PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

KENILWORTH

TAMÁS MÁTRAI

ABSTRACT. We construct a G_δ σ-ideal $\mathcal{I}$ of compact subsets of 2^ω such that $\mathcal{I}$ contains all the singletons but there is *no* dense G_δ set $D \subseteq 2^\omega$ such that $\{K \subseteq D\colon K \text{ compact}\} \subseteq \mathcal{I}$. This answers a question of A. S. Kechris in the negative.

1. The Question of Kechris

Let X be a Polish space and let $\mathcal{I}$ be a σ-ideal of compact subsets of X. By the Dichotomy Theorem (see [4, Theorem 7 pp. 268]) if $\mathcal{I}$ is $\mathbf{\Pi}^1_1$ it is either G_δ or $\mathbf{\Pi}^1_1$-complete. Thus, in view of the two alternatives, a G_δ σ-ideal may be considered as an extremely simple object. Moreover, for compact X, one can refine the classification of the G_δ case of the Dichotomy Theorem: with the notation $\mathcal{K}(A) = \{K \subseteq A\colon K \text{ compact}\}$, a G_δ σ-ideal $\mathcal{I} \subseteq \mathcal{K}(X)$ is either $\mathbf{\Pi}^0_2$-complete, or of the form $\mathcal{K}(A)$ where A is $\mathrm{D}_2(\mathbf{\Pi}^0_1)$, $\mathbf{\Pi}^0_1$, $\mathbf{\Sigma}^0_1$ or $\mathbf{\Delta}^0_1$, in which case $\mathcal{I}$ is $\mathrm{D}_2(\mathbf{\Pi}^0_1)$-, $\mathbf{\Pi}^0_1$-, $\mathbf{\Sigma}^0_1$-complete and $\mathbf{\Delta}^0_1$, respectively (see [5, Theorem 1.4]).

However, a $\mathbf{\Pi}^0_2$-complete σ-ideal is not necessarily of the form $\mathcal{K}(D)$ for a G_δ set $D \subseteq X$: the σ-ideals of compact meager sets or compact Lebesgue null sets are obvious examples. On the other hand, for these two σ-ideals, which are comeager subsets of $\mathcal{K}(X)$ e.g. by the fact that they contain all the finite sets, it is at least true that they contain $\mathcal{K}(D)$ for a dense G_δ set $D \subseteq X$. This property turned out to be very useful (see e.g. [2] and [3]), and in [5, Remark 4.17] conditions on the construction of G_δ σ-ideals were formulated which guarantee that a G_δ σ-ideal $\mathcal{I}$ containing all the singletons satisfies $\mathcal{K}(D) \subseteq \mathcal{I}$ for some dense G_δ set $D \subseteq X$. Nevertheless, the problem, originally posed by A. S. Kechris ([2, Problem pp. 191] and [3, Problem 3 pp. 121], see also [5, Problem 6.1]), whether this property holds for arbitrary G_δ σ-ideals in $\mathcal{K}(2^\omega)$ remained open. The purpose of the present paper is to give a negative solution to this problem.

Theorem 1.1. *There exists a G_δ σ-ideal $\mathcal{I}$ of compact subsets of 2^ω such that $\mathcal{I}$ contains all the singletons but there is no dense G_δ set $D \subseteq 2^\omega$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.*

Once such an example found it is natural to ask whether the construction carries over all (perfect) Polish spaces and whether we can work in the hyperspace of closed sets instead of compact sets; but most importantly, whether we can include into $\mathcal{I}$ more than just singletons and avoid other families of compact sets than just $\mathcal{K}(D)$ for dense G_δ sets $D \subseteq X$.

The extension of the construction to arbitrary perfect Polish spaces will be carried out in Section 3 without serious difficulties. Moreover, the construction of the example in Section 2 will allow us

2000 *Mathematics Subject Classification.* Primary 03E15; Secondary 54H05, 28A05.

Key words and phrases. G_δ σ-ideal of compact sets, singleton, ideal extension, covering property.

The research was partially supported by the OTKA grants F 43620, K 49786, K 61600 and by the József Öveges Program of the NKTH and KPI.

to put into $\mathcal{I}$ any fixed family of compact sets of the form $\mathcal{K}(Z)$ where Z is a meager $\mathbf{\Sigma}^0_2$ set. Unfortunately we cannot prove a general theorem allowing us to extend any $\mathbf{\Pi}^1_1$ σ-ideal $\mathcal{J}$ to a G_δ σ-ideal $\mathcal{I}$ in such a way that $\mathcal{K}(D) \not\subseteq \mathcal{J} \implies \mathcal{K}(D) \not\subseteq \mathcal{I}$ for every D in a reasonable collection of dense G_δ sets (see Problem 3.9). Another property of our construction is that it does not produce a σ-ideal with the covering property. A nontrivial G_δ σ-ideal with the covering property would be a natural and well-understood way to give a negative answer to the question of Kechris (see [1, Problem 12 pp. 137] and the comments after [5, Problem 6.1]).

We are grateful to S. Solecki and M. Zelený for their helpful remarks on the construction of our example.

2. The example

In this section we prove the following result, which is slightly more general than Theorem 1.1. As above, 2^ω stands for the Cantor space with is usual topology. For every $A \subseteq 2^\omega$ we set $\mathcal{K}(A) = \{K \subseteq A \colon K \text{ compact}\}$ and $\mathcal{L}(A) = \{K \in \mathcal{K}(2^\omega) \colon K \cap A \neq \emptyset\}$. The space $\mathcal{K}(2^\omega)$ is endowed with the Vietoris topology which makes it a compact Polish space (see e.g. [1, (4.25) Theorem pp. 26]). The closure and the interior of a set $A \subseteq X$ is denoted by $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$.

Theorem 2.1. *Let $Z \subseteq 2^\omega$ be a meager $\mathbf{\Sigma}^0_2$ set. Then there exists a G_δ σ-ideal $\mathcal{I}$ of compact subsets of 2^ω such that $\mathcal{I}$ contains all the singletons and $\mathcal{K}(Z) \subseteq \mathcal{I}$, but there is no dense G_δ set $D \subseteq 2^\omega$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.*

For the construction, we pursue the following simple strategy. We aim to construct the complement of our G_δ σ-ideal $\mathcal{I}$. We start by finding a closed set $\mathcal{P} \subseteq \mathcal{K}(2^\omega)$ such that $\mathcal{P}$ does not contain countable compact sets or sets in $\mathcal{K}(Z)$, and for every closed nowhere dense set $A \subseteq 2^\omega$ we have $\mathcal{L}(A) \cap \mathcal{P}$ is nowhere dense in $\mathcal{P}$. Then for every dense G_δ set $D \subseteq 2^\omega$ we have $\mathcal{P} \cap \mathcal{K}(D) \neq \emptyset$ since $\mathcal{P} \cap \mathcal{K}(D) = \mathcal{P} \setminus \mathcal{L}(2^\omega \setminus D)$ is comeager in $\mathcal{P}$. Thus already $\mathcal{P} \subseteq \mathcal{K}(2^\omega) \setminus \mathcal{I}$ guarantees $\mathcal{K}(D) \setminus \mathcal{I} \neq \emptyset$ for every dense G_δ set $D \subseteq 2^\omega$. Finding our $\mathcal{P}$ is the crucial step; once found, it is easy to accompany it with a countable collection of closed subsets of $\mathcal{K}(2^\omega)$ such that together with $\mathcal{P}$ they form the complement of a σ-ideal.

