we would like to thank Pandelis Dodos for the helpful discussions, and for drawing our attention to [2].

2. Preliminary results

In this section we introduce some notation, we recall some fundamental results about analytic ideals and submeasures, and we prove a property of relative σ-ideals of compact sets which seems to play the key role in their Tukey theory. Our reference for the basic notions of descriptive set theory is [14]. As above, a subset of a partially ordered set is called *(un)bounded* if it is (un)bounded from above.

For every set V, $\mathcal{P}(V)$ denotes the power set of V. For every $s, t \in \omega^{<\omega}$, $|s|$ denotes the length of s and $s^\frown t$ stands for the sequence $(s(0)\dots s(|s|-1)t(0)\dots t(|t|-1))$. For every $s, t \in \omega^{<\omega}$, we write $s \sqsubseteq t$ if $t|_{|s|} = s$. We define $\mathbb{R}^+ = \{x \in \mathbb{R} \colon x \geq 0\}$, and for every $s \in 2^{<\omega}$ we set $N_s = \{x \in 2^\omega \colon s \sqsubseteq x\}$.

2.1. **Analytic p-ideals.** We will make use of the connection between analytic ideals and submeasures.

Definition 2.1. A function $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$ is a *submeasure* if

(1) $\varphi(\emptyset) = 0$;
(2) for every $A, B \in [\omega]^{<\omega}$, $A \subseteq B$ implies $\varphi(A) \leq \varphi(B)$;
(3) for every $A, B \in [\omega]^{<\omega}$, $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

Given a submeasure $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$, we define $\overline{\varphi} \colon \mathcal{P}(\omega) \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$\overline{\varphi}(H) = \sup\{\varphi(A) \colon A \in [H]^{<\omega}\} \ (H \in \mathcal{P}(\omega)),$$

and we set

$$\mathrm{Fin}(\varphi) = \{H \in \mathcal{P}(\omega) \colon \overline{\varphi}(H) < +\infty\},$$
$$\mathrm{Exh}(\varphi) = \{H \in \mathcal{P}(\omega) \colon \lim_{n \to \infty} \overline{\varphi}(H \setminus n) = 0\}.$$

A submeasure $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$ is *exhaustive* if $\mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi)$.

The first statement of the following characterization theorem is due to K. Mazur [23, Lemma 1.2 p. 104], the second and third statements are results of S. Solecki [27, Theorem 3.1 p. 58 and Theorem 3.4 p. 62].

Theorem 2.2. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be arbitrary.*

(1) $\mathcal{I}$ is an F_σ ideal if and only if there is a submeasure $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$ such that $\mathcal{I} = \mathrm{Fin}(\varphi)$.
(2) $\mathcal{I}$ is an analytic p-ideal if and only if there is a submeasure $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$.
(3) $\mathcal{I}$ is an F_σ p-ideal if and only if there is a submeasure $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$ such that $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$.

It is customary to isolate two special classes of analytic p-ideals (see [29] and [30]).

Definition 2.3. Let $\mathcal{I}$ be an analytic p-ideal. Then $\mathcal{I}$ is *density-like* if $\mathcal{I} = \mathrm{Exh}(\varphi)$ for a submeasure $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$ satisfying the following: for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $A_n \in [\omega]^{<\omega}$ with $\varphi(A_n) < \delta$, $\max A_n < \min A_{n+1}$ $(n < \omega)$ there exist $I \in [\omega]^\omega$ such that $\varphi(\bigcup_{n \in I} A_n) < \varepsilon$.

On the other hand, $\mathcal{I}$ is *summable-like* if $\mathcal{I} = \mathrm{Exh}(\varphi)$ for a submeasure $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$ satisfying the following: there is an $\varepsilon > 0$ such that for every $\delta > 0$ there are $A_n \in [\omega]^{<\omega}$ with $\varphi(A_n) < \delta$, $\max A_n < \min A_{n+1}$ $(n < \omega)$ such that for some $k < \omega$, every $I \in [\omega]^k$ satisfies $\varphi(\bigcup_{n \in I} A_n) \geq \varepsilon$.

It is easy to see that these properties do not depend on the choice of the representing submeasures. We remark that for Corollary 4.4, which recovers arrow (13) on the Tukey picture, we construct density-like ideals, while the ideals constructed for Theorem 1.5 are *neither density-like nor summable-like.*

Analytic p-ideals can be endowed with the *submeasure topology*, as follows. Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an analytic p-ideal, and let $\varphi \colon [\omega]^{<\omega} \to \mathbb{R}^+$ be a submeasure satisfying $\mathcal{I} = \mathrm{Exh}(\varphi)$. By replacing φ with $\varphi' \colon [\omega]^{<\omega} \to \mathbb{R}^+$, $\varphi'(A) = \varphi(A) + \sum_{n \in A} 2^{-n}$, we can assume $\varphi(\{n\}) > 0$ $(n < \omega)$. Define

$d\colon \mathcal{I}\times\mathcal{I} \to \mathbb{R}^+$ by setting $d(H,H') = \overline{\varphi}(H\Delta H')$ $(H,H' \in \mathcal{I})$. In [27, Theorem 3.1 p. 58] it is proved that d is a metric which generates a Polish topology on $\mathcal{I}$. Note that the submeasure topology is finer than the Cantor space topology inherited from $\mathcal{P}(\omega)$. In the sequel, every analytic p-ideal will be endowed with this topology; in particular, notions referring to Baire category are to be understood in the submeasure topology.

2.2. **Relative σ-ideals of compact sets.** Recall that $\mathcal{K}$ denotes the family of compact subsets of 2^ω endowed with the Vietoris topology, partially ordered by inclusion. Note that this is a Polish topology metrized by the Hausdorff-distance. For every $\mathcal{H} \subseteq \mathcal{K}$, set $\mathcal{H}^{\downarrow} = \{L \in \mathcal{K}\colon \exists K \in \mathcal{H}\ (L \subseteq K)\}$. A set $\mathcal{H} \subseteq \mathcal{K}$ is *hereditary* if $\mathcal{H} = \mathcal{H}^{\downarrow}$.

The following simple property plays a key role in the Tukey theory of relative σ-ideals of compact sets.

Proposition 2.4. *With the notation of Definition 1.3, let $\mathcal{F} \subseteq \mathcal{K}$ be a closed set which is closed under taking finite unions, and let $\mathcal{I} \subseteq \mathcal{F}$ be a σ-ideal of compact sets relatively to $\mathcal{F}$. Let $V \subseteq 2^\omega$ be an open set and let $\mathcal{P}(j) \subseteq \mathcal{P}(V)\cap\mathcal{K}$ $(j < \omega)$ be hereditary families such that $\mathcal{P}(V)\cap\mathcal{I} \subseteq \bigcup_{j<\omega} \mathcal{P}(j)$. Then for every $K \in \mathcal{P}(V)\cap\mathcal{I}$ there is a clopen set $U \subseteq 2^\omega$ and a $j < \omega$ such that $K \subseteq U$ and $\mathcal{P}(U)\cap\mathcal{I} \subseteq \mathcal{P}(j)$.*

Proof. Let $K \in \mathcal{P}(V)\cap\mathcal{I}$ be arbitrary. Let $U(j) \subseteq V$ $(j < \omega)$ be clopen sets such that $K = \bigcap_{j<\omega} U(j)$. We show that $\mathcal{P}(U(j))\cap\mathcal{I} \subseteq P(j)$ for some $j < \omega$.

Suppose $\mathcal{P}(U(j))\cap\mathcal{I} \not\subseteq \mathcal{P}(j)$ $(j < \omega)$. Then for every $j < \omega$ let $K(j) \subseteq U(j)$ be such that $K(j) \in \mathcal{I}\setminus\mathcal{P}(j)$. Set $L = K \cup \bigcup_{j<\omega} K(j)$. Since $\mathcal{F} \subseteq \mathcal{K}$ is closed and it is also closed under taking finite unions, $K, K(j) \in \mathcal{F}$ $(j < \omega)$ implies $L \in \mathcal{F}$. Thus by $\mathcal{I}$ being a σ-ideal relatively to $\mathcal{F}$, we get $L \in \mathcal{P}(V)\cap\mathcal{I}$, as well. Since $\mathcal{P}(j)$ $(j < \omega)$ are hereditary, $K(j) \subseteq L$ implies $L \notin \mathcal{P}(j)$ $(j < \omega)$. This contradicts $\mathcal{P}(V)\cap\mathcal{I} \subseteq \bigcup_{j<\omega}\mathcal{P}(j)$, so the proof is complete. ☐

Corollary 2.5. *Let $\mathcal{F}$ and $\mathcal{I}$ be as in Proposition 2.4. Let $V \subseteq 2^\omega$ be an open set and let $\mathcal{P}(j) \subseteq \mathcal{P}(V)\cap\mathcal{K}$ $(j < \omega)$ be hereditary families such that $\mathcal{P}(V)\cap\mathcal{I} \subseteq \bigcup_{j<\omega}\mathcal{P}(j)$. Then*

$$\bigcup(\mathcal{P}(V)\cap\mathcal{I}) \subseteq \bigcup\{U \subseteq 2^\omega\colon U \textit{ is clopen, } \exists j < \omega\ (\mathcal{P}(U)\cap\mathcal{I} \subseteq \mathcal{P}(j))\}.$$

Proof. By Proposition 2.4, for every $K \in \mathcal{P}(V)\cap\mathcal{I}$ there is a clopen set $U \subseteq 2^\omega$ and a $j < \omega$ such that $K \subseteq U$ and $\mathcal{P}(U)\cap\mathcal{I} \subseteq \mathcal{P}(j)$. So the statement follows ☐

For finite covers we can say even more.

Lemma 2.6. *With the notation of Definition 1.3, let $\mathcal{F} \subseteq \mathcal{K}$ be a closed set which is closed under taking finite unions, and let $\mathcal{I} \subseteq \mathcal{F}$ be a σ-ideal of compact sets relatively to $\mathcal{F}$. Let $V \subseteq 2^\omega$ be an open set, let $n < \omega$ and let $\mathcal{P}(j) \subseteq \mathcal{K}$ $(j < n)$ be hereditary families such that $\mathcal{P}(V)\cap\mathcal{I} \subseteq \bigcup_{j<n}\mathcal{P}(j)$. Then there is a $j < n$ such that $\mathcal{P}(V)\cap\mathcal{I} \subseteq \mathcal{P}(j)$.*

Proof. Suppose that $\mathcal{P}(V)\cap\mathcal{I} \not\subseteq \mathcal{P}(j)$ $(j < n)$. Then for every $j < n$ there is a $K_j \in \mathcal{P}(V)\cap\mathcal{I}$ such that $K_j \notin \mathcal{P}(j)$. Let $L = \bigcup_{j<n} K_j$; then $L \in \mathcal{P}(V)\cap\mathcal{I}$. Now $K_j \subseteq L$ witnesses $L \notin \mathcal{P}(j)$ $(j < n)$, which is a contradiction. ☐

3. PARTITION CALCULUS AND THE SPORADIC NON-REDUCTION RESULTS

We introduce partition schemes.

Definition 3.1. Let T be a set such that $\emptyset \in T$, and let $\tau \subseteq T \times \mathcal{P}(T)$, $\mathcal{S} \subseteq \mathcal{P}(T)$ be arbitrary. Then $(T, \tau, \mathcal{S})$ is called a *partition scheme*.

Let P be an arbitrary set. A function $D \colon T \to \mathcal{P}(P)$ is a *τ-partition* if $D(\emptyset) = P$ and for every $(t, H) \in \tau$ we have $D(t) = \bigcup_{h \in H} D(h)$.

Let P be a partially ordered set. We write $(T, \tau, \mathcal{S}) \to P$ if for every τ-partition $D \colon T \to \mathcal{P}(P)$ there are $S \in \mathcal{S}$ and $d \colon S \to P$ such that $d(s) \in D(s)$ $(s \in S)$ and $\{d(s) \colon s \in S\} \subseteq P$ is unbounded.

The arrow notation is intentional, it alludes to the Ramsey theoretic flavor of the forthcoming results. The relevance of partition schemes for Tukey reductions comes from the following simple observation.

Proposition 3.2. *Let $(T, \tau, \mathcal{S})$ be a partition scheme and let P, Q be partially ordered sets. Then $(T, \tau, \mathcal{S}) \to P$ and $P \leq_T Q$ imply $(T, \tau, \mathcal{S}) \to Q$.*

Proof. Let $f \colon P \to Q$ be a Tukey map. Let $D_Q \colon T \to \mathcal{P}(Q)$ be an arbitrary τ-partition. Set $D_P \colon T \to \mathcal{P}(P)$, $D_P(t) = f^{-1}(D_Q(t))$ $(t \in T)$. Since $D_P(\emptyset) = P$ and for every $(t, H) \in \tau$ we have

$$D_P(t) = f^{-1}(D_Q(t)) = f^{-1}(\textstyle\bigcup_{h \in H} D_Q(h)) = \bigcup_{h \in H} f^{-1}(D_Q(h)) = \bigcup_{h \in H} D_P(h),$$

D_P is a τ-partition. Since $(T, \tau, \mathcal{S}) \to P$, we have an $S \in \mathcal{S}$ and a $d_P \colon S \to P$ such that $d_P(s) \in D_P(s)$ $(s \in S)$ and $\{d_P(s) \colon s \in S\} \subseteq P$ is unbounded.

Define $d_Q \colon S \to Q$ by $d_Q(s) = f(d_p(s))$ $(s \in S)$. We have $\{d_Q(s) \colon s \in S\} = f[\{d_P(s) \colon s \in S\}]$. By f being a Tukey map, we get $\{d_Q(s) \colon s \in S\} \subseteq Q$ is unbounded, as required. This completes the proof. □

Partition schemes can be used to witness non-reducibility, as follows. If $(T, \tau, \mathcal{S})$ is a partition scheme and P, Q are partially ordered sets such that $(T, \tau, \mathcal{S}) \to P$ but $(T, \tau, \mathcal{S}) \not\to Q$ then by Proposition 3.2 we have $P \not\leq_T Q$. Moreover, every non-reduction can be witnessed by a suitable partition scheme, as follows.

Proposition 3.3. *Let P be a directed partially ordered set and let κ be an infinite cardinal. Set $T = P \times \kappa$,*

$$\tau = \{(\langle p, 0\rangle, \{\langle p, \alpha\rangle\}) \colon p \in P, \alpha < \kappa\} \cup \{(\emptyset, \{\langle p, 0\rangle \colon p \in P\})\}$$

and $\mathcal{S} = \{S \subseteq P \times \kappa \colon \mathrm{Pr}_P(S) \subseteq P$ is bounded$\}$. Let Q be a directed partially ordered set satisfying that every unbounded subset of Q has an unbounded subset of cardinality $\leq \kappa$. Then $Q \leq_T P$ if and only if $(T, \tau, \mathcal{S}) \not\to Q$.

Proof. The partition $D \colon P \times \kappa \to \mathcal{P}(P)$, $D(\langle p, \alpha\rangle) = \{p\}$ $(p \in P, \alpha < \kappa)$ witnesses $(T, \tau, \mathcal{S}) \not\to P$. Thus by Proposition 3.2, $Q \leq_T P$ implies $(T, \tau, \mathcal{S}) \not\to Q$.

To see the converse, suppose $(T, \tau, \mathcal{S}) \not\to Q$; then there is a τ-partition $D \colon P \times \kappa \to \mathcal{P}(Q)$ witnessing this. Define $f \colon Q \to P$ by setting $f(q)$ to be any element of P such that $q \in D(\langle f(q), 0\rangle)$.

To see that f is a Tukey map, let $H \subseteq Q$ be unbounded; by our assumption on Q, we can assume $|H| \leq \kappa$. By the definition of τ, we have $D(\langle p, 0\rangle) = D(\langle p, \alpha\rangle)$ $(\alpha < \kappa)$. So by the definition of f, there is a function $d \colon f[H] \times \kappa \to Q$ such that $d(\langle p, \alpha\rangle) \in D(\langle p, \alpha\rangle)$ $(p \in f[H], \alpha < \kappa)$ and $\{d(\langle p, \alpha\rangle) \colon p \in f[H], \alpha < \kappa\} = H$. Thus by $H \subseteq Q$ being unbounded, we have $f[H] \times \kappa \notin \mathcal{S}$; i.e. $\mathrm{Pr}_P(f[H] \times \kappa) = f[H] \subseteq P$ is unbounded, as required. □

We present the partition schemes which witness the non-reduction parts of arrows (3), (4), (a) and (b) on the Tukey picture. We remark that it is not too hard to translate [30, Proposition 6.3 p. 1894], [30, Lemma 6.7 p. 1895] and [30, Proposition 6.8 p. 1896] into partition schemes witnessing the non-reduction parts of arrows (5)-(7). According to the complexity of the corresponding partition scheme, one gets the impression that the characterization of the cofinal type of ω^ω is trivial, the distinction between analytic p-ideals and relative σ-ideals of compact sets is easy, separating density-like p-ideals from F_σ p-ideals is more complicated, $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{K}_\mathcal{N}$ is involved, and $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{Z}_0$ is very subtle.

3.1. **The cofinal type of ω^ω.** Let $T = \omega^{<\omega}$, $\tau = \{(t, \{t^\frown(n) \colon n < \omega\}) \colon t \in \omega^{<\omega}\}$ and

$$\mathcal{S} = \{S \subseteq \omega^{<\omega} \colon \exists \sigma \in \omega^\omega \ (s(n) \leq \sigma(n) \ (s \in S, n < |s|))\}.$$

Proposition 3.4. *Let P be a directed partially ordered set such that every unbounded subset of P has a countable unbounded subset. Then $P \leq_T \omega^\omega$ if and only if $(T, \tau, \mathcal{S}) \not\to P$.*

Proof. The partition $D \colon \omega^{<\omega} \to \mathcal{P}(\omega^\omega)$, $D(s) = \{\sigma \in \omega^\omega \colon s \sqsubseteq \sigma\}$ shows $(T, \tau, \mathcal{S}) \not\to \omega^\omega$. Thus by Proposition 3.2, $P \leq_T \omega^\omega$ implies $(T, \tau, \mathcal{S}) \not\to P$.

To see the converse, suppose $(T, \tau, \mathcal{S}) \not\to P$; then there is a τ-partition $D \colon \omega^{<\omega} \to \mathcal{P}(P)$ witnessing this. For every $p \in P$ we define $f(p) \in \omega^\omega$ by induction, as follows. By the definition of τ, there is a $\sigma(0) < \omega$ such that $p \in D((\sigma(0)))$. If $n < \omega$ and $\sigma(i)$ $(i \leq n)$ are defined such that $p \in D((\sigma(i))_{i \leq n})$ then again by the definition of τ, there is a $\sigma(n+1) < \omega$ such that $p \in D((\sigma(i))_{i \leq n+1})$. This completes the inductive step of the definition of σ. We set $f(p) = (\sigma(i))_{i < \omega}$.

We show that f is a Tukey map. Let $H \subseteq P$ be unbounded; by our condition on P, we can assume H is countable. Let $\sigma \in \omega^\omega$ be arbitrary; set $S = \{s \in \omega^{<\omega} \colon s(n) \leq \sigma(n) \ (n < w)\}$. If $f[H]$ is bounded by σ then for every $h \in H$ we have $\{s \in S \colon h \in D(s)\}$ is an infinite set. So by an easy induction one can define $d \colon S \to P$ such that $d(s) \in D(s)$ $(s \in S)$ and $H \subseteq d[S]$, hence $d[S] \subseteq P$ is unbounded. This contradicts the assumption that D witnesses $(T, \tau, \mathcal{S}) \not\to P$, and completes the proof. ☐

In [30, Proposition 4.3 p. 1886] and [30, Proposition 6.5 p. 1895], fairly general conditions are given on a directed partially ordered set P which imply $\omega^\omega \leq_T P$; Proposition 3.4 may be considered as a counterpart of these result.

It is not too hard to see that $(T, \tau, \mathcal{S}) \to \mathcal{Z}_0$ and $(T, \tau, \mathcal{S}) \to \mathcal{K}_\mathcal{N}$ (see [12] for $\mathcal{Z}_0 \not\leq_T \omega^\omega$ and [10] for $\mathcal{K}_\mathcal{N} \not\leq_T \omega^\omega$). For general analytic p-ideals $\mathcal{I}$, [30, Proposition 6.3.(ii) p. 1894] gives a characterization of $\mathcal{I} \leq_T \omega^\omega$. We give the analogous characterization for σ-ideals of compact sets.

Proposition 3.5. *With the notation of Definition 1.3, let $\mathcal{I}$ be a σ-ideal of compact sets. Then $\mathcal{I} \leq_T \omega^\omega$ if and only if there is a G_δ set $A \subseteq 2^\omega$ such that $\mathcal{I} = \mathcal{P}(A) \cap \mathcal{K}$.*

Proof. First suppose there is a G_δ set $A \subseteq 2^\omega$ such that $\mathcal{I} = \mathcal{P}(A) \cap \mathcal{K}$. Then the statement follows from [9, Theorem 15 p. 38], but for the sake of completeness we present a simple direct proof. Let $\varphi \colon A \to \omega^\omega$ be a homeomorphism of A with a closed subspace of ω^ω (see e.g. [14, Theorem 7.8 p. 38]). For every $K \in \mathcal{I}$, $\varphi[K] \subseteq \omega^\omega$ is compact hence bounded. So we can define $f \colon \mathcal{I} \to \omega^\omega$ to be such that for every $K \in \mathcal{I}$, $\sigma \leq f(K)$ $(\sigma \in \varphi[K])$.

We show that the pre-image of bounded subsets of ω^ω under f is bounded in $\mathcal{I}$. For every $\sigma \in \omega^\omega$, $f(K) \leq \sigma$ implies $\varphi[K] \subseteq \{\sigma' \in \varphi[A] \colon \sigma' \leq \sigma\}$; i.e. $K \subseteq \varphi^{-1}(\varphi[A] \cap \{\sigma' \in \omega^\omega \colon \sigma' \leq \sigma\})$, which is a compact subset of A, as required.

Now suppose $\mathcal{I} \leq_T \omega^\omega$. By Proposition 3.4, this means $(T, \tau, \mathcal{S}) \not\to \mathcal{I}$. Let $D \colon \omega^{<\omega} \to \mathcal{P}(\mathcal{I})$ be a τ-partition witnessing this. For every $s \in \omega^{<\omega}$, set $\mathcal{P}(s) = D(s)^\downarrow$ and

$$U(s) = \bigcup\{U \subseteq 2^\omega \colon U \text{ is clopen}, \mathcal{P}(U) \cap \mathcal{I} \subseteq \mathcal{P}(s)\}.$$

We show that $A = \mathrm{cl}_{2^\omega}(\bigcup \mathcal{I}) \cap \bigcap_{n<\omega} \bigcup\{U(s) \colon s \in \omega^{<\omega}, |s| = n\}$ fulfills the requirements.

By Corollary 2.5 with $V = 2^\omega$, $\bigcup\mathcal{I} \subseteq \bigcup\{U(s)\colon s \in \omega^{<\omega}, |s| = n\}$ $(n < \omega)$. This shows $\bigcup\mathcal{I} \subseteq A$, i.e. $\mathcal{I} \subseteq \mathcal{P}(A) \cap \mathcal{K}$.

To see the other inclusion, let $K \in \mathcal{P}(A) \cap \mathcal{K}$ be arbitrary. By K being compact, there is an $S \in \mathcal{S}$ such that for every $n < \omega$, $K \subseteq \bigcup\{U(s)\colon s \in S,\ |s| = n\}$. For every $s \in \omega^{<\omega}$ and $x \in U(s) \cap \bigcup\mathcal{I}$ we have $\{x\} \in \mathcal{I}$, hence $\{x\} \in \mathcal{P}(s)$ and so $x \in \bigcup D(s)$. Thus $U(s) \cap \bigcup\mathcal{I} \subseteq \bigcup D(s)$ $(s \in \omega^{<\omega})$. Then for every $n < \omega$ there is a finite set $\mathcal{L}_n \subseteq \bigcup\{D(s)\colon s \in S,\ |s| = n\}$ such that the 2^{-n}-neighborhood of $\bigcup\mathcal{L}_n$ contains K. Thus $K \subseteq \mathrm{cl}_{2^\omega}(\bigcup\bigcup_{n<\omega}\mathcal{L}_n)$.

