Tamás Mátrai,* Department of Mathematics, Central European University, Budapest, Nádor utca 9., H-1051 Hungary (e-mail: matrait@renyi.hu)

The graph of Gâteaux derivatives is $w^\star$-connected

Abstract

We show that if $(X, \|.\|)$ is a separable Banach space, $\Omega \subset X$ is open, connected and $f : \Omega \to \mathbb{R}$ is an everywhere Gâteaux differentiable Lipschitz continuous function, then the graph of the derivative of f is connected in $(\Omega, \|.\|) \times (X^\star, w^\star)$.

1 Introduction

As a generalization of classical Darboux property, J. Malý has proved ([1], Theorem 1, page 168) that the range of the derivative of a Fréchet differentiable function is connected in $X^\star$ endowed with the norm topology:

Theorem 1.1. *Let f be a Fréchet differentiable function defined on an open subset D of the Banach space X. Then for any closed convex set $K \subset D$ with nonempty interior, $f'(K)$ is a connected subset of $(X^\star, \|.\|_{X^\star})$.*

On the other hand, a result of R. Deville and P. Hájek ([5], Theorem 1) shows that the above mentioned theorem of J. Malý does not hold if the condition of Fréchet differentiability is weakened to Gâteaux differentiability (for more details and other related results see the remarks at the end of this note). In view of this fact, D. Azagra asked whether it is true that the range of Gâteaux derivatives is connected at least if $X^\star$ is endowed with the $w^\star$ topology. We answer his question in positive:

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Theorem 1.2. *Let* $(X, \|.\|)$ *be a separable Banach space,* $\Omega \subset X$ *be open, connected and let* $f : \Omega \to \mathbb{R}$ *be an everywhere Gâteaux differentiable locally Lipschitz function. Then the graph of the derivative,*

$$Graph(f') = \{(x, f'(x)) : x \in \Omega\} \subset (\Omega, \|.\|) \times (X^\star, w^\star)$$

is connected.

Our reference for the basic notions concerning differentiability is [7], for topological notions we refer to [9] and [7].

For a Banach space X, $B_X(x, r)$ denotes the open ball in X centered at x with radius r. For the unit ball we write B_X. We denote the norm in $X^\star$ by $\|.\|_{X^\star}$.

The symbol cl_X stands always for norm closure in X, while ∂_X indicates the corresponding boundary.

For the value at $v \in X$ of a functional $x^\star \in X^\star$ we use $x^\star(v)$.

We denote by $\mathbb{N}$ the set of nonnegative integers.

F_σ stands for the sets which can be obtained as countable union of closed sets.

2 Proof of Theorem

We will use the following two well-known theorems. For the proof of the first, see [9], Volume 1., Chapter II., §27, page 301., while the second can be found in [7], Chapter 3., Proposition 62, page 56 and in [8], 1.9.37.

Theorem 2.1. *Let* h *be a function of the first Baire class between metric spaces, that is so that the inverse image of open sets under* h *is* F_σ. *Then the set of discontinuity points of* h *is of first category.*

Theorem 2.2. *If the Banach space* X *is separable, then* $(B_{X^\star}, w^\star)$ *is compact, metrizable, so specially separable.*

The new element of our proof is contained in the following result.

Theorem 2.3. *Let* $(X, \|.\|)$ *be a separable Banach space,* $\Omega \subset X$ *be open, connected and let* $F : \Omega \to (X^\star, w^\star)$ *be a function with the following properties:*

1. F *is of the first Baire class;*
2. *if* V *is a finite dimensional linear subspace of* X, $w \in V$, $r > 0$ *satisfying* $cl_V B_V(w, r) \subset \Omega$, *then for every* $u \in \partial_V B_V(w, r)$ *we have* $F(u) \in cl_{V^\star} F(B_V(w, r))$.

Then $Graph(F) \subset (\Omega, \|.\|) \times (X^\star, w^\star)$ *is connected.*

The connection between Theorem 1.2 and Theorem 2.3 is established in the following two lemmas.

Lemma 2.4. *Let $(X, \|.\|)$ be a separable Banach space, $\Omega \subset X$ open and $f : \Omega \to \mathbb{R}$ an everywhere Gâteaux differentiable locally Lipschitz function. Then $f' : (\Omega, \|.\|) \to (X^\star, w^\star)$ is of the first Baire class.*

Lemma 2.5. *Let V be a finite dimensional Banach space, $\Omega \subset V$ open. Let $w \in V$, $r > 0$ satisfying $cl_V B_V(w,r) \subset \Omega$, and let $u \in \partial_V B_V(w,r)$. Suppose that $f : \Omega \to \mathbb{R}$ is everywhere Fréchet differentiable. Then $f'(u) \in cl_{V^\star} f'(B_V(w,r))$.*

Before giving the proofs we note that Lemma 2.5 is a straightforward corollary of Theorem 1.1. A more general statement has been observed by R. Deville and P. Hájek ([5], Proposition on page 2.) where a proof based on the above mentioned result of Malý was given. In the form as stated below, this result is a very easy consequence of Ekeland's variational principle (see [1], Lemma 2, page 168). However, since this independent proof would be mainly technical, we give only a proof based on Theorem 1.1.

