

Hurewicz test sets for generalized separation and reduction

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Abstract

We prove a Hurewicz-type theorem for generalized separation: we present a method which allows us to test if for a sequence of Borel sets $(A_i)_{i < \omega}$ satisfying $\bigcap_{i < \omega} A_i = \emptyset$ there is a sequence $(B_i)_{i < \omega}$ of Π_ξ^0 sets such that $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = \emptyset$ or not. We also prove an analogous result for generalized reduction. The results of the paper are motivated by a Hurewicz-type theorem of A. Louveau and J. Saint Raymond on ordinary separation of analytic sets.



1. *Introduction*

It is well known that the analytic sets have the separation property, i.e. a pair of disjoint analytic sets can be separated by a Borel set. If we need to estimate the Borel class of this separating set then the following theorem is very useful (see [2]); $(2^\omega, \tau_{2^\omega})$ stands for the Cantor set with its usual product topology.

THEOREM 1. *(A. Louveau, J. Saint Raymond) Let $3 \leq \xi < \omega_1$ and (X, τ) be a Polish space. If $P_\xi \subseteq 2^\omega$ is $\Pi_\xi^0(\tau_{2^\omega})$ but not $\Sigma_\xi^0(\tau_{2^\omega})$ and $A_0, A_1 \subseteq X$ is any pair of disjoint analytic sets, then either A_0 can be separated from A_1 by a $\Sigma_\xi^0(\tau)$ set or there is a continuous one-to-one map $\varphi: (2^\omega, \tau_{2^\omega}) \rightarrow X$ with $\varphi(P_\xi) \subseteq A_0$ and $\varphi(2^\omega \setminus P_\xi) \subseteq A_1$.*

The same conclusion holds for $\xi = 2$ if $P_2 \subseteq 2^\omega$ is the complement of a dense countable set.

Thus for our fixed $0 < \xi < \omega_1$, either A_0 can be separated from A_1 by a Σ_ξ^0 set, and this property can be witnessed by the separating set itself; or A_0 cannot be separated from A_1 by a Σ_ξ^0 set, a property which is a priori unwitnessable but from Theorem 1 we get that in this case the inclusions $\varphi(P_\xi) \subseteq A_0$, $\varphi(2^\omega \setminus P_\xi) \subseteq A_1$ show that the separation by a Σ_ξ^0 set is impossible.

Now the analytic sets also have the generalized reduction property, that is for every sequence $(A_i)_{i < \omega}$ of analytic sets with $\bigcap_{i < \omega} A_i = \emptyset$ there is a sequence $(B_i)_{i < \omega}$ of Borel sets such that $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = \emptyset$. But of what complexity must B_i ($i < \omega$) be? That is if ξ is fixed, when can each B_i ($i < \omega$) be chosen in Π_ξ^0 ? To answer this question, we will prove in particular the following result.

THEOREM 2. *For every $0 < \xi < \omega_1$, there exists a Polish space (D_ξ, τ_{D_ξ}) homeomorphic to $(2^\omega, \tau_{2^\omega})$ and $\Delta_{\xi+1}^0(\tau_{D_\xi})$ sets $R_\xi(i) \subseteq D_\xi$ ($i < \omega$) such that the following hold.*

- (i) *If $B_i \subseteq D_\xi$ is a $\Pi_\xi^0(\tau_{D_\xi})$ set and $R_\xi(i) \subseteq B_i$ ($i < \omega$) then $\bigcap_{i < \omega} B_i \neq \emptyset$.*
- (ii) *If (X, τ) is a Polish space and $(A_i)_{i < \omega}$ is a sequence of Borel sets in X such that*

$$\bigcap_{i < \omega} A_i = \emptyset$$

(a) either there is a sequence $(B_i)_{i < \omega}$ of $\Pi_\xi^0(\tau)$ sets such that $A_i \subseteq B_i$ ($i < \omega$)

and $\bigcap_{i < \omega} B_i = \emptyset$,

(b) or there is a continuous map $\varphi: D_\xi \rightarrow X$ such that $\varphi(R_\xi(i)) \subseteq A_i$ ($i < \omega$).

Observe that if (ii)(b) holds then the map φ indeed shows that (ii)(a) fails: if $(B_i)_{i < \omega}$ is a sequence of $\Pi_\xi^0(\tau)$ sets such that $A_i \subseteq B_i \subseteq X$ ($i < \omega$) then $\bigcap_{i < \omega} \varphi^{-1}(B_i) \neq \emptyset$ follows from (i) using $R_\xi(i) \subseteq \varphi^{-1}(B_i)$ ($i < \omega$), thus $\bigcap_{i < \omega} B_i \neq \emptyset$. So this theorem has the same feature of testing as does Theorem 1. We call it Hurewicz-like because Theorem 1 for $\xi = 2$ is the so-called Hurewicz dichotomy theorem.

The concept of our approach is similar to that of [2], that is we will define games. On the other hand we do not follow the spirit of [2], that is we do not define closed games, neither will the reducing map φ in Theorem 2.(ii)(b) be one-to-one. We will have to refer to Borel determinacy, so in particular this proof is not effective. We also restrict our attention to sequences of Borel sets. We dare to accept this because in case of generalized separation for analytic sets the separating sequence of Borel sets can be obtained constructively, as done in the 2E.1 Strong Separation Theorem of [5].

We note that the problem of finding a Hurewicz test for generalized separation has an aspect which does not appear in ordinary separation: namely that for Theorem 2.(i) we have to find a property which can guarantee that the intersection of a countable family of Borel sets is nonempty. We do this using the topologized Hurewicz test sets of [4]. The results of this paper can be considered as an application of the theory initiated in [4] and developed in [3].

2. Preliminaries

Our terminology and notation follow [1]. Let ξ, ϑ_i ($i < \omega$) be ordinals. We write $\vartheta_i \rightarrow \xi$ if ξ is successor and $\vartheta_i + 1 = \xi$ ($i < \omega$) or if ξ is limit, $\vartheta_i \leq \vartheta_j$ ($i \leq j < \omega$) and $\sup_{i < \omega} \vartheta_i = \xi$.