We define $S' \subseteq \omega^{<\omega}$, as follows. For every $s \in S$ and $L \in D(s) \cap \mathcal{L}_{|s|}$ pick an $n(s, L) < \omega$ such that $L \in D(s^\frown(n(s, L)))$, and put $s^\frown(n(s, L)) \in S'$. Then $S' \in \mathcal{S}$ and we have a $d\colon S' \to \mathcal{I}$ such that $d(s) \in D(s)$ $(s \in S')$ and $\bigcup_{n<\omega}\mathcal{L}_n \subseteq \{d(s)\colon s \in S'\}$. Then by $\{d(s)\colon s \in S'\}$ being bounded, we have $\mathrm{cl}_{2^\omega}(\bigcup\{d(s)\colon s \in S'\}) \in \mathcal{I}$. Since $K \subseteq \mathrm{cl}_{2^\omega}(\bigcup\{d(s)\colon s \in S'\})$, we get $K \in \mathcal{I}$ so the proof is complete. ☐

Note that in Proposition 3.5 we did not assume any definability property for $\mathcal{I}$. We also mention that Proposition 3.5 can be extended to suitable relative σ-ideals of compact sets, as follows. We used that $\mathcal{I}$ is a σ-ideal of compact sets only when we approximated the portions $K \cap U(s)$ $(s \in S)$ with elements of $\mathcal{I}$. This can be carried out for relative σ-ideals of compact sets with the additional property that in Proposition 2.4, the clopen set U can be chosen to satisfy $U \cap \bigcup\mathcal{I} = \bigcup(\mathcal{P}(U) \cap \mathcal{I})$. E.g., in Section 5 we prove such an amended version of Proposition 2.4 for the so-called monotone σ-ideal of compact sets (see Lemma 5.8). So with the notation of Definition 1.3, if $\mathcal{I}$ is such a σ-ideal of compact sets relative to $\mathcal{F}$, then $\mathcal{I} \leq_T \omega^\omega$ if and only if there is a G_δ set $A \subseteq 2^\omega$ such that $\mathcal{I} = \mathcal{P}(A) \cap \mathcal{F}$.

3.2. $\mathcal{K}_\mathcal{M}$ **vs.** $\mathcal{K}_\mathcal{N}$**.** Observe that $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{K}_\mathcal{N}$ follows from $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{Z}_0$ and $\mathcal{K}_\mathcal{N} \leq_T \mathcal{Z}_0$. Despite, we prove $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{K}_\mathcal{N}$ directly because the partition calculus behind this non-reducibility result contains important information about $\mathcal{K}_\mathcal{M}$. In particular, (2) of Proposition 3.6 is the natural counterpart of [10, Theorem 3B (c) p. 198].

Let $T = (\omega \times \omega)^{<\omega}$,

$$\tau = \{(s, \{s^\frown(\langle i, j\rangle)\colon j < \omega\})\colon s \in (\omega \times \omega)^{<\omega}, i < \omega\}$$

and let $\mathcal{S}$ be the family of those subtrees $S \subseteq T$ which satisfy

(i) for every $s \in S$, $\{\langle i, j\rangle\colon s^\frown(\langle i, j\rangle) \in S\}$ is finite;
(ii) for every $s \in S$ and $i < \omega$, $|\{j\colon s^\frown(\langle i, j\rangle) \in S\}| \leq 1$.

By Proposition 3.2, $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{K}_\mathcal{N}$ is the corollary of the following.

Proposition 3.6. *We have*

(1) $(T, \tau, \mathcal{S}) \not\to \mathcal{K}_\mathcal{N}$*;*
(2) $(T, \tau, \mathcal{S}) \to \mathcal{K}_\mathcal{M}$*.*

We need the following corollary of Proposition 2.4.

Corollary 3.7. *Let* $V \subseteq 2^\omega$ *be a clopen set. Let* $\mathcal{P}_i(j) \subseteq \mathcal{P}(V) \cap \mathcal{K}_\mathcal{M}$ $(i, j < \omega)$ *be hereditary families such that for every* $i < \omega$, $\mathcal{P}(V) \cap \mathcal{K}_\mathcal{M} \subseteq \bigcup_{j<\omega}\mathcal{P}_i(j)$*. Then there are* $I \in [\omega]^{<\omega}$*, clopen sets* $U_i \subseteq V$ *and* $j_i < \omega$ $(i \in I)$ *such that* $V = \bigcup_{i\in I} U_i$ *and* $\mathcal{P}(U(i)) \cap \mathcal{K}_\mathcal{M} \subseteq \mathcal{P}_i(j_i)$ $(i \in I)$*.*

Proof. Let $\{x_i\colon 0 < i < \omega\} \subseteq V$ be dense. By Proposition 2.4, for every $0 < i < \omega$ there is a $j_i < \omega$ and a clopen set $U_i \subseteq V$ with $\{x_i\} \subseteq U_i$ and $\mathcal{P}(U_i) \cap \mathcal{K}_\mathcal{M} \subseteq \mathcal{P}_i(j_i)$. Then $K = V \setminus \bigcup_{i<\omega} U_i$ satisfies $K \in \mathcal{P}(V) \cap \mathcal{K}_\mathcal{M}$. So again by Lemma 2.4, there is a $j_0 < \omega$ and a clopen set $U_0 \subseteq V$

with $K \subseteq U_i$ and $\mathcal{P}(U_0) \cap \mathcal{K}_{\mathcal{M}} \subseteq \mathcal{P}_0(j_0)$. That is, $V = \bigcup_{i<\omega} U_i$; i.e. by compactness, there is an $I \in [\omega]^{<\omega}$ satisfying $V = \bigcup_{i \in I} U_i$. By the choice of j_i and U_i $(i \in I)$, the proof is complete. $\square$

Proof of Proposition 3.6. To see (1), we have to define a τ-partition $D\colon (\omega \times \omega)^{<\omega} \to \mathcal{P}(\mathcal{K}_{\mathcal{N}})$ witnessing $(T, \tau, \mathcal{S}) \not\to \mathcal{K}_{\mathcal{N}}$. For every $n < \omega$ and $i < \omega$, let $(U^n_i(j))_{j<\omega}$ be an enumeration of all clopen sets $U \subseteq 2^\omega$ such that for every $s \in 2^n$, the measure of $U \cap N_s$ is 2^{-n-i-2}.

We define $D(s)$ $(s \in (\omega \times \omega)^{<\omega})$ by induction on $|s|$. We set $D(\emptyset) = \mathcal{K}_{\mathcal{N}}$. Let $0 < n < \omega$ and suppose that $D(s)$ is defined for every $s \in (\omega \times \omega)^{n-1}$. Let $\varphi_n\colon (\omega \times \omega)^{n-1} \times \omega \to \omega$ be an arbitrary injection. For every $s \in (\omega \times \omega)^{n-1}$, set

$$D(s^\frown(\langle i, j\rangle)) = \{K \in D(s)\colon K \subseteq U^n_{\varphi_n(s,i)}(j)\} \ (i, j < \omega).$$

This completes the inductive step of the definition of D.

Since for fixed $n, i < \omega$ we have $\mathcal{K}_{\mathcal{N}} = \bigcup_{j<\omega} \mathcal{P}(U^n_i(j)) \cap \mathcal{K}_{\mathcal{N}}$, D is a τ-partition. To finish the proof of (1), let $S \in \mathcal{S}$ and $d\colon S \to \mathcal{K}_{\mathcal{N}}$ satisfy $d(s) \in D(s)$ $(s \in S)$. We show that $K = \mathrm{cl}_{2^\omega}(\bigcup\{d(s)\colon s \in S\})$ is of measure zero. By the Lebesgue density theorem, it is enough to show that for every $t \in 2^{<\omega} \setminus \{\emptyset\}$ the measure of $K \cap N_t$ does not exceed $2^{-|t|-1}$. Let $t \in 2^{<\omega} \setminus \{\emptyset\}$ be arbitrary, say $n = |t|$. Set

$$K^- = \bigcup\{d(s)\colon s \in S, |s| < n\}, \ K^+ = \bigcup\{d(s)\colon s \in S, |s| \geq n\}.$$

By (i), K^- is a finite union of measure zero sets, so it is of measure zero. We have

$$K^+ \subseteq \bigcup \left\{U^n_{\varphi_n(s,i)}(j)\colon s^\frown(\langle i, j\rangle) \in S, |s| = n-1\right\}.$$

The union on the right hand side is finite. By (ii), for fixed $s \in S$ with $|s| = n-1$ and $i < \omega$ there is at most one j with $s^\frown\langle i, j\rangle \in S$. The measure of $U^n_{\varphi_n(s,i)}(j) \cap N_t$ is $2^{-n-\varphi_n(s,i)-2}$. So since $\varphi_n(s, i) \neq \varphi_n(s', i')$ for $(s, i) \neq (s', i')$, the measure of the clopen set

$$\bigcup \left\{U^n_{\varphi_n(s,i)}(j)\colon s^\frown(\langle i, j\rangle) \in S, |s| = n-1\right\} \cap N_t$$

does not exceed $\sum_{i<\omega} 2^{-n-i-2} = 2^{-n-1}$. Since $K \subseteq K^- \cup \mathrm{cl}_{2^\omega}(K^+)$, the the proof of (1) is complete.

To see (2), let $D\colon (\omega \times \omega)^{<\omega} \to \mathcal{P}(\mathcal{K}_{\mathcal{M}})$ be a τ-partition; we construct a tree $S \subseteq (\omega \times \omega)^{<\omega}$ satisfying (i)-(ii) and a $d\colon S \to \mathcal{K}_{\mathcal{M}}$ with $d(s) \in D(s)$ $(s \in S)$ such that $\{d(s)\colon s \in S\}$ is unbounded.

We define S by successive extensions. For every $n < \omega$, let $S_n = \{s \in S\colon |s| = n\}$. We define d in parallel with S; moreover, in addition, we define clopen sets $U_s \subseteq 2^\omega$ $(s \in S)$ such that $\mathcal{P}(U_s) \cap \mathcal{K}_{\mathcal{M}} \subseteq D(s)^\downarrow$ $(s \in S)$ and for every $n < \omega$, $\bigcup_{s \in S_n} U_s = 2^\omega$.

Set $S_0 = \{\emptyset\}$, $U_\emptyset = 2^\omega$ and let $d(\emptyset) \in \mathcal{K}_{\mathcal{M}} \setminus \{\emptyset\}$ be arbitrary. Let $0 < n < \omega$ be arbitrary and suppose that S_{n-1} and $d(s)$, U_s $(s \in S_{n-1})$ are already defined. Take an arbitrary $s \in S_{n-1}$. Set

$$\mathcal{P}_i(j) = \mathcal{P}(U_s) \cap D(s^\frown(\langle i, j\rangle))^\downarrow \ (i, j < \omega).$$

By $\mathcal{P}(U_s) \cap \mathcal{K}_{\mathcal{M}} \subseteq D(s)^\downarrow$ and by D being a τ-partition, for every $i < \omega$ we have $\mathcal{P}(U_s) \cap \mathcal{K}_{\mathcal{M}} \subseteq \bigcup_{j<\omega} \mathcal{P}_i(j)$. So by Corollary 3.7, there are $I \in [\omega]^{<\omega}$, clopen sets $U_i \subseteq U_s$ and $j_i < \omega$ $(i \in I)$ such that $U_s = \bigcup_{i \in I} U_i$ and $\mathcal{P}(U_i) \cap \mathcal{K}_{\mathcal{M}} \subseteq \mathcal{P}_i(j_i)$ $(i \in I)$. Put $s^\frown\langle i, j\rangle \in S$ if and only if $i \in I$ and $j = j_i$, and define $U_{s^\frown(\langle i, j_i\rangle)} = U_i$ $(i \in I)$. Finally let $d(s^\frown(\langle i, j_i\rangle)) \in D(s^\frown(\langle i, j_i\rangle))$ be arbitrary such that the 2^{-n} neighborhood of $d(s^\frown(\langle i, j_i\rangle))$ in 2^ω covers $U_{s^\frown(\langle i, j_i\rangle)}$. Such a set exists by $\mathcal{P}(U_{s^\frown(\langle i, j_i\rangle)}) \cap \mathcal{K}_{\mathcal{M}} \subseteq D(s^\frown(\langle i, j_i\rangle))^\downarrow$. This competes the inductive step of the construction.

We show that $\bigcup\{d(s)\colon s \in S\}$ is dense in 2^ω. Indeed, for every $n < \omega$ and $s \in S_n$, the 2^{-n} neighborhood of $d(s)$ covers U_s, so by $\bigcup_{s \in S_n} U_s = 2^\omega$ the 2^{-n} neighborhood of $\bigcup\{d(s)\colon s \in S_n\}$ covers 2^ω. So $\{d(s)\colon s \in S\}$ is unbounded, as required. □

3.3. $\mathcal{K}_\mathcal{M}$ **vs.** $\mathcal{Z}_0$**.** Since we couldn't find a partition property witnessing $\mathcal{K}_\mathcal{M} \not\leq_T \mathcal{Z}_0$ significantly simpler than $\mathcal{Z}_0$ itself, we prove this non-reducibility directly.

We introduce some notation. Let $I_j = [2^j, 2^{j+1})$ $(j < \omega)$. For every $j < \omega$ and $H \subseteq \omega$, set

$$\varphi_j(H) = \frac{|H \cap I_j|}{2^j}, \ \varphi(H) = \sup_{j<\omega} \varphi_j(H).$$

We say $a \in 2^{<\omega}$ is *regular* if $|a| = 2^j$ for some $j < \omega$. For every $s, t \in 2^{<\omega}$, $s \vee t \in 2^{<\omega}$ is defined by $|s \vee t| = \max\{|s|, |t|\}$, $[s \vee t = 1] = [s = 1] \cup [t = 1]$. Similarly, for every $s, t \in 2^{<\omega}$ we write $s \subseteq t$ if $[s = 1] \subseteq [t = 1]$, and $s \setminus t \in 2^{<\omega}$ is defined by $|s \setminus t| = |s|$, $[s \setminus t = 1] = [s = 1] \setminus [t = 1]$.

For every $a \in 2^{<\omega}$ and $\varepsilon > 0$, let

$$N(a, \varepsilon) = \{H \in \mathcal{P}(\omega)\colon H \cap |a| \subseteq [a = 1], \varphi(H \setminus |a|) \leq \varepsilon\}.$$

Lemma 3.8. $\mathcal{Z}_0 = \{H \in \mathcal{P}(\omega)\colon \lim_{j<\omega} \varphi_j(H) = 0\}$.

Proof. It is obvious that for every $H \in \mathcal{Z}_0$ we have $\lim_{j<\omega} \varphi_j(H) = 0$. To see the converse, let $H \in \mathcal{P}(\omega)$ be arbitrary. If $\lim_{j<\omega} \varphi_j(H) \neq 0$ then there is an $\varepsilon > 0$ and a $J \in [\omega]^\omega$ such that $\varphi_j(H) \geq \varepsilon$ $(j \in J)$. For every $j \in J$ we have $|H \cap 2^{j+1}|/2^{j+1} \geq \varepsilon/2$ so $H \notin \mathcal{Z}_0$. □

Our key lemma is the following.

Lemma 3.9. *Let $f\colon \mathcal{K}_\mathcal{M} \to \mathcal{Z}_0$ be an arbitrary function. Let $V \subseteq 2^\omega$ be a clopen set, let $a \in 2^{<\omega}$ be regular and let $\varepsilon > 0$ be such that $\mathcal{P}(V) \cap \mathcal{K}_\mathcal{M} \subseteq [f^{-1}(N(a, \varepsilon))]^\downarrow$. Let $(\varepsilon_i)_{i<\omega} \subseteq \mathbb{R}^+$ be arbitrary such that $\sum_{i<\omega} \varepsilon_i \leq \varepsilon$. Then there are $I \in [\omega]^{<\omega}$, clopen sets $U_i \subseteq V$ $(i \in I)$ and $b \in 2^{<\omega}$ such that*

1. *$a \sqsubseteq b$, $\varphi(b \setminus a) \leq 3\varepsilon$;*
2. *$V = \bigcup_{i \in I} U_i$;*
3. *$\mathcal{P}(U(i)) \cap \mathcal{K}_\mathcal{M} \subseteq [f^{-1}(N(b, \varepsilon_i))]^\downarrow$ $(i \in I)$.*

Proof. Let $\{y_i\colon 0 < i < \omega\} \subseteq V$ be dense. By induction, we define clopen sets $U_i \subseteq V$ $(0 < i < \omega)$ and $c_i, b_i \in 2^{<\omega}$ $(0 < i < \omega)$ such that with $b_0 = a$ we have

- (i) b_{i+1}, c_{i+1} are regular, $b_i \sqsubseteq b_{i+1}$, $b_i \sqsubseteq c_{i+1}$, $|b_{i+1}| = |c_{i+1}|$, $\varphi(c_{i+1} \setminus b_i) \leq \varepsilon$ and $\varphi(b_{i+1} \setminus b_i) \leq \varepsilon$ $(i < \omega)$;
- (ii) $\mathcal{P}(V) \cap \mathcal{K}_\mathcal{M} \subseteq [f^{-1}(N(b_i, \varepsilon))]^\downarrow$ $(i < \omega)$;
- (iii) $y_{i+1} \in U_{i+1}$ and $\mathcal{P}(U_{i+1}) \cap \mathcal{K}_\mathcal{M} \subseteq [f^{-1}(N(c_{i+1}, \varepsilon_{i+1}))]^\downarrow$ $(i < \omega)$;

as follows. Let $i < \omega$ and suppose that b_i is found such that (ii) holds. Since

$$N(b_i, \varepsilon) \subseteq \bigcup\{N(c, \varepsilon_{i+1})\colon c \in 2^{<\omega}, b_i \sqsubseteq c, \varphi(c \setminus b_i) \leq \varepsilon, c \text{ is regular}\},$$

by Lemma 2.4 there is a clopen set $y_{i+1} \in U_{i+1} \subseteq V$ and a regular $c_{i+1} \in 2^{<\omega}$ such that $b_i \sqsubseteq c_{i+1}$, $\varphi(c_{i+1} \setminus b_i) \leq \varepsilon$ and

$$\mathcal{P}(U_{i+1}) \cap \mathcal{K}_\mathcal{M} \subseteq [f^{-1}(N(c_{i+1}, \varepsilon_{i+1}))]^\downarrow.$$

Similarly, we have

$$N(b_i, \varepsilon) \subseteq \bigcup\{N(b, \varepsilon)\colon b \in 2^{|c_{i+1}|}, b_i \sqsubseteq b, \varphi(b \setminus b_i) \leq \varepsilon\},$$

so by (ii) and by Lemma 2.6 there is a $b_{i+1} \in 2^{|c_{i+1}|}$ such that $b_i \sqsubseteq b_{i+1}$, $\varphi(b_{i+1} \setminus b_i) \leq \varepsilon$ and $\mathcal{P}(V) \cap \mathcal{K}_\mathcal{M} \subseteq [f^{-1}(N(b_{i+1}, \varepsilon))]^\downarrow$. Since (i), (ii) and (iii) are satisfied, the inductive step is complete.

Let $K = V \setminus \bigcup_{i<\omega} U_i$; then $K \in \mathcal{P}(V) \cap \mathcal{K}_{\mathcal{M}}$. So again by Lemma 2.4, using

$$N(b_0,\varepsilon) \subseteq \bigcup\{N(c,\varepsilon_0)\colon c \in 2^{<\omega}, b_0 \sqsubseteq c, \varphi(c \setminus b_0) \le \varepsilon, c \text{ is regular}\},$$

there is a clopen set $K \subseteq U_0 \subseteq V$ and a regular $c_0 \in 2^{<\omega}$ such that $b_0 \sqsubseteq c$, $\varphi(c_0 \setminus b_0) \le \varepsilon$ and

$$\mathcal{P}(U_0) \cap \mathcal{K}_{\mathcal{M}} \subseteq [f^{-1}(N(c_0,\varepsilon_0))]^{\downarrow}. \tag{3.1}$$

We have $V = \bigcup_{i<\omega} U_i$. So by compactness, there is an $I \in [\omega]^{<\omega}$ satisfying $V = \bigcup_{i\in I} U_i$; i.e. (2) holds.

To find our b, let $n = \max_{i\in I} |c_i|$. For every $i \in I$ we have

$$N(c_i,\varepsilon_i) \subseteq \bigcup\{N(d,\varepsilon_i)\colon d \in 2^n, c_i \sqsubseteq d, \varphi(d \setminus c_i) \le \varepsilon_i\},$$

so by (iii), (3.1) and Lemma 2.6 there is a $d_i \in 2^n$ such that $c_i \sqsubseteq d_i$, $\varphi(d_i \setminus c_i) \le \varepsilon_i$ and

$$\mathcal{P}(U_i) \cap \mathcal{K}_{\mathcal{M}} \subseteq [f^{-1}(N(d_i,\varepsilon_i))]^{\downarrow}. \tag{3.2}$$

Set $b = \bigvee_{i\in I} b_i \vee \bigvee_{i\in I} d_i$; we show that this choice fulfills the requirements. It is obvious that $a \sqsubseteq b$. Let $(i_l)_{l<|I|}$ be a strictly increasing enumeration of I. Then

$$b \setminus a = (b_{i_0} \setminus a) \vee \left(\bigvee_{0<l<|I|}(b_{i_l} \setminus b_{i_{l-1}})\right) \vee (c_{i_0} \setminus a) \vee \left(\bigvee_{0<l<|I|}(c_{i_l} \setminus c_{i_{l-1}})\right) \vee \left(\bigvee_{l<|I|}(d_{i_l} \setminus c_{i_l})\right).$$

By (i), we have

$$\varphi\left((b_{i_0} \setminus a) \vee \left(\bigvee_{0<l<|I|}(b_{i_l} \setminus b_{i_{l-1}})\right)\right) \le \max\{\varphi(b_{i_0} \setminus a), \varphi(b_{i_l} \setminus b_{i_{l-1}})\colon 0 < l < |I|\} \le \varepsilon;$$

and

$$\varphi\left((c_{i_0} \setminus a) \vee \left(\bigvee_{0<l<|I|}(c_{i_l} \setminus c_{i_{l-1}})\right)\right) \le \max\{\varphi(c_{i_0} \setminus a), \varphi(c_{i_l} \setminus c_{i_{l-1}})\colon 0 < l < |I|\} \le \varepsilon.$$

By the definition of $(d_i)_{i\in I}$, we have $\varphi(\bigvee_{l<|I|}(d_{i_l} \setminus c_{i_l})) \le \sum_{i\in I} \varepsilon_i \le \varepsilon$. To summarize, $\varphi(b \setminus a) \le 3\varepsilon$, i.e. (1) holds. Since (3) follows from (3.2) and $d_i \subseteq b$ ($i \in I$), the proof is complete. $\square$

Proposition 3.10. *$\mathcal{K}_{\mathcal{M}} \not\le_T \mathcal{Z}_0$.*

Proof. Let $f\colon \mathcal{K}_{\mathcal{M}} \to \mathcal{Z}_0$ be an arbitrary function. Let $\eta\colon (\omega^{<\omega}, \sqsubseteq) \to \omega$ be an increasing injective function such that $\eta(\emptyset) = 0$. We construct a finitely branching infinite tree $T \subseteq \omega^{<\omega}$, a strictly increasing function $b\colon (T,\sqsubseteq) \to 2^{<\omega}$ and a family of clopen sets $U(t) \subseteq 2^\omega$ ($t \in T$) with the following properties:

(i) $b(t)$ is regular ($t \in T$) and $\varphi(b(t^\frown(i)) \setminus b(t)) \le 3 \cdot 2^{-\eta(t^\frown(i))}$ ($t^\frown(i) \in T$);
(ii) $2^\omega = \bigcup\{U(t)\colon t \in T, |t| = n\}$ ($n < \omega$);
(iii) $\mathcal{P}(U(t^\frown(i))) \cap \mathcal{K}_{\mathcal{M}} \subseteq [f^{-1}(N(b(t), 2^{-\eta(t^\frown(i))}))]^{\downarrow}$ ($t^\frown(i) \in T$).