We recall some notation following [1]. For every $s, t \in 2^{<\omega}$, $|s|$ denotes the length of s and $s^\frown t$ stands for the sequence $s(0) \dots s(|s|-1)t(0) \dots t(|t|-1)$. We set $N_s = \{x \in 2^\omega \colon s \subseteq x\}$. If $T \subseteq 2^{<\omega}$ is a tree

(i) the maximal branches, or terminal nodes, of T are denoted by $\mathfrak{T}(T)$;
(ii) for $s \in 2^{<\omega}$, $T_s = \{t \in 2^{<\omega} \colon s^\frown t \in T\}$ and $s^\frown T = \{t \in 2^\omega \colon \exists u \in T \ (t \subseteq s^\frown u)\}$;
(iii) $[T] = \{x \in 2^\omega \colon \forall n < \omega \ (x|_n \in T)\}$.

For every $n < \omega$, we identify 2^n with the maximal branches of the full binary tree $2^{<n}$, i.e. $2^n = \mathfrak{T}(2^{<n})$, indexed according to the lexicographic order. If $\sigma \colon 2^n \to 2$ is given, $T(\sigma)$ is the subtree of $2^{<n}$ generated by $\bigcup \sigma^{-1}(1)$.

For every $\mathcal{P} \subseteq \mathcal{K}(2^\omega)$ we define

$$\mathcal{P}^\uparrow = \{A \in \mathcal{K}(2^\omega) \colon \exists P \in \mathcal{P} \ (P \subseteq A)\}. \tag{2.1}$$

For $2 \leq n < \omega$ and $\sigma \colon 2^n \to 2$ we set (see (2.2) below)

$$g_l(\sigma) = \min\{i < 2^n \colon \sigma(i) = 1\},$$
$$g_r(\sigma) = 2^n - 1 - \max\{i < 2^n \colon \sigma(i) = 1\},$$
$$b(\sigma) = \max\{d \leq 2^n \colon \forall i \in [g_l(\sigma), g_l(\sigma) + d) \ (\sigma(i) = 1)\},$$

and let $n(s)$ denote the length of the longest sequence of consecutive 0's in $[2^{n-2}, 2^n - 2^{n-2} - 1]$.

$$n = 5 \colon \sigma = \underbrace{000}_{g_l(\sigma)=3}\underbrace{1111}_{b(\sigma)=4}001\underbrace{00}_{n(\sigma)=2}11011101111011\underbrace{000000}_{g_r(\sigma)=6} \tag{2.2}$$

For every $2 \leq n < \omega$ set

$$\Sigma_n = \{\sigma \in 2^{2^n} \colon g_l(\sigma) \leq 2^{n-2},\ g_r(\sigma) \leq 2^{n-2} \text{ and } n(\sigma) \leq b(\sigma)\}; \tag{2.3}$$

e.g. the sequence of (2.2) is in Σ_5.

For every $\eta \colon \omega \to \omega$ fixed increasing function we define a σ-ideal $\mathcal{I}$, as follows. Consider the following inductive construction of a sequence of finite trees $T^n \subseteq 2^{<\omega}$ $(n < \omega)$. Set $T^0 = \{\emptyset\}$; let $0 < n < \omega$ and suppose that T^{n-1} is already defined. For every $t \in \mathfrak{T}(T^{n-1})$ take an arbitrary $n(t) < \omega$ satisfying $\max\{2, \eta(|t|)\} \leq n(t)$ and pick an arbitrary sequence $\sigma_t \in \Sigma_{n(t)}$. We define T_n by extending T^{n-1} at every $t \in \mathfrak{T}(T^{n-1})$ by $T(\sigma_t)$, that is

$$T^n = \left\{ t^\frown s \in 2^{<\omega} \colon t \in \mathfrak{T}(T^{n-1}), s \in T(\sigma_t) \right\}.$$

Such a sequence $(T^n)_{n<\omega}$ is called *η-admissible.* A tree $T \subseteq 2^{<\omega}$ is *η-admissible* if $T = \bigcup_{n<\omega} T^n$ for some η-admissible sequence $(T^n)_{n<\omega}$. For every $s \in 2^{<\omega}$ set

$$\mathcal{P}_s = \{[s^\frown T] \colon T \text{ is } \eta\text{-admissible}\}, \tag{2.4}$$

and with the notation of (2.1) let

$$\mathcal{I} = \mathcal{K}(2^\omega) \setminus \bigcup_{s \in 2^{<\omega}} \mathcal{P}_s^{\uparrow}. \tag{2.5}$$

First we show that for every $\eta \colon \omega \to \omega$, $\mathcal{I}$ is a G_δ σ-ideal containing all the singletons such that $\mathcal{K}(D) \not\subseteq \mathcal{I}$ for every dense G_δ set $D \subseteq 2^\omega$. Next we show that for every meager $\mathbf{\Sigma}^0_2$ set $Z \subseteq 2^\omega$ there is an increasing η such that $\mathcal{K}(Z) \subseteq \mathcal{I}$. The propositions we use will be proved at the end of this section.

To see that $\mathcal{I}$ is a G_δ subset of $\mathcal{K}(2^\omega)$, we need the following observations.

Proposition 2.2. *Let X be a compact Polish space and let $\mathcal{P} \subseteq \mathcal{K}(X)$ be a closed set. Then $\mathcal{P}^{\uparrow} \subseteq \mathcal{K}(X)$ is closed, as well.*

Proposition 2.3. *For every $s \in 2^{<\omega}$ and $K \in \mathcal{K}(2^\omega)$, if $K \in cl_{\mathcal{K}(2^\omega)}(\mathcal{P}_s)$ then either $K \in \mathcal{P}_s$ or* $\mathrm{int}_{2^\omega}(K) \neq \emptyset$.