Proof of Lemma 2.5 Take an $x \in B_V(w,r)$ and $\rho < r$ such that $\partial_V B_V(x,\rho) \cap \partial_V B_V(w,r) = \{u\}$. Then Theorem 1.1 for $K = \mathrm{cl}_V B_V(x,\rho)$ gives that $f'(\mathrm{cl}_V B_V(x,\rho))$ is connected, specially

$$f'(u) \in \mathrm{cl}_{V^\star} f'(K \setminus \{u\}) \subset \mathrm{cl}_{V^\star} f'(B_V(w,r)).$$

This proves the statement.■

Proof of Lemma 2.4 We have to show that whenever $B \subset (X^\star, w^\star)$ is open, $\{x \in \Omega : f'(x) \in B\}$ is F_σ.

Suppose first that f is Lipschitz with constant L. We can assume that $L < 1$. From the continuity of f we have that for every direction $w \in \partial_X B_X$, closed set $J \subset \mathbb{R}$ and real $T > 0$ the set

$$A_{w,J,T} = \left\{ x \in \Omega : \frac{f(x+tw) - f(x)}{t} \in J,\ 0 < |t| \le T \right\}$$

is relatively closed.

Let now $I \subset \mathbb{R}$ be an open interval. Since f is Gâteaux differentiable everywhere, whenever $J_i \subset I$, $i \in \mathbb{N}$ is a sequence of closed intervals with $J_i \subset J_k$ for $i \le k$ and $\bigcup_{i\in\mathbb{N}} J_i = I$, we have

$$\{x \in \Omega : (f'(x))(w) \in I\} = \bigcup_{i\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A_{w,J_i,\frac{1}{n+1}},$$

an F_σ set. Note that from $L < 1$ we have $f'(x) \in B_{X^\star}$ for every $x \in \Omega$.

Thus if

$$w_1, \dots, w_m \in \partial_X B_X$$

are directions and $I_1 \dots, I_m \subset \mathbb{R}$ are open intervals, for the basic open set

$$B = \{x^\star \in B_{X^\star} : x^\star(w_j) \in I_j\}$$

we have that the set

$$\{x \in \Omega : f'(x) \in B\} = \bigcap_{j=1}^{m} \{x \in \Omega : (f'(x))(w_j) \in I_j\}$$

is F_σ. Since X is separable, we have from Theorem 2.2 that $B_{X^\star}$ is metrizable and separable, so every open set in $B_{X^\star}$ is the countable union of basic open sets, which proves the statement for Lipschitz functions.

Consider now the general case. Since X is separable, there is a countable collection of open sets $\{\Omega_i : i \in \mathbb{N}\}$ such that

$$\Omega = \bigcup_{i \in \mathbb{N}} \Omega_i$$

and f is Lipschitz on Ω_i. Then for any open set $B \subset X^\star$,

$$\{x \in \Omega : f'(x) \in B\} = \bigcup_{i \in \mathbb{N}} \{x \in \Omega_i : f'(x) \in B\}.$$

From the preceding we know that $\{x \in \Omega_i : f'(x) \in B\}$ is F_σ in Ω_i, so it is F_σ in Ω, that the inverse image of open sets under f' is F_σ. This completes the proof.■

Proof of Theorem 2.3. Suppose that $\mathrm{Graph}(F)$ is not connected, that is there are disjoint open sets $A, B \subset \Omega \times X^\star$ such that $\mathrm{Graph}(F) \subset A \cup B$ and $A \cap \mathrm{Graph}(F) \neq \emptyset$, $B \cap \mathrm{Graph}(F) \neq \emptyset$. Using Pr_Ω for the projection to Ω, we set

$$\mathcal{A} = Pr_\Omega(A \cap \mathrm{Graph}(F)),$$

$$\mathcal{B} = Pr_\Omega(B \cap \mathrm{Graph}(F)).$$

The sets $\mathcal{A}$ and $\mathcal{B}$ are nonempty disjoint subsets of the connected set Ω, so $P = \mathrm{cl}_X \mathcal{A} \cap \mathrm{cl}_X \mathcal{B} \cap \Omega$ is nonempty.

The set P is relatively closed in Ω, that is of second category. Since X is separable, Theorem 2.2 implies that $(X^\star, w^\star)$ is metrizable. Since F is of the first Baire class, by Lemma 2.1 we can find a point of continuity x_0 of $F|_P$. By symmetry, we can assume that $(x_0, F(x_0)) \in A$.