For every ordinal $\xi < \omega_1$ we fix once and for all a sequence $(\vartheta_i)_{i < \omega}$ such that $\vartheta_i \rightarrow \xi$. To avoid complicated notations, we do not indicate the dependence of the sequence on ξ , it will be always clear which pair of ordinal and sequence is considered.

In this note we will need to refine Polish topologies by turning countably many pairwise disjoint closed sets into open sets. We do this as described in [1], that is the open sets of the ancient topology together with their portion on the members of our collection of closed sets serve as a base of the new, finer topology. We will use that the topology obtained in this way is also Polish.

DEFINITION 3. Let (X, τ) be a Polish space, $\mathcal{P} = \{P_i : i < \omega\}$ be a countable collection of pairwise disjoint $\Pi_1^0(\tau)$ sets. Then $\tau[\mathcal{P}]$ denotes that Polish topology refining τ where each P_i ($i < \omega$) is turned successively into an open set.

It is easy to see that the resulting finer topology $\tau[\mathcal{P}]$ is independent from the enumeration of \mathcal{P} . This will be clear shortly when we fix a base of $\tau[\mathcal{P}]$. We also use the notation $\tau[\mathcal{P}]$ when the countable collection of not necessarily $\Pi_1^0(\tau)$ sets \mathcal{P} can be enumerated on such a way that P_n is $\Pi_1^0(\tau[\{P_i : i < n\}])$.

We also need a precise notion of basic open sets in our spaces.

DEFINITION 4. Let (X_i, τ_i) ($i \in I$) be Polish spaces; if a basis \mathcal{G}_i is fixed in the spaces (X_i, τ_i) ($i \in I$), which are meant to be the basic open sets in (X_i, τ_i) , then the *basic*

open sets of $(\prod_{i \in I} X_i, \prod_{i \in I} \tau_i)$ are the open sets of the form $\prod_{i \in J} G_i \times \prod_{i \in I \setminus J} X_i$, where $J \subseteq I$ is finite and $G_i \in \mathcal{G}_i$ for every $i \in J$.

If the basic open sets \mathcal{G} are fixed in the Polish space (X, τ) and $\tau[\mathcal{P}]$ makes sense for a countable collection \mathcal{P} of subsets of X , then *the basic open sets* of $\tau[\mathcal{P}]$ are of the form $G \cap F$ or G with $G \in \mathcal{G}$, $F \in \mathcal{P}$.

Observe that the basic open sets defined on this way form a basis of $\prod_{i \in I} \tau_i$ and $\tau[\mathcal{P}]$, respectively. From now on whenever a Polish space (X, τ) appears we assume that a countable basis comprised of basic τ -open sets is fixed with respect to the convention of Definition 4. The closure in (X, τ) is denoted by cl_τ .

As usual, $\Pi_\xi^0(\tau)$ ($\Sigma_\xi^0(\tau)$, $\Delta_\xi^0(\tau)$ resp.) ($0 < \xi < \omega$) stands for the ξ^{th} multiplicative (additive, ambiguous resp.) Borel class in the Polish space (X, τ) , starting with $\Pi_1^0(\tau) =$ closed sets, $\Sigma_1^0(\tau) =$ open sets.

We recall some definitions related to separation.

DEFINITION 5. Let X be an arbitrary set. For $A_0, A_1 \subseteq X$ we say that $G \subseteq X$ *separates* A_0 from A_1 if we have $A_0 \subseteq G \subseteq X \setminus A_1$.

A class $\Gamma \subseteq 2^X$ has the *separation property* if for every $A_0, A_1 \in \Gamma$ there is a set $G \in \Gamma$ such that $X \setminus G \in \Gamma$ and G separates A_0 from A_1 . A class $\Gamma \subseteq 2^X$ has the *generalized separation property* if for every sequence $(A_i)_{i < \omega} \subseteq \Gamma$ with $\bigcap_{i < \omega} A_i = \emptyset$ there is a sequence $(G_i)_{i < \omega} \subseteq \Gamma$ such that $(X \setminus G_i)_{i < \omega} \subseteq \Gamma$, $A_i \subseteq G_i$ ($i < \omega$) and $\bigcap_{i < \omega} G_i = \emptyset$.

A class $\Gamma \subseteq 2^X$ has the *reduction property* if for every $A, A' \in \Gamma$ there are $B, B' \in \Gamma$ such that $B \subseteq A$, $B' \subseteq A'$, $A \cup A' = B \cup B'$ and $B \cap B' = \emptyset$.

A $\Gamma \subseteq 2^X$ has the *generalized reduction property* if for every sequence $(A_i)_{i < \omega} \subseteq \Gamma$

there is a sequence $(G_i)_{i < \omega} \subseteq \Gamma$ such that $G_i \subseteq A_i$, $G_i \cap G_j = \emptyset$ ($i, j < \omega$, $i \neq j$) and

$$\bigcup_{i < \omega} G_i = \bigcup_{i < \omega} A_i.$$

3. Spaces, sets and topologies

In our first definition we recall the spaces, the sets and the topologies of [4], Section 3 on pages 162-163.

DEFINITION 6. We set $(C_1, \tau_{C_1}) = (2^\omega, \tau_{2^\omega})$, $P_1 = \{x \in C_1 : \forall m \in \omega (x(m) = 1)\}$ and $\tau_1 = \omega$. We set $\tau_{P_1} = \tau_{C_1}$ on C_1 .