To see that $\mathcal{I}$ is a G_δ subset of $\mathcal{K}(2^\omega)$ observe that if $K \in \mathcal{K}(2^\omega)$ satisfies $\mathrm{int}_{2^\omega}(K) \neq \emptyset$, say for an $s \in 2^{<\omega}$ we have $N_s \subseteq K$, then by (2.4) we have $K \in \mathcal{P}_s^{\uparrow}$. Hence by Proposition 2.3,

$$\bigcup_{s \in 2^{<\omega}} \mathcal{P}_s^{\uparrow} = \bigcup_{s \in 2^{<\omega}} \left[\mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_s)\right]^{\uparrow}, \tag{2.6}$$

and by Proposition 2.2 this latter is an F_σ subset of $\mathcal{K}(2^\omega)$. Hence $\mathcal{I}$ is a G_δ subset of $\mathcal{K}(2^\omega)$, as required. Notice that there are sets $P \in \mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_s)$ which have isolated points. This phenomenon is quite disturbing but as we will see in Proposition 3.6 it is inevitable.

Next we observe that $\mathcal{I}$ contains all the singletons. By (2.5) it is enough to prove the following.

Proposition 2.4. *Let $s \in 2^{<\omega}$ and $P \in \mathcal{P}_s$, say $P = [s^\frown T]$ where T is η-admissible. Then for every $t \in T$, $N_t \cap P$ is a nonempty perfect subset of 2^ω.*

We have to show that $\mathcal{I}$ is a σ-ideal. By (2.5), $\mathcal{I}$ is closed under taking compact subsets, thus we only need the following.

Proposition 2.5. *Let $K \in \mathcal{K}(2^\omega)$ and suppose that $K = \bigcup_{i<\omega} K_i$ for some $K_i \in \mathcal{I}$ $(i < \omega)$. Then $K \in \mathcal{I}$.*

To see that $\mathcal{K}(D) \not\subseteq \mathcal{I}$ for every dense G_δ set $D \subseteq 2^\omega$ we will prove the following.

Proposition 2.6. *Let $A \subseteq 2^\omega$ be a nowhere dense closed set. Then $\mathcal{L}(A) \cap \mathcal{P}_\emptyset$ is nowhere dense in $\mathcal{P}_\emptyset$.*

Let $D \subseteq 2^\omega$ be a dense G_δ set, say $D = 2^\omega \setminus \bigcup_{n<\omega} A_n$ where $A_n \subseteq 2^\omega$ is a nowhere dense closed set $(n < \omega)$. We have $\mathcal{K}(D) = \mathcal{K}(2^\omega) \setminus \bigcup_{n<\omega} \mathcal{L}(A_n)$. By Proposition 2.6, $\mathcal{L}(A_n) \cap \mathcal{P}_\emptyset$ is nowhere dense in $\mathcal{P}_\emptyset$ $(n < \omega)$. Since $\mathcal{L}(A_n)$ is closed, this implies $\mathcal{L}(A_n) \cap \mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_\emptyset)$ is nowhere dense in $\mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_\emptyset)$ $(n < \omega)$. Hence $\bigcup_{n<\omega} \mathcal{L}(A_n) \cap \mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_\emptyset)$ is meager in $\mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_\emptyset)$. This means $\mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_\emptyset) \cap \mathcal{K}(D) \neq \emptyset$ so by (2.6), $\mathcal{K}(D) \setminus \mathcal{I} \neq \emptyset$, as required.

To obtain an $\eta\colon \omega \to \omega$ which guarantees $\mathcal{K}(Z) \subseteq \mathcal{I}$, we need the following.

Proposition 2.7. *Let $A \subseteq 2^\omega$ be a nowhere dense set and let $s \in 2^{<\omega}$ be arbitrary. Then there exists an $n(A, s) < \omega$ such that for every $n \geq n(A, s)$ and for every $\sigma \in \Sigma_n$ there exists a $t \in T(\sigma)$ such that $N_{s^\frown t} \cap A = \emptyset$.*

For every nowhere dense set $A \subseteq 2^\omega$ we define

$$\zeta_A\colon \omega \to \omega,\ \zeta_A(n) = \max\{n(A, s)\colon s \in 2^{<n}\}. \tag{2.7}$$

Proposition 2.8. *Let $Z \subseteq 2^\omega$ be a meager $\mathbf{\Sigma}^0_2$ set, say $Z = \bigcup_{n<\omega} Z_n$ where $Z_n \subseteq 2^\omega$ is a nowhere dense closed set $(n < \omega)$. If $\eta\colon \omega \to \omega$ is an increasing function and for every $n < \omega$, $\zeta_{Z_k}(m) \leq \eta(n)$ $(k, m \leq 2n)$ then $\mathcal{K}(Z) \subseteq \mathcal{I}$.*

It remains to prove the propositions. Before doing so we observe some simple properties of our construction.

Lemma 2.9. *Let $(n_k)_{k<\omega} \subseteq \omega$ be a strictly increasing sequence, let $\sigma_k \in \Sigma_{n_k}$ $(k < \omega)$ and suppose that for a $K \in \mathcal{K}(2^\omega)$ we have $\lim T(\sigma_k) = K$. Then $\mathrm{int}_{2^\omega}(K) \neq \emptyset$.*

Proof. We distinguish two cases. If $\liminf_{k\to\infty} 2^{-n_k} b(\sigma_k) = 0$ then by (2.3) $\liminf_{k\to\infty} 2^{-n_k} n(\sigma_k) = 0$, as well. By $\lim T(\sigma_k) = K$, this implies $N_{01} \cup N_{10} \subseteq K$ so $\mathrm{int}_{2^\omega}(K) \neq \emptyset$.

Else for an $\varepsilon > 0$ we have $\liminf_{k\to\infty} 2^{-n_k} b(\sigma_k) > \varepsilon$. By the fact $(T(\sigma_k))_{k<\omega}$ is convergent it is possible only if for $n = \lceil -\log_2(\varepsilon) \rceil$ there is a $t \in 2^n$ such that for every sufficiently large $k < \omega$ and $s \in 2^{n_k - n}$ we have $\sigma_k(t^\frown s) = 1$. Then by $\lim T(\sigma_k) = K$, $N_t \subseteq K$ so again $\mathrm{int}_{2^\omega}(K) \neq \emptyset$, as required. $\square$

Lemma 2.10. *Let $A \subseteq 2^\omega$ be a nowhere dense set, let $s \in 2^{<\omega}$ and $M < \omega$ be arbitrary. Then there exists an $n < \omega$ and a $\sigma \in \Sigma_n$ such that $M \leq n$ and*

$$\bigcup\{N_{s^\frown t}\colon t \in 2^n,\ \sigma(t) = 1\} \subseteq 2^\omega \setminus A. \tag{2.8}$$

Proof. By identifying N_s with 2^ω, it is enough to prove the statement in the $s = \emptyset$ case. Since A is nowhere dense there is an $s_0 \in 2^{<\omega}$ satisfying $00 \subseteq s_0$ and $N_{s_0} \subseteq 2^\omega \setminus A$. Let $n \geq |s_0|$ be such that $M \leq n$ and

$$\forall t \in 2^{|s_0|}\ \exists t' \in 2^{n-1}\ (t \subseteq t',\ N_{t'} \subseteq 2^\omega \setminus A); \tag{2.9}$$

such an n exists again by A being nowhere dense. We define $\sigma\colon 2^n \to 2$ by setting, for every $t \in 2^{n+1}$, $\sigma(t) = 1$ if and only if $N_t \subseteq 2^\omega \setminus A$. Then (2.8) holds so it remains to show $\sigma \in \Sigma_n$.

