Infinite dimensional perfect set theorems

Tamás Mátrai

University of Toronto

Madison

April 3, 2009

Just before the coffee break...

A one dimensional perfect set theorem

$$\left.\begin{array}{c} A \subseteq \mathbb{R} \\ \text{analytic,} \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ H \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ P \subseteq A \end{array}\right.$$

A two dimensional perfect set theorem (?)

$$\left.\begin{array}{c} A \subseteq \mathbb{R} \\ \text{analytic,} \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ H \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ P \subseteq A \end{array}\right.$$

A two dimensional perfect set theorem (?)

$$\left.\begin{array}{c} A \subseteq [\mathbb{R}]^2 \\ \text{analytic,} \\ \\ \exists H \in [\mathbb{R}]^{\omega_1} : \\ H \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ P \subseteq A \end{array}\right.$$

A two dimensional perfect set theorem (?)

$$\left.\begin{array}{c} A \subseteq [\mathbb{R}]^2 \\ \text{analytic,} \\ \\ \exists H \in [\mathbb{R}]^{\omega_1} : \\ H \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ [P]^2 \subseteq A \end{array}\right.$$

A two dimensional perfect set theorem (?)

$$\left.\begin{array}{c} A \subseteq [\mathbb{R}]^2 \\ \text{analytic,} \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ [H]^2 \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ [P]^2 \subseteq A \end{array}\right.$$

A two dimensional perfect set theorem (?)

$$\left.\begin{array}{c} A \subseteq [\mathbb{R}]^2 \\ \text{analytic,} \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ [H]^2 \subseteq A \end{array}\right\} \not\Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ [P]^2 \subseteq A \end{array}\right.$$

FALSE!

Folklore counterexample

Theorem

$\exists A \subseteq [\mathbb{R}]^2$ F_σ *set:*

- $\exists H \in [\mathbb{R}]^{\omega_1} : [H]^2 \subseteq A$;
- $\forall P \subseteq \mathbb{R}$ *nonempty perfect* $\exists Q \subseteq P$ *nonempty perfect:*

$$[Q]^2 \cap A = \emptyset.$$

Folklore counterexample

Theorem

$\exists A \subseteq [\mathbb{R}]^2$ F_σ *set:*

- $\exists H \in [\mathbb{R}]^{\omega_1} : [H]^2 \subseteq A$;
- $\forall P \subseteq \mathbb{R}$ *nonempty perfect* $\exists Q \subseteq P$ *nonempty perfect:*

$$[Q]^2 \cap A = \emptyset.$$

Terminology: $\alpha \leq \omega$, $A \subseteq \mathbb{R}^\alpha$, $H \subseteq \mathbb{R}$,

Folklore counterexample

Theorem

$\exists A \subseteq [\mathbb{R}]^2$ F_σ set:

- *$\exists H \in [\mathbb{R}]^{\omega_1} : [H]^2 \subseteq A$;*
- *$\forall P \subseteq \mathbb{R}$ nonempty perfect $\exists Q \subseteq P$ nonempty perfect:*

$$[Q]^2 \cap A = \emptyset.$$

Terminology: $\alpha \leq \omega$, $A \subseteq \mathbb{R}^\alpha$, $H \subseteq \mathbb{R}$,

⊣— $[H]^\alpha$: α element subsets; Inconvenient for $\alpha = \omega$!

Folklore counterexample

Theorem

$\exists A \subseteq [\mathbb{R}]^2$ F_σ set:

- *$\exists H \in [\mathbb{R}]^{\omega_1} : [H]^2 \subseteq A$;*
- *$\forall P \subseteq \mathbb{R}$ nonempty perfect $\exists Q \subseteq P$ nonempty perfect:*

$$[Q]^2 \cap A = \emptyset.$$

Terminology: $\alpha \leq \omega$, $A \subseteq \mathbb{R}^\alpha$, $H \subseteq \mathbb{R}$,

- $[H]^\alpha$: α element subsets;
- $IS_\alpha(H)$ or $H^{[\alpha]}$: injective $\alpha \to H$ functions;

Folklore counterexample

Theorem

$\exists A \subseteq [\mathbb{R}]^2$ F_σ set:

- *$\exists H \in [\mathbb{R}]^{\omega_1} : [H]^2 \subseteq A$;*
- *$\forall P \subseteq \mathbb{R}$ nonempty perfect $\exists Q \subseteq P$ nonempty perfect:*

$$[Q]^2 \cap A = \emptyset.$$

Terminology: $\alpha \leq \omega$, $A \subseteq \mathbb{R}^\alpha$, $H \subseteq \mathbb{R}$,

- $[H]^\alpha$: α element subsets;
- $IS_\alpha(H)$ or $H^{[\alpha]}$: injective $\alpha \to H$ functions;
- A is symmetric: coordinate permutation invariant;

Significant constraint for $\alpha = \omega$!

Folklore counterexample

Theorem

$\exists A \subseteq [\mathbb{R}]^2$ F_σ set:

- $\exists H \in [\mathbb{R}]^{\omega_1} : [H]^2 \subseteq A$;
- $\forall P \subseteq \mathbb{R}$ *nonempty perfect* $\exists Q \subseteq P$ *nonempty perfect:*

$$[Q]^2 \cap A = \emptyset.$$

Terminology: $\alpha \le \omega$, $A \subseteq \mathbb{R}^\alpha$, $H \subseteq \mathbb{R}$,

- $[H]^\alpha$: α element subsets;
- $IS_\alpha(H)$ or $H^{[\alpha]}$: injective $\alpha \to H$ functions;
- A is symmetric: coordinate permutation invariant;
- H is A-homogeneous: $IS_\alpha(H) \subseteq A$ (or $[H]^\alpha \subseteq A$ if $\alpha < \omega$).

Folklore counterexample

Theorem

$\exists A \subseteq [\mathbb{R}]^2$ F_σ set:

- $\exists H \in [\mathbb{R}]^{\omega_1} : [H]^2 \subseteq A$;
- $\forall P \subseteq \mathbb{R}$ *nonempty perfect* $\exists Q \subseteq P$ *nonempty perfect:*

$$[Q]^2 \cap A = \emptyset.$$

Terminology: $\alpha \leq \omega$, $A \subseteq \mathbb{R}^\alpha$, $H \subseteq \mathbb{R}$,

- $[H]^\alpha$: α element subsets;
- $IS_\alpha(H)$ or $H^{[\alpha]}$: injective $\alpha \longrightarrow H$ functions;
- A is symmetric: coordinate permutation invariant;
- H is A-homogeneous: $IS_\alpha(H) \subseteq A$ (or $[H]^\alpha \subseteq A$ if $\alpha < \omega$).

$\exists A \subseteq [\mathbb{R}]^2$ F_σ set:

$\exists$ uncountable A-homogeneous $\wedge$ $\nexists$ nonempty perfect A-homogeneous

Folklore counterexample

Theorem

$\exists A \subseteq [\mathbb{R}]^2$ F_σ set:

- *$\exists H \in [\mathbb{R}]^{\omega_1}$: $[H]^2 \subseteq A$;*
- *$\forall P \subseteq \mathbb{R}$ nonempty perfect $\exists Q \subseteq P$ nonempty perfect:*

$$[Q]^2 \cap A = \emptyset.$$

Terminology: $\alpha \leq \omega$, $A \subseteq \mathbb{R}^\alpha$, $H \subseteq \mathbb{R}$,

- $[H]^\alpha$: α element subsets;
- $IS_\alpha(H)$ or $H^{[\alpha]}$: injective $\alpha \to H$ functions;
- A is symmetric: coordinate permutation invariant;
- H is A-homogeneous: $IS_\alpha(H) \subseteq A$ (or $[H]^\alpha \subseteq A$ if $\alpha < \omega$).

$\exists A \subseteq [\mathbb{R}]^2$ F_σ set:

$\exists$ uncountable A-homogeneous $\wedge \nexists$ nonempty perfect A-homogeneous

W. Kubiś, S. Shelah (MA): uncountable $\rightsquigarrow$ arbitrary $\aleph_\gamma < \mathfrak{c}$ $(\gamma < \omega_1)$

Positive results

W. Kubiś:

$$\left.\begin{array}{c} A \subseteq [\mathbb{R}]^n \\ G_\delta, \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ [H]^n \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ [P]^n \subseteq A \end{array}\right.$$

Positive results

W. Kubiś:

$$\left.\begin{array}{c} A \subseteq [\mathbb{R}]^n \\ G_\delta, \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ [H]^n \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ [P]^n \subseteq A \end{array}\right.$$

J. Mycielski:

$$\left.\begin{array}{c} A \subseteq [\mathbb{R}]^n \\ \\ \\ \text{comeager or} \\ \text{of full measure} \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ [P]^n \subseteq A \end{array}\right.$$

Infinite dimensional perfect set theorems I.