Let now $1 < \xi < \omega_1$ and suppose that the space $(C_\vartheta, \tau_{C_\vartheta})$, the set P_ϑ , the topology τ_{P_ϑ} and the ordinal τ_ϑ are defined for every $\vartheta < \xi$. Then with $\vartheta_i \rightarrow \xi$ let

$$C_\xi = \prod_{i < \omega} C_{\vartheta_i}, \quad \tau_{C_\xi} = \prod_{i < \omega} \tau_{C_{\vartheta_i}},$$

$$P_\xi = \{x \in C_\xi : \forall i < \omega (x(i, \cdot) \in C_{\vartheta_i} \setminus P_{\vartheta_i})\}, \quad (3.1)$$

$$\tau_{P_\xi}^< = \prod_{i < \omega} \tau_{P_{\vartheta_i}}, \quad \tau_\xi = \sum_{i < \omega} \tau_{\vartheta_i}. \quad (3.2)$$

Finally let $\tau_{P_\xi} = \tau_{P_\xi}^<[\{U_{\xi, n} : n < \omega\}]$ where

$$\begin{aligned} U_{\xi, n} &= \prod_{i < n} (C_{\vartheta_i} \setminus P_{\vartheta_i}) \times P_{\vartheta_n} \times \prod_{n < i < \omega} C_{\vartheta_i} \subseteq \\ &\subseteq \prod_{i < n} C_{\vartheta_i} \times C_{\xi_n} \times \prod_{n < i < \omega} C_{\vartheta_i} = C_\xi (n < \omega). \end{aligned} \quad (3.3)$$

The spaces and sets appearing in Theorem 2 are the following.

DEFINITION 7. For $\xi = 1$ let $(D_1, \tau_{D_1}) = (\omega + 1, \mathfrak{o}_{\omega+1})$ where $\mathfrak{o}_{\omega+1}$ is the order topology on $\omega + 1$. Let $\lambda_1 : \omega \times \omega \rightarrow \omega$ be the bijection defined by

$$\lambda_1(n, i) = \frac{1}{2}(i + n + 1)(i + n) + n \quad (n, i < \omega)$$

and set $R_1(n) = \{\lambda_1(n, i) : i < \omega\}$ ($n < \omega$) and $R_1 = \{\omega\}$.

For $1 < \xi < \omega_1$, let $(C_\xi(i), \tau_{C_\xi(i)})$, $\{P_\xi(i), \tau_{P_\xi(i)}\}$ ($i < \omega$) be copies of the space (C_ξ, τ_{C_ξ})

and pair $\{P_\xi, \tau_{P_\xi}\}$ defined in Definition 6. Set

$$D_\xi = \prod_{i < \omega} C_\xi(i), \quad \tau_{D_\xi} = \prod_{i < \omega} \tau_{C_\xi(i)}, \quad (3.4)$$

$$\hat{P}_\xi(n) = \prod_{i < n} C_\xi(i) \times P_\xi(n) \times \prod_{n < i < \omega} C_\xi(i), \quad (3.5)$$

$$\tau_{\hat{P}_\xi(n)} = \prod_{i < n} \tau_{P_\xi(i)}^\prec \times \tau_{P_\xi(n)} \times \prod_{n < i < \omega} \tau_{P_\xi(i)}^\prec \quad (3.6)$$

and

$$R_\xi(n) = (D_\xi \setminus \hat{P}_\xi(n)) \cap \bigcap_{i < \omega, i \neq n} \hat{P}_\xi(i) = \prod_{i < n} P_\xi(i) \times (D_\xi \setminus P_\xi(n)) \times \prod_{n < i < \omega} P_\xi(i), \quad (3.7)$$

$$R_\xi = \bigcap_{i < \omega} \hat{P}_\xi(i) = \prod_{i < \omega} P_\xi(i), \quad \tau_{R_\xi}^\prec = \prod_{i < \omega} \tau_{P_\xi(i)}^\prec. \quad (3.8)$$

We state a trivial lemma on the Borel class of the preceding sets.

LEMMA 8. *We have $C_\xi = 2^{\tau_\xi}$ ($0 < \xi < \omega_1$). For every $0 < \xi < \omega_1$, P_ξ is a $\Pi_\xi^0(\tau_{C_\xi})$ set, $R_1(n)$ is $\Delta_2^0(\tau_{D_1})$ ($n < \omega$), and if $1 < \xi < \omega_1$, $\hat{P}_\xi(n)$ is a $\Pi_\xi^0(\tau_{D_\xi})$ set and $R_\xi(n)$ is a $\Delta_{\xi+1}^0(\tau_{D_\xi})$ set ($n < \omega$).*

Proof. The first statement follow immediately from Definition 6 by induction. The second statement is Claim 7 in [4]. From this the remaining statements are automatic. \square

To close this section, we state some claims and a lemma from [4] which are the heart of the proof of the results of this paper (see [4], Claim 8 on page 163, Claim 9 on page 164 and Lemma 13.(ii) on page 165).

CLAIM 9.

- (i) *For every $1 \leq \xi < \omega_1$, $P_\xi \subseteq C_\xi$ is a τ_{P_ξ} -nowhere dense hence τ_{P_ξ} -meager $\Pi_1^0(\tau_{P_\xi})$ set.*

- (ii) For $\xi \geq 2$, $P_\xi \subseteq C_\xi$ is a $\tau_{P_\xi}^<$ -dense $\Pi_2^0(\tau_{P_\xi}^<)$ hence $\tau_{P_\xi}^<$ -residual set.
- (iii) For $\xi \geq 2$, $\tau_{P_\xi}|_{P_\xi} = \tau_{P_\xi}^<|_{P_\xi}$.

LEMMA 10. Fix $1 \leq \xi < \omega_1$. Let (Z, σ) be an arbitrary Polish space, $G \subseteq Z \times C_\xi$ be a basic $\sigma \times \tau_{P_\xi}$ -open set with $G \cap (Z \times P_\xi) \neq \emptyset$. If for a set $W \in \Sigma_\xi^0(\sigma \times \tau_{C_\xi})$, $W \cap (Z \times P_\xi)$ is $\sigma \times (\tau_{P_\xi}|_{P_\xi})$ -residual relatively in $G \cap (Z \times P_\xi)$, then W is $\sigma \times \tau_{P_\xi}$ -residual in a $\sigma \times \tau_{P_\xi}$ -open set $H \subseteq Z \times C_\xi$ satisfying that $G \cap (Z \times P_\xi)$ is contained in the $\sigma \times \tau_{P_\xi}$ -closure of H .