By the definition of s_0 we have $g_l(\sigma) \leq 2^{n-2}$ and $b(\sigma) \geq 2^{n-|s_0|}$. By (2.9) applied for $t \in 2^{|s_0|}$, $t(i) = 1$ $(i < |s_0|)$, we have $\sigma(t') = 1$ for some $t' \in 2^n$ with $t'(i) = 1$ $(i < |s_0|)$ so

$$g_r(\sigma) \leq 2^{n-|s_0|} - 1 \leq 2^{n-2}.$$

Again by (2.9), $n(\sigma) \leq 2(2^{n-1-|s_0|} - 1) = 2^{n-|s_0|} - 2 \leq b(\sigma)$, so the proof is complete. □

Proof of Proposition 2.2. Let $(Q_n)_{n<\omega} \subseteq \mathcal{P}^{\uparrow}$ be convergent, $Q = \lim_{n<\omega} Q_n$; we show $Q \in \mathcal{P}^{\uparrow}$. Let $P_n \subseteq Q_n$ satisfy $P_n \in \mathcal{P}$ $(n < \omega)$. By passing to a subsequence we can assume $(P_n)_{n<\omega}$ is convergent, say $P = \lim_{n<\omega} P_n$. Then $P \in \mathcal{P}$ and clearly $P \subseteq Q$, so the proof is complete. □

Proof of Proposition 2.3. Fix $s \in 2^{<\omega}$, let $K \in \mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_s)$, say $[s^\frown T(k)] \to K$ as $k \to \infty$ where $T(k)$ $(k < \omega)$ are η-admissible trees, i.e. $T(k) = \bigcup_{n<\omega} T^n(k)$ with $(T^n(k))_{n<\omega}$ η-admissible $(k < \omega)$. We distinguish two cases. Suppose first that for every $n < \omega$ the set $\{T^n(k) \colon k < \omega\}$ is finite. By a diagonalization, we can find a sequence $(k_i)_{i<\omega}$ and trees $T^n(\infty)$ $(n < \omega)$ such that

$$\forall n < \omega \ \forall^\infty i < \omega \ T^n(k_i) = T^n(\infty).$$

Then $(T^n(\infty))_{n<\omega}$ is η-admissible and $K = \left[\bigcup_{n<\omega} s^\frown T^n(\infty)\right]$, hence $K \in \mathcal{P}_s$.

Else, for an $n < \omega$ we have $\{T^n(k) \colon k < \omega\}$ is infinite. Let n_0 be smallest such n. By passing to a subsequence, for each $n < n_0$ we can find a tree $T^n(\infty)$ such that for all $k < \omega$ and $n < n_0$ we have $T^n(k) = T^n(\infty)$.

We show that $\mathrm{int}_{2^\omega}(K) \neq \emptyset$ in this case. Set $t = \emptyset$ if $n_0 = 0$, else let $t \in \mathfrak{T}(T^{n_0-1}(\infty))$ be such that $\{T^{n_0}(k)_t \colon k < \omega\}$ is infinite. Let $n_k(t) < \omega$ and $\sigma_k(t) \in \Sigma_{n_k(t)}$ be the parameters for which $T^{n_0}(k)_t = T(\sigma_k(t))$. By passing to a subsequence we can assume $n_k(t)$ is strictly increasing as $k \to \infty$. Then $([T(k)])_{k<\omega}$ is convergent implies $([T(\sigma_k(t))])_{k<\omega}$ is convergent, as well. Thus from Lemma 2.9 we get that $\mathrm{int}_{2^\omega}(K \cap N_{s^\frown t}) \neq \emptyset$, which completes the proof. □

Proof of Proposition 2.4. By (2.4) it is enough to prove the statement for $s = \emptyset$. By the definition of Σ_n in (2.3), for every $2 \leq n < \omega$ and $\sigma \in \Sigma_n$ the tree $T(\sigma)$ has at least one splitting node. Therefore an η-admissible tree is a perfect tree, so the statement follows. □

Proof of Proposition 2.5. Let $K, K_i \in \mathcal{K}(2^\omega)$ $(i < \omega)$ such that $K = \bigcup_{i<\omega} K_i$. Suppose $K \notin \mathcal{I}$, that is $K \in \mathcal{P}_s^{\uparrow}$ for some $s \in 2^{<\omega}$, say $[s^\frown T] \subseteq K$ for some η-admissible tree $T = \bigcup_{n<\omega} T^n$ with $(T^n)_{n<\omega}$ η-admissible. We show that $K_i \notin \mathcal{I}$ for some $i < \omega$.

By the Baire Category Theorem, there exists $i < \omega$ and $t \in 2^{<\omega}$ such that

$$\emptyset \neq [s^\frown T] \cap N_{s^\frown t} \subseteq K_i. \tag{2.10}$$

By non-emptiness of the intersection on the left of (2.10) and by extending t we can assume $t \in \mathfrak{T}(T^m)$ for some $m < \omega$. For every $n < \omega$ set $\hat{T}^n = T_t^{m+n}$. Since η is increasing, $(\hat{T}^n)_{n<\omega}$ is η-admissible. Thus $\hat{T} = \bigcup_{n<\omega} \hat{T}^n$ is η-admissible, as well. We have $[s^\frown t^\frown \hat{T}] = N_{s^\frown t} \cap [T] \subseteq K_i$, so $K_i \in \mathcal{P}_{s^\frown t}^{\uparrow}$. Hence $K_i \notin \mathcal{I}$, as stated. □

Proof of Proposition 2.6. Let $P \in \mathcal{P}_\emptyset$ be arbitrary, say $P = [T]$ where $T = \bigcup_{n<\omega} T^n$ with $(T^n)_{n<\omega}$ η-admissible. We construct a sequence $(P_k)_{k<\omega} \subseteq \mathcal{P}_\emptyset$ such that $P_k \cap A = \emptyset$ $(k < \omega)$ and $\lim_{k<\omega} P_k = P$. This will complete the proof.