Suppose first that the construction is done; we show that it witnesses that f is not a Tukey map. For every $t^\frown(i) \in T$, by (iii) we can find a $K(t^\frown(i)) \in \mathcal{K}_{\mathcal{M}}$ such that the $2^{-|t|}$ neighborhood of $K(t^\frown(i))$ covers $U(t^\frown(i))$ and $f(K(t^\frown(i))) \in N(b(t), 2^{-\eta(t^\frown(i))})$. Then by (ii),

$$\{K(t^\frown(i))\colon t \in T, i < \omega, t^\frown(i) \in T\}$$

is unbounded in $\mathcal{K}_{\mathcal{M}}$. On the other hand,

$$\{f(K(t^\frown(i)))\colon t \in T, i < \omega, t^\frown(i) \in T\} \subseteq \bigvee_{t\in T} b(t) \vee \bigvee\{f(K(t^\frown(i))) \setminus b(t)\colon t \in T, i < \omega, t^\frown(i) \in T\}.$$

We have $b(t), f(K(t^\frown(i))) \in \mathcal{Z}_0$ ($t, i < \omega, t^\frown(i) \in T$). By (i),

$$\varphi(\bigvee\{b(t)\colon t \in T, |t| > n\} \setminus \bigvee\{b(t)\colon t \in T, |t| \le n\}) \le 3 \cdot \sum\{2^{-\eta(t)}\colon t \in T, |t| > n\} \le 3 \cdot 2^{-n},$$

and by (iii),

$$\varphi(\bigvee\{f(K(t^\frown(i)))\setminus b(t)\colon t\in T, |t|\geq n, i<\omega, t^\frown(i)\in T\})\leq \\ \sum\{2^{-\eta(t^\frown(i))}\colon t\in T, |t|\geq n, i<\omega, t^\frown(i)\in T\}\leq 2^{-n}.$$

So $\bigvee_{t\in T} b(t)\in\mathcal{Z}_0$ and $\bigvee\{f(K(t^\frown(i)))\setminus b(t)\colon t\in T, t^\frown(i)\in T\}\in\mathcal{Z}_0$, i.e.

$$\{f(K(t^\frown(i)))\colon t, i<\omega, t^\frown(i)\in T\}$$

is bounded in $\mathcal{Z}_0$, as required.

It remains to perform the construction. We define T by successive extensions and we define b and U in parallel with T. For every $n<\omega$, let $T_n=\{t\in T\colon |t|=n\}$. Set $T_0=\{\emptyset\}$ and $U_\emptyset=2^\omega$.

Let $n<\omega$ and suppose that T_n, $U(t)$ $(t\in T_n)$ are already defined such that (ii) and (iii) hold; and if $n>0$ then $b(t)$ $(t\in T_{n-1})$ are also defined such that (i) holds. Let $t\in T_n$ be arbitrary. We apply Lemma 3.9 with $V=U(t)$, $\varepsilon=2^{-\eta(t)}$, $\varepsilon_i=2^{-\eta(t^\frown(i))}$ $(i<\omega)$, and $a=b(t|_{|t|-1})$ if $|t|>0$ and $a=\emptyset$ if $|t|=0$. Since $\mathcal{P}(U(t))\cap\mathcal{K}_{\mathcal{M}}\subseteq[f^{-1}(N(a,2^{-\eta(t)}))]^\downarrow$ holds by the inductive assumption on (iii) for $|t|>0$ and by $\eta(\emptyset)=0$ for $|t|=0$, we get $I\in[\omega]^{<\omega}$, clopen sets $U_i\subseteq U(t)$ and $b\in 2^{<\omega}$ such that (1)-(3) of Lemma 3.9 hold. Set $b(t)=b$, put $t^\frown(i)\in T_{n+1}$ if and only if $i\in I$ and let $U(t^\frown(i))=U_i$ $(i\in I)$. Then (1), (2) and (3) of Lemma 3.9 imply (i), (ii) and (iii), so the proof is complete. □

4. COFINAL DIVERSITY OF ANALYTIC p-IDEALS

In this section we recover [17, Theorem 6 p. 183], which yields arrow (13) on the Tukey picture (see Corollary 4.4), and we prove Theorem 1.5.

4.1. **Cofinal diversity from one submeasure.**

In [17] the authors obtain that the structure $(\mathcal{P}(\omega),\subseteq^\star)$ embeds into the family of density-like $F_{\sigma\delta}$ p-ideals on ω partially ordered by $\leq_T$. To this end, they construct *one* density-like $F_{\sigma\delta}$ p-ideal $\mathcal{I}$ with the property that if $S,T\subseteq\omega$ are "sufficiently different" then $\mathcal{I}\cap\mathcal{P}(S)$ and $\mathcal{I}\cap\mathcal{P}(T)$ are Tukey incomparable. Every subsequent result establishing cofinal diversity uses the same idea (see e.g. [22] or [18, Proposition 5.29]).

Here as well, we follow the same strategy. We formulate a combinatorial condition on a submeasure φ which guarantees that $(\mathcal{P}(\omega),\subseteq^\star)$ embeds into the family

$$\{\mathcal{I}\subseteq\mathcal{P}(\omega)\colon \mathcal{I}\text{ is an analytic }p\text{-ideal}, \mathcal{I}\leq_T \mathrm{Exh}(\varphi)\}$$

partially ordered by $\leq_T$. Our present result is the counterpart of [18, Proposition 5.29] for ideals of the form $\mathrm{Exh}(\varphi)$. We use the following notation.

Definition 4.1. Let $\{d_n\colon n<\omega\}\subseteq\omega\setminus\{0\}$ be arbitrary. Set $m_0=0$, $m_{n+1}=m_n+d_n$ $(n<\omega)$ and $I_{\{n\}}=[m_n,m_{n+1})$ $(n<\omega)$. For every $S\subseteq\omega$, let $I_S=\bigcup_{n\in S}I_{\{n\}}$. If $\varphi\colon[\omega]^{<\omega}\to\mathbb{R}^+$ is a submeasure, for every $S\subseteq\omega$ set $\mathcal{I}_S(\varphi)=\{H\in\mathrm{Exh}(\varphi)\colon H\subseteq I_S\}$.

Proposition 4.2. *Let $\varphi\colon[\omega]^{<\omega}\to\mathbb{R}^+$ be a submeasure, and let $S,T\in[\omega]^\omega$ satisfy $S\subseteq^\star T$. Then, with the notation of Definition 4.1, $\mathcal{I}_S(\varphi)\leq_T\mathcal{I}_T(\varphi)$.*

Proof. Set $f\colon\mathcal{I}_S(\varphi)\to\mathcal{I}_T(\varphi)$, $f(H)=H\cap I_T$ $(H\in\mathcal{I}_S(\varphi))$; we show that f is a Tukey map. Let $\mathcal{H}\subseteq\mathcal{I}_S(\varphi)$ be unbounded, i.e. $\lim_{n<\omega}\overline{\varphi}((\bigcup\mathcal{H})\setminus n)\neq 0$. Since $\bigcup f[\mathcal{H}]=I_T\cap\bigcup\mathcal{H}$ and $S\subseteq^\star T$ implies $I_S\subseteq^\star I_T$, we have $\bigcup\mathcal{H}\subseteq^\star\bigcup f[\mathcal{H}]$. Thus $\lim_{n<\omega}\overline{\varphi}((\bigcup f[\mathcal{H}])\setminus n)\neq 0$, i.e. $f[\mathcal{H}]\subseteq\mathcal{I}_T(\varphi)$ is unbounded, as required. □

Proposition 4.3. *Let $\varphi\colon[\omega]^{<\omega}\to\mathbb{R}^+$ be a submeasure. With the notation of Definition 4.1, suppose there exist $C>1$ and $\{c_n\colon n<\omega\}\subseteq\mathbb{R}^+$ such that*

(1) $\varphi(\{n\}) > 0$ *(*$n < \omega$*)*, $\lim_{n<\omega} \varphi(\{n\}) = 0$*;*
(2) $\varphi(I_{\{n\}}) \geq 1/C$ *(*$n < \omega$*);*
(3) $\sum_{n<\omega} c_n < +\infty$*;*
(4) for every $n < \omega$ *and* $J \leq 2^{m_n}$, $H_j \in \mathrm{Exh}(\varphi)$ *and* $H_j \subseteq \omega \setminus m_n$ *(*$j < J$*) imply*

$$\overline{\varphi}(\bigcup_{j<J} H_j) \leq C \cdot \max\{\overline{\varphi}(H_j) \colon j < J\} + c_n.$$

Then for every $S, T \in [\omega]^\omega$, $S \cap T = \emptyset$ *implies* $\mathcal{I}_S(\varphi) \not\leq_T \mathcal{I}_S(\varphi)$.

Proof. Let $f \colon \mathcal{I}_S(\varphi) \to \mathcal{I}_T(\varphi)$ be an arbitrary function; we show that f is not a Tukey map. For every $\varepsilon > 0$, $m < \omega$ and $V \subseteq m$, let

$$\mathcal{A}(\varepsilon, m) = \{H \in \mathcal{I}_S(\varphi) \colon \overline{\varphi}(H) < 1,\ \overline{\varphi}(f(H) \setminus m) \leq \varepsilon\},$$
$$\mathcal{A}(\varepsilon, m, V) = \{H \in \mathcal{I}_S(\varphi) \colon \overline{\varphi}(H) < 1,\ \overline{\varphi}(f(H) \setminus m) \leq \varepsilon,\ f(H) \cap m \subseteq V\}.$$

By induction on $i < \omega$, we define $n_i < \omega$, $V_i \subseteq m_{n_i}$ and $G_i \subseteq \mathcal{I}_S(\varphi)$ such that

(i) $n_i \in S$, $n_i < n_{i+1}$ $(i < \omega)$;
(ii) $V_i \cap m_{n_{i-1}} = V_{i-1}$ and $\varphi(V_i \setminus m_{n_{i-1}}) \leq 2^{-i+1}$ $(0 < i < \omega)$;
(iii) for every $k \in I_{\{n_i\}}$ we have $G_i \cap \{H \in \mathcal{I}_S(\varphi) \colon k \in H\} \neq \emptyset$ $(i < \omega)$;
(iv) $\mathcal{A}(2^{-i}, m_{n_i}, V_i)$ is of second category and $\varphi(I_{\{n_i\}} \cap \bigcup \mathcal{A}(2^{-i}, m_{n_i}, V_i)) \geq 1/C^2 - c_{n_i}/C$ $(i < \omega)$.

Suppose first the construction is done; we prove that it witnesses that f is not a Tukey map. We have $|I_{\{n_i\}}| = d_{n_i}$ $(i < \omega)$, so for every $i < \omega$ there are $H_i(j) \in \mathcal{A}(2^{-i}, m_{n_i}, V_i)$ $(j < d_{n_i})$ such that

$$I_{\{n_i\}} \cap \bigcup \mathcal{A}(2^{-i}, m_{n_i}, V_i) = I_{\{n_i\}} \cap \bigcup_{j<d_{n_i}} H_i(j).$$

Thus by (iv),

$$\varphi(I_{\{n_i\}} \cap \bigcup_{j<d_{n_i}} H_i(j)) \geq 1/C^2 - c_{n_i}/C \ (i < w). \tag{4.1}$$

Set $\mathcal{H} = \{H_i(j) \colon i < \omega,\ j < d_{n_i}\}$. Then by (4.1), for every $i < \omega$, $\overline{\varphi}((\bigcup \mathcal{H}) \setminus m_{n_i}) \geq 1/C^2 - c_{n_i}/C$. So by (3) we get $\lim_{n<\omega} \overline{\varphi}((\bigcup \mathcal{H}) \setminus n) \neq 0$, i.e. $\mathcal{H} \subseteq \mathcal{I}_S(\varphi)$ is unbounded.

We show that $f[\mathcal{H}] \subseteq \mathcal{I}_T(\varphi)$ is bounded. We have

$$\bigcup f[\mathcal{H}] = \bigcup_{i<\omega, j<d_{n_i}} f(H_i(j)) = \bigcup_{i<\omega} \left(\left(\bigcup_{j<d_{n_i}} f(H_i(j)) \cap m_{n_i} \right) \cup \left(\bigcup_{j<d_{n_i}} f(H_i(j)) \setminus m_{n_i} \right) \right).$$

By definition, $H_i(j) \in \mathcal{A}(2^{-i}, m_{n_i}, V_i)$ implies $f(H_i(j)) \cap m_{n_i} \subseteq V_i$ $(i < \omega, j < d_{n_i})$; in particular, $\bigcup_{j<d_{n_i}} f(H_i(j)) \cap m_{n_i} \subseteq V_i$ $(i < \omega)$. Set $R_i = \bigcup_{j<d_{n_i}} f(H_i(j)) \setminus m_{n_i}$ $(i < \omega)$. Thus

$$\bigcup f[\mathcal{H}] \subseteq \bigcup_{i<\omega} V_i \cup \bigcup_{i<\omega} R_i. \tag{4.2}$$

For every $i < \omega$, $H_i(j) \in \mathcal{A}(2^{-i}, m_{n_i}, V_i)$ implies $\overline{\varphi}(f(H_i(j)) \setminus m_{n_i}) \leq 2^{-i}$ $(j < d_{n_i})$. By (i) and $S \cap T = \emptyset$ we have $n_i \notin T$; in particular, by $f(H_i(j)) \subseteq I_T$ we get

$$f(H_i(j)) \setminus m_{n_i} = f(H_i(j)) \setminus m_{n_i+1} \ (j < d_{n_i}).$$

By applying (4) with $n = n_i + 1$ and $J = d_{n_i}$, we get

$$\overline{\varphi}(R_i) = \overline{\varphi}(\bigcup_{j<d_{n_i}} f(H_i(j)) \setminus m_{n_i+1}) \leq C \cdot 2^{-i} + c_{n_i+1} \ (i < \omega). \tag{4.3}$$

For every $k \leq l < \omega$, by $V_i \subseteq m_{n_i}$ $(i < \omega)$ from (4.2) we get

$$\bigcup f[\mathcal{H}] \setminus m_{n_l} \subseteq \left(\bigcup_{l<i<\omega} V_i \setminus m_{n_l} \right) \cup \left(\bigcup_{i \leq k} R_i \setminus m_{n_l} \right) \cup \left(\bigcup_{k<i<\omega} R_i \right).$$

By (ii),

$$\overline{\varphi}(\bigcup_{l<i<\omega} V_i \setminus m_{n_l}) = \overline{\varphi}(\bigcup_{l<i<\omega} (V_i \setminus m_{n_{i-1}})) \leq \sum_{l<i<\omega} \varphi(V_i \setminus m_{n_{i-1}}) \leq 2^{-l+1} \ (l < \omega).$$

By (4.3),

$$\overline{\varphi}(\bigcup_{k<i<\omega} R_i) \leq C \cdot 2^{-k} + \sum_{k<i<\omega} c_{n_i+1} \ (k < \omega).$$

Thus for every $k \leq l < \omega$,

$$\overline{\varphi}(\bigcup f[\mathcal{H}] \setminus m_{n_l}) \leq \overline{\varphi}(\bigcup_{l<i<\omega} V_i \setminus m_{n_l}) + \overline{\varphi}(\bigcup_{i\leq k} R_i \setminus m_{n_l}) + \overline{\varphi}(\bigcup_{k<i<\omega} R_i) \leq$$
$$2^{-l+1} + \overline{\varphi}(\bigcup_{i\leq k} R_i \setminus m_{n_l}) + C \cdot 2^{-k} + \sum_{k<i<\omega} c_{n_i+1}.$$

By $f(H_i(j)) \in \mathcal{I}_T(\varphi)$ $(i < \omega,\ j < d_{n_i})$ we have $R_i \in \mathcal{I}_T(\varphi)$ $(i < \omega)$. So for every $k < \omega$,

$$\lim_{l<\omega} \overline{\varphi}(\bigcup_{i\leq k} R_i \setminus m_{n_l}) = 0.$$

Thus for every $k < \omega$,

$$\limsup_{l<\omega} \overline{\varphi}(\bigcup f[\mathcal{H}] \setminus m_{n_l}) \leq C \cdot 2^{-k} + \sum_{k<i<\omega} c_{n_i+1}.$$

By (3), this implies $\lim_{n<w} \overline{\varphi}(\bigcup f[\mathcal{H}] \setminus n) = 0$; thus $f[\mathcal{H}] \subseteq \mathcal{I}_T(\varphi)$ is bounded, as required.

It remains to perform the construction. Let first $i = 0$. We have $\mathcal{A}(1, m) \subseteq \mathcal{A}(1, m+1)$ $(m < \omega)$ and

$$\bigcup_{m<\omega} \mathcal{A}(1, m) = \{H \in \mathcal{I}_S(\varphi) \colon \overline{\varphi}(H) < 1\},$$

which is an open set. So by the Baire category theorem, there is an $m < \omega$ such that $\mathcal{A}(1, m)$ is of second category. Let $G_0 \subseteq \mathcal{I}_S(\varphi)$ be a non-empty open set such that $\mathcal{A}(1, m_{n_0})$ is of second category everywhere in G_0. By (1), there exist $H \in G_0$ and $\varepsilon > 0$ such that $\{H' \in \mathcal{I}_S(\varphi) \colon \overline{\varphi}(H\Delta H') < \varepsilon\} \subseteq G_0$. By increasing m if necessary, using $S \in [\omega]^\omega$ and (1), we can assume

(a) $m = m_{n_0}$ for some $n_0 \in S$, then (i) holds;
(b) $\varphi(\{k\}) < \varepsilon$ $(k \in I_{\{n_0\}})$.

Set $V_0 = m_{n_0}$. Statement (ii) is empty for $i = 0$, and (b) implies (iii). Since $\mathcal{A}(1, m_{n_0})$ is of second category everywhere in G_0, by (iii) we have $I_{\{n_0\}} \subseteq \bigcup \mathcal{A}(1, m_{n_0}, V_0))$. So (iv) follows from (2). This completes the $i = 0$ case of the construction.

Let now $i < \omega$ be arbitrary and suppose n_i, V_i and G_i have already been defined such that (iv) holds. We have $\mathcal{A}(2^{-i-1}, m, V_i \cup (m \setminus m_{n_i})) \subseteq \mathcal{A}(2^{-i-1}, m+1, V_i \cup (m+1 \setminus m_{n_i}))$ $(m < \omega)$ and

$$\mathcal{A}(2^{-i}, m_{n_i}, V_i) \subseteq \bigcup_{m_{n_i}<m<\omega} \mathcal{A}(2^{-i-1}, m, V_i \cup (m \setminus m_{n_i})),$$

so there is an $m_{n_i} < m < \omega$ such that $\mathcal{A}_i = \mathcal{A}(2^{-i-1}, m, V_i \cup (m \setminus m_{n_i})) \cap \mathcal{A}(2^{-i}, m_{n_i}, V_i)$ is of second category. Let $G_{i+1} \subseteq \mathcal{I}_S(\varphi)$ be a non-empty open set such that $\mathcal{A}_i$ is of second category everywhere in G_{i+1}. By (1), there exist $H \in G_{i+1}$ and $\varepsilon > 0$ such that

$$\{H' \in \mathcal{I}_S(\varphi) \colon \overline{\varphi}(H\Delta H') < \varepsilon\} \subseteq G_{i+1}.$$

By increasing m if necessary, using $S \in [\omega]^\omega$ and (1), we can assume

(c) $m = m_{n_{i+1}}$ for some $n_{i+1} \in S$; then (i) holds;
(d) $\varphi(\{k\}) < \varepsilon$ $(k \in I_{\{n_{i+1}\}})$.

Then (d) implies (iii). Since $\mathcal{A}_i$ is of second category everywhere in G_{i+1}, by (iii) we have

$$I_{\{n_{i+1}\}} \subseteq \bigcup \mathcal{A}_i. \tag{4.4}$$

It remains to find V_{i+1} satisfying (iv). Set

$$\mathcal{V} = \{V \subseteq m_{n_{i+1}} \colon V \cap m_{n_i} = V_i,\ \varphi(V \setminus m_{n_i}) \leq 2^{-i},\ \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V) \text{ is of second category}\}.$$

If we find a $V \in \mathcal{V}$ such that

$$\varphi(I_{\{n_{i+1}\}} \cap \bigcup \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V)) \geq 1/C^2 - c_{n_{i+1}}/C;$$

then $V_{i+1} = V$ fulfills the requirements.

So suppose no such a V exists, i.e. for every $V \in \mathcal{V}$ we have

$$\varphi(I_{\{n_{i+1}\}} \cap \bigcup \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V)) < 1/C^2 - c_{n_{i+1}}/C. \tag{4.5}$$

By definition, for every $H \in \mathcal{A}_i$ we have $H \in \mathcal{A}(2^{-i}, m_{n_i}, V_i)$ hence $\overline{\varphi}(H \setminus m_{n_i}) \leq 2^{-i}$ and $H \cap m_{n_i} \subseteq V_i$. So for every $H \in \mathcal{A}_i$, $V_H = V_i \cup H \cap m_{n_{i+1}} \subseteq m_{n_{i+1}}$ satisfies $V_H \cap m_{n_i} = V_i$, $\varphi(V_H \setminus m_{n_i}) \leq 2^{-i}$ and $H \in \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V_H)$. Thus

$$\mathcal{A}_i \setminus \bigcup_{V \in \mathcal{V}} \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V)$$

is of first category. In particular, by (iii), (4.4) and since $\mathcal{A}_i$ is of second category everywhere in G_{i+1},

$$I_{\{n_{i+1}\}} \cap \bigcup \bigcup_{V \in \mathcal{V}} \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V) = I_{\{n_{i+1}\}} \cap \bigcup \mathcal{A}_i = I_{\{n_{i+1}\}}.$$

So (2) implies

$$\overline{\varphi}(I_{\{n_{i+1}\}} \cap \bigcup \bigcup_{V \in \mathcal{V}} \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V)) \geq 1/C. \tag{4.6}$$

On the other hand, $|\mathcal{V}| \leq 2^{m_{n_{i+1}}}$. So by (4), (4.5) implies

$$\varphi(I_{\{n_{i+1}\}} \cap \bigcup \bigcup_{V \in \mathcal{V}} \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V)) \leq \\ C \cdot \max\{\varphi(I_{\{n_{i+1}\}} \cap \bigcup \mathcal{A}(2^{-i-1}, m_{n_{i+1}}, V)) \colon V \in \mathcal{V}\} + c_{n_{i+1}} < 1/C,$$

which contradicts (4.6). This completes the inductive step of the construction and the proof. □

From Proposition 4.3, [17, Theorem 6 p. 183] immediately follows.