4. Testing generalized separation and reduction

We aim to prove the following three results. The first says that $R_\xi(i)$ ($i < \omega$) has the property we need.

CLAIM 11. Let $0 < \xi < \omega_1$ be fixed and suppose that $B_i \subseteq D_\xi$ is a $\Pi_\xi^0(\tau_{D_\xi})$ set and $R_\xi(i) \subseteq B_i$ ($i < \omega$). Then $R_\xi \cap \bigcap_{i < \omega} B_i \neq \emptyset$.

The second generalizes slightly Theorem 2.

THEOREM 12. Let $0 < \xi < \omega_1$. In the Polish space (X, τ) let $(A_i)_{i < \omega}$ be a sequence of Borel sets. Then

- (i) either there is a sequence $(B_i)_{i < \omega}$ of $\Pi_\xi^0(\tau)$ set such that $A_i \subseteq B_i$ ($i < \omega$) and

$$\bigcap_{i < \omega} B_i = \bigcap_{i < \omega} A_i,$$

- (ii) or there is a continuous map $\varphi: D_\xi \rightarrow X$ such that $\varphi(R_\xi) \cap \bigcap_{i < \omega} A_i = \emptyset$ and $\varphi(R_\xi(i)) \subseteq A_i$ ($i < \omega$).

The third handles the problem of generalized reduction.

THEOREM 13. Let $0 < \xi < \omega_1$. In the Polish space (X, τ) let $(A_i)_{i < \omega}$ be a sequence of Borel sets. Then

- (i) either there is a sequence $(B_i)_{i < \omega}$ of $\Sigma_\xi^0(\tau)$ set such that $B_i \subseteq A_i$, $B_i \cap B_j = \emptyset$ ($i, j < \omega$, $i \neq j$) and $\bigcup_{i < \omega} B_i = \bigcup_{i < \omega} A_i$,
- (ii) or there is a continuous map $\varphi: D_\xi \rightarrow X$ such that $\varphi(R_\xi) \subseteq \bigcup_{i < \omega} A_i$ and $\varphi(R_\xi(i)) \cap A_i = \emptyset$. ($i < \omega$).

Let us discuss first these results. Theorem 2 is the $\bigcap_{i < \omega} A_i = \emptyset$ case of Theorem 12 together with Claim 11.

Next observe that in Theorem 12, (ii) excludes (i) (as (ii)(b) excluded (ii)(a) in Theorem 2), as follows. If a map φ satisfies the conditions of (ii) then by Claim 11, $R_\xi \cap \bigcap_{i < \omega} \varphi^{-1}(B_i) \neq \emptyset$. This implies $\varphi(R_\xi) \cap \bigcap_{i < \omega} B_i \neq \emptyset$ so since $\varphi(R_\xi) \subseteq X \setminus \bigcap_{i < \omega} A_i$, we have $\bigcap_{i < \omega} B_i \neq \bigcap_{i < \omega} A_i$. So Theorem 12 also provides a kind of Hurewicz test.

Note also that Theorem 13 follows from Theorem 12 by taking complements.

Proof of Theorem 13. By repeating the previous argument, one easily obtains that by Claim 11, (ii) excludes (i). Set $A'_i = X \setminus A_i$ ($i < \omega$). By Theorem 12, either there is a sequence $(B'_i)_{i < \omega}$ of $\Pi_\xi^0(\tau)$ set such that $A'_i \subseteq B'_i$ and $\bigcap_{i < \omega} A'_i = \bigcap_{i < \omega} B'_i$ or there is a continuous map $\varphi: D_\xi \rightarrow X$ such that $\varphi(R_\xi) \cap \bigcap_{i < \omega} A'_i = \emptyset$ and $\varphi(R_\xi(i)) \subseteq A'_i$ ($i < \omega$).

In the first case let $B''_i = X \setminus B'_i$ ($i < \omega$). The class Σ_ξ^0 has the generalized reduction property in Polish spaces (see e.g. [1], (22.16) Theorem on page 172) so there is a $\Sigma_\xi^0(\tau)$ set $B_i \subseteq B''_i$ ($i < \omega$) such that $B_i \cap B_j = \emptyset$ ($i, j < \omega$, $i \neq j$) and $\bigcup_{i < \omega} B_i = \bigcup_{i < \omega} B''_i = \bigcup_{i < \omega} A_i$; which is (i).

In the second case we have $\varphi(R_\xi(i)) \cap A_i = \emptyset$ ($i < \omega$) and $\varphi(R_\xi) \subseteq X \setminus \bigcap_{i < \omega} A_i = \bigcup_{i < \omega} A_i$; which is (ii). So the proof is complete. \square

4.1. *Why Π_ξ^0 ?*

It remains to prove Claim 11 and Theorem 12. Before doing this we feel obliged to clarify why the testing of generalized separation is set up only for Π_ξ^0 sets and not for Σ_ξ^0 sets. We have three reasons for this, the first is a corollary of Theorem 2.

COROLLARY 14. *If in Theorem 2 case (ii)(a) holds the sequence $(B_i)_{i < \omega}$ can be chosen such that $B_i \in \Delta_\xi^0(\tau)$ ($i < \omega$).*

Proof. Since the class Π_ξ^0 has the generalized separation property in Polish spaces (see e.g. [1], (22.16) Theorem on page 172), the statement follows. \square

That is generalized separation with Π_ξ^0 sets incorporates generalized separation with Σ_ξ^0 sets. The following result says that in a particular case Theorem 1 gives an answer to the problem with Σ_ξ^0 sets.