For every $k < \omega$ we find an η-admissible sequence $((T^n(k))_{n<\omega})$ satisfying

(1) $T^n(k) = T^n$ $(n \leq k < \omega)$,
(2) $\bigcup\{N_s \colon s \in \mathfrak{T}(T^{k+1}(k))\} \subseteq 2^\omega \setminus A$,

as follows. Fix $k < \omega$. For $n \le k$, $T^n(k)$ is given by 1; observe that η-admissibility holds for $T^n(k)$ $(n \le k)$. Since A is nowhere dense we can apply Lemma 2.10 for every $s \in \mathfrak{T}(T^k(k))$ with $M = \eta(|s|)$. We get that $T^k(k)$ can be extended to a finite tree $T^{k+1}(k)$ such that 2 holds and η-admissibility holds for $T^n(k)$ $(n \le k+1)$. Finally we construct $T^n(k)$ $(k+1 < n < \omega)$ arbitrarily such that $(T^n(k))_{n<\omega}$ is η-admissible; this is clearly possible.

Set $P_k = [\bigcup_{n<\omega} T^n(k)]$; then $P_k \in \mathcal{P}_\emptyset$ $(k < \omega)$. By 1, $\lim_{k<\omega} P_k = P$ while by 2, $P_k \cap A = \emptyset$ $(k < \omega)$. This completes the proof. □

Proof of Proposition 2.7. Suppose the statement fails for an $s \in 2^{<\omega}$, i.e. for every $n < \omega$ there is a sequence $\sigma_n \in \Sigma_n$ satisfying $N_{s^\frown t} \cap A \neq \emptyset$ $(t \in T(\sigma_n))$. We can pass to a subsequence $(n_k)_{k<\omega}$ such that $([T(\sigma_{n_k})])_{k<\omega}$ is convergent in $\mathcal{K}(2^\omega)$, say $K = \lim_{k<\omega}[s^\frown T(\sigma_{n_k})]$. By Lemma 2.9, $\mathrm{int}_{2^\omega}(K) \neq \emptyset$. However, $K \subseteq \mathrm{cl}_{2^\omega}(A)$ which contradicts A is nowhere dense. □

Proof of Proposition 2.8. Since $\mathcal{I}$ is a σ-ideal, it is enough to show $Z_k \in \mathcal{I}$ $(k < \omega)$. Fix $k < \omega$ and $s \in 2^{<\omega}$; we show $Z_k \notin \mathcal{P}_s^\uparrow$, that is we show $[s^\frown T] \not\subseteq Z_k$ for every $T = \bigcup_{n<\omega} T^n$ with η-admissible $(T^n)_{n<\omega}$.

Let $n < \omega$ be such that there is a terminal node $t \in T^{n-1}$ satisfying $k \le |t|$, $|s| \le |t|$. By η-admissibility, the parameter n_t in the construction of T^n_t satisfies $\eta(|t|) \le n_t$. Thus

$$\zeta_{Z_k}(|s^\frown t|) = \zeta_{Z_k}(|s| + |t|) \le \zeta_{Z_k}(2|t|) \le \eta(|t|) \le n_t.$$

By the definition of ζ_{Z_k} in (2.7) and by Proposition 2.7, we have $N_{s^\frown t^\frown u} \cap Z_k = \emptyset$ for some $u \in T(\sigma_t)$. Since $t^\frown u \in T^n$ and by Proposition 2.4 ,$[T] \cap N_{t^\frown u} \neq \emptyset$, we have $[s^\frown T] \cap N_{s^\frown t^\frown u} \neq \emptyset$ hence $[s^\frown T] \not\subseteq Z_k$. This completes the proof. □

3. Analysis

In this section we extend the construction in Section 2 for 2^ω to perfect Polish spaces. We also discuss some particularities of our construction, i.e. the role of isolated points.

3.1. **Example in perfect Polish spaces.**

First we handle compact perfect Polish spaces. Note that requiring the Polish space to be perfect is not a superfluous assumption since in a Polish space containing a dense open countable set Theorem 1.1 cannot hold. Our main tool is Proposition 3.2. We set

$$I = \{\sigma \in 2^\omega \colon \forall n < \omega\ \exists i, j < \omega\ (n \le i, j,\ \sigma(i) = 0,\ \sigma(j) = 1)\}, \tag{3.1}$$

$\mathbb{H} = (2^\omega)^\omega$ and $\mathbb{I} = I^\omega \subseteq \mathbb{H}$. We start with a folklore lemma.

Lemma 3.1. *Let $C \subseteq [1/3, 2/3]^\omega \subseteq [0,1]^\omega$ be a countable set and let $D \subseteq [0,1]^\omega$ be a dense G_δ set. Then there is an $x = (x_n)_{n<\omega} \in (0, 1/3)^\omega$ such that*

$$C + x = \{(c_n + x_n)_{n<\omega} \colon (c_n)_{n<\omega} \in C\} \subseteq D.$$

Proof. For every $c \in C$, $\{x \in (0,1/3)^\omega \colon x + c \in [0,1]^\omega \setminus D\} \subseteq (0,1/3)^\omega$ is a meager set so the statement follows from the Baire Category Theorem. □

Proposition 3.2. *If X is a compact perfect Polish space there is a dense G_δ set $G \subseteq X$ and continuous surjective map $\varphi \colon 2^\omega \to X$ such that φ is one-to-one on $\varphi^{-1}(G)$ and for every nowhere dense closed set $Z \subseteq X$ we have $\varphi^{-1}(Z) \subseteq 2^\omega$ is nowhere dense.*

Proof. Let $f\colon 2^\omega \to [0,1]$ and $F\colon \mathbb{H} \to [0,1]^\omega$,

$$f(\sigma) = \sum_{i<\omega} \frac{\sigma(i)}{2^{i+1}},\ F((\sigma_n)_{n<\omega}) = (f(\sigma_n))_{n<\omega}\ (\sigma, \sigma_n \in 2^\omega\ (n<\omega))$$

be the usual continuous surjections. Since f is a homeomorphism on I of (3.1), F is also a homeomorphism on $\mathbb{I}$ and $F(\mathbb{I}) \subseteq [0,1]^\omega$ is a dense G_δ set.