Corollary 4.4. *The structure $(\mathcal{P}(\omega), \subseteq^\star)$ embeds into the family of density-like $F_{\sigma\delta}$ p-ideals on ω partially ordered by $\leq_T$.*

Proof. We follow the notation of Definition 4.1. Define $\{d_n \colon n < \omega\} \subseteq \omega \setminus \{0\}$ by $d_0 = 1$, $d_n = 2^{(n+1)^2 2^{\sum_{i<n} d_i}}$ $(0 < n < \omega)$. Set

$$\varphi_n(A) = \frac{\log_2(|A \cap I_{\{n\}}| + 1)}{\log_2(d_n + 1)} \ (n < \omega), \ \varphi(A) = \max_{n<\omega} \varphi_n(A) \ (A \in [\omega]^{<\omega}).$$

It is clear that φ_n $(n < \omega)$ are submeasures, so φ is also a submeasure. It is also immediate that $\mathcal{I}_S(\varphi)$ $(S \in [\omega]^\omega)$ are density-like and $F_{\sigma\delta}$. We show that for every $S, T \in [\omega]^\omega$, $S \subseteq^\star T$ if and only if $\mathcal{I}_S(\varphi) \leq_T \mathcal{I}_S(\varphi)$. By Proposition 4.2, $S \subseteq^\star T$ implies $\mathcal{I}_S(\varphi) \leq_T \mathcal{I}_T(\varphi)$. To see the converse, it is enough to verify the conditions of Proposition 4.3; then for $S \not\subseteq^\star T$, by $\mathcal{I}_{S \setminus T}(\varphi) \leq_T \mathcal{I}_S(\varphi)$ and $\mathcal{I}_{S \setminus T}(\varphi) \not\leq_T \mathcal{I}_T(\varphi)$ we get $\mathcal{I}_S(\varphi) \not\leq_T \mathcal{I}_T(\varphi)$, as required.

Set $C = 1$ and $c_n = 1/(n+1)^2$ $(n < \omega)$. Since $\lim_{n<\omega} d_n = +\infty$, (1) of Proposition 4.3 holds. We have

$$\varphi(I_{\{n\}}) = \varphi_n(I_{\{n\}}) = \frac{\log_2(|I_{\{n\}}| + 1)}{\log_2(d_n + 1)} = \frac{\log_2(d_n + 1)}{\log_2(d_n + 1)} = 1 \ (n < \omega),$$

so (2) also holds. Statement (3) is well-known. For (4), observe that for every $J < \omega$ and $H_j \subseteq \omega$ $(j < J)$ we have

$$\varphi_n(\bigcup_{j<J} H_j) \leq \max\{\varphi_n(H_j) \colon j < J\} + \frac{J}{\log_2(d_n + 1)} \ (n < \omega).$$

Thus for every $n<\omega$, $J\leq 2^{m_n}$ and $H_j\subseteq\omega\setminus m_n$ $(j<J)$ we have

$$\overline{\varphi}(\textstyle\bigcup_{j<J}H_j)=\sup\{\varphi_k(\bigcup_{j<J}H_j))\colon n\leq k<\omega\}\leq$$
$$\sup\{\max\{\varphi_k(H_j)+J/\log_2(d_k+1)\colon j<J\}\colon n\leq k<\omega\}\leq$$
$$\max\{\overline{\varphi}(H_j)\colon j<J\}+\sup\{2^{m_n}/\log_2(d_k+1)\colon n\leq k<\omega\}.$$

We have $m_n=\sum_{j<n}d_j$ $(0<n<\omega)$, so

$$2^{m_n}\leq\log_2(d_n+1)/(n+1)^2\leq\log_2(d_k+1)/(n+1)^2\ (n\leq k<\omega).$$

Thus $\sup\{2^{m_n}/\log_2(d_k+1)\colon n\leq k<\omega\}\leq 1/(n+1)^2=c_n$, i.e.

$$\overline{\varphi}(\textstyle\bigcup_{j<J}H_j)\leq\max\{\overline{\varphi}(H_j)\colon j<J\}+c_n,$$

as required. □

4.2. F_σ **p-ideals.** Similarly to Corollary 4.4, in order to prove Theorem 1.5 it is enough to construct one *exhaustive* submeasure $\varphi\colon[\omega]^{<\omega}\to\mathbb{R}^+$ which satisfies the conditions of Proposition 4.3. This is significantly more complicated than the submeasure for Corollary 4.4. The strategy of the construction, similar in spirit to the Banach space construction in [21], is as follows.

The crucial condition of Proposition 4.3 is (4). First we will prove the easy fact that it is enough to verify (4) for finite sets.

Lemma 4.5. *Let $\varphi\colon[\omega]^{<\omega}\to\mathbb{R}^+$ be a submeasure. Suppose there exist $C>0$ and $\{c_n\colon n<\omega\}\subseteq\mathbb{R}^+$ such that*

(4') for every $n<\omega$, $J\leq 2^{m_n}$ and $H_j\in[\omega\setminus m_n]^{<\omega}$ $(j<J)$ we have

$$\varphi(\textstyle\bigcup_{j<J}H_j)\leq C\cdot\max\{\varphi(H_j)\colon j<J\}+c_n.$$

Then $\overline{\varphi}$ satisfies (4) of Proposition 4.3.

Given a submeasure φ, condition (4') of Lemma 4.5 yields a countable set of inequalities on the values of φ. So we may define φ on $[\omega]^1$, and then set φ to be the maximal submeasure which satisfies (4'). With a bit of luck, the resulting submeasure will be automatically exhaustive. Indeed, this is what happens.

We need some terminology in advance. A partially ordered set $(R,\sqsubseteq)$ is a *rooted tree* if R has a $\sqsubseteq$-minimal element, and for every $r\in R$ the set $\{r'\in R\colon r'\sqsubseteq r\}$ is well-ordered by $\sqsubseteq$. The minimal element of a rooted tree R will always be denoted by $\emptyset_R$. If R is a rooted tree and $r\in R$ then $\mathrm{succ}_R(r)=\{r'\in R\colon r\sqsubset r',\ \forall s\in R\ (r\not\sqsubset s\vee s\not\sqsubset r')\}$, $\deg_R(r)=|\mathrm{succ}_R(r)|$, $R_r=\{r'\in R\colon r\sqsubseteq r'\}$ and $\mathrm{term}(R)=\{r\in R\colon\mathrm{succ}_R(r)=\emptyset\}$. A tree R is *binary* if $\deg_R(r)=2$ $(r\in R\setminus\mathrm{term}(R))$.

If R is a tree then every $T\subseteq R$ is a tree with the partial order $\sqsubseteq|_T$. We say $T\subseteq R$ is a *subtree of* R if T is closed under taking $\sqsubseteq$-predecessors in R.

In order to keep track of the inequalities resulting from (4') of Lemma 4.5, we introduce the following partition schemes.

Definition 4.6. Let $A\in[\omega]^{<\omega}\setminus\{\emptyset\}$ be arbitrary. A *resolution on A* is a pair $\mathcal{R}=(R,\rho)$ where R is a finite rooted tree, $\rho\colon R\to\mathcal{P}(A)\setminus\{\emptyset\}$ and

1. $2\leq\deg_R(r)$ $(r\in R\setminus\mathrm{term}(R))$;
2. $\rho(\emptyset_R)=A$;
3. $\rho(r)=\bigcup\{\rho(r')\colon r'\in\mathrm{succ}_R(r)\}$ and $\rho(r')\cap\rho(r'')=\emptyset$ $(r',r''\in\mathrm{succ}_R(r),r'\neq r'')$;
4. $|\rho(r)|=1$ $(r\in\mathrm{term}(R))$.

For every $r \in R$ we set $\mathcal{R}_r = (R_r, \rho|_{R_r})$.

Given a function $\ell\colon \omega \to \omega \setminus 2$, (R, ρ) is an *ℓ-resolution on A* if, in addition to (1)-(4), we have

(5) $\deg_R(r) \leq \ell(\min \rho(r))$ $(r \in R \setminus \operatorname{term}(R))$.

If $w\colon [A]^1 \to \mathbb{R}^+$ is an arbitrary function and $\mathcal{R} = (R, \rho)$ is a resolution on A, then the *w-norm of $\mathcal{R}$* is

$$\|\mathcal{R}\|_w = \max\left\{\textstyle\sum_{t \in S \cap \operatorname{term}(R)} w(t)\colon S \subseteq R \text{ is a binary subtree}\right\}.$$

We define the *function $\varphi\colon [\omega]^{<\omega} \to \mathbb{R}^+$ associated to w and ℓ* by $\varphi(\emptyset) = 0$,

$$\varphi(A) = \min\{\|\mathcal{R}\|_w\colon \mathcal{R} \text{ is an } \ell\text{-resolution on } A\} \ (A \in [\omega]^{<\omega} \setminus \{\emptyset\}). \tag{4.7}$$

Observe that by (1) and (3), there are only finitely many non-isomorphic resolutions on a finite set; so the definition of φ in (4.7) makes sense. Now we can define the main object of this section.

Definition 4.7. In accordance with Definition 4.1, set $m_0 = 0$, $d_n = 2^{n \cdot m_n}$ and $m_{n+1} = m_n + d_n$ $(n < \omega)$. For every $n < \omega$ and $k \in I_{\{n\}} = [m_n, m_{n+1})$, set $w(k) = 2^{-n}$ and $\ell(k) = 2^{\max\{m_n, 1\}}$. Define $\varphi\colon [\omega]^{<\omega} \to \mathbb{R}^+$ by $\varphi(\emptyset) = 0$,

$$\varphi(A) = \min\{\|\mathcal{R}\|_w\colon \mathcal{R} \text{ is an } \ell\text{-resolution on } A\} \ (A \in [\omega]^{<\omega} \setminus \{\emptyset\}). \tag{4.8}$$

Set $C = 2$ and $c_n = 0$ $(n < \omega)$.

We show that the φ of Definition 4.7 is an exhaustive submeasure which satisfies the conditions of Proposition 4.3. The main difficulty is to prove that φ is exhaustive; this necessitates subtle estimates on w-norms of resolutions. The following lemma shows another way to compute $\|\mathcal{R}\|_w$.

Lemma 4.8. *Let $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$, let $\mathcal{R} = (R, \rho)$ be a resolution on A and let $w\colon [A]^1 \to \mathbb{R}^+$ be an arbitrary function. Define $w_{\mathcal{R}}\colon \mathcal{R} \to \mathbb{R}^+$ inductively, by setting $w_{\mathcal{R}}(r) = w(\rho(r))$ $(r \in \operatorname{term}(R))$ and*

$$w_{\mathcal{R}}(r) = \max\{w_{\mathcal{R}}(r') + w_{\mathcal{R}}(r'')\colon r', r'' \in \operatorname{succ}_R(r),\ r' \neq r''\} \ (r \in R \setminus \operatorname{term}(R)).$$

Then $\|\mathcal{R}_r\|_w = w_{\mathcal{R}}(r)$ $(r \in \mathcal{R})$.

Proof. We prove the statement by induction on $|R|$. If $|R| = 1$, i.e. $R = \{\emptyset_R\}$, then the only non-empty binary subtree of R is $\{\emptyset_R\}$, hence $\|\mathcal{R}\|_w = \|\mathcal{R}_{\emptyset_R}\|_w = w(\rho(\emptyset_R)) = w_{\mathcal{R}}(\emptyset_R)$, as stated.

Suppose now $|R| > 1$. For every $r \in R \setminus \{\emptyset_R\}$, $\mathcal{R}_r = (R_r, \rho|_{R_r})$ is a resolution on $\rho(r)$, and $|R_r| < |R|$. So by the inductive hypothesis we have

$$\|\mathcal{R}_r\|_w = w_{\mathcal{R}_r}(\emptyset_{R_r}) = w_{\mathcal{R}}(r) \ (r \in R \setminus \{\emptyset_R\}).$$

So it remains to show $\|\mathcal{R}\|_w = \|\mathcal{R}_{\emptyset_R}\|_w = w_{\mathcal{R}}(\emptyset_R)$.

Let $r', r'' \in \operatorname{succ}_R(\emptyset_R)$, $r' \neq r''$ be such that $w_{\mathcal{R}}(\emptyset_R) = w_{\mathcal{R}}(r') + w_{\mathcal{R}}(r'')$. Let $S' \subseteq R_{r'}$ and $S'' \subseteq R_{r''}$ be binary subtrees satisfying

$$\|\mathcal{R}_{r'}\|_w = \textstyle\sum_{t \in S' \cap \operatorname{term}(R_{r'})} w(t),\ \|\mathcal{R}_{r''}\|_w = \sum_{t \in S'' \cap \operatorname{term}(R_{r''})} w(t).$$

Then $S = \{\emptyset_R\} \cup S' \cup S''$ is a binary subtree of R and

$$\textstyle\sum_{t \in S \cap \operatorname{term}(R)} w(t) = \sum_{t \in S' \cap \operatorname{term}(R_{r'})} w(t) + \sum_{t \in S'' \cap \operatorname{term}(R_{r''})} w(t) = w_{\mathcal{R}}(r') + w_{\mathcal{R}}(r'') = w_{\mathcal{R}}(\emptyset_R),$$

i.e. $w_{\mathcal{R}}(\emptyset_R) \leq \|\mathcal{R}\|_w$.

To see the converse, let S be a binary subtree of R such that $\|\mathcal{R}\|_w = \sum_{t \in S \cap \operatorname{term}(R)} w(t)$. Let $\operatorname{succ}_S(\emptyset_S) = \{r', r''\}$, $S' = S_{r'}$ and $S'' = S_{r''}$. Then $S' \subseteq R_{r'}$, $S'' \subseteq R_{r''}$ are binary subtrees so

$$\sum_{t \in S \cap \operatorname{term}(R)} w(t) = \sum_{t \in S' \cap \operatorname{term}(R_{r'})} w(t) + \sum_{t \in S'' \cap \operatorname{term}(R_{r''})} w(t) \leq \\ \|\mathcal{R}_{r'}\|_w + \|\mathcal{R}_{r''}\|_w = w_{\mathcal{R}}(r') + w_{\mathcal{R}}(r'') \leq w_{\mathcal{R}}(\emptyset_R),$$

i.e. $\|\mathcal{R}\|_w \leq w_{\mathcal{R}}(\emptyset_R)$, as required. ☐

4.2.1. *Restrictions and sums of resolutions.* We introduce two operations on resolutions.

Definition 4.9. Let $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$ and let $\mathcal{R} = (R, \rho)$ be a resolution on A. Let $B \subseteq A$, $B \neq \emptyset$ and define $\rho'(r) = \rho(r) \cap B$ $(r \in R)$, and $P = \{r \in R \colon \rho'(r) \neq \emptyset\}$, $R' = \{r \in P \colon \deg_P(r) \neq 1\}$. Then *the restriction of* $\mathcal{R}$ *to* B is $\mathcal{R}|_B = (R', \rho')$.

Let $1 < n < \omega$, let $A_i \in [\omega]^{<\omega} \setminus \{\emptyset\}$ $(i < n)$ be pairwise disjoint, and for every $i < n$ let $\mathcal{R}_i = (R_i, \rho_i)$ be a resolution on A_i. Set $R = \{\emptyset_R\} \cup \bigcup_{i<n} R_i$, and define the partial order $\sqsubseteq$ on R such that R_i $(i < n)$ inherit their partial orders, $\emptyset_R \sqsubset \emptyset_{R_i}$ $(i < n)$, and $\{\emptyset_{R_i} \colon i < n\}$ are pairwise $\sqsubseteq$-incomparable. Define $\rho \colon R \to \mathcal{P}(\bigcup_{i<n} A_i)$ by $\rho(\emptyset_R) = \bigcup_{i<n} A_i$, $\rho(r) = \rho_i(r)$ $(r \in R_i,\ i < n)$. Then $\bigoplus_{i<n} \mathcal{R}_i = (R, \rho)$ is *the sum of the resolutions* $\{\mathcal{R}_i \colon i < n\}$.

We show that these operations yield resolutions and we compute their norms.

Lemma 4.10. *Let* $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$, *let* $B \subseteq A$, $B \neq \emptyset$ *and let* $\mathcal{R} = (R, \rho)$ *be a resolution on* A. *Then* $\mathcal{R}|_B = (R', \rho')$ *is a resolution on* B.

Let $\ell \colon \omega \to \omega \setminus 2$ *be an increasing function and let* $w \colon [A]^1 \to \mathbb{R}^+$ *be arbitrary. If* $\mathcal{R}$ *is an* ℓ*-resolution on* A *then* $\mathcal{R}|_B$ *is also an* ℓ*-resolution on* B. *Moreover,* $\|\mathcal{R}|_B\|_w \leq \|\mathcal{R}\|_w$.

Proof. We follow the notation of Definition 4.9. Consider first the pair $\mathcal{P} = (P, \rho')$. Conditions (2)-(4) for $\mathcal{R}$ immediately imply (2)-(4) for $\mathcal{P}$. However, (1) may fail for $\mathcal{P}$. For $p \in P$, we have $\deg_P(p) = 1$, say $\operatorname{succ}_P(p) = \{r\}$, if and only for every $r' \in \operatorname{term}(R)$ satisfying $p \sqsubset r'$ and $r \not\sqsubseteq r'$ we have $\rho(r') \not\subseteq B$. Then by (3) for $\mathcal{R}$, $\rho(p) \cap B = \rho(r) \cap B$. So by removing $\{p \in P \colon \deg_P(p) = 1\}$ from P we get that $\mathcal{R}|_B$ satisfies (1), and (2)-(4) remain valid. Thus $\mathcal{R}|_B$ is a resolution.

If $\mathcal{R}$ is an ℓ-resolution, then by

$$\deg_{R'}(r) = \deg_P(r) \leq \deg_R(r) \leq \ell(\min \rho(r)) \leq \ell(\min(\rho(r) \cap B)) = \ell(\min \rho'(r)) \ (r \in R'),$$

$\mathcal{R}|_B$ is an ℓ-resolution on B, as required. Moreover, with the notation of Lemma 4.8, $w_{\mathcal{R}|_B}(r) \leq w_{\mathcal{R}}(r)$ $(r \in R')$ follows from the inductive definitions of $w_{\mathcal{R}}$ and $w_{\mathcal{R}|_B}$. So by Lemma 4.8,

$$\|\mathcal{R}|_B\|_w = w_{\mathcal{R}|_B}(\emptyset_{\mathcal{R}|_B}) \leq w_{\mathcal{R}}(\emptyset_{\mathcal{R}|_B}) \leq w_{\mathcal{R}}(\emptyset_{\mathcal{R}}) = \|\mathcal{R}\|_w,$$

which completes the proof. ☐

Lemma 4.11. *Let* $1 < n < \omega$, *let* $A_i \in [\omega]^{<\omega} \setminus \{\emptyset\}$ $(i < n)$ *be pairwise disjoint, and for every* $i < n$ *let* $\mathcal{R}_i = (R_i, \rho_i)$ *be a resolution on* A_i. *Then* $\bigoplus_{i<n} \mathcal{R}_i$ *is a resolution on* $\bigcup_{i<n} A_i$.

Let $\ell \colon \omega \to \omega \setminus 2$ *and* $w \colon \bigcup_{i<n} [A_i]^1 \to \mathbb{R}^+$ *be arbitrary. If* $\mathcal{R}$ *is an* ℓ*-resolution on* A *and* $n \leq \ell(\min \bigcup_{i<n} A_i)$ *then* $\bigoplus_{i<n} \mathcal{R}_i$ *is also an* ℓ*-resolution on* $\bigcup_{i<n} A_i$. *Moreover,*

$$\|\bigoplus_{i<n} \mathcal{R}_i\|_w = \max\{\|\mathcal{R}_k\|_w + \|\mathcal{R}_l\|_w \colon k < l < n\}.$$

Proof. We follow the notation of Definition 4.9. Conditions (1)-(4) of Definition 4.6 obviously hold for $\bigoplus_{i<n} \mathcal{R}_i$. Condition (5) of Definition 4.6 has to be verified only for $r = \emptyset_R$. Since $\deg_R(\emptyset_R) = n$, the statement follows.

By Lemma 4.8,

$$\|\bigoplus_{i<n}\mathcal{R}_i\|_w = w_{\bigoplus_{i<n}\mathcal{R}_i}(\emptyset_R) = \max\{w_{\bigoplus_{i<n}\mathcal{R}_i}(\emptyset_{R_k}) + w_{\bigoplus_{i<n}\mathcal{R}_i}(\emptyset_{R_l})\colon k<l<n\} = \max\{w_{\mathcal{R}_k}(\emptyset_{R_k}) + w_{\mathcal{R}_l}(\emptyset_{R_l})\colon k<l<n\} = \max\{\|\mathcal{R}_k\|_w + \|\mathcal{R}_l\|_w\colon k<l<n\},$$

as required. ☐

Corollary 4.12. *Let $B \in [\omega]^{<\omega}\setminus\{\emptyset\}$, $w\colon B\to\mathbb{R}^+$ and $\ell\colon B\to\omega\setminus 2$ be arbitrary. For some $l<\omega$, let $B_i\subseteq B$, $B_i\neq\emptyset$ $(i<l)$ be pairwise disjoint such that $B=\bigcup_{i<l}B_i$. For every $i<l$, let $\mathcal{R}_i=(R_i,\rho_i)$ be an ℓ-resolution on B_i. Then there is an ℓ-resolution $\mathcal{R}$ on B such that $\|\mathcal{R}\|_w=\sum_{i<l}\|\mathcal{R}_i\|_w$.*

Proof. For $l=1$ the statement is trivial. If $l>1$, let $\mathcal{R}=\mathcal{R}_0\oplus(\dots(\mathcal{R}_{l-2}\oplus\mathcal{R}_{l-1})\dots)$. Then by Lemma 4.11, $\mathcal{R}$ is an ℓ-resolution on B and $\|\mathcal{R}\|_w=\sum_{i<l}\|\mathcal{R}_i\|_w$. ☐

Corollary 4.13. *Let $1<m<\omega$, $c>0$, $B\in[\omega]^{<\omega}\setminus\{\emptyset\}$, $w\colon B\to\mathbb{R}^+$ and $\ell\colon B\to\omega\setminus m$ be arbitrary. With the notation of Definition 4.6, let $B_s\subseteq B$, $B_s\neq\emptyset$ $(s\in S)$ be pairwise disjoint such that $B=\bigcup_{s\in S}B_s$, $\varphi(B_s)\le c/25$ $(s\in S)$ and $\sum_{s\in S}\varphi(B_s)\le m\cdot c/5$. Then there is an ℓ-resolution $\mathcal{R}$ on B such that $\|\mathcal{R}\|_w\le 12c/25$.*

Proof. By (4.7), for every $s\in S$ there is an ℓ-resolution $\mathcal{R}_s$ on B_s such that $\varphi(B_s)=\|\mathcal{R}_s\|_w$. Let $\{s_k\colon k<|S|\}$ be an enumeration of S such that $\|\mathcal{R}_{s_k}\|_w\ge\|\mathcal{R}_{s_{k+1}}\|_w$ $(k<|S|-1)$. For every $i<|S|/m$, let $\mathcal{R}(i)=\bigoplus\{\mathcal{R}_{s_l}\colon i\cdot m\le l<\min\{(i+1)\cdot m,|S|\}\}$. By Lemma 4.11, $\mathcal{R}(i)$ is an ℓ-resolution on $\bigcup\{B_{s_l}\colon i\cdot m\le l<\min\{(i+1)\cdot m,|S|\}\}$ $(i<|S|/m)$. So by Corollary 4.12, there is an ℓ-resolution $\mathcal{R}$ on B such that $\|\mathcal{R}\|_w=\sum_{i<|S|/m}\|\mathcal{R}(i)\|_w$. We show that this $\mathcal{R}$ fulfills the requirements.