CLAIM 15. *Let $A_0, A_1 \subseteq X$ be disjoint analytic sets. Then there are disjoint $\Sigma_\xi^0(\tau)$ sets $B_0, B_1 \subseteq X$ such that $A_i \subseteq B_i$ ($i = 0, 1$) if and only if A_0 can be separated from A_1 by a $\Sigma_\xi^0(\tau)$ set and vice versa, A_1 can be separated from A_0 by a $\Sigma_\xi^0(\tau)$ set.*

Proof. If we have the disjoint $\Sigma_\xi^0(\tau)$ sets $B_0, B_1 \subseteq X$ such that $A_i \subseteq B_i$ ($i = 0, 1$) then the separation is obviously possible in both directions. Suppose now that a $\Sigma_\xi^0(\tau)$ set B'_i separates A_i from A_{1-i} ($i = 0, 1$). The class Σ_ξ^0 has the reduction property in Polish spaces (see again [1], (22.16) Theorem on page 172), thus we have disjoint $\Sigma_\xi^0(\tau)$ sets $B_0, B_1 \subseteq X$ such that $B_i \subseteq B'_i$ ($i = 0, 1$) and $B_0 \cup B_1 = B'_0 \cup B'_1$. Since $B'_i \cap A_{1-i} = \emptyset$ ($i = 0, 1$), we have $B_i \cap A_{1-i} = \emptyset$ ($i = 0, 1$). But $A_0 \cup A_1 \subseteq B'_0 \cup B'_1 = B_0 \cup B_1$, so we get $A_i \subseteq B_i$ ($i = 0, 1$). The proof is complete. \square

The next claim is simply trivial.

CLAIM 16. Let $0 < \xi < \omega_1$ be fixed and let $\vartheta_i \rightarrow \xi$. Let $(A_i)_{i < \omega}$ be a decreasing sequence of analytic sets in X such that $\bigcap_{i < \omega} A_i = \emptyset$. If there is a sequence $(B_i)_{i < \omega}$ of $\Pi_\xi^0(\tau)$ sets such that $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = \emptyset$ there is a sequence $(B'_i)_{i < \omega}$ satisfying $A_i \subseteq B'_i$ ($i < \omega$) and $\bigcap_{i < \omega} B'_i = \emptyset$ such that B'_i is $\Sigma_{\vartheta_i}^0(\tau)$ ($i < \omega$).

Proof. Write $B_i = \bigcap_{j < \omega} B_i(j)$ where $B_i(j)$ is $\Sigma_{\vartheta_j}^0(\tau)$ ($j < \omega$) and $B_i(j) \subseteq B_i(k)$ ($k \leq j < \omega, i < \omega$). Set $B'_i = \bigcap_{j \leq i} B_j(i)$, then B'_i is $\Sigma_{\vartheta_i}^0(\tau)$ ($i < \omega$). Since $A_i \subseteq A_j \subseteq B_j \subseteq B_j(i)$ ($j \leq i < \omega$) we have $A_i \subseteq B'_i$ ($i < \omega$). We also have $\bigcap_{i < \omega} B'_i = \bigcap_{i, j < \omega} B_j(i) = \bigcap_{i < \omega} B_i = \emptyset$, which completes the proof. \square

Claim 15 and Claim 16 handle the two extreme situations in generalized separation: when we have only two sets which are disjoint and when we have infinitely many sets which are decreasing. The cases in-between can be handled with similar tricks but the separation conditions become numerous and complicated. For reduction by Π_ξ^0 sets the same remarks apply as for separation by Σ_ξ^0 sets. So we leave this and turn to the proof of Claim 11. We start with a claim.

CLAIM 17. Let $1 < \xi < \omega_1$ be fixed. Then

- (i) R_ξ is $\tau_{R_\xi}^<$ -residual in D_ξ ;
- (ii) $R_\xi(i)$ is $\tau_{P_\xi(i)}$ -residual in D_ξ ($i < \omega$);
- (iii) if $B \subseteq D_\xi$ is $\Pi_\xi^0(\tau_{D_\xi})$ and it is $\tau_{P_\xi(i)}$ -residual in D_ξ for some $i < \omega$ it is $\tau_{R_\xi}^<$ -residual in D_ξ .

Proof. By Claim 9.(ii), $P_\xi(i)$ is $\tau_{P_\xi(i)}^<$ -residual in $C_\xi(i)$ ($i < \omega$). Thus by (3.4) and (3.8), R_ξ is $\tau_{R_\xi}^<$ -residual in D_ξ , which proves (i).

For (ii) fix an $i < \omega$. By Claim 9.(i) we have that $P_\xi(i)$ is $\tau_{P_\xi(i)}$ -nowhere dense in $C_\xi(i)$

hence $C_\xi(i) \setminus P_\xi(i)$ is $\tau_{P_\xi(i)}$ -residual in $C_\xi(i)$. As we have seen in the proof of (i), $P_\xi(i)$ is $\tau_{P_\xi(i)}^\prec$ -residual in $C_\xi(i)$ ($i < \omega$) so by (3.4), (3.6) and (3.7), $R_\xi(i)$ is $\tau_{P_\xi(i)}$ -residual in D_ξ .

For (iii) fix an $i < \omega$ and let $B \subseteq D_\xi$ be an arbitrary $\Pi_\xi^0(\tau_{D_\xi})$ set which is $\tau_{P_\xi(i)}$ -residual in D_ξ . Suppose that B is not $\tau_{R_\xi}^\prec$ -residual in D_ξ , that is for a nonempty basic $\tau_{R_\xi}^\prec$ -open set $G \subseteq D_\xi$ we have that $W = D_\xi \setminus B$ is residual in G . Set

$$(Z, \sigma) = \left(\prod_{j < \omega, j \neq i} C_\xi(j), \prod_{j < \omega, j \neq i} \tau_{P_\xi(j)}^\prec \right),$$

then we have $D_\xi = Z \times C_\xi(i)$, $\tau_{R_\xi}^\prec = \sigma \times \tau_{P_\xi(i)}^\prec$, $\tau_{P_\xi(i)} = \sigma \times \tau_{P_\xi(i)}$, and since $\sigma \times \tau_{C_\xi(i)}$ refines τ_{D_ξ} we have that W is $\Sigma_\xi^0(\sigma \times \tau_{C_\xi(i)})$. Moreover by Claim 9.(iii),

$$\tau_{R_\xi}^\prec|_{Z \times P_\xi(i)} = \sigma \times \left(\tau_{P_\xi(i)}^\prec|_{P_\xi(i)} \right) = \sigma \times \left(\tau_{P_\xi(i)}|_{P_\xi(i)} \right) = \tau_{P_\xi(i)}|_{Z \times P_\xi(i)}.$$