Every compact Polish space is homeomorphic to a closed subset of $[1/3, 2/3]^\omega$ (see e.g. [1, (4.14) Theorem pp. 22]) so we regard X as a subset of $[1/3,2/3]^\omega \subseteq [0,1]^\omega$. By applying Lemma 3.1 for a countable dense subset of X and $D = F(\mathbb{I})$ we can assume that $X \cap F(\mathbb{I})$ is a dense G_δ subset of X.

Since X is perfect, $X \cap F(\mathbb{I})$ is perfect, as well. Therefore $\mathrm{cl}_{\mathbb{H}}(F^{-1}(X) \cap \mathbb{I})$ is a nonempty zero dimensional compact perfect Polish space, so we have a homeomorphism $i\colon 2^\omega \to \mathrm{cl}_{\mathbb{H}}(F^{-1}(X)\cap\mathbb{I})$. We show that $G = X \cap F(\mathbb{I})$ and $\varphi\colon 2^\omega \to X$, $\varphi = F|_{\mathrm{cl}_{\mathbb{H}}(F^{-1}(X)\cap\mathbb{I})} \circ i$ fulfill the requirements. Since φ is continuous and $X \cap F(\mathbb{I}) \subseteq \varphi(2^\omega)$, φ is surjective and $\varphi^{-1}(G) = i^{-1}(F^{-1}(X) \cap \mathbb{I})$ is a dense G_δ set in 2^ω. Since i and $F|_{\mathbb{I}}$ are homeomorphisms, $\varphi|_{\varphi^{-1}(G)}\colon \varphi^{-1}(G) \to G$ is a homeomorphism, as well. Let $Z \subseteq X$ be a nowhere dense closed set. Then

$$\varphi^{-1}(Z) = (\varphi^{-1}(Z) \setminus \varphi^{-1}(G)) \cup \varphi^{-1}(Z \cap G) \subseteq (2^\omega \setminus \varphi^{-1}(G)) \cup \varphi^{-1}(Z\cap G). \tag{3.2}$$

The first term of the union on the right of (3.2) is meager since $\varphi^{-1}(G)$ is a dense G_δ set in 2^ω. For the second term, we have $Z \cap G$ is nowhere dense in $X \cap G$ and $\varphi|_{\varphi^{-1}(G)}$ is a homeomorphism, so $\varphi^{-1}(Z\cap G)$ is nowhere dense in $\varphi^{-1}(G)$ hence in 2^ω as well. So $\varphi^{-1}(Z) \subseteq 2^\omega$ is a closed meager hence nowhere dense set, as required. □

Corollary 3.3. *Let X be a compact perfect Polish space and let $Z \subseteq X$ be a meager $\mathbf{\Sigma}^0_2$ set. Then there exists a G_δ σ-ideal $\mathcal{I}$ of compact subsets of X such that $\mathcal{I}$ contains all the singletons and $\mathcal{K}(Z) \subseteq \mathcal{I}$, but there is no dense G_δ set $D \subseteq X$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.*

Proof. Let $G \subseteq X$ be the dense G_δ set and $\varphi\colon 2^\omega \to X$ be the map of Proposition 3.2. Let $\mathcal{P}_s$ $(s \in 2^{<\omega})$ be as in (2.4) with an η as in Proposition 2.8 for the meager $\mathbf{\Sigma}^0_2$ set $\varphi^{-1}(Z \cup (X \setminus G))$. Let

$$\mathcal{X}_s = \{\varphi(K)\colon K \in \mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_s)\}\ (s \in 2^{<\omega})$$

and set $\mathcal{I} = \mathcal{K}(X) \setminus \bigcup_{s\in 2^{<\omega}} \mathcal{X}_s^{\uparrow}$. We show that $\mathcal{I}$ fulfills the requirements.

Since φ is continuous, $\mathcal{X}_s \subseteq \mathcal{K}(X)$ is closed hence by Proposition 2.2, $\mathcal{X}_s^{\uparrow}$ is also closed $(s \in 2^{<\omega})$. Thus $\mathcal{I} \subseteq \mathcal{K}(X)$ is a G_δ family of compact sets. By the choice of η, for every $s \in 2^{<\omega}$ and $K \in \mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_s)$ we have $K \cap \varphi^{-1}(G)$ is uncountable and $K \not\subseteq \varphi^{-1}(Z)$. Since φ is one-to-one on $\varphi^{-1}(G)$, this implies $\mathcal{X}_s^{\uparrow}$ does not contain singletons or subsets of Z $(s \in 2^{<\omega})$. Hence $\mathcal{I}$ contains all the singletons and $\mathcal{K}(Z) \subseteq \mathcal{I}$.

Next we show that $\mathcal{I}$ is a σ-ideal. Clearly, $\mathcal{I}$ is hereditary. Let $K \in \mathcal{K}(X) \setminus \mathcal{I}$, say $\varphi(L) \subseteq K$ for some $L \in \mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_s)$ and $s \in 2^{<\omega}$. Suppose $K = \bigcup_{n<\omega} K_n$ for some $K_n \in \mathcal{K}(X)$ $(n<\omega)$. We have $L \subseteq \bigcup_{n<\omega} \varphi^{-1}(K_n)$ so for an $n<\omega$, $\varphi^{-1}(K_n) \notin \mathcal{K}(2^\omega) \setminus \bigcup_{s \in 2^{<\omega}} \mathcal{P}_s^{\uparrow}$, say $L' \subseteq \varphi^{-1}(K_n)$ for some $L' \in \mathcal{P}_{s'}$, $s' \in 2^{<\omega}$. Then $\varphi(L') \subseteq K_n$ which shows $K_n \notin \mathcal{I}$ and proves the statement.

Finally let $D \subseteq X$ be a dense G_δ set. By the choice of φ, $\varphi^{-1}(D)$ is a dense G_δ subset of 2^ω. By Proposition 2.6 there is a $K \in \mathrm{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_\emptyset) \cap \mathcal{K}(\varphi^{-1}(D))$. Then $\varphi(K) \in \mathcal{X}_\emptyset \cap \mathcal{K}(D)$ hence $\mathcal{K}(D) \not\subseteq \mathcal{I}$, which completes the proof. □

Before extending the construction to non-compact Polish spaces let us point out that in such spaces the question of Kechris is practically answered by the family of closed σ-compact sets $\mathcal{K}_\sigma(X)$.

We recall some notation in advance. If X is a Polish space we denote the family of closed subsets of X by $\mathcal{F}(X)$ and we endow $\mathcal{F}(X)$ with the Vietoris topology. For $D \subseteq X$ we set $\mathcal{F}(D) = \{F \in \mathcal{F}(X) \colon F \subseteq D\}$. The following is an immediate corollary of [5, Example 3.17 and Theorem 3.19].