By Lemma 4.11,

$$\|\mathcal{R}(i)\|_w\le 2\cdot\|\mathcal{R}_{i\cdot m}\|_w\le 2c/25\ (i<|S|/m),$$

so for $|S|\le 2m$ the statement is obvious. Suppose $|S|>2m$; then

$$\textstyle\sum_{i<|S|/m}\|\mathcal{R}(i)\|_w\le 2\cdot\sum_{i<|S|/m}\|\mathcal{R}_{i\cdot m}\|_w\le 2c/25+2\cdot\sum_{0<i<|S|/m}\|\mathcal{R}_{i\cdot m}\|_w;$$

i.e. it is enough to show

$$\textstyle\sum_{0<i<|S|/m}\|\mathcal{R}_{i\cdot m}\|_w\le c/5. \tag{4.9}$$

For every $0<i<(|S|/m)$ and $0<l\le m$ we have $\|\mathcal{R}_{i\cdot m}\|_w\le\|\mathcal{R}_{s_{(i-1)\cdot m+l}}\|_w$, thus

$$m\cdot\|\mathcal{R}_{i\cdot m}\|_w\le\textstyle\sum_{0<l\le m}\|\mathcal{R}_{s_{(i-1)\cdot m+l}}\|_w.$$

This implies

$$\textstyle\sum_{0<i<|S|/m}m\cdot\|\mathcal{R}_{i\cdot m}\|_w\le\sum_{0<i<(|S|/m)}\sum_{0<l\le m}\|\mathcal{R}_{s_{(i-1)\cdot m+l}}\|_w\le\sum_{s\in S}\|\mathcal{R}_s\|_w\le m\cdot c/5.$$

This completes the proof. ☐

4.2.2. *More subtle w-norm estimates for resolutions.*

Lemma 4.14. *Let $B\in[\omega]^{<\omega}\setminus\{\emptyset\}$ be arbitrary, let $\mathcal{R}=(R,\rho)$ be a resolution on B and let $w\colon B\to\mathbb{R}^+$ be arbitrary. Let $T\subseteq R$ be a subtree satisfying $\deg_T(t)\le 2$ $(t\in T)$ and let $q\in R\setminus T$ be arbitrary. Let $p\sqsubset q$ be $\sqsubseteq$-maximal satisfying $p\in T$.*

1. *If $\deg_T(p)=1$ then $\|\mathcal{R}\|_w\ge\sum_{t\in T\cap\mathrm{term}(R)}w(t)+\|\mathcal{R}_q\|_w$.*
2. *If $\deg_T(p)=2$ then*

$$\|\mathcal{R}\|_w\ge\textstyle\sum_{t\in T\cap\mathrm{term}(R)}w(t)+\|\mathcal{R}_q\|_w-\sum_{t\in T\cap\mathrm{term}(R_p)}w(t)/2$$

(3) $\|\mathcal{R}\|_w \geq \|\mathcal{R}_q\|_w + \sum_{t \in T \cap \mathrm{term}(R)} w(t)/2$.
(4) $\|\mathcal{R}\|_w \geq \|\mathcal{R}_q\|_w + \|\mathcal{R}|_{B \setminus \rho(q)}\|_w/2$.

Proof. Suppose first $\deg_T(p) = 1$. Then T can be extended to a binary subtree $S \subseteq R$ such that $q \in S$, $\sum_{t \in S \cap \mathrm{term}(R_q)} w(t) = \|\mathcal{R}_q\|$. By definition,

$$\|\mathcal{R}\|_w \geq \sum_{t \in S \cap \mathrm{term}(R)} w(t) \geq \sum_{t \in T \cap \mathrm{term}(R)} w(t) + \|\mathcal{R}_q\|_w,$$

so (1) follows.

To see (2), let $\mathrm{succ}_T(p) = \{t', t''\}$. By symmetry, we can assume

$$\sum_{t \in T_{t''} \cap \mathrm{term}(R)} w(t) \leq \sum_{t \in T_{t'} \cap \mathrm{term}(R)} w(t). \tag{4.10}$$

Then (1) can be applied to the tree $T \setminus T_{t''}$; we get

$$\|\mathcal{R}\|_w \geq \sum_{t \in (T \setminus T_{t''}) \cap \mathrm{term}(R)} w(t) + \|\mathcal{R}_q\|_w.$$

By $\sum_{t \in T_{t'} \cap \mathrm{term}(R)} w(t) = \sum_{t \in (T \setminus T_{t''}) \cap \mathrm{term}(R)} w(t)$ and

$$\sum_{t \in T \cap \mathrm{term}(R)} w(t) = \sum_{t \in T_{t'} \cap \mathrm{term}(R)} w(t) + \sum_{t \in T_{t''} \cap \mathrm{term}(R)} w(t),$$

(2) follows from (4.10).

Statement (3) is immediate from (1) or (2). To see (4), with the notation of Definition 4.9, set $\mathcal{R}|_{B \setminus \rho(q)} = (R', \rho')$. Let $S \subseteq R'$ be a binary subtree such that $\|\mathcal{R}|_{B \setminus \rho(q)}\|_w = \sum_{t \in S \cap \mathrm{term}(R')} w(t)$. Let $T \subseteq R$ be the subtree generated by $S \subseteq R$; then $q \notin T$. Since

$$\|\mathcal{R}|_{B \setminus \rho(q)}\|_w = \sum_{t \in S \cap \mathrm{term}(R)} w(t) = \sum_{t \in T \cap \mathrm{term}(R)} w(t),$$

(4) follows from (3). ☐

The following three corollaries will be useful when we prove that φ is exhaustive.

Corollary 4.15. *Let $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$ be arbitrary, let $\mathcal{R} = (R, \rho)$ be a resolution on A and let $w \colon A \to \mathbb{R}^+$ be arbitrary. Let $T \subseteq R$ be a subtree satisfying $\deg_T(t) \leq 2$ $(t \in T)$. Suppose for some $c > 0$, $\|\mathcal{R}\|_w < 26c/25$ and $\sum_{t \in T \cap \mathrm{term}(R)} w(t) > 24c/25$. Then $\{q \in R \colon \|\mathcal{R}_q\|_w \geq 14c/25\}$ is a subset of T and it forms a $\sqsubseteq$-chain.*

Proof. First suppose there is a $q \in R \setminus T$ such that $\|\mathcal{R}_q\| \geq 14c/25$. We can apply (3) of Lemma 4.14 for A, $\mathcal{R}$, T and q. We get

$$\|\mathcal{R}\|_w \geq \|\mathcal{R}_q\|_w + \sum_{t \in T \cap \mathrm{term}(R)} w(t)/2 \geq 14c/25 + 12c/25 = 26c/25,$$

a contradiction. If $q', q'' \in T$ with $\|\mathcal{R}_{q'}\|, \|\mathcal{R}_{q''}\| \geq 14c/25$ are $\sqsubseteq$-incomparable then for the $\sqsubseteq$-maximal $q \in T$ satisfying $q \sqsubset q', q''$ we have $w_{\mathcal{R}}(q) \geq w_{\mathcal{R}}(q') + w_{\mathcal{R}}(q'') \geq 28c/25$. By Lemma 4.8, $\|\mathcal{R}\|_w = w_{\mathcal{R}}(\emptyset_R) \geq w_{\mathcal{R}}(q)$, which is again a contradiction. ☐

Corollary 4.16. *Let $l < \omega$, $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$, $w \colon A \to \mathbb{R}^+$ and $\ell \colon A \to \omega \setminus 2$ be arbitrary. Let $\mathcal{R} = (R, \rho)$ be an ℓ-resolution on A and let $T \subseteq R$ be a subtree satisfying $\deg_T(t) \leq 2$ and $\ell(\min \rho(t)) \leq l$ $(t \in T)$. Suppose for some $c > 0$, $\|\mathcal{R}\|_w < 26c/25$ and $\sum_{t \in T \cap \mathrm{term}(R)} w(t) > 24c/25$. Then $\{q \in R \setminus T \colon w_{\mathcal{R}}(q) \geq 12c/25\}$ contains no $\sqsubseteq$-antichain of size $l + 1$.*

Proof. Set $Q = \{q \in R \setminus T \colon w_{\mathcal{R}}(q) \geq 12c/25\}$. For every $q \in Q$, let $p(q) \in T$ be $\sqsubseteq$-maximal with $p(q) \sqsubset q$. First we show $p(q') = p(q'')$ $(q', q'' \in Q)$. Fix an arbitrary $q \in Q$. By (1) of Lemma 4.14, $\deg_T(p(q)) = 1$ is not possible. Then by (2) of Lemma 4.14,

$$26c/25 > \|\mathcal{R}\|_w \geq \textstyle\sum_{t \in T \cap \mathrm{term}(R)} w(t) + \|\mathcal{R}_q\|_w - \sum_{t \in T \cap \mathrm{term}(R_{p(q)})} w(t)/2 >$$
$$24c/25 + 12c/25 - \textstyle\sum_{t \in T \cap \mathrm{term}(R_{p(q)})} w(t)/2,$$

i.e. $\sum_{t \in T \cap \mathrm{term}(R_{p(q)})} w(t) > 4c/5$.

Since $\|\mathcal{R}_{p(q)}\|_w \geq \sum_{t \in T \cap \mathrm{term}(R_{p(q)})} w(t) \geq 14c/25$, by Corollary 4.15 we get that if $q', q'' \in Q$ then $p(q')$, $p(q'')$ are $\sqsubseteq$-comparable. So if $p(q') \neq p(q'')$ then we can assume $p(q') \sqsubset p(q'')$. Then

$$\|\mathcal{R}\|_w \geq w_{\mathcal{R}}(p(q')) \geq w_{\mathcal{R}}(p(q'')) + w_{\mathcal{R}}(q) \geq 4c/5 + 14c/25 > 26c/25,$$

a contradiction.

Finally suppose $\{q_j \colon j \leq l\} \subseteq Q$ is a $\sqsubseteq$-antichain. Let $p = p(q_j)$ $(j \leq l)$. Since $\mathcal{R}$ is an ℓ-resolution, by $p \in T$, we have $\deg_R(p) \leq l$. Thus there is an $r \in \mathrm{succ}_R(p)$ and $i < j \leq l$ such that $r \sqsubset q_i, q_j$. By the definition of $p(q)$ $(q \in Q)$, we have $r \notin T$. However,

$$\|\mathcal{R}_r\|_w = w_{\mathcal{R}}(r) \geq w_{\mathcal{R}}(q_i) + w_{\mathcal{R}}(q_j) \geq 24c/25,$$

which contradicts Corollary 4.15. ☐

Corollary 4.17. *Let $B \in [\omega]^{<\omega} \setminus \{\emptyset\}$ and $w \colon B \to \mathbb{R}^+$ be arbitrary. Let $\mathcal{R} = (R, \rho)$ be a resolution on B, let $r \in R$ and let $\{p_i \colon i < k\} \subseteq R_r$ be pairwise $\sqsubseteq$-comparable.*

1. *Let $s_i \in \mathrm{succ}_R(p_i)$ be such that $s_i \not\sqsubseteq p_j$ $(j < k)$. Then $\sum_{i<k} w_{\mathcal{R}}(s_i) \leq \|\mathcal{R}_r\|_w$.*
2. *If for some $c > 0$ we have $w_{\mathcal{R}}(r) < 26c/25$ and for every $i < k$ there is an $s_i \in \mathrm{succ}_R(p_i)$ such that $s_i \not\sqsubseteq p_j$ $(j < k)$ and $w_{\mathcal{R}}(s_i) > c/25$, then $k \leq 25$.*

Proof. By re-indexing, we can assume $p_i \sqsubseteq p_{i+1}$ $(i < k-1)$. It is obvious that $w_{\mathcal{R}}(p_{k-1}) \geq w_{\mathcal{R}}(s_{k-1})$ and $w_{\mathcal{R}}(p_i) \geq w_{\mathcal{R}}(p_{i+1}) + w_{\mathcal{R}}(s_i)$ $(i < k-1)$. Thus

$$\|\mathcal{R}_r\|_w = w_{\mathcal{R}}(r) \geq w_{\mathcal{R}}(p_0) \geq \textstyle\sum_{i<k} w_{\mathcal{R}}(s_i),$$

so (1) follows. For (2), this gives $26c/25 > \sum_{i<k} w_{\mathcal{R}}(s_i) > k \cdot c/25$, so the statement follows. ☐

Proposition 4.18. *With the notation of Definition 4.6, suppose ℓ is increasing. Then φ is a submeasure.*

Proof. Condition (1) of Definition 2.1 holds. To see (2), let $A, B \in [\omega]^{<\omega} \setminus \{\emptyset\}$ satisfy $B \subseteq A$. Let $\mathcal{R} = (R, \rho)$ be an ℓ-resolution on A. By Lemma 4.10, $\mathcal{R}|_B$ is an ℓ-resolution on B and $\|\mathcal{R}|_B\|_w \leq \|\mathcal{R}\|_w$. Since $\mathcal{R}$ was arbitrary, $\varphi(B) \leq \varphi(A)$ follows.

To see (3), let $A, B \in [\omega]^{<\omega}$ be arbitrary. If $A \subseteq B$ or $B \subseteq A$ then the statement is obvious, so suppose $A \not\subseteq B$, $B \not\subseteq A$. Then by (2), we can assume $A \cap B = \emptyset$, $A, B \neq \emptyset$. Let $\mathcal{R}' = (R', \rho')$ and $\mathcal{R}'' = (R'', \rho'')$ be ℓ-resolutions on A and B.

Consider $\mathcal{R}' \oplus \mathcal{R}''$. By Lemma 4.11, using $\ell(k) \geq 2$ $(k < \omega)$, this is an ℓ-resolution on $A \cup B$, and $\|\mathcal{R}' \oplus \mathcal{R}''\|_w = \|\mathcal{R}'\|_w + \|\mathcal{R}''\|_w$. Since $\mathcal{R}'$, $\mathcal{R}''$ were arbitrary, this yields $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$, as required. This completes the proof. ☐

4.2.3. *The constant weight case.* In order to show (2) of Proposition 4.3, we have to compute $\varphi(I_{\{n\}})$ $(n < \omega)$. The following lemma contains the key ingredient of the computation.

Lemma 4.19. *For every $n < \omega$, define the polynomial $P_n \in \mathbb{Z}[x]$ by setting $P_0(x) = 0$, $P_1(x) = 1$, $P_{2^n+d}(x) = P_{2^n}(x) + (x-1) \cdot P_d(x)$ $(0 < d \leq 2^n,\ n < \omega)$. Then for every $x \in \omega \setminus 2$,*

(1) $P_{2^n}(x) = x^n$ $(n < \omega)$
(2) $P_d(x) \leq P_{i+d}(x) - P_i(x)$ $(d, i < \omega)$;
(3) $P_{i+d}(x) - P_i(x) \leq P_{2^n}(x) - P_{2^n-d}(x)$ $(i + d \leq 2^n,\ n < \omega)$;
(4) $P_i(x) + (x-1) \cdot P_j(x) \leq P_{i+j}(x)$ $(j \leq i < \omega)$.

Proof. Statement (1) follows by a straightforward induction on $n < \omega$, so we work only for (2)-(4). We prove (2)-(4) simultaneously, by induction. Observe that $P_2(x) = x$. Thus (2)-(4) hold for every $i, j, d < \omega$ satisfying $i + j, i + d \leq 2$.

Let $0 < n < \omega$ be arbitrary and suppose (2)-(4) hold for every $i, j, d < \omega$ satisfying $i+j, i+d \leq 2^n$; we verify (2)-(4) for every $i, j, d < \omega$ satisfying $i + j, i + d \leq 2^{n+1}$. To see (2), by symmetry we can assume $d \leq i$. We distinguish several cases. If $i + d \leq 2^n$ then the statement follows from the inductive assumption on (2). If $2^n \leq i$, say $i = 2^n + a$ then

$$P_i(x) = P_{2^n}(x) + (x-1) \cdot P_a(x),\ P_{i+d}(x) = P_{2^n}(x) + (x-1) \cdot P_{a+d}(x),$$

i.e.

$$P_{i+d}(x) - P_i(x) = (x-1) \cdot (P_{a+d}(x) - P_a(x)).$$

Since $i + d \leq 2^{n+1}$, we have $a + d \leq 2^n$ so $P_d(x) \leq P_{a+d}(x) - P_a(x)$ by the inductive assumption on (2). Since $x > 1$, this implies

$$P_d(x) \leq (x-1) \cdot (P_{a+d}(x) - P_a(x)),$$

as required. Finally if $d \leq i < 2^n$ and $2^n < i + d$ then let $a, b < \omega$ such that $i + a = 2^n$, $a + b = d$, i.e. $i + d = 2^n + b$. We have

$$P_{i+d}(x) - P_i(x) = (P_{2^n+b}(x) - P_{2^n}(x)) + (P_{2^n}(x) - P_i(x)).$$

By the previous case, $P_b(x) \leq P_{2^n+b}(x) - P_{2^n}(x)$. By the inductive assumption on (3), using $d = a + b$, $P_d(x) - P_b(x) \leq P_{2^n}(x) - P_i(x)$. So

$$P_d(x) = (P_d(x) - P_b(x)) + P_b(x) \leq (P_{2^n}(x) - P_i(x)) + (P_{2^n+b}(x) - P_{2^n}(x)) = P_{i+d}(x) - P_i(x).$$

This completes the proof of (2).

We prove (3) also by distinguishing several cases. Suppose first $d \leq 2^n$, say $2^{n+1} - d = 2^n + a$. Then

$$P_{2^{n+1}}(x) - P_{2^{n+1}-d}(x) = (x-1) \cdot (P_{2^n}(x) - P_a(x)).$$

If $i + d \leq 2^n$ then by the inductive assumption on (3), using $2^n - d = a$, we have

$$P_{i+d}(x) - P_i(x) \leq P_{2^n}(x) - P_a(x)$$

so the statement follows from $x > 1$. If $2^n \leq i$ then

$$P_{i+d}(x) - P_i(x) = (x-1) \cdot (P_{i+d-2^n}(x) - P_{i-2^n}(x)).$$

By the inductive assumption on (3),

$$P_{i+d-2^n}(x) - P_{i-2^n}(x) \leq P_{2^n}(x) - P_a(x)$$

so the statement follows again from $x > 1$.

If $i < 2^n < i + d$ then

$$P_{i+d}(x) - P_i(x) = (P_{i+d}(x) - P_{2^n}(x)) + (P_{2^n}(x) - P_i(x)).$$

By the previous case, $P_{2^n}(x) - P_i(x) \leq P_{2^{n+1}}(x) - P_{2^n+i}(x)$. By the inductive assumption on (2),

$$P_{i+d}(x) - P_{2^n}(x) = (x-1) \cdot P_{i+d-2^n}(x) \leq (x-1) \cdot (P_i(x) - P_{2^n-d}(x)) = P_{2^n+i}(x) - P_{2^{n+1}-d}(x),$$

so the statement follows.

Finally suppose $2^n < d$. Observe that $i \leq 2^{n+1} - d < 2^n$, hence

$$P_{i+d}(x) - P_i(x) = (P_{i+d}(x) - P_{2^{n+1}-d}(x)) + (P_{2^{n+1}-d}(x) - P_i(x)),$$
$$P_{2^{n+1}}(x) - P_{2^{n+1}-d}(x) = (P_{2^{n+1}}(x) - P_{i+d}(x)) + (P_{i+d}(x) - P_{2^{n+1}-d}(x)).$$

As we have seen above, $P_{2^{n+1}-d}(x) - P_i(x) \leq P_{2^{n+1}}(x) - P_{i+d}(x)$, so the proof of (3) is complete.

It remains to prove (4). By the inductive assumption on (4), we can assume $2^n < i + j$; in particular $2^{n-1} < i$. If in addition $2^n \leq i$ then

$$P_i(x) = P_{2^n}(x) + (x-1) \cdot P_{i-2^n}(x),\ P_{i+j}(x) = P_{2^n}(x) + (x-1) \cdot P_{i+j-2^n(x)}.$$

By the inductive assumption on (2) we have $P_{i-2^n}(x) + P_j(x) \leq P_{i+j-2^n}(x)$, so the statement follows.

If $2^{n-1} < i < 2^n$ then

$$P_i(x) + (x-1) \cdot P_j(x) = P_{2^{n-1}}(x) + (x-1) \cdot (P_{i-2^{n-1}}(x) + P_j(x))$$

and

$$P_{i+j}(x) = P_{2^n}(x) + (x-1) \cdot P_{i+j-2^n}(x) = P_{2^{n-1}}(x) + (x-1) \cdot (P_{2^{n-1}}(x) + P_{i+j-2^n}(x)).$$

So it is enough to show $P_{i-2^{n-1}}(x) + P_j(x) \leq P_{2^{n-1}}(x) + P_{i+j-2^n}(x)$, i.e.

$$P_j(x) - P_{i+j-2^n}(x) \leq P_{2^{n-1}}(x) - P_{i-2^{n-1}}(x).$$

For $j \leq 2^{n-1}$, this follows from (3). If $2^{n-1} < j$ then $P_j(x) = P_{2^{n-1}}(x) + (x-1) \cdot P_{j-2^{n-1}}(x)$, thus it is enough to show

$$(x-1) \cdot P_{j-2^{n-1}}(x) + P_{i-2^{n-1}}(x) \leq P_{i+j-2^n}(x).$$

This follows from the inductive assumption on (4), so the proof is complete. □

Corollary 4.20. *Fix $m \in \omega \setminus 2$, $c > 0$ and let $w\colon \omega \to \omega$, $\ell\colon \omega \to \omega$ be defined by $w(\{k\}) = c$, $\ell(k) = m$ $(k < \omega)$. Then for every $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$, with the notation of Definition 4.6,*

$$\varphi(A) = c \cdot \min\{n < \omega\colon |A| \leq P_n(m)\}. \tag{4.11}$$

Proof. By linearity, we can assume $c = 1$. First we prove $\varphi(A) \leq \min\{n < \omega\colon |A| \leq P_n(m)\}$. Let $n < \omega$ be such that $|A| \leq P_n(m)$. By Proposition 4.18, φ is a submeasure so it is enough to show that there is an ℓ-resolution $\mathcal{R}_n$ on $P_n(m)$ such that $\|\mathcal{R}_n\|_w = n$. We prove this by induction on $n < \omega$. The statement is obvious for $n = 1$. Let $n < \omega$ and suppose $\mathcal{R}_i = (R_i, \rho_i)$ $(i \leq 2^n)$ are constructed.

We will use the following notation. If $A \in [\omega]^{<\omega}$, $\mathcal{R} = (R, \rho)$ is a resolution on A and $k < \omega$ then $\mathcal{R}\dot{+}k$ is the resolution (R, ρ') on $\{a + k\colon a \in A\}$ where $\rho'(r) = \{a + k\colon a \in \rho(r)\}$ $(r \in R)$.

Let $2^n < i \leq 2^{n+1}$ be arbitrary, say $i = 2^n + d$. We have $P_i(m) = P_{2^n}(m) + (m-1) \cdot P_d(m)$. For every $l < m - 1$, let $\mathcal{R}_d(l) = \mathcal{R}_d\dot{+}(P_{2^n}(m) + l \cdot P_d(m))$ and set $\mathcal{R}_i = \mathcal{R}_{2^n} \oplus \bigoplus_{l<m-1} \mathcal{R}_d(l)$. By Lemma 4.11, using that ℓ is constant, we get that $\mathcal{R}_i$ is an ℓ-resolution; moreover, by $\|\mathcal{R}_{2^n}\|_w = 2^n$ and $\|\mathcal{R}_d\|_w = d$ we have $\|\mathcal{R}_i\|_w = 2^n + d = i$, as required.