Thus by Claim 9.(ii), $Z \times P_\xi$ is a $\tau_{R_\xi}^\prec$ -residual $\Pi_2^0(\tau_{R_\xi}^\prec)$ set in D_ξ , in particular $G \cap (Z \times P_\xi(i)) \neq \emptyset$ and $W \cap (Z \times P_\xi(i))$ is $\sigma \times (\tau_{P_\xi(i)}|_{P_\xi(i)})$ -residual in $G \cap (Z \times P_\xi(i))$. So we can apply Lemma 10 and we obtain that W is of $\sigma \times \tau_{P_\xi(i)}$ -second category, that is of $\tau_{P_\xi(i)}$ -second category in D_ξ . Since B is $\tau_{P_\xi(i)}$ -residual in D_ξ , this contradicts $B \cap W = \emptyset$. The proof is complete. \square

Proof of Claim 11. Consider first $\xi = 1$. If B_i is $\Pi_1^0(\tau_{D_1})$ then $\omega \in \text{cl}_{\tau_{D_1}}(R_1(i)) \subseteq B_i$ ($i < \omega$) so indeed $\omega \in R_1 \cap \bigcap_{i < \omega} B_i$.

Let now $1 < \xi < \omega_1$. By Claim 17.(i), R_ξ is $\tau_{R_\xi}^\prec$ -residual in D_ξ . We show that B_i ($i < \omega$) is also $\tau_{R_\xi}^\prec$ -residual in D_ξ , this will complete the proof.

Fix some $i < \omega$. By Claim 17.(ii), $R_\xi(i)$ is $\tau_{P_\xi(i)}$ -residual in D_ξ . Thus by $R_\xi(i) \subseteq B_i$, B_i is also $\tau_{P_\xi(i)}$ -residual in D_ξ . Since B_i is $\Pi_\xi^0(\tau_{D_\xi})$, Claim 17.(iii) can be applied and gives that B_i is $\tau_{R_\xi}^\prec$ -residual in D_ξ , as stated. \square

4.2. *The game*

The next task is to find the appropriate image of $R_\xi(n)$ ($n < \omega$) if generalized separation by Π_ξ^0 sets is not possible. Fix a $1 < \xi < \omega_1$. Let $\lambda_\xi: \omega \rightarrow \omega \cdot \tau_\xi$ be a bijection. By (3.4) and Lemma 8 we have $D_\xi = (2^{\tau_\xi})^\omega$; so there is a homeomorphism $\Lambda_\xi: (2^\omega, \tau_{2^\omega}) \rightarrow (D_\xi, \tau_{D_\xi})$ corresponding to λ_ξ . We define the Borel game \mathfrak{G}_ξ we need in the proof of Theorem 12.

Let $A_i, G, H \subseteq D_\xi$ be Borel sets such that $A_i \subseteq G$ ($i < \omega$). In the game

$$\mathfrak{G}_\xi((A_i)_{i < \omega}, G, H)$$

players I and II play

$$\begin{array}{ccc} \text{I} & \alpha(0) & \alpha(1) \\ & & \dots \\ & \beta(0) & \beta(1) \\ \text{II} & & \end{array}$$

where $\alpha(i), \beta(i) \in \{0, 1\}$ ($i < \omega$), and II wins if and only if

$$\Lambda_\xi(\beta) \in G \text{ and } \begin{cases} \Lambda_\xi(\alpha) \in R_\xi(i) & \implies \Lambda_\xi(\beta) \in A_i \ (i < \omega), \\ \Lambda_\xi(\alpha) \in R_\xi & \implies \Lambda_\xi(\beta) \notin H; \end{cases}$$

else I wins.

We associate maps to the strategies of I.

DEFINITION 18. Fix a $1 < \xi < \omega_1$ and let σ be a strategy of I in \mathfrak{G}_ξ . Then $\rho_\sigma: D_\xi \rightarrow D_\xi$ denotes the function mapping to a $y \in D_\xi$ the unique $x \in D_\xi$ such that the run $\{\Lambda_\xi^{-1}(x), \Lambda_\xi^{-1}(y)\}$ is according to σ .

Note that ρ_σ is continuous. In the sequel we will use this property in the form that ρ_σ^{-1} keeps the Borel class of sets.

By the following claim the determinacy of the game \mathfrak{G}_ξ indeed proves Theorem 12.

CLAIM 19. *For every $1 < \xi < \omega_1$, in the game \mathfrak{G}_ξ a winning strategy*

(i) *for player I gives a sequence $(B_i)_{i < \omega}$ of $\Pi_\xi^0(\tau_{D_\xi})$ sets such that $A_i \subseteq B_i$ ($i < \omega$)*

and $(G \setminus H) \cap \bigcap_{i < \omega} B_i = \emptyset$;

(ii) *for player II gives a continuous map $\varphi: D_\xi \rightarrow G$ such that $\varphi(R_\xi) \cap H = \emptyset$ and*

$\varphi(R_\xi(i)) \subseteq A_i$ ($i < \omega$).

Proof. Consider first a winning strategy σ for I. Let $B_i = \rho_\sigma^{-1}(\hat{P}_\xi(i))$ ($i < \omega$). By Lemma 8, B_i is $\Pi_\xi^0(\tau_{D_\xi})$ ($i < \omega$). So it remains to show that $A_i \subseteq B_i$ ($i < \omega$) and that $(G \setminus H) \cap \bigcap_{i < \omega} B_i = \emptyset$.