Proposition 3.4. *Let X be an arbitrary Polish space. Then $\mathcal{K}_\sigma(X)$ is a G_δ subset of $\mathcal{F}(X)$ which contains all the singletons. If $\operatorname{int}_X(K) = \emptyset$ for every $K \in \mathcal{K}(X)$ then for every dense G_δ set $D \subseteq X$ we have $\mathcal{F}(D) \not\subseteq \mathcal{K}_\sigma(X)$.*

Finally Corollary 3.3 easily yields the following.

Corollary 3.5. *Let X be an arbitrary perfect Polish space and let $Z \subseteq X$ be a meager $\mathbf{\Sigma}^0_2$ set. Then there exists a G_δ σ-ideal $\mathcal{I}$ of closed subsets of X such that $\mathcal{I}$ contains all the singletons and $\mathcal{F}(Z) \subseteq \mathcal{I}$, but there is no dense G_δ set $D \subseteq X$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.*

The same result holds for compact sets instead of closed sets, as well.

Proof. Let $\hat{X}$ be a Polish compactification of X. Let $\hat{\mathcal{I}} \subseteq \mathcal{K}(\hat{X})$ be the σ-ideal of Corollary 3.3 applied for $\hat{X}$ with the meager $\mathbf{\Sigma}^0_2$ set $Z \cup (\hat{X} \setminus X)$. Set $\mathcal{I} = \{K \cap X \colon K \in \hat{\mathcal{I}}\} \subseteq \mathcal{F}(X)$. It is obvious that $\mathcal{I}$ is a hereditary G_δ family in $\mathcal{F}(X)$ and contains all the singletons. Since $\mathcal{K}(\hat{X} \setminus X) \subseteq \hat{\mathcal{I}}$, $\mathcal{I}$ is closed under taking countable unions. By $\mathcal{K}(Z \cup \hat{X} \setminus X) \subseteq \hat{\mathcal{I}}$ we have $\mathcal{F}(Z) \subseteq \mathcal{I}$. Finally X is a dense G_δ subset of $\hat{X}$ thus for every dense G_δ set $D \subseteq X \subseteq \hat{X}$ we have $\{K \in \mathcal{K}(\hat{X}) \colon K \subseteq D\} \not\subseteq \hat{\mathcal{I}}$ hence $\mathcal{K}(D) \not\subseteq \mathcal{I}$, as well.

To have a σ-ideal of compact sets with the same properties we only have to take $\mathcal{I} \cap \mathcal{K}(X)$. This completes the proof. □

3.2. **The role of isolated points.** It is easy to check that even if $\mathcal{P}_s$ contains only nonempty perfect sets $\operatorname{cl}_{\mathcal{K}(2^\omega)}(\mathcal{P}_s)$ contains sets with isolated points. This is somehow disturbing: singletons are to be avoided and it would be nice to construct $\mathcal{K}(2^\omega) \setminus \mathcal{I}$ in such a way that it is closed under taking nonempty clopen portions. However, this is impossible. Recall $\mathcal{L}(A) = \{K \in \mathcal{K}(2^\omega) \colon K \cap A \neq \emptyset\}$ $(A \subseteq 2^\omega)$.

Proposition 3.6. *Set*

$$\mathcal{P} = \{K \in \mathcal{K}(2^\omega) \setminus \{\emptyset\} \colon K \text{ is perfect}\}. \tag{3.3}$$

Then $\mathcal{P}$ is a dense G_δ subset of $\mathcal{K}(2^\omega) \setminus \{\emptyset\}$ but for every closed set $\mathcal{L} \subseteq \mathcal{K}(2^\omega)$, if $\mathcal{L} \subseteq \mathcal{P}$ there is a nowhere dense closed set $A \subseteq 2^\omega$ such that $\mathcal{L} \subseteq \mathcal{L}(A)$. In particular, if $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ is a G_δ set and $\mathcal{K}(2^\omega) \setminus \mathcal{I} \subseteq \mathcal{P}$ then there is a dense G_δ set $D \subseteq 2^\omega$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.

Thus sets with isolated points play a crucial role in every construction providing a negative answer to the question of Kechris. But Proposition 3.6 has another much more interesting corollary. First we need to recall the so-called *covering property* (see e.g. [3, Definition 9 pp. 135], [5, Section 3] or [6]).

Definition 3.7. Let X be a Polish space. A family $\mathcal{F}$ of closed subsets of X has the *covering property* if for every $\mathbf{\Sigma}^1_1$ set $A \subseteq X$

(1) either there is a countable subfamily $\mathcal{A} \in [\mathcal{F}]^{\leq\omega}$ such that $A \subseteq \bigcup \mathcal{A}$;
(2) or there is a closed set $H \subseteq X$ such that $H \subseteq A$ and $F \cap H$ is nowhere dense in H for every $F \in \mathcal{F}$.

The relevance of the covering property comes from the following observation. Let $\mathcal{J} \subseteq \mathcal{K}(2^\omega)$ be a $\mathbf{\Pi}^1_1$ σ-ideal such that for every dense G_δ set $D \subseteq 2^\omega$ we have $\mathcal{K}(D) \not\subseteq \mathcal{J}$. Consider the family

$$\mathfrak{F} = \{\mathcal{L}(A) \colon A \in \mathcal{K}(2^\omega) \text{ is nowhere dense}\} \subseteq \mathcal{K}(\mathcal{K}(2^\omega)).$$

By assumption, the $\mathbf{\Sigma}^1_1$ set $\mathcal{K}(2^\omega)\setminus\mathcal{I}$ cannot be covered by the union of countably many members of $\mathfrak{F}$. So if the family $\mathfrak{F}$ had the covering property we would have a compact set $\mathcal{P}\subseteq\mathcal{K}(2^\omega)$ for which $\mathcal{L}(A)\cap\mathcal{P}$ is nowhere dense in $\mathcal{P}$ for every nowhere dense set $A\subseteq 2^\omega$. As we have seen in Section 1 for $\mathcal{J}=\{K\in\mathcal{K}(2^\omega)\colon K \text{ is countable}\}$ the existence of such a $\mathcal{P}$ is the key to the construction of a G_δ σ-ideal $\mathcal{I}\subseteq\mathcal{K}(2^\omega)$ such that $\mathcal{J}\subseteq\mathcal{I}$ and still for every dense G_δ set $D\subseteq 2^\omega$ we have $\mathcal{K}(D)\not\subseteq\mathcal{I}$. However, as an immediate corollary of Proposition 3.6 we have the following.