Infinite dimensional perfect set theorems I.

$$\left.\begin{array}{c} A \subseteq \mathbb{R}^{\omega} \\ ??? \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ IS_{\omega}(H) \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ IS_{\omega}(P) \subseteq A \end{array}\right.$$

Infinite dimensional perfect set theorems I.

$$\left.\begin{array}{c} A \subseteq \mathbb{R}^\omega \\ \text{open}, \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ IS_\omega(H) \subseteq A \end{array}\right\} \overset{?}{\Longrightarrow} \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ IS_\omega(P) \subseteq A \end{array}\right.$$

Infinite dimensional perfect set theorems I.

$$\left.\begin{array}{c} A \subseteq \mathbb{R}^\omega \\ \text{open,} \\ \\ \exists H \in [\mathbb{R}]^{\omega_1}: \\ IS_\omega(H) \subseteq A \end{array}\right\} \not\Longrightarrow \left\{\begin{array}{c} \exists P \subseteq \mathbb{R} \text{ nonempty} \\ \text{perfect:} \\ IS_\omega(P) \subseteq A \end{array}\right.$$

Theorem

$$A \subseteq \mathbb{R}^2 \ F_\sigma,\ H \subseteq \mathbb{R}\ A\text{-homogeneous}$$

$$\rightsquigarrow$$

$$A^+ \subseteq \mathbb{R}^\omega \text{ symmetric open:}$$

- *H is A^+-homogeneous;*
- *$\nexists\ \neg\emptyset$ perfect A-homog. $\Rightarrow$ $\nexists\ \neg\emptyset$ perfect A^+-homogeneous*

Infinite dimensional perfect set theorems II.

Theorem

$A \subseteq \mathbb{R}^\omega$ has the Baire property (analytic or coanalytic),

$$\exists H \subseteq \mathbb{R} \text{ non-(perfectly)-meager } A\text{-homogeneous}$$

$$\Downarrow$$

$$\exists P \subseteq \mathbb{R} \ \neg\emptyset \text{ perfect } A\text{-homogeneous}$$

Infinite dimensional perfect set theorems II.

Theorem

$A \subseteq \mathbb{R}^\omega$ has the Baire property (analytic or coanalytic),

$$\exists H \subseteq \mathbb{R} \text{ non-(perfectly)-meager } A\text{-homogeneous}$$

$$\Downarrow$$

$$\exists P \subseteq \mathbb{R} \ \neg\emptyset \text{ perfect } A\text{-homogeneous}$$

However:

$$A \subseteq \mathbb{R}^\omega \text{ open, dense and of full measure}$$

$$\not\Downarrow$$

$$\exists P \subseteq \mathbb{R} \ \neg\emptyset \text{ perfect } A\text{-homogeneous}$$

Motivation: analytic iDeals

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

$$? \Downarrow ?$$

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

$$? \Downarrow ?$$

$$\exists \mathcal{P} \subseteq \mathcal{I} \ \neg\emptyset \text{ perfect} : \forall \mathcal{J} \in [\mathcal{P}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$\mathcal{H}$: strongly unbounded

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \boxed{\forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}}$$

$$? \Downarrow ?$$

$$\exists \mathcal{P} \subseteq \mathcal{I} \ \neg\emptyset \text{ perfect} : \forall \mathcal{J} \in [\mathcal{P}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

$\mathcal{H}$: strongly unbounded

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \boxed{\forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}}$$

$$? \Downarrow ?$$

$$\exists \mathcal{P} \subseteq \mathcal{I} \ \neg\emptyset \ \text{perfect} : \forall \mathcal{J} \in [\mathcal{P}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