Suppose that for some $i < \omega$ we have a $y \in A_i \setminus B_i$, that is $y \in A_i \setminus \rho_\sigma^{-1}(\hat{P}_\xi(i))$. Since $A_i \setminus \rho_\sigma^{-1}(\hat{P}_\xi(i)) = A_i \cap \rho_\sigma^{-1}(D_\xi \setminus \hat{P}_\xi(i))$, there is some $x \in D_\xi \setminus \hat{P}_\xi(i)$ such that the run $\{\Lambda_\xi^{-1}(x), \Lambda_\xi^{-1}(y)\}$ in \mathfrak{G}_ξ is according to σ . We have

$$D_\xi \setminus \hat{P}_\xi(i) = R_\xi(i) \cup \bigcup_{j < \omega, j \neq i} \left(D_\xi \setminus \left(\hat{P}_\xi(j) \cup \hat{P}_\xi(i) \right) \right).$$

But if $x \in R_\xi(i)$ then II wins since $y \in A_i \subseteq G$; while if $x \notin R_\xi(i)$ and

$$x \in \bigcup_{j < \omega, j \neq i} \left(D_\xi \setminus \left(\hat{P}_\xi(j) \cup \hat{P}_\xi(i) \right) \right)$$

then $x \notin R_\xi(j)$ ($j < \omega$) and $x \notin R_\xi$ so II wins already by $y \in G$. This contradicts the definition of σ and proves that $A_i \subseteq B_i$ ($i < \omega$).

Suppose now that $y \in (G \setminus H) \cap \bigcap_{i < \omega} B_i$, that is $y \in G \setminus H$ and

$$y \in \rho_\sigma^{-1}(\hat{P}_\xi(i)) \quad (i < \omega). \tag{4.1}$$

Let $x \in D_\xi$ be arbitrary such that the run $\{\Lambda_\xi^{-1}(x), \Lambda_\xi^{-1}(y)\}$ in \mathfrak{G}_ξ is according to σ .

Then by (4.1), $x \in \bigcap_{i < \omega} \hat{P}_\xi(i)$ so $x \notin R_\xi(j)$ ($j < \omega$). Thus II wins by $y \in G \setminus H$, which again contradicts the choice of σ and proves $(G \setminus H) \cap \bigcap_{i < \omega} B_i = \emptyset$.

Let now σ be a winning strategy of II. Let $\varphi: D_\xi \rightarrow D_\xi$ map to an $x \in D_\xi$ the unique $y \in D_\xi$ such that the run $\{\Lambda_\xi^{-1}(x), \Lambda_\xi^{-1}(y)\}$ in \mathfrak{G}_ξ is according to σ . This map is clearly continuous. We have $\varphi(D_\xi) \subseteq G$, $\varphi(R_\xi) \cap H = \emptyset$ and if $i < \omega$ and $x \in R_\xi(i)$ then $\varphi(x) \in A_i$ since the strategy σ is winning for II. This completes the proof. \square

4.3. *Passing to arbitrary Polish spaces*

From this Theorem 12 will follow by standard arguments (see [2], Corollary 2 and Theorem 3), that we give in the remaining part of the paper. We have to extend the results to arbitrary Polish spaces; for this we need the following result (see e.g. [2], Theorem on page 455).

THEOREM 20. *(Saint Raymond) Let $(E, \tau_E), (F, \tau_F)$ be compact metrizable spaces and (Q, τ_Q) be a Polish space. If $h: E \rightarrow F$ is a continuous surjection and $f: E \rightarrow Q$ is a Baire-1 function then h has a Baire-1 section (i.e. there is a Baire-1 function $s: F \rightarrow E$ such that $h(s(y)) = y$ ($y \in F$)) such that $f \circ s: F \rightarrow Q$ is also a Baire-1 function.*

This theorem is applicable for a class of sets Γ if this Γ can be built up from Σ_2^0 sets on some canonical way.

DEFINITION 21. Let $D \subseteq 2^\omega$ and let $(C(n))_{n < \omega}$ be a sequence of arbitrary subsets of some base set X . The the *Hausdorff operation associated to D* acts on the sequence $(C(n))_{n < \omega}$ as $D((C(n))_{n < \omega}) = \{x \in X: \{n < \omega: x \in C(n)\} \in D\}$.

A class Γ is called a Σ_2^0 *generated Hausdorff class* if there is a basis $D \subseteq 2^\omega$ such that in every Polish space (X, τ) the members of Γ are the sets of the form $D((C(n))_{n < \omega})$ where $(C(n))_{n < \omega} \subseteq \Sigma_2^0(\tau)$.

We recall that each class Σ_ξ^0 ($1 < \xi < \omega_1$) is a Σ_2^0 generated Hausdorff class (see e.g. [1], (23.5) Exercise on page 180.) The following corollary is a variant of the so-called transfer lemma (for a version, see e.g. [2], Corollary 2 on page 455). It will allow us to extend our results to arbitrary compact spaces.

COROLLARY 22. *Let (E, τ_E) and (F, τ_F) be compact metrizable spaces, $h: E \rightarrow F$ be a continuous surjection and Γ be a Σ_2^0 generated Hausdorff class. Suppose that $(A_i)_{i < \omega}$ is a sequence of subsets of F , $T \subseteq F$ and there is a sequence $(H_i)_{i < \omega} \subseteq \Gamma$ of subsets of E satisfying $h^{-1}(A_i) \cap H_i = \emptyset$ ($i < \omega$) and $\bigcup_{i < \omega} H_i = h^{-1}(T)$. Then there is a sequence $(B_i)_{i < \omega} \subseteq \Gamma$ of subsets of F satisfying $A_i \cap B_i = \emptyset$ ($i < \omega$) and $\bigcup_{i < \omega} B_i = T$.*