Corollary 3.8. *The family $\mathfrak{F}$ does not have the covering property in the Polish space $\mathcal{K}(2^\omega)$.*

Proof. By Proposition 3.6, the family $\mathcal{P}$ of (3.3) is a dense G_δ subset of $\mathcal{K}(2^\omega)$. It is obvious that $\mathcal{L}(A)\cap\mathcal{P}$ is nowhere dense in $\mathcal{P}$ for every nowhere dense set $A\subseteq 2^\omega$. In particular $\mathcal{P}$ cannot be covered by the union of countably many members of $\mathfrak{F}$. But again by Proposition 3.6, for every closed set $\mathcal{L}\subseteq\mathcal{P}$ there is a nowhere dense closed set $A\subseteq 2^\omega$ such that $\mathcal{L}\subseteq\mathcal{L}(A)$. Hence the covering property fails for $\mathfrak{F}$. ☐

Hence such a far reaching generalization of our construction is not possible *via the covering property of* $\mathfrak{F}$. Since $\{A\in\mathcal{K}(2^\omega)\colon A \text{ is nowhere dense}\}\subseteq\mathcal{K}(2^\omega)$ is a G_δ set it is easy to check that $\mathfrak{F}$ is a G_δ subset of $\mathcal{K}(\mathcal{K}(2^\omega))$, as well. We note that by [5, Corollary 3.5] every F_σ family of compact sets has the covering property. Accordingly, for every F_σ set $\mathcal{A}\subseteq\mathcal{K}(2^\omega)$ the construction of Section 1 goes through for $\mathfrak{F}_\mathcal{A}=\{\mathcal{L}(A)\colon A\in\mathcal{A}\}$. Nevertheless, the following problem remains open. Just as for the Question of Kechris, we expect a possibly easy negative answer here, as well.

Problem 3.9. *Let $\mathcal{J}\subseteq\mathcal{K}(2^\omega)$ be an arbitrary $\mathbf{\Pi}^1_1$ σ-ideal such that for every dense G_δ set $D\subseteq 2^\omega$ we have $\mathcal{K}(D)\not\subseteq\mathcal{J}$. Can we find a G_δ σ-ideal $\mathcal{I}\subseteq\mathcal{K}(2^\omega)$ such that $\mathcal{J}\subseteq\mathcal{I}$ and still for every dense G_δ set $D\subseteq 2^\omega$ we have $\mathcal{K}(D)\not\subseteq\mathcal{I}$?*

We close this paper with the proof of Proposition 3.6.

Proof of Proposition 3.6. For every $n<\omega$ set

$$\mathcal{S}_n=\{S\in\mathcal{K}(2^\omega)\colon \exists s\in 2^n\ (|S\cap N_s|=1)\}.$$

Then $\mathcal{S}_n\subseteq\mathcal{K}(2^\omega)$ is a nowhere dense closed set $(n<\omega)$ and $\mathcal{K}(2^\omega)\setminus\mathcal{P}=\bigcup_{n<\omega}\mathcal{S}_n$. So $\mathcal{P}\subseteq\mathcal{K}(2^\omega)$ is a dense G_δ set. Let $\mathcal{L}\subseteq\mathcal{P}$ be a closed set. Since $\mathcal{L}\cap\mathcal{S}_n=\emptyset$ $(n<\omega)$, by the definition of Vietoris open neighborhoods we have a function $\kappa\colon\omega\to\omega$ such that $n<\kappa(n)$ $(n<\omega)$ and if $K\in\mathcal{K}(2^\omega)$, $S\in\mathcal{S}_n$ satisfy

$$\forall s\in 2^{\kappa(n)}\ (K\cap N_s\neq\emptyset \iff S\cap N_s\neq\emptyset) \tag{3.4}$$

then $K\notin\mathcal{L}$.

Set $d_0=0$, $d_{n+1}=\kappa(d_n)$ $(n<\omega)$ and let

$$A=\{\sigma\in 2^\omega\colon \forall n<\omega\ \exists i\in[d_n,d_{n+1})\ (\sigma(i)\neq 0)\}.$$

Then $A\subseteq 2^\omega$ is a nowhere dense closed set. We show $\mathcal{L}\subseteq\mathcal{L}(A)$. To this end, pick $K\in\mathcal{K}(2^\omega)\setminus\{\emptyset\}$ with $K\cap A=\emptyset$; we have to show $K\notin\mathcal{L}$.

For every $n<\omega$ let

$$U_n=\{\sigma\in 2^\omega\colon \sigma|_{[d_n,d_{n+1})}\equiv 0 \text{ and } \forall j<n\ \exists i\in[d_j,d_{j+1})\ (\sigma(i)\neq 0)\}.$$

The sets U_n $(n<\omega)$ are open, pairwise disjoint and $2^\omega\setminus A=\bigcup_{n<\omega}U_n$. Hence there is a minimal $N<\omega$ such that $K\subseteq\bigcup_{n\le N}U_n$. Set $\mathbb{O}\in 2^\omega$, $\mathbb{O}(i)=0$ $(i<\omega)$. We define $S\subseteq 2^\omega$ by $S\cap A=\emptyset$,

$S \cap U_n = K \cap U_n$ $(n \neq N)$ and for $s \in 2^{d_{N+1}}$ with $N_s \subseteq U_N$ let

$$S \cap N_s = \begin{cases} s^\frown \mathbb{O} & \text{if } K \cap N_s \neq \emptyset, \\ \emptyset & \text{if } K \cap N_s = \emptyset. \end{cases} \tag{3.5}$$

Clearly, $S \in \mathcal{K}(2^\omega)$. By the minimality of N we have $U_N \cap S \neq \emptyset$, hence by (3.5), $S \in \mathcal{S}_{d_N}$. By definition, (3.4) holds for $n = d_N$ hence $K \notin \mathcal{L}$, as required. □

References

[1] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics 156 (Springer-Verlag, 1994).
[2] A. S. Kechris, *Hereditary properties of the class of closed sets of uniqueness for trigonometric series*, Israel J. Math. 73 (1991), no. 2, 189–198.
[3] A. S. Kechris, *The descriptive set theory of σ-ideals of compact sets*, Logic Colloq. '88 (Padova, 1988), 117–138, Stud. Logic Found. Math., 127, North-Holland, Amsterdam, 1989.
[4] A. S. Kechris, A. Louveau, W. H. Woodin, *The structure of σ-ideals of compact sets*, Trans. Amer. Math. Soc. 301 (1987), no. 1, 263–288.
[5] E. Matheron, M. Zelený, *Descriptive set theory of families of small sets*, J. Symbolic Logic, to appear.
[6] A. C. E. Uzcátegui, *The covering property for σ-ideals of compact sets*, Fund. Math. 141 (1992), no. 2, 119–146.

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda street 13-15, H-1053 Budapest, Hungary
E-mail address: matrait@renyi.hu