$\mathcal{A} \subseteq \mathcal{P}(\omega)^\omega$, $(A_n)_{n<\omega} \in \mathcal{A} \Leftrightarrow \bigcup_{n<\omega} A_n \notin \mathcal{I}$, $\mathcal{A} \subseteq (2^\omega)^\omega$ coanalytic

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

$\mathcal{H}$: strongly unbounded

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \boxed{\forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}}$$

$$?\Downarrow?$$

$$\exists \mathcal{P} \subseteq \mathcal{I} \ \neg\emptyset \ \text{perfect} : \forall \mathcal{J} \in [\mathcal{P}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

$\mathcal{A} \subseteq \mathcal{P}(\omega)^\omega$, $(A_n)_{n<\omega} \in \mathcal{A} \Leftrightarrow \bigcup_{n<\omega} A_n \notin \mathcal{I}$, $\mathcal{A} \subseteq (2^\omega)^\omega$ coanalytic

Answer:

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$\mathcal{H}$: strongly unbounded

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \boxed{\forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}}$$

$$? \Downarrow ?$$

$$\exists \mathcal{P} \subseteq \mathcal{I} \ \neg\emptyset \text{ perfect} : \forall \mathcal{J} \in [\mathcal{P}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

$\mathcal{A} \subseteq \mathcal{P}(\omega)^\omega$, $(A_n)_{n<\omega} \in \mathcal{A} \Leftrightarrow \bigcup_{n<\omega} A_n \notin \mathcal{I}$, $\mathcal{A} \subseteq (2^\omega)^\omega$ coanalytic

Answer: NO!

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$\mathcal{H}$: strongly unbounded

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \boxed{\forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}}$$

$$? \Downarrow ?$$

$$\exists \mathcal{P} \subseteq \mathcal{I} \ \neg\emptyset \text{ perfect} : \forall \mathcal{J} \in [\mathcal{P}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

$\mathcal{A} \subseteq \mathcal{P}(\omega)^\omega$, $(A_n)_{n<\omega} \in \mathcal{A} \Leftrightarrow \bigcup_{n<\omega} A_n \notin \mathcal{I}$, $\mathcal{A} \subseteq (2^\omega)^\omega$ coanalytic

Answer: NO! $\exists \mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal:

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

$\mathcal{H}$: strongly unbounded

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \boxed{\forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}}$$

$$? \Downarrow ?$$

$$\exists \mathcal{P} \subseteq \mathcal{I} \ \neg\emptyset \ \text{perfect} : \forall \mathcal{J} \in [\mathcal{P}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

$\mathcal{A} \subseteq \mathcal{P}(\omega)^\omega$, $(A_n)_{n<\omega} \in \mathcal{A} \Leftrightarrow \bigcup_{n<\omega} A_n \notin \mathcal{I}$, $\mathcal{A} \subseteq (2^\omega)^\omega$ coanalytic

Answer: NO! $\exists \mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal:

- $\mathcal{I}$ has an uncountable strongly unbounded set;

Motivation: analytic iDeals

$\mathcal{I} \subseteq \mathcal{P}(\omega)$

ideal:

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$;
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

analytic:

- $\mathcal{I} \subseteq 2^\omega$ analytic.

Question: $\mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal

↓ $\mathcal{H}$: strongly unbounded

$$\exists \mathcal{H} \in [\mathcal{I}]^{\omega_1} : \boxed{\forall \mathcal{J} \in [\mathcal{H}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}}$$

$$?\Downarrow?$$

$$\exists \mathcal{P} \subseteq \mathcal{I} \ \neg\emptyset \text{ perfect} : \forall \mathcal{J} \in [\mathcal{P}]^\omega \ \bigcup \mathcal{J} \notin \mathcal{I}$$

$\mathcal{A} \subseteq \mathcal{P}(\omega)^\omega$, $(A_n)_{n<\omega} \in \mathcal{A} \Leftrightarrow \bigcup_{n<\omega} A_n \notin \mathcal{I}$, $\mathcal{A} \subseteq (2^\omega)^\omega$ coanalytic

Answer: NO! $\exists \mathcal{I} \subseteq \mathcal{P}(\omega)$ analytic ideal:

- $\mathcal{I}$ has an uncountable strongly unbounded set;
- $\mathcal{I}$ has no $\neg\emptyset$ perfect strongly unbounded set.