We prove the converse inequality by induction on $|A|$. The statement is obvious for $|A| = 1$, so let $|A| \geq 2$. Let $\mathcal{R}$ be an ℓ-resolution on A such that $\|\mathcal{R}\|_w = \varphi(A)$. Set $A_r = \rho(r)$ $(r \in \mathrm{succ}_R(\emptyset_R))$. By Lemma 4.8 we have $r', r'' \in \mathrm{succ}_R(\emptyset_R)$ such that

$$\|\mathcal{R}_{r'}\|_w \geq \|\mathcal{R}_{r''}\|_w \geq \|\mathcal{R}_r\|_w \ (r \in \mathrm{succ}_R(\emptyset_R) \setminus \{r', r''\})$$

and

$$\|\mathcal{R}\|_w = w_{\mathcal{R}}(\emptyset_{\mathcal{R}}) = w_{\mathcal{R}}(r') + w_{\mathcal{R}}(r'') = \|\mathcal{R}_{r'}\|_w + \|\mathcal{R}_{r''}\|_w.$$

By the inductive assumption,

$$|A_{r'}| \leq P_{\|\mathcal{R}_{r'}\|_w}(m),\ |A_r| \leq P_{\|\mathcal{R}_{r''}\|_w}(m) \ (r \in \mathrm{succ}_R(\emptyset_R) \setminus \{r'\}),$$

so

$$|A| = \textstyle\sum_{r \in \mathrm{succ}_R(\emptyset_R)} |A_r| \leq P_{\|\mathcal{R}_{r'}\|_w}(m) + (m-1) \cdot P_{\|\mathcal{R}_{r''}\|_w}(m).$$

By (4) of Lemma 4.19,

$$P_{\|\mathcal{R}_{r'}\|_w}(m) + (m-1) \cdot P_{\|\mathcal{R}_{r''}\|_w}(m) \leq P_{\|\mathcal{R}_{r'}\|_w + \|\mathcal{R}_{r''}\|_w}(m) = P_{\|\mathcal{R}\|_w}(m).$$

Thus $|A| \leq P_{\|\mathcal{R}\|_w}(m)$, which completes the proof. □

Before verifying the conditions of Proposition 4.3, we have to prove Lemma 4.5.

Proof of Lemma 4.5. Let $n < \omega$, $J \leq 2^{m_n}$ and $H_j \in \mathrm{Exh}(\varphi)$, $H_j \subseteq \omega \setminus m_n$ $(j < J)$ be arbitrary. For every $\varepsilon > 0$ there exist $H'_j \in [H_j]^{<\omega}$ $(j < J)$ such that $\overline{\varphi}(\bigcup_{j<J} H_j) - \varphi(\bigcup_{j<J} H'_j) < \varepsilon$. Then from (4') we get

$$\overline{\varphi}(\textstyle\bigcup_{j<J} H_j) \leq \varphi(\bigcup_{j<J} H'_j) + \varepsilon \leq C \cdot \max\{\varphi(H'_j) \colon j < J\} + c_n + \varepsilon \leq C \cdot \max\{\overline{\varphi}(H_j) \colon j < J\} + c_n + \varepsilon.$$

Since ε was arbitrary, the proof is complete. □

Proposition 4.21. *With the notation of Definition 4.7, φ satisfies the conditions of Proposition 4.3.*

Proof. We have $\varphi(\{k\}) = 2^{-n}$ $(k \in I_{\{n\}},\ n < \omega)$, so (1) follows. To see (2), we show $\varphi(I_{\{n\}}) = 1$ $(n < \omega)$. For $n = 0$ we have $I_{\{0\}} = \{0\}$, $w(0) = 1$ so $\varphi(I_{\{0\}}) = 1$ follows. For $0 < n < \omega$, recall that $|I_{\{n\}}| = d_n = 2^{n \cdot m_n}$, $w(k) = 2^{-n}$ and $\ell(k) = 2^{\max\{m_n, 1\}} = 2^{m_n}$ $(k \in I_{\{n\}})$. So by Corollary 4.20,

$$\varphi(I_{\{n\}}) = 2^{-n} \cdot \min\{N < \omega \colon |I_{\{n\}}| \leq P_N(2^{m_n})\} = 2^{-n} \cdot \min\{N < \omega \colon 2^{n \cdot m_n} \leq P_N(2^{m_n})\}.$$

By (1) and (2) of Lemma 4.19, $\min\{N < \omega \colon 2^{n \cdot m_n} \leq P_N(2^{m_n})\} = 2^n$, so $\varphi(I_{\{n\}}) = 1$, as stated.

Statement (3) is trivial. By Lemma 4.5, (4) follows from (4'). To see (4'), let $n < \omega$, $J \leq 2^{m_n}$ and $H_j \in [\omega \setminus m_n]^{<\omega}$ $(j < J)$ be arbitrary. By Proposition 4.18, φ is a submeasure so we can assume H_j $(j < J)$ are pairwise disjoint. Let $J' = \{j < J \colon H_j \neq \emptyset\}$; then $\bigcup_{j \in J'} H_j = \bigcup_{j<J} H_j$.

For every $j \in J'$, let $\mathcal{R}_j$ be an ℓ-resolution on H_j such that $\|\mathcal{R}_j\|_w = \varphi(H_j)$. Set $\mathcal{R} = \bigoplus_{j \in J'} \mathcal{R}_j$. Since $\ell(k) \geq 2^{m_n}$ $(k \in \omega \setminus m_n)$, by Lemma 4.11, $\mathcal{R}$ is an ℓ-resolution on $\bigcup_{j \in J'} H_j$, and

$$\begin{aligned}\varphi(\textstyle\bigcup_{j \in J'} H_j) \leq \|\mathcal{R}\|_w = \max\{\|\mathcal{R}_i\|_w + \|\mathcal{R}_j\|_w \colon i, j \in J',\ i \neq j\} \leq \\ 2 \cdot \max\{\|\mathcal{R}_j\|_w \colon j \in J'\} = 2 \cdot \max\{\varphi(H_j) \colon j < J\},\end{aligned}$$

as stated. This completes the proof. □

4.2.4. *Exhaustiveness.* The only remaining task is to show that φ is exhaustive. We prove a last easy lemma on the size of trees.

Lemma 4.22. *Let (R,ρ) be a resolution on a set $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$. Then $|R| \leq 2 \cdot |A| - 1$.*

Proof. We prove the statement by induction on $|A|$. For $|A| = 1$ the statement is obvious. Let $|A| > 1$ and let (R,ρ) be a resolution on A. Then by the inductive hypothesis, using $\deg_R(\emptyset_R) \geq 2$,

$$|R| = 1 + \textstyle\sum_{r \in \operatorname{succ}_R(\emptyset_R)} |R_r| \leq 1 + \sum_{r \in \operatorname{succ}_R(\emptyset_R)} (2 \cdot |\rho(r)| - 1) \leq 2 \cdot |A| - 1,$$

as required. □

Proposition 4.23. *With the notation of Definition 4.6, suppose ℓ is increasing and unbounded. Then φ is exhaustive.*

Proof. Suppose φ is not exhaustive. Since $\mathrm{Exh}(\varphi) \subseteq \mathrm{Fin}(\varphi)$ holds for every submeasure φ, φ being not exhaustive implies that there exist $H \subseteq \omega$ and $c > 0$ with $\overline{\varphi}(H) < +\infty$ and $\lim_{n<\omega} \overline{\varphi}(H \setminus n) = c$. Let $k < \omega$ satisfy $\overline{\varphi}(H \setminus k) < 26c/25$. By induction on $n < \omega$, we define $A_n \in [\omega]^{<\omega}$ $(n < \omega)$ such that

1. $A_n \subseteq H \setminus k$, $\max A_n < \min A_{n+1}$ $(n < \omega)$;
2. $\varphi(A_n) > 24c/25$ $(n < \omega)$;
3. $27 \cdot (2 \cdot |\bigcup_{i \leq n} A_i| - 1) \cdot \max\{\ell(k) \colon k \in \bigcup_{i \leq n} A_i\} \leq \ell(\min A_{n+1})$ $(n < \omega)$;

as follows.

Since $\overline{\varphi}(H \setminus k) \geq c$, there is an $A_0 \in [H \setminus k]^{<\omega}$ with $\varphi(A_0) > 24c/25$. Let $n < \omega$ and suppose A_i $(i \leq n)$ are already defined. Let $k_n < \omega$ satisfy $\max A_n < k_n$ and

$$27 \cdot (2 \cdot |\textstyle\bigcup_{i \leq n} A_i| - 1) \cdot \max\{\ell(k) \colon k \in \bigcup_{i \leq n} A_i\} \leq \ell(k_n).$$

Since $\overline{\varphi}(H \setminus k_n) \geq c$, there is an $A_{n+1} \in [H \setminus k_n]^{<\omega}$ with $\varphi(A_{n+1}) > 24c/25$. This completes the inductive step of the construction.

Set $l_n = \max\{\ell(k) \colon k \in \bigcup_{i \leq n} A_i\}$ $(n < w)$. We show $\varphi(\bigcup_{i \leq l_0+1} A_i) \geq 26c/25$; this contradicts $\bigcup_{i \leq l_0+1} A_i \subseteq H \setminus k$, so the proof will be complete. Set $A = \bigcup_{i \leq l_0+1} A_i$ and let $\mathcal{R} = (R,\rho)$ be an arbitrary ℓ-resolution on A. It is enough to show that $\|\mathcal{R}\|_w \geq 26c/25$. So suppose $\|\mathcal{R}\|_w < 26c/25$.

With the notation of Definition 4.9, let $\mathcal{R}|_{A_0} = (R_0', \rho_0')$. Let $S \subseteq R_0'$ be a binary subtree such that $\|\mathcal{R}|_{A_0}\|_w = \sum_{t \in S \cap \mathrm{term}(R_0')} w(t)$. Note that $\|\mathcal{R}|_{A_0}\|_w \geq \varphi(A_0) > 24c/25$. Let $T \subseteq R$ be the subtree generated by $S \subseteq R$. Corollary 4.15 and Corollary 4.16 apply to A, $\mathcal{R}$, T and $l = l_0$. So we have

- (i) $\{q \in R \colon \|\mathcal{R}_q\| \geq 14c/25\}$ is a subset of T and it forms a $\sqsubseteq$-chain;
- (ii) $\{q \in R \setminus T \colon w_{\mathcal{R}}(q) \geq 12c/25\}$ contains no $\sqsubseteq$-antichain of size $l_0 + 1$.

Fix an arbitrary $n \leq l_0$. Set $Z_n = \{r \in R \colon \rho(r) \cap \bigcup_{i \leq n} A_i = \emptyset\}$, and let $S_n \subseteq Z_n$ be the set of $\sqsubseteq$-minimal elements of Z_n. We will show that there is an $s_n \in S_n$ such that $\varphi(\rho(s_n) \cap A_{n+1}) \geq 12c/25$. Now we assume this and finish the proof.

We show that for every $n < n' \leq l_0$, s_n and $s_{n'}$ are $\sqsubseteq$-incomparable in R. Indeed, if $s_n, s_{n'}$ are $\sqsubseteq$-comparable for some $n < n' \leq l_0$, then by $\rho(s_n) \cap A_{n+1} \neq \emptyset$ and $\rho(s_{n'}) \cap A_{n+1} = \emptyset$ we have $s_n \sqsubset s_{n'}$. We apply (4) of Lemma 4.14 for $B = \rho(s_n)$, $\mathcal{R}_{s_n}$ and $q = s_{n'}$. We have $\rho(s_n) \cap A_{n+1} \cap \rho(s_{n'}) = \emptyset$, so by $\varphi(\rho(s_n) \cap A_{n+1}) \geq 12c/25$ we get $\|\mathcal{R}_{s_n}|_{B \setminus \rho(s_{n'})}\|_w \geq 12c/25$. Similarly, $\|\mathcal{R}_{s_{n'}}\|_w \geq \varphi(\rho(s_{n'}) \cap A_{n'+1}) \geq 12c/25$. So we get

$$\|\mathcal{R}_{s_n}\|_w \geq \|\mathcal{R}_{s_n}|_{B \setminus \rho(s_{n'})}\|_w/2 + \|\mathcal{R}_{s_{n'}}\|_w \geq 6c/25 + 12c/25 > 14c/25.$$

By $\rho(s_n) \cap A_0 = \emptyset$ we have $s_n \notin T$, which contradicts (i). Thus $\{s_n : n \leq l_0\}$ is an antichain in R. By (ii) this is impossible, so the proof is complete.

So it remains to find $s_n \in S_n$ such that $\varphi(\rho(s_n) \cap A_{n+1}) \geq 12c/25$ $(n \leq l_0)$. Fix $n \leq l_0$ and suppose no such s_n exists. We build an ℓ-resolution $\mathcal{R}(n)$ on A_{n+1} such that $\|\mathcal{R}(n)\|_w \leq 24c/25$; this contradiction will complete the proof.

Our strategy is the following. By definition, $\{\rho(s) \cap A_{n+1} : s \in S_n\}$ is a partition of A_{n+1}. So we only need to control the size of S_n, since $\varphi(\rho(s) \cap A_{n+1}) < 12c/25$ $(s \in S_n)$ is assumed. To this end, with the notation of Definition 4.9, let $\mathcal{R}|_{\bigcup_{i \leq n} A_i} = (R', \rho')$ and let $P = \{r \in R : \rho(r) \cap \bigcup_{i \leq n} A_i \neq \emptyset\}$; then $P \subseteq R$ is the subtree generated by $R' \subseteq R$. For every $s \in S_n$, let $p(s) \in R$ be such that $s \in \operatorname{succ}_R(p(s))$. By $p(s) \sqsubset s$ we have $\rho(p(s)) \cap \bigcup_{i \leq n} A_i \neq \emptyset$, so $p(s) \in P$ $(s \in S_n)$. We know $\deg_R(p) \leq l_n$ $(p \in P)$, so it would be enough to control the size of P.

However, this is impossible: we have no control on the cardinality of $\{p \in P : \deg_P(p) = 1\}$. On the other hand, by Lemma 4.22 we have $|R'| \leq 2 \cdot |\bigcup_{i \leq n} A_i| - 1$, and we also know that $P \setminus R'$ is the union of at most $|R'|$ many chains of $\sqsubseteq$-consecutive elements of P having degree 1 in P. This idea motivates the following definitions.

For every $s \in S_n$ with $\deg_P(p(s)) \neq 1$, let $B_s = \rho(s) \cap A_{n+1}$. By $R' = \{p \in P : \deg_P(p) \neq 1\}$, we have

$$|\{s \in S : \deg_P(p(s)) \neq 1\}| \leq |R'| \cdot l_n,$$

so the cardinality of the resulting B_ss is at most $|R'| \cdot l_n$. By $\varphi(\rho(s) \cap A_{n+1}) < 12c/25$, we can find an ℓ-resolution $\mathcal{R}(s)$ on B_s such that $\|\mathcal{R}(s)\|_w < 12c/25$ $(s \in S_n, \deg_P(p(s)) \neq 1)$.

For every $s \in S_n$ with $\deg_P(p(s)) = 1$, let $r(s) \in R'$ be $\sqsubseteq$-maximal such that $r(s) \sqsubset p(s)$. Fix an $r \in R'$ and consider the sets

$$S_n(r) = \{s \in S_n : \deg_P(p(s)) = 1,\ r(s) = r\},$$

$$S_n^<(r) = \{s \in S_n(r) : \varphi(\rho(s) \cap A_{n+1}) \leq c/25\},$$

$$S_n^>(r) = \{s \in S_n(r) : \varphi(\rho(s) \cap A_{n+1}) > c/25\}.$$

Then $S_n(r) = S_n^<(r) \cup S_n^>(r)$. Since $w_{\mathcal{R}}(r(s)) < 26c/25$ and $\{p(s) : s \in S_n^>(r)\}$ are pairwise $\sqsubseteq$-comparable, by (2) of Corollary 4.17 we have $|\{p(s) : s \in S_n^>(r)\}| \leq 25$, hence $|S_n^>(r)| \leq 25 \cdot l_n$. So we can set $B_s = \rho(s) \cap A_{n+1}$ $(s \in S_n^>(r))$; altogether the cardinality of the resulting B_ss is at most $|R'| \cdot 25 \cdot l_n$. Again by $\varphi(\rho(s) \cap A_{n+1}) < 12c/25$, we can find an ℓ-resolution $\mathcal{R}(s)$ on B_s such that $\|\mathcal{R}(s)\|_w < 12c/25$ $(s \in S_n^>(r), r \in R')$.

To cover $s \in S_n^<(r)$ $(r \in R')$, for every $r \in R'$ set

$$B_r(s) = \rho(s) \cap A_{n+1} \ (s \in S_n^<(r)),\ B_r = \textstyle\bigcup_{s \in S_n^<(r)} B_r(s).$$

By (1) of Corollary 4.17 for $\{p(s) : s \in S_n^<(r)\} \subseteq R_r$,

$$\textstyle\sum_{s \in S_n^<(r)} \varphi(B_r(s)) \leq \sum_{s \in S_n^<(r)} w_{\mathcal{R}}(s) \leq w_{\mathcal{R}}(r) \cdot \max_{s \in S_n^<(r)} \deg_R(p(s)) < 2c \cdot l_n.$$

For every $r \in R'$ with $B_r \neq \emptyset$, we can apply Corollary 4.13 with $m = 10 \cdot l_n < \ell(\min A_{n+1})$, $B = B_r$ and $B_r(s)$ $(s \in S_n^<(r))$; we get an ℓ-resolution $\mathcal{R}(r)$ on B_r such that $\|\mathcal{R}(r)\|_w \leq 12c/25$. The cardinality of the resulting B_rs is at most $|R'|$.

Let $\mathcal{R}$ be the sum of the resolutions $\mathcal{R}(s)$ and $\mathcal{R}(r)$ found above. By (3) we have

$$|R'| \cdot l_n + |R'| \cdot 25 \cdot l_n + |R'| \leq 27 \cdot l_n \cdot (2 \cdot |\textstyle\bigcup_{i \leq n} A_i| - 1) \leq \ell(\min A_{n+1}).$$

So by Lemma 4.11, $\mathcal{R}$ is an ℓ-resolution on A_{n+1} satisfying $\|\mathcal{R}\|_w \leq 24c/25$, as required. □

Proof of Theorem 1.5. With the notation of Definition 4.7, by Proposition 4.18 and Proposition 4.23, φ is an exhaustive submeasure. By Proposition 4.2, $S \subseteq^\star T$ implies $\mathcal{I}_S(\varphi) \leq_T \mathcal{I}_S(\varphi)$.

By Proposition 4.21, φ satisfies the conditions of Proposition 4.3. Thus if $S \not\subseteq^\star T$ then by $\mathcal{I}_{S\setminus T}(\varphi) \leq_T \mathcal{I}_S(\varphi)$ and $\mathcal{I}_{S\setminus T}(\varphi) \not\leq_T \mathcal{I}_T(\varphi)$ we get $\mathcal{I}_S(\varphi) \not\leq_T \mathcal{I}_T(\varphi)$, as required. □

4.3. **Analysis.** The construction presented in the previous section is very flexible. The local and global properties of the resulting submeasure can be controlled by the properties of the resolutions and by the definition of the w-norm. As an illustration, we show how Tsirelson-like submeasures can be constructed this way. This example is especially interesting because unlike in Definition 4.6, here resolutions will capture lower bound-like constraints.

We perform this construction only for the simplest Tsirelson-like submeasures; in particular, we follow the notation of [35, Section 1]. We remark that modern Banach space theory makes use of more general Tsirelson-like spaces (see e.g. [2]), which yield generalized Tsirelson-like submeasures. Not too surprisingly, the norm estimates in these spaces and our norm estimates for resolution are technically similar (see e.g. the analysis of functionals introduced in [2, Notation p. 51]). Nevertheless, as we will see below, the submeasure φ of Definition 4.7 is not Tsirelson-like. So it would be interesting to examine the Banach space theoretic implications of the construction carried out in Section 4.2.

Definition 4.24. For every $E, F \in [\omega]^{<\omega} \setminus \{\emptyset\}$, we write $E \leq F$ if $\max E \leq \min F$, and $E < F$ if $\max E < \min F$. Let $w\colon \omega \to \mathbb{R}^+$ be arbitrary and $\ell\colon \omega \to \omega \setminus 2$ be a strictly increasing function. Then for every $1 < l < \omega$, $\{E_i\colon i < l\} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ is *ℓ-admissible* if $E_i < E_{i+1}$ $(i < l)$ and $l \leq \ell(\min E_0)$.

We define $\varphi_n\colon [\omega]^{<\omega} \to \mathbb{R}^+$ $(n < \omega)$ by induction, as follows. Set $\varphi_0(A) = \max_{a\in A} w(a)$ $(A \in [\omega]^{<\omega})$. Let $n < \omega$ be arbitrary and suppose φ_n is defined. Then set $\varphi_n(\emptyset) = 0$,

$$\varphi_{n+1}(A) = \max\left\{\varphi_n(A), \frac{1}{2}\sum\nolimits_{i<l}\varphi_n(E_i)\colon \bigcup\nolimits_{i<l} E_i \subseteq A,\ \{E_i\colon i<l\} \text{ is } \ell\text{-admissible}\right\}\ (A \in [\omega]^{<\omega}).$$

The *Tsirelson submeasure associated to w and ℓ* is the function $\varphi_T\colon [\omega]^{<\omega} \to \mathbb{R}^+$ defined by $\varphi_T(A) = \lim_{n<\omega}\varphi_n(A)$ $(A \in [\omega]^{<\omega})$.

The property characterizing Tsirelson submeasures is

$$\varphi_T(A) = \max\left\{\max\nolimits_{a\in A} w(a), \frac{1}{2}\sum\nolimits_{i<l}\varphi_T(E_i)\colon \bigcup\nolimits_{i<l} E_i \subseteq A,\ \{E_i\colon i<l\} \text{ is } \ell\text{-admissible}\right\}\ (A \in [\omega]^{<\omega}) \tag{4.12}$$

(see e.g. [35, Proposition 1.2 p. 262]). The main characteristic of φ of Definition 4.7 is (4) of Proposition 4.2, so φ is not Tsirelson. Moreover, with the terminology of Definition 2.3, if w is non-trivial then $\mathrm{Exh}(\varphi_T)$ is summable-like; while $\mathrm{Exh}(\varphi)$ is not density-like by being exhaustive, and not summable-like by (4) of Proposition 4.2.

We define the resolutions which yield Tsirelson submeasures.

Definition 4.25. Let $\ell\colon \omega \to \omega \setminus 2$ be a strictly increasing function, and $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$ be arbitrary. An *ℓ-Tsirelson resolution on A* is a pair $\mathcal{R} = (R, \rho)$ where R is a finite rooted tree, $\rho\colon R \to \mathcal{P}(A) \setminus \{\emptyset\}$ and

(1) $2 \leq \deg_R(r)$ $(r \in R \setminus \mathrm{term}(R))$;
(2) $\rho(r') \subseteq \rho(r) \subseteq A$ $(r \in R, r' \in \mathrm{succ}_R(r))$;

(3) for every $r \in R \setminus \operatorname{term}(R)$ and $r', r'' \in \operatorname{succ}_R(r)$ with $r' \neq r''$, either $\rho(r') < \rho(r'')$ or $\rho(r'') < \rho(r')$;

(4) $\deg_R(r) \leq \ell(\min \rho(r))$ $(r \in R \setminus \operatorname{term}(R))$.

If $w\colon A \to \mathbb{R}^+$ is an arbitrary function and $\mathcal{R} = (R, \rho)$ is a resolution on A, then we define $\omega_{\mathcal{R}}\colon R \to \mathbb{R}^+$ inductively by $w_{\mathcal{R}}(r) = \max_{a \in \rho(r)} w(a)$ $(r \in \operatorname{term}(R))$,

$$w_{\mathcal{R}}(r) = \frac{1}{2} \textstyle\sum_{r' \in \operatorname{succ}_R(r)} w_{\mathcal{R}}(r') \ (r \in R \setminus \operatorname{term}(R)).$$

Finally set $\|\mathcal{R}\|_w = w_{\mathcal{R}}(\emptyset_R)$.