Proof. Let $H_i = D((C_i(n))_{n < \omega})$ where D is a basis for Γ and $C_i(n)$ is $\Sigma_2^0(\tau_E)$ ($i, n < \omega$). Let $f_{i,n}: E \rightarrow [0, 1]$ be a Baire-1 function such that $C_i(n) = \{x \in E: f_{i,n}(x) > 0\}$ ($i, n < \omega$). Set $f: E \rightarrow [0, 1]^{\omega \times \omega}$, $f = \prod_{i,n < \omega} f_{i,n}$. From Theorem 20 for h and f we get a Baire-1 section s of h such that $f \circ s$ is also of Baire-1, that is $V_i(n) = \{y \in F: f_{i,n}(s(y)) > 0\}$ is a $\Sigma_2^0(\tau_F)$ set. Thus for $B_i = D((V_i(n))_{n < \omega})$ we have $B_i \in \Gamma$ ($i < \omega$). It is clear that $s^{-1}(H_i) = B_i$ ($i < \omega$). Since $s^{-1}(h^{-1}(T)) = T$ and $\bigcup_{i < \omega} H_i = h^{-1}(T)$, we obtain $\bigcup_{i < \omega} B_i = T$. From $s^{-1}(H_i) = B_i$ we have $s(B_i) \subseteq H_i$, which gives $B_i = h(s(B_i)) \subseteq h(H_i)$ ($i < \omega$). Since $H_i \cap h^{-1}(A_i) = \emptyset$ implies $h(H_i) \cap A_i = \emptyset$ ($i < \omega$), we have $B_i \cap A_i = \emptyset$ ($i < \omega$) which completes the proof. \square

It remains to finish with the proof of Theorem 12.

Proof of Theorem 12. Set $H = \bigcap_{i < \omega} A_i$, and let first $\xi = 1$. If $\bigcap_{i < \omega} \text{cl}_\tau(A_i) = H$ we have (i). Else let $x_\omega \in \bigcap_{i < \omega} \text{cl}_\tau(A_i) \setminus H$. Fix a metric d on X generating the topology τ .

Since $x_\omega \in \text{cl}_\tau(A_i)$ ($i < \omega$), we have

$$A_i \cap B_d\left(x_\omega, \frac{1}{n+i}\right) \neq \emptyset \quad (i, n < \omega)$$

where $B_d(x_\omega, 1/n)$ is the open ball around x_ω with radius $1/n$ ($n < \omega$). So we can pick points $x_{\lambda_1(i,n)} \in A_i \cap B_d(x_\omega, 1/(n+i))$ ($i, n < \omega$). Let $\varphi: D_1 \rightarrow X$ be defined by $\varphi(i) = x_i$ ($i \leq \omega$). This map is clearly continuous, $\varphi(R_1(i)) \subseteq A_i$ ($i < \omega$) and $\varphi(R_1) \notin H$, as required.

Let now $1 < \xi < \omega_1$. Let $(\hat{X}, \hat{\tau})$ be a Polish compactification of (X, τ) (see e.g. [1], (4.14) Theorem); then $X \subseteq \hat{X}$ is a $\Pi_2^0(\hat{\tau})$ set. Take a continuous surjection $h: D_\xi \rightarrow \hat{X}$ and consider the game $\mathfrak{G}_\xi((h^{-1}(A_i))_{i < \omega}, h^{-1}(X), h^{-1}(H))$. This game is clearly Borel so by Borel determinacy either player I or player II has a winning strategy.

If player I has a winning strategy then by Claim 19.(i) we have a sequence $(U_i)_{i < \omega}$ of $\Pi_\xi^0(\tau_{D_\xi})$ sets such that $h^{-1}(A_i) \subseteq U_i$ ($i < \omega$) and

$$(h^{-1}(X) \setminus h^{-1}(H)) \cap \bigcap_{i < \omega} U_i = \emptyset.$$

That is by $h^{-1}(H) \subseteq h^{-1}(A_i) \subseteq h^{-1}(X)$ ($i < \omega$) we have

$$h^{-1}(X) \cap \bigcap_{i < \omega} U_i = h^{-1}(H). \tag{4.2}$$

Let $(E, \tau_E) = (D_\xi, \tau_{D_\xi})$, $(F, \tau_F) = (\hat{X}, \hat{\tau})$, $\Gamma = \Sigma_\xi^0$, $T = \hat{X} \setminus H$ and $H_i = D_\xi \setminus (h^{-1}(X) \cap U_i)$ ($i < \omega$). Since $h^{-1}(A_i) \subseteq h^{-1}(X) \cap U_i$ implies $h^{-1}(A_i) \cap H_i = \emptyset$ ($i < \omega$) and

$$\bigcup_{i < \omega} H_i = D_\xi \setminus \left(h^{-1}(X) \cap \bigcap_{i < \omega} U_i \right) = D_\xi \setminus h^{-1}(H) = h^{-1}(T)$$

by (4.2), we can apply Corollary 22 to get a sequence $(V_i)_{i < \omega}$ of $\Sigma_\xi^0(\hat{\tau})$ sets such that $A_i \cap V_i = \emptyset$ and $\bigcup_{i < \omega} V_i = \hat{X} \setminus H$. Then $B_i = X \setminus V_i$ is $\Pi_\xi^0(\tau)$, $A_i \subseteq B_i$ ($i < \omega$) and $\bigcap_{i < \omega} B_i = H$, as required.

If Player II has a winning strategy by Claim 19.(ii) we have a continuous map $\psi: D_\xi \rightarrow$

$h^{-1}(X)$ such that $\psi(R_\xi) \cap h^{-1}(H) = \emptyset$ and $\psi(R_\xi(i)) \subseteq h^{-1}(A_i)$ ($i < \omega$). Then for $\varphi = h \circ \psi$, $\varphi: D_\xi \rightarrow X$ is also continuous and satisfies $\varphi(R_\xi) \cap H = \emptyset$, $\varphi(R_\xi(i)) \subseteq A_i$ ($i < \omega$). This completes proof. \square

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