Proposition 4.26. *With the notation of Definition 4.24 and Definition 4.25,*

$$\varphi_T(A) = \max\{\|\mathcal{R}\|_w \colon \mathcal{R} \textit{ is an } \ell\textit{-Tsirelson resolution on } A\} \ (A \in [\omega]^{<\omega} \setminus \{\emptyset\}).$$

Proof. Fix an $A \in [\omega]^{<\omega} \setminus \{\emptyset\}$. First we show $\varphi_T(A) \geq \|\mathcal{R}\|_w$ for every ℓ-Tsirelson resolution $\mathcal{R}$ on A. Let $\mathcal{R}$ be an ℓ-Tsirelson resolution on A. We inductively show $\varphi_T(\rho(r)) \geq w_{\mathcal{R}}(r)$ $(r \in R)$; then the statement follows from $\rho(\emptyset_R) \subseteq A$ and $\|\mathcal{R}\|_w = w_{\mathcal{R}}(\emptyset_R)$. For $r \in \operatorname{term}(R)$, by (4.12) we have $\varphi_T(\rho(r)) \geq \max_{a \in \rho(r)} w(a) = w_{\mathcal{R}}(r)$. If $r \in R$ is arbitrary and we have $\varphi_T(\rho(r')) \geq w_{\mathcal{R}}(r')$ $(r' \in \operatorname{succ}_R(r))$ then by $\mathcal{R}$ being an ℓ-Tsirelson resolution and by (4.12),

$$\varphi_T(\rho(r)) \geq \frac{1}{2} \textstyle\sum_{r' \in \operatorname{succ}_R(r)} \varphi_T(\rho(r')) \geq \frac{1}{2} \sum_{r' \in \operatorname{succ}_R(r)} w_{\mathcal{R}}(r') = w_{\mathcal{R}}(r),$$

as required.

We prove the converse inequality by induction on $|A|$. For $|A| = 1$ the statement is obvious; so suppose $|A| > 1$. If $\varphi_T(A) = \max_{a \in A} w(a)$ then the resolution $\mathcal{R} = (R, \rho)$ defined by $R = \{\emptyset_R\}$, $\rho(\emptyset_R) = A$ satisfies $\varphi_T(A) \leq \|\mathcal{R}\|_w$. Else by (4.12) there are $l < \omega$ and ℓ-admissible $\{E_i \colon i < l\}$ such that $\bigcup_{i<l} E_i \subseteq A$ and

$$\varphi_T(A) = \frac{1}{2} \textstyle\sum_{i<l} \varphi_T(E_i).$$

By the inductive hypothesis, for every $i < l$ there is an ℓ-Tsirelson resolution $\mathcal{R}(i)$ on E_i such that $\varphi_T(E_i) \leq \|\mathcal{R}(i)\|_w$. Set $\mathcal{R} = \bigoplus_{i<l} \mathcal{R}(i)$. It is immediate that $\mathcal{R}$ is an ℓ-Tsirelson resolution on A and

$$\|\mathcal{R}\|_w = \frac{1}{2} \textstyle\sum_{i<l} \|\mathcal{R}(i)\|_w.$$

This completes the inductive step and the proof. □

5. Cofinal diversity of relative σ-ideals of compact sets

In an attempt to establish the cofinal diversity of relative σ-ideals of compact sets, in [30, Section 7 p. 1899] an operator was introduced which maps analytic p-ideals to monotone σ-ideals of compact sets. In order to simplify notation, in this section we identify $\mathcal{P}(\omega)$ and 2^ω; in particular, we understand set operations and the partial order $\subseteq$ on 2^ω via this identification.

Definition 5.1. A set $K \subseteq 2^\omega$ is *monotone* if for every $x, y \in 2^\omega$, $y \subseteq x \in K$ implies $y \in K$. The family of monotone compact subsets of 2^ω is denoted by $\mathcal{K}_{\text{mon}}$.

For every $\mathcal{I} \subseteq \mathcal{P}(\omega)$, set

$$D(\mathcal{I}) = \{K \in \mathcal{K}_{\text{mon}} \colon \exists x \in \mathcal{I} \ (x \setminus n \notin K \ (n < \omega))\}.$$

The nontrivial parts of the following proposition are obtained in [30, Theorem 7.2 p. 1900], [30, Theorem 7.4 p. 1902] and [30, Theorem 7.6 p. 1904].

Proposition 5.2. *With the notation of Definition 5.1, $\mathcal{K}_{mon} \subseteq \mathcal{K}$ is closed and it is closed under taking finite unions.*

1. *If $\mathcal{I}$ is an analytic p-ideal then $D(\mathcal{I})$ is an analytic σ-ideal of compact sets relatively to $\mathcal{K}_{mon}$, and $D(\mathcal{I}) \leq_T \mathcal{K}_{\mathcal{M}}$.*
2. *If $\mathcal{I}$ is an analytic p-ideal then $\omega^\omega <_T \mathcal{I}$ implies $D(\mathcal{I}) \leq_T \mathcal{I}$.*
3. *Let $\mathcal{I}, \mathcal{J}$ be an analytic p-ideals such that $\omega^\omega \leq_T \mathcal{I} \leq_T \mathcal{J}$. Then $D(\mathcal{I}) \leq_T D(\mathcal{J})$.*

It was asked in [30] whether the D operator keeps Tukey non-reducibility in general, or at least for the cofinal types accounting for arrow (13) on the Tukey picture, which were recovered in Corollary 4.4 (see [30, Question 4 p. 1909]). Here we answer this question in the affirmative.

Proposition 5.3. *Let $\varphi\colon [\omega]^{<\omega} \to \mathbb{R}^+$ be a submeasure satisfying (1)-(3) of Proposition 4.3, and*

(4^+) for every $n < \omega$, $J \leq 2^{m_n}$ and $H_j \in \mathrm{Exh}(\varphi)$ ($j < J$) with $H_j \cap m_n = H_{j'} \cap m_n$ ($j, j' < J$) we have

$$\overline{\varphi}(\textstyle\bigcup_{j<J} H_j) \leq \max\{\overline{\varphi}(H_j)\colon j < J\} + c_n.$$

Then for every $S, T \in [\omega]^\omega$, $S \cap T = \emptyset$ implies $D(\mathcal{I}_S(\varphi)) \not\leq_T D(\mathcal{I}_T(\varphi))$.

We will see that with the notation of Proposition 4.3, the $C = 1$ assumption is crucial. In fact, we think that D maps the cofinal types accounting for Theorem 1.5 into the same cofinal type; in particular, D does not keep Tukey non-reducibility in general. However, this result would necessitate further careful w-norm estimates of resolutions, so we do not work for it here.

Theorem 1.6 immediately follows from Proposition 5.3.

Proof of Theorem 1.6. Let $\varphi\colon [\omega]^{<\omega} \to \mathbb{R}^+$ be the submeasure of Corollary 4.4. As we have seen in the proof of Corollary 4.4, φ satisfies (4^+) of Proposition 5.3. We show that the mapping $S \mapsto D(\mathcal{I}_S(\varphi))$ ($S \in [\omega]^\omega$) fulfills the requirements. By Corollary 4.4, for every $S, T \in [\omega]^\omega$ we have that $\mathcal{I}_S(\varphi)$ is an analytic p-ideal and $\mathcal{I}_S(\varphi) \leq_T \mathcal{I}_T(\varphi)$ if and only if $S \subseteq^\star T$. By [30, Proposition 4.3 p. 1886] or [17, Theorem 2 p. 178], $\omega^\omega \leq_T \mathcal{I}_S(\varphi)$ ($S \in [\omega]^\omega$). So by Proposition 5.2 and Proposition 5.3, for every $S, T \in [\omega]^\omega$ we have that $D(\mathcal{I}_S(\varphi))$ is a σ-ideal of monotone compact sets, $D(\mathcal{I}_S(\varphi)) \leq \mathcal{K}_{\mathcal{M}}$, and $D(\mathcal{I}_S(\varphi)) \leq_T D(\mathcal{I}_T(\varphi))$ if and only if $S \subseteq^\star T$. This completes the proof. $\square$

The proof of Proposition 5.3 is based on a subtle analysis of σ-ideals of monotone compact sets. We introduce some notation. For every $s \in 2^{\leq\omega}$ and $i < 2$, $[s = i] = \{n \in \mathrm{dom}\, s\colon s(n) = i\}$. For every $A \in [\omega]^{<\omega}$, let $U(A) = \{x \in 2^\omega\colon [x = 0] \cap A \neq \emptyset\}$. For $\mathcal{A} \subseteq [\omega]^{<\omega}$, we set $U(\mathcal{A}) = \bigcap_{A \in \mathcal{A}} U(A)$; in particular, $U(\emptyset) = 2^\omega$. Let $\mathfrak{M}$ denote the family of monotone clopen subsets of 2^ω. Recall that $N_s = \{x \in 2^\omega\colon s \sqsubseteq x\}$ ($s \in 2^{<\omega}$); and for every $s, t \in 2^{<\omega}$, $s \vee t \in 2^{<\omega}$ is defined by $|s \vee t| = \max\{|s|, |t|\}$, $[s \vee t = 1] = [s = 1] \cup [t = 1]$.

Let $\varphi\colon [\omega]^{<\omega} \to \mathbb{R}^+$ be a submeasure. Set $R(\varphi) = \{\varphi(A)\colon A \in [\omega]^{<\omega}\}$. With the notation of Definition 4.1, for every $\alpha \in \mathbb{R}^+$ and $S \in [\omega]^\omega$ let

$$E_\alpha(S) = \{x \subseteq I_S\colon \overline{\varphi}(x) \leq \alpha\},\ \mathbb{V}_\alpha(S) = \{V \in [I_S]^{<\omega}\colon \varphi(V) \leq \alpha\}.$$

Note that for every $\alpha, \beta \in \mathbb{R}^+$ and $S \in [\omega]^\omega$, $E_\alpha(S) \subseteq 2^\omega$ is a compact set and $\alpha < \beta$ implies $E_\alpha(S) \subseteq E_\beta(S)$.

We start with a characterization of monotone clopen sets.

Lemma 5.4. *(1) For every $\mathcal{A}, \mathcal{B} \in [[\omega]^{<\omega}]^{<\omega}$ we have $U(\mathcal{A} \cup \mathcal{B}) = U(\mathcal{A}) \cap U(\mathcal{B})$.*

(2) $\mathfrak{M} = \{U(\mathcal{A})\colon \mathcal{A} \in [[\omega]^{<\omega}]^{<\omega}\}$.

Proof. If $\mathcal{A}, \mathcal{B} \in [[\omega]^{<\omega}]^{<\omega}$ then $U(\mathcal{A} \cup \mathcal{B}) = \bigcap_{A \in \mathcal{A} \cup \mathcal{B}} U(A) = U(\mathcal{A}) \cap U(\mathcal{B})$, so (1) follows.

For (2), it is clear that $U(A)$ is monotone and clopen for every $A \in [\omega]^{<\omega}$. Thus $U(\mathcal{A})$ is also monotone and clopen for every $\mathcal{A} \in [[\omega]^{<\omega}]^{<\omega}$.

To see the converse, let $U \in \mathfrak{M}$, say $U = 2^\omega \setminus \bigcup_{s \in S} N_s$ for some finite set $S \subseteq 2^{<\omega}$. We show that $\mathcal{A} = \{[s = 1] \colon \in S\}$ fulfills the requirements. If $x \in U(\mathcal{A})$, then clearly $s \not\subseteq x$ $(s \in S)$ so $x \in U$.

Now pick an $x \in U$. If there is an $s \in S$ with $[s = 1] \subseteq [x = 1]$ then by U being monotone we get $U \cap N_s \neq \emptyset$, a contradiction. So for every $s \in S$ we have $[x = 0] \cap [s = 1] \neq \emptyset$; which implies $x \in U(\mathcal{A})$. This completes the proof. ☐

Next we characterize $D(\mathcal{I}_S(\varphi))$.

Lemma 5.5. *Let $\varphi\colon [\omega]^{<\omega} \to \mathbb{R}^+$ be a submeasure such that $\varphi(\{n\}) > 0$ $(n < \omega)$. Let $S \in [\omega]^\omega$ be fixed. For every $K \in \mathcal{K}_{\mathrm{mon}}$, the following are equivalent:*

(1) $K \in D(\mathcal{I}_S(\varphi))$;

(2) for every $\alpha \in \mathbb{R}^+ \setminus \{0\}$, $E_\alpha(S) \not\subseteq K$;

(3) for every $\alpha \in \mathbb{R}^+ \setminus R(\varphi)$, $K \cap E_\alpha(S)$ is meager in $E_\alpha(S)$.

Proof. $1 \Rightarrow 2$: Let $x \in \mathcal{I}_S(\varphi)$ witness $K \in D(\mathcal{I}_S(\varphi))$, and fix an $\alpha \in \mathbb{R}^+ \setminus \{0\}$. Let $n < \omega$ be such that $\varphi(x \setminus n) \leq \alpha$. Then $x \setminus n \in E_\alpha(S) \setminus K$, as required.

$2 \Rightarrow 3$: Suppose that for an $\alpha \in \mathbb{R}^+ \setminus R(\varphi)$, $K \cap E_\alpha(S)$ is not meager in $E_\alpha(S)$, say for some $s \in 2^{<\omega}$, $N_s \cap E_\alpha(S) \neq \emptyset$ and $N_s \cap E_\alpha(S) \subseteq K \cap E_\alpha(S)$. By $N_s \cap E_\alpha(S) \neq \emptyset$ and $\alpha \notin R(\varphi)$ we have $\varphi([s = 1]) < \alpha$ and $[s = 1] \subseteq I_S$. Let $\alpha' \in \mathbb{R}^+ \setminus R(\varphi)$, $\alpha' \leq \alpha - \varphi([s = 1])$ be so small that for every $x \in 2^\omega$, $\overline{\varphi}(x) < \alpha'$ implies $x|_{|s|} = 0$. Then for every $x \subseteq I_S$ with $\overline{\varphi}(x) \leq \alpha'$ we have $s \cup x \in N_s \cap E_\alpha(S) \subseteq K$. Since K is monotone, this implies $E_{\alpha'}(S) \subseteq K$ which contradicts 2.

$3 \Rightarrow 1$: Let $(\alpha_i)_{i<\omega} \subseteq \mathbb{R}^+ \setminus R(\varphi)$ satisfy $\sum_{i<\omega} \alpha_i < +\infty$. For every $i < \omega$ pick $s_i \in 2^{<\omega}$ such that $N_{s_i} \cap E_{\alpha_i}(S) \neq \emptyset$ and $N_{s_i} \cap K = \emptyset$. By $N_{s_i} \cap E_{\alpha_i}(S) \neq \emptyset$ we have $\varphi([s_i = 1]) \leq \alpha_i$ and $[s_i = 1] \subseteq I_S$ $(i < \omega)$. By passing to a subsequence, we can assume that for every $i < \omega$ and $n \leq |s_i|$, $s_{i+1}(n) = 0$. Let $x = \bigvee_{i<\omega} s_i$; then $x \in \mathcal{I}_S(\varphi)$. Since K is monotone, for every $n < \omega$, $x \setminus n \notin K$ is witnessed by s_i for i sufficiently large. Thus $K \in D(\mathcal{I}_S(\varphi))$, as required. ☐

Now we are in position to outline the strategy of the proof of Proposition 5.3. Let $f\colon D(\mathcal{I}_S(\varphi)) \to D(\mathcal{I}_T(\varphi))$ be an arbitrary function. Similarly to the proof of (2) of Proposition 3.6, one is tempted to find a family $\mathcal{H} \subseteq D(\mathcal{I}_S(\varphi))$ such that $f[\mathcal{H}] \subseteq D(\mathcal{I}_T(\varphi))$ is bounded but $\bigcup \mathcal{H}$ is dense in 2^ω. However, this is impossible. Instead, at each step of the iterative construction, we will have to pass to an $E_\alpha(S)$ for smaller and smaller α; for the details, see Lemma 5.10 and Lemma 5.11.

We will need the following simple observations.

Lemma 5.6. *Let $\varphi\colon [\omega]^{<\omega} \to \mathbb{R}^+$ be a submeasure satisfying (1) and (2) of Proposition 4.3. Let $\mathcal{B} \in [[\omega]^{<\omega}]^{<\omega}$ be fixed. If $y \in U(\mathcal{B})$ and $\overline{\varphi}(y) < 1/C$ then $K = \{x \in 2^\omega \colon x \subseteq y\}$ is a monotone compact set, $y \in K \subseteq U(\mathcal{B})$ and $K \in D(\mathcal{I}_S(\varphi))$ $(S \in [\omega]^\omega)$. In particular, for every $S \in [\omega]^\omega$ we have $U(\mathcal{B}) = \bigcup(\mathcal{P}(U(\mathcal{B})) \cap D(\mathcal{I}_S(\varphi)))$.*

Proof. It is obvious that $K \in \mathcal{K}_{\mathrm{mon}}$ and $y \in K \subseteq U(\mathcal{B})$. By $\varphi(y) < 1/C$, for every $j < \omega$ sufficiently large there is an $n_j \in I_{\{j\}}$ such that $y(n_j) = 0$. Let $S \in [\omega]^\omega$ be arbitrary and set $z \in [\{n_j \colon j \in S\}]^\omega$ be such that $\sum_{n \in z} \varphi(\{n\}) < +\infty$. Then $z \in \mathcal{I}_S(\varphi)$ and $z \setminus n \notin K$ $(n < \omega)$, so $K \in D(\mathcal{I}_S(\varphi))$. The second statement immediately follows, so the proof is complete. ☐

Lemma 5.7. *Let $T \in [\omega]^\omega$, $\mathcal{W} \in [[I_T]^{<\omega}]^{<\omega}$ and $\delta \in \mathbb{R}^+ \setminus \{0\}$ be fixed. Then*

$$\mathcal{P}(U(\mathcal{W})) \cap D(\mathcal{I}_T(\varphi)) = \bigcup_{V \in \mathbb{V}_\delta(T)} \mathcal{P}(U(\mathcal{W} \cup \{V\})) \cap D(\mathcal{I}_T(\varphi)).$$

Proof. Let $K \in \mathcal{P}(U(\mathcal{W})) \cap D(\mathcal{I}_T(\varphi))$ be arbitrary; we have to find a $V \in \mathbb{V}_\delta(T)$ such that $K \subseteq U(\mathcal{W} \cup \{V\})$. Let $K \in D(\mathcal{I}_T(\varphi))$ be witnessed by $x \in \mathcal{I}_T(\varphi)$. By passing to $x \setminus n$ for some $n < \omega$ we can assume $\varphi(x) \leq \delta$. Let $V_i = [x = 1] \cap i$ $(i < \omega)$. For every $y \in K$ we have $x \not\subseteq y$, thus $K \subseteq \bigcup_{i<\omega} U(\mathcal{W} \cup \{V_i\})$. By $V_i \subseteq V_j$ we get $U(V_i) \subseteq U(V_j)$ $(i \leq j < \omega)$, so by compactness we have $K \subseteq V_i$ for some $i < \omega$. Then $V = V_i$ fulfills the requirements. □

5.1. **Compactness arguments.** We need the following amended version of Lemma 2.4. The improvement is that we find the clopen set U in $\mathfrak{M}$.

Lemma 5.8. *Let $S \in [\omega]^\omega$ and $\mathcal{B} \in [[I_S]^{<\omega}]^{<\omega}$ be fixed. Let $\mathcal{R}(j) \subseteq D(\mathcal{I}_S(\varphi))$ $(j < \omega)$ be hereditary families such that $\mathcal{P}(U(\mathcal{B})) \cap D(\mathcal{I}_S(\varphi)) \subseteq \bigcup_{j<\omega} \mathcal{R}(j)$. Then for every $K \in \mathcal{P}(U(\mathcal{B})) \cap D(\mathcal{I}_S(\varphi))$ there is an $\mathcal{A} \in [[I_S]^{<\omega}]^{<\omega}$ and a $j < \omega$ with $K \subseteq U(\mathcal{A})$ and $\mathcal{P}(U(\mathcal{A})) \cap D(\mathcal{I}_S(\varphi)) \subseteq \mathcal{R}(j)$.*

Proof. Let $K \in \mathcal{P}(U(\mathcal{B})) \cap D(\mathcal{I}_S(\varphi))$ be arbitrary. Set $K^+ = \{x \in 2^\omega : \exists y \in K\ (x|_{I_S} = y|_{I_S})\}$. By $B \subseteq I_S$ $(B \in \mathcal{B})$ we have $K \subseteq K^+ \subseteq U(\mathcal{B})$. If $x \in \mathcal{I}_S(\varphi)$ satisfies $x \setminus n \notin K$ $(n < \omega)$ then by $x \subseteq I_S$ we have $x \setminus n \notin K^+$ $(n < \omega)$. Since $K^+ \in \mathcal{K}_{\mathrm{mon}}$, we have $K^+ \in D(\mathcal{I}_S(\varphi))$.

Let $U(j) \in \mathfrak{M}$, $U(j) \subseteq U(\mathcal{B})$ $(j < \omega)$ be such that $K^+ = \bigcap_{j<\omega} U(j)$. By (2) of Lemma 5.4, there are $\mathcal{A}_j \in [[\omega]^{<\omega}]^{<\omega}$ $(j < \omega)$ such that $U(j) = U(\mathcal{A}_j)$ $(j < \omega)$. Set $\mathcal{A}_j^- = \{A \cap I_S : A \in \mathcal{A}_j\} \cup \mathcal{B}$ $(j < \omega)$. Then $\mathcal{A}_j^- \in [[I_S]^{<\omega}]^{<\omega}$ $(j < \omega)$; and by the definition of K^+ and $K^+ \subseteq U(\mathcal{A}_j) \subseteq U(\mathcal{B})$ $(j < \omega)$, we have $K^+ \subseteq U(\mathcal{A}_j^-)$ $(j < \omega)$.

If $\mathcal{P}(U(\mathcal{A}_j^-)) \cap D(\mathcal{I}_S(\varphi)) \not\subseteq \mathcal{R}(j)$ $(j < \omega)$ then for every $j < \omega$ let $K(j) \subseteq U(\mathcal{A}_j^-)$ be such that $K(j) \in D(\mathcal{I}_S(\varphi)) \setminus \mathcal{R}(j)$. Let $L = K^+ \cup \bigcup_{j<\omega} K(j)$; then $L \in \mathcal{P}(U(\mathcal{B})) \cap D(\mathcal{I}_S(\varphi))$. Since $\mathcal{R}(j)$ $(j < \omega)$ are hereditary, $K(j) \subseteq L$ implies $L \notin \mathcal{R}(j)$ $(j < \omega)$. This contradicts $\mathcal{P}(U(\mathcal{B})) \cap D(\mathcal{I}_S(\varphi)) \subseteq \bigcup_{j<\omega} \mathcal{R}(j)$. So $\mathcal{P}(U(\mathcal{A}_j^-)) \cap D(\mathcal{I}_S(\varphi)) \subseteq \mathcal{R}(j)$ for some $j < \omega$, which completes the proof. □

Corollary 5.9. *Let $\varphi : [\omega]^{<\omega} \to \mathbb{R}^+$ be a submeasure satisfying (1) and (2) of Proposition 4.3. Let $S \in [\omega]^\omega$ and $\mathcal{B} \in [[I_S]^{<\omega}]^{<\omega}$ be fixed. Let $\mathcal{R}(j) \subseteq D(\mathcal{I}_S(\varphi))$ $(j < \omega)$ be hereditary families such that $\mathcal{P}(U(\mathcal{B})) \cap D(\mathcal{I}_S(\varphi)) \subseteq \bigcup_{j<\omega} \mathcal{R}(j)$. Then for every $\alpha < 1/C$ there is an $n < \omega$ and there are $\mathcal{A}_i \in [[I_S]^{<\omega}]^{<\omega}$ $(i < n)$ such that*

(1) $E_\alpha(S) \cap U(\mathcal{B}) \subseteq \bigcup_{i<n} U(\mathcal{A}_i)$,
(2) for every $i < n$ there is a $j_i < \omega$ satisfying $\mathcal{P}(U(\mathcal{A}_i)) \cap D(\mathcal{I}_S(\varphi)) \subseteq \mathcal{R}_{j_i}$.

Proof. Set $\mathbb{A} = \{\mathcal{A} \in [[I_S]^{<\omega}]^{<\omega} : \exists j < \omega\ (\mathcal{P}(U(\mathcal{A})) \cap D(\mathcal{I}_S(\varphi)) \subseteq \mathcal{R}_j)\}$. If $E_\alpha(S) \cap U(\mathcal{B}) \subseteq \bigcup\{U(\mathcal{A}) : \mathcal{A} \in \mathbb{A}\}$ then the statement follows from the compactness of $E_\alpha(S)$.

Suppose $E_\alpha(S) \cap U(\mathcal{B}) \setminus \bigcup\{U(\mathcal{A}) : \mathcal{A} \in \mathbb{A}\} \neq \emptyset$, say y is an element of this set. By Lemma 5.6, there is a $K \in \mathcal{P}(U(\mathcal{B})) \cap D(\mathcal{I}_S(\varphi))$ with $y \in K$. By Lemma 5.8, there is an $\mathcal{A} \in [[I_S]^{<\omega}]^{<\omega}$ and a $j < \omega$ with $K \subseteq U(\mathcal{A})$ and $\mathcal{P}(U(\mathcal{A})) \cap D(\mathcal{I}_S(\varphi)) \subseteq \mathcal{R}(j)$. Then $\mathcal{A} \in \mathbb{A}$, which contradicts the choice of y. □

5.2. **Combinatorial arguments.** From now on we suppose that the submeasure $\varphi : [\omega]^{<\omega} \to \mathbb{R}^+$ and the sequence $\{c_n : n < \omega\} \subseteq \mathbb{R}^+$ satisfy the conditions of Proposition 5.3.

Lemma 5.10. *Let $\alpha \in \mathbb{R}^+$ and $S, T \in [\omega]^\omega$ be fixed satisfying $S \cap T = \emptyset$. Let $\delta \in \mathbb{R}^+$, $n < \omega$, $(V_i)_{i<n} \subseteq \mathbb{V}_\delta(T)$, $\mathcal{B} \in [[I_S]^{<\omega}]^{<\omega}$ and $(\mathcal{A}_i)_{i<n} \subseteq [[I_S]^{<\omega}]^{<\omega}$ satisfy $E_\alpha(S) \cap U(\mathcal{B}) \subseteq \bigcup_{i<n} U(\mathcal{A}_i)$. Let $j \in T$ be such that $\max(\bigcup \mathcal{B}) < m_j$. Then there is a $W \in \mathcal{P}(I_{\{j\}}) \cap \mathbb{V}_{\delta+c_j}(T)$ such that*

$$E_{\alpha - c_{j+1}}(S) \cap U(\mathcal{B}) \subseteq \bigcup\{U(\mathcal{A}_i) : i < n, V_i \cap I_{\{j\}} \subseteq W\}. \tag{5.1}$$

Proof. Let $\mathcal{V} = \{V_i \cap I_{\{j\}} \colon i < n\}$. If there is a $W \in \mathcal{V}$ such that

$$E_{\alpha - c_{j+1}}(S) \cap U(\mathcal{B}) \subseteq \bigcup \{U(\mathcal{A}_i) \colon i < n, V_i \cap I_{\{j\}} = W\}$$

then we are done. So suppose this is not the case, i.e. for every $V \in \mathcal{V}$ the set

$$\Sigma(V) = \{s \in 2^{<\omega} \colon N_s \cap E_{\alpha - c_{j+1}}(S) \cap U(\mathcal{B}) \neq \emptyset,\ N_s \cap \bigcup \{U(\mathcal{A}_i) \colon i < n, V_i \cap I_{\{j\}} = V\} = \emptyset\}$$

is non-empty.

Let $\mathcal{V}' \subseteq \mathcal{V}$ be a minimal subfamily with the property that for every $t \in 2^{m_j}$, if there is a $V \in \mathcal{V}$ such that $t \notin \{s|_{m_j} \colon s \in \Sigma(V)\}$ then there is a $Z(V) \in \mathcal{V}'$ such that $t \notin \{s|_{m_j} \colon s \in \Sigma(Z(V))\}$. Then $|\mathcal{V}'| \leq 2^{m_j}$. Set $W = \bigcup \mathcal{V}'$; we prove that W fulfills the requirements.

Since for every $V \in \mathcal{V}'$ we have $V = I_{\{j\}} \cap V_i$ for some $i < n$, we have $V \subseteq \omega \setminus m_j$ and $\varphi(V) \leq \delta$ $(V \in \mathcal{V}')$. So by (4^+) of Proposition 5.3 we get $\varphi(W) \leq \delta + c_j$, that is $W \in \mathcal{P}(I_{\{j\}}) \cap \mathbb{V}_{\delta + c_j}(T)$. It remains to prove (5.1).

Suppose (5.1) does not hold, i.e. there is an $s \in 2^{<\omega}$ such that $N_s \cap E_{\alpha - c_j}(S) \cap U(\mathcal{B}) \neq \emptyset$ and

$$N_s \cap \bigcup \{U(\mathcal{A}_i) \colon i < n, V_i \cap I_{\{j\}} \subseteq W\} = \emptyset.$$

By extending s we can assume $|s| \geq m_j$. By $V \subseteq W$ $(V \in \mathcal{V}')$, for every $V \in \mathcal{V}'$ we have $s \in \Sigma(V)$. So for every $V \in \mathcal{V}$ there is an $s(V) \in \Sigma(V)$ such that $s|_{m_j} = s(V)|_{m_j}$. Set

$$z = s|_{m_j} \vee \bigvee\nolimits_{V \in \mathcal{V}} s(V) = s|_{m_j} \vee \bigvee\nolimits_{V \in \mathcal{V}} (s(V) \setminus m_j).$$

We prove that $N_z \cap E_\alpha(S) \cap U(\mathcal{B}) \neq \emptyset$ and that $N_z \cap \bigcup_{i<n} U(\mathcal{A}_i) = \emptyset$; this contradiction will complete the proof.

By $N_s \cap U(\mathcal{B}) \neq \emptyset$ and $\max(\bigcup \mathcal{B}) < m_j$ we have $B \not\subseteq [s|_{m_j} = 1]$ $(B \in \mathcal{B})$. Similarly, by $s(V) \in \Sigma(V)$ we have $N_{s(V)} \cap E_{\alpha - c_{j+1}}(S) \neq \emptyset$, so $\varphi(s(V)) \leq \alpha - c_j$ and $[s(V) = 1] \subseteq I_S$ $(V \in \mathcal{V})$. Observe that by $j \notin S$ we have

$$s|_{m_j} \vee s(V) = s|_{m_j} \vee (s(V) \setminus m_j) = s|_{m_j} \vee (s(V) \setminus m_{j+1})\ (V \in \mathcal{V}),$$

and $|\mathcal{V}| \leq 2^{d_j} \leq 2^{m_{j+1}}$. So we can apply (4^+) of Proposition 5.3 with $n = j+1$, $J = |\mathcal{V}|$ for $s|_{m_j} \vee s(V) \setminus m_{j+1}$ $(V \in \mathcal{V})$. We get $\varphi(z) \leq \alpha - c_{j+1} + c_{j+1} = \alpha$. Since $\max(\bigcup \mathcal{B}) < m_j$, we have $B \not\subseteq [z = 1]$ $(B \in \mathcal{B})$; i.e. $N_z \subseteq U(\mathcal{B})$. This proves $N_z \cap E_\alpha(S) \cap U(\mathcal{B}) \neq \emptyset$.

Let $i < n$ be arbitrary. Let $V = V_i \cap I_{\{j\}}$. By definition, $N_{s(V)} \cap U(\mathcal{A}_i) = \emptyset$, i.e. there is an $A \in \mathcal{A}_i$ such that $A \subseteq [s(V) = 1]$. Since $[s(V) = 1] \subseteq [z = 1]$, we have $N_z \cap U(\mathcal{A}_i) = \emptyset$. This completes the proof. $\square$

Lemma 5.11. *Let $\alpha \in \mathbb{R}^+$, $S, T \in [\omega]^\omega$ and $\mathcal{B} \in [[I_S]^{<\omega}]^{<\omega}$ be fixed satisfying $S \cap T = \emptyset$. Then for every $\varepsilon \in \mathbb{R}^+ \setminus \{0\}$ there is a $\delta \in \mathbb{R}^+ \setminus \{0\}$, $\delta \leq \varepsilon$ with the following property: if $n < \omega$, $(V_i)_{i<n} \subseteq \mathbb{V}_\delta(T)$ and $(\mathcal{A}_i)_{i<n} \subseteq [[I_S]^{<\omega}]^{<\omega}$ satisfy $E_\alpha(S) \cap U(\mathcal{B}) \subseteq \bigcup_{i<n} U(\mathcal{A}_i)$ then there is a $W \in \mathbb{V}_{\delta+\varepsilon}(T)$ such that $E_{\alpha-\varepsilon}(S) \cap U(\mathcal{B}) \subseteq \bigcup \{U(\mathcal{A}_i) \colon i < n, V_i \subseteq W\}$.*

Proof. Let $\delta \in \mathbb{R}^+ \setminus \{0\}$, $\delta \leq \varepsilon$ be so small that for every $x \subseteq \omega$, $\overline{\varphi}(x) \leq \delta$ implies the following:

(i) $x \cap I_{\{j\}} \neq \emptyset$ implies $\max(\bigcup \mathcal{B}) < m_j$;
(ii) $x \cap I_{\{j\}} \neq \emptyset$ implies $\sum_{j \leq i < \omega} c_i \leq \varepsilon/2$.

We show that this δ fulfills the requirements. To this end, let $n < \omega$, $(V_i)_{i<n} \subseteq \mathbb{V}_\delta(T)$ and $(\mathcal{A}_i)_{i<n} \subseteq [[I_S]^{<\omega}]^{<\omega}$ satisfy $E_\alpha(S) \cap U(\mathcal{B}) \subseteq \bigcup_{i<n} U(\mathcal{A}_i)$. Let $N = \{j < \omega \colon I_{\{j\}} \cap \bigcup_{i<n} V_i \neq \emptyset\}$; then $N \subseteq T$. By condition (i) on δ, we have $\max(\bigcup \mathcal{B}) < m_{\min N}$. For every $j \in N$, we define $W \cap I_{\{j\}}$ inductively, as follows.

Let $j_0 = \min N$. Let $j \in N$ be arbitrary and suppose that either $j = j_0$ or for every $l \in N \cap j$ we found $W_l \in \mathcal{P}(I_{\{l\}}) \cap \mathbb{V}_{\delta+\sum_{j_0\le i\le l} c_i}(T)$ such that

$$E_{\alpha-\sum_{j_0\le i\le l+1} c_i}(S) \cap U(\mathcal{B}) \subseteq \bigcup\{U(\mathcal{A}_i)\colon i<n, V_i \cap I_{\{l'\}} \subseteq W_{l'}\ (l' \in N,\ l' \le l)\}.$$

Set $n_j = \{i < n\colon V_i \cap I_{\{l\}} \subseteq W_l\ (l \in N \cap j)\}$. Apply Lemma 5.10 to $\alpha - \sum_{j_0\le i\le j} c_i$, S, T, $\mathcal{B}$, $\delta + \sum_{j_0\le i<j} c_i$, $|n_j|$, $(V_i)_{i\in n_j}$, $(\mathcal{A}_i)_{i\in n_j}$ and $j \in T$. We get a $W_j \in \mathcal{P}(I_{\{j\}}) \cap \mathbb{V}_{\delta+\sum_{j_0\le i\le j} c_i}(T)$ such that

$$E_{\alpha-\sum_{j_0\le i\le j+1} c_i}(S) \cap U(\mathcal{B}) \subseteq \bigcup\{U(\mathcal{A}_i)\colon i<n, V_i \cap I_{\{l\}} \subseteq W_l\ (l \in N,\ l \le j)\}.$$

This completes the inductive step of the construction.

Set $W = \bigcup_{j\in N} W_j$. By a repeated application of (4^+) of Proposition 5.3 we get

$$\varphi(W) \le \max\{\varphi(W_{j_0}), \varphi(\bigcup_{j\in N\setminus\{j_0\}} W_j)\} + c_{j_0} \le \dots \le \max\{\varphi(W_j)\colon j \in N\} + \sum_{j\in N} c_j \le \delta + 2\cdot\sum_{j\in N} c_j.$$

So by condition (ii) on δ, $E_{\alpha-\varepsilon} \subseteq E_{\alpha-\sum_{j_0\le i\le(\max N)+1} c_i}$ and $W \in \mathbb{V}_{\delta+\varepsilon}(T)$. So W fulfills the requirements. □

Observe that in the proof of Lemma 5.11 we do not have control on the number of applications of Lemma 5.10. This is the deep reason why our present approach is not able to prove that D keeps non-reducibility for the ideals constructed for Theorem 1.5. In fact, we think this indicates that D maps these ideals into one cofinal type.

Proof of Proposition 5.3. Let $f\colon D(\mathcal{I}_S(\varphi)) \to D(\mathcal{I}_T(\varphi))$ be an arbitrary function. For every $\mathcal{W} \in [[I_T]^{<\omega}]^{<\omega}$, let $\Phi(\mathcal{W}) = \{K \in D(\mathcal{I}_S(\varphi))\colon f(K) \subseteq U(\mathcal{W})\}^{\downarrow}$.

By Lemma 5.7, for every $\delta \in \mathbb{R}^+ \setminus \{0\}$ and $\mathcal{W} \in [[I_T]^{<\omega}]^{<\omega}$ we have

$$\Phi(\mathcal{W}) = \bigcup_{V\in\mathbb{V}_\delta(T)} \Phi(\mathcal{W} \cup \{V\}). \tag{5.2}$$

We construct a locally finite tree $Q \subseteq \omega^{<\omega}$, families $\{\mathcal{A}_q\colon q \in Q\} \subseteq [[I_S]^{<\omega}]^{<\omega}$ and $\{W_j\colon j < \omega\} \subseteq [I_T]^{<\omega}$ with the following properties:

(i) $\varphi(W_j) \le 2^{-j}$ $(j < \omega)$;
(ii) for every $n < \omega$, $E_{1/2+2^{-n}}(S) \subseteq \bigcup\{U(\mathcal{A}_q)\colon q \in Q,\ |q| = n\}$;
(iii) for every $q \in Q$, $\mathcal{P}(U(\mathcal{A}_q)) \cap D(\mathcal{I}_S(\varphi)) \subseteq \Phi(\{W_j\colon j < |q|\})$.

Suppose first that the construction is done; we prove that it witnesses that f is not a Tukey map. By Lemma 5.6 and (iii), for every $q \in Q$ there is a $K_q \in D(\mathcal{I}_S(\varphi))$ such that $f(K_q) \subseteq U(\{W_j\colon j < |q|\})$ and the $2^{-|q|}$ neighborhood of K_q contains $U(\mathcal{A}_q)$. Then by 2, $E_{1/2}(S) \subseteq \mathrm{cl}_{2^\omega}(\bigcup\{K_q\colon q \in Q\})$, i.e. by Lemma 5.5, $\{K_q\colon q \in Q\}$ is unbounded in $D(\mathcal{I}_S(\varphi))$. On the other hand, since Q is locally finite,

$$\mathrm{cl}_{2^\omega}\left(\bigcup\{f(K_q)\colon q \in Q\}\right) \subseteq \left(\bigcap_{n<\omega} U(\{W_j\colon j < n\})\right) \cup \bigcup\{f(K_q)\colon q \in Q\}. \tag{5.3}$$

By (i) we have $E_\alpha(T) \not\subseteq \bigcap_{n<\omega} U(\{W_j\colon j < n\})$ $(\alpha \in \mathbb{R}^+ \setminus \{0\})$. So by Lemma 5.5,

$$\bigcap_{n<\omega} U(\{W_j\colon j < n\}) \in D(\mathcal{I}_T(\varphi)).$$

Thus again by Lemma 5.5, for every $\alpha \in \mathbb{R}^+ \setminus R(\varphi)$, the set on the right hand side of (5.3) is meager on $E_\alpha(T)$. Hence $\bigcup\{f(K_q)\colon q \in Q\}$ is bounded in $D(\mathcal{I}_T(\varphi))$, as required.

It remains to perform the construction. We do this by induction on height. Put $\emptyset \in Q$ and set $\mathcal{A}_\emptyset = \emptyset$; then (ii) holds by $U(\emptyset) = 2^\omega$. Let $n < \omega$ and suppose that $Q \cap \omega^n$ is constructed, and

$\{\mathcal{A}_q\colon q\in Q\cap\omega^n\}\subseteq[[I_S]^{<\omega}]^{<\omega}$ and $\{W_j\colon j<n\}\subseteq[I_T]^{<\omega}$ are defined such that (i)-(iii) hold up to level n.

Let $q\in Q$, $|q|=n$ be arbitrary. Let $\varepsilon_n\in\mathbb{R}^+\setminus\{0\}$ satisfy $\varepsilon_n\le 2^{-n-2}$, and that for every $W\in[\omega]^{<\omega}$ and $j<\omega$, $\varphi(W)\le 2\varepsilon_n$ and $W\cap I_{\{j\}}\neq\emptyset$ implies $|Q\cap\omega^n|\le 2^{m_j}$ and $c_j\le 2^{-n-1}$. Then by (4^+) of Proposition 5.3, for every $\{W_i\colon i<|Q\cap\omega^n|\}\subseteq\omega^{<\omega}$ with $\varphi(W_i)\le 2\varepsilon_n$ $(i<|Q\cap\omega^n|)$ we have $\varphi(\bigcup_{i<|Q\cap\omega^n|}W_i)\le 2^{-n}$. Let δ_q satisfy the statement of Lemma 5.11 with $\alpha=1/2+2^{-n}$, S, T, $\mathcal{B}=\mathcal{A}_q$ and $\varepsilon=\varepsilon_n$. Since $\mathcal{P}(U(\mathcal{A}_q))\cap D(\mathcal{I}_S(\varphi))\subseteq\Phi(\{W_j\colon j<|q|\})$, by (5.2) we have

$$\mathcal{P}(U(\mathcal{A}_q))\cap D(\mathcal{I}_S(\varphi))\subseteq\bigcup\nolimits_{V\in\mathbb{V}_{\delta_q}(T)}\Phi(\{W_j\colon j<|q|\}\cup\{V\}).$$

So by Corollary 5.9 for $\alpha=1/2+2^{-n}$, S and $\mathcal{B}=\mathcal{A}_q$, there is an $n_q<\omega$ and there are $\mathcal{A}_q(i)\in[[I_S]^{<\omega}]^{<\omega}$ $(i<n_q)$, $V_q(i)\in\mathbb{V}_{\delta_q}(T)$ $(i<n_q)$ such that

$$E_{1/2+2^{-n}}(S)\cap U(\mathcal{A}_q)\subseteq\bigcup\nolimits_{i<n}U(\mathcal{A}_q(i))$$

and

$$\mathcal{P}(U(\mathcal{A}_q(i)))\cap D(\mathcal{I}_S(\varphi))\subseteq\Phi(\{W_j\colon j<|q|\}\cup\{V_q(i)\})\ (i<n_q).\tag{5.4}$$

Thus we are in the situation of Lemma 5.11, i.e. there is a $W_q\in\mathbb{V}_{\delta_q+\varepsilon_n}(T)$ such that

$$E_{1/2+2^{-n}-\varepsilon_n}(S)\cap U(\mathcal{A}_q)\subseteq\bigcup\{U(\mathcal{A}_q(i))\colon i<n_q, V_q(i)\subseteq W_q\}.\tag{5.5}$$

For every $i<n_q$, we put $q^\frown(i)\in Q$ if and only if $V_q(i)\subseteq W_q$. We set $W_n=\bigcup_{q\in Q\cap\omega^n}W_q$, and $\mathcal{A}_{q^\frown(i)}=\mathcal{A}_q(i)$ $(q^\frown(i)\in Q\cap\omega^{n+1})$. We show that this definition fulfills the requirements.

To see (i), observe that $\varphi(W_q)\le 2\varepsilon_n$ $(q\in Q\cap\omega^n)$, so by the choice of ε_n, $\varphi(W)\le 2^{-n}$. Since $\varepsilon_n\le 2^{-n-1}$, by the inductive hypothesis and (5.5) we have

$$E_{1/2+2^{-n-1}}(S)\subseteq\bigcup\{U(\mathcal{A}_q)\colon q\in Q\cap\omega^{n+1}\},$$

which shows (ii).

For (iii), pick an arbitrary $q^\frown(i)\in Q\cap\omega^{n+1}$. Then by (5.4),

$$\mathcal{P}(U(\mathcal{A}_{q^\frown(i)}))\cap D(\mathcal{I}_S(\varphi))=\mathcal{P}(U(\mathcal{A}_q(i)))\cap D(\mathcal{I}_S(\varphi))\subseteq\Phi(\{W_j\colon j<n\}\cup\{V_q(i)\}).$$

Since $V_q(i)\subseteq W_q\subseteq W_n$, we have $U(\{W_j\colon j<n\}\cup\{V_q(i)\})\subseteq U(\{W_j\colon j<n\}\cup\{W_n\})$ and so

$$\Phi(\{W_j\colon j<n\}\cup\{V_q(i)\})\subseteq\Phi(\{W_j\colon j<n\}\cup\{W_n\}).$$

This completes the inductive step and the proof. □

6. Problems

We believe that Proposition 4.3 is essentially optimal, in the sense that the combinatorics in its proof accounts for all non-reducibility results for analytic p-ideals. So we think that the most important problem is to find the optimal condition for Tukey reducibility between analytic p-ideals.

Problem 6.1. *Find optimal conditions on submeasures* $\varphi,\varphi'\colon[\omega]^{<\omega}\to\mathbb{R}^+$ *for* $\mathrm{Exh}(\varphi)\le_T\mathrm{Exh}(\varphi')$.

It would be interesting to study whether it is possible to construct exotic non-exhaustive submeasures using the resolution approach presented in Section 4.2. In particular, the following would be of special importance (see also [17, Conjecture 1 p. 194]).

Problem 6.2. *Apply the techniques of Section 4.2 to construct a submeasure* $\varphi\colon[\omega]^{<\omega}\to\mathbb{R}^+$ *such that* $\mathrm{Fin}(\varphi)$ *and* ω^ω *are* $\le_T$*-incomparable.*

Also, similarly to the interplay between Tsirelson spaces and Tsirelson submeasures, there may be Banach space theoretic implications of the construction carried out in Section 4.2.

It would be enlightening to know more about the cofinal diversity of G_δ σ-ideals of compact sets.

Problem 6.3. *Find the maximal cofinal type among G_δ σ-ideals of compact sets. Prove that the structure $(\mathcal{P}(\omega), \subseteq^\star)$ embeds into the family of G_δ σ-ideals of compact sets.*

We think that the following question is likely to have an affirmative answer.

Problem 6.4. *Let $\mathcal{J}, \mathcal{J}'$ be analytic p-ideals or G_δ σ-ideals of compact sets satisfying $\omega^\omega \leq_T \mathcal{J} <_T \mathcal{J}'$. Is it true that the structure $(\mathcal{P}(\omega), \subseteq^\star)$ embeds into the family of analytic p-ideals or analytic relative σ-ideals of compact sets $\mathcal{I}$ satisfying $\mathcal{J} \leq_T \mathcal{I} \leq_T \mathcal{J}'$?*

An enthusiastic reader may like to venture into the study of Tukey reducibility between all the $\mathcal{I}_S(\varphi)$s and $D(\mathcal{I}_T(\varphi))$s mentioned in the present paper. However, such an endeavor is likely to encounter considerable technical difficulties without shedding light to the broader Tukey picture. Moreover, if e.g. Problem 6.4 has an affirmative answer, then such particular investigations can turn out to be out of interest.

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