Since Ω and A are open, there is a $\rho > 0$ and directions

$$x_1, x_2, \ldots, x_m \in \partial_X B_X$$

such that $B_X(x_0, \rho) \subset \Omega$ and for every $y \in X$ and $y^\star \in X^\star$, $\|y - x_0\| < \rho$ and $|y^\star(x_j) - (F(x_0))(x_j)| < \rho$, $j = 1, \ldots, m$ imply $(y, y^\star) \in A$.

By the continuity of $F|_P$ at x_0, there is an $0 < \varepsilon < \rho$ such that for every $x \in P \cap B_X(x_0, \varepsilon)$ and $1 \leq j \leq m$ we have

$$|(F(x))(x_j) - (F(x_0))(x_j)| < \frac{\rho}{2}. \tag{1}$$

So $P \cap B_X(x_0, \varepsilon) \subset \mathcal{A}$, that is $\mathcal{B}$ is open in $B_X(x_0, \varepsilon)$.

Since $x_0 \in \mathrm{cl}_X\mathcal{B}$, we can find an $x_{m+1} \in \mathcal{B} \cap B_X(x_0, \varepsilon)$. Let be $V \leq X$ be the linear space spanned by $x_0, x_1, \ldots, x_m, x_{m+1}$. From $x_0 \in B_V(x_0, \varepsilon) \cap \mathcal{A}$ and $x_{m+1} \in B_V(x_0, \varepsilon) \cap \mathcal{B}$ we have that the set

$$W = B_V(x_0, \varepsilon) \cap \mathcal{B}$$

is nonempty, $W \neq B_V(x_0, \varepsilon)$ and, as noticed above, W is open.

Since V is finite dimensional, W is open and $W \neq B_V(x_0, \varepsilon)$, we can find a ball contained in W touching $\partial_V(V \cap \mathcal{B})$, that is a $w \in W$ and an $r > 0$ such that $B_V(w, r) \subset W$ and $\partial_V B_V(w, r) \cap \partial_V W$ is a nonempty subset of $B_V(x_0, \varepsilon)$. Let $u \in \partial_V B_V(w, r) \cap \partial_V W$. Note that from $u \in \mathcal{A}$ and $u \in \mathrm{cl}_V W$ we have $u \in P \cap B_V(x_0, \varepsilon)$.

From $\varepsilon < \rho$ we have that $\mathrm{cl}_V B_V(w, r) \subset \Omega$, and $B_V(w, r) \subset W \subset \mathcal{B}$ implies that

$$|(F(x))(x_j) - (F(x_0)(x_j)| \geq \rho$$

for at least one index $j \in \{1, 2, \ldots, m\}$ whenever $x \in B_V(w, r)$. On the other hand, it follows from $u \in P \cap B_X(x_0, \varepsilon)$ and (1) that

$$|(F(u))(x_j) - (F(x_0)(x_j)| < \frac{\rho}{2}$$

for every $j = 1, \ldots, m$. Thus

$$F(u) \notin \mathrm{cl}_{V^\star} F(B_V(w, r)),$$

which contradicts to the second condition. This proves the statement.■

Proof of Theorem 1.2 Let $F = f'$. According to Lemma 2.4, F is of the first Baire class. Since in finite dimensional spaces the notion of Fréchet and Gâteaux differentiability coincides for locally Lipschitz functions, for every finite dimensional subspace $V \leq X$, the restriction $f|_V$ is everywhere Fréchet differentiable, that is from Lemma 2.5 we have that $F|_V = f'|_V$ satisfies the second condition of Theorem 2.3. Thus $\mathrm{Graph}(f') \subset (\Omega, \|.\|) \times (X^\star, w^\star)$ is connected, as stated.■

Remarks.

1. R. Deville and P. Hájek ([5], Theorem 1) have constructed an everywhere Gâteaux differentiable function $f : l_1 \to \mathbb{R}$ such that f' is norm to $w^\star$ continuous and for which $f'(l_1) \subset (l_1^\star, \|.\|_{l_1^\star})$ has an isolated point. This shows that the range of Gâteaux differentiable functions need not to be connected in the norm topology.

R. Deville and P. Hájek ([5], Theorem 2) have also constructed a mapping $f : l_1 \to \mathbb{R}^2$ such that

$$\|f'(x) - f'(y)\|_{L(l_1, \mathbb{R}^2)} \geq 1$$

whenever $x, y \in l_1,\ x \neq y$. Thus the range of the derivative of vector valued functions can be very disconnected in the norm topology.

2. J. Saint Raymond ([6], Example 14.) constructed an everywhere Fréchet differentiable mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, such that the Jacobi determinant $\det f'(x)$ admits exactly two values. Therefore, $f'(\mathbb{R}^2)$ is not connected. This shows that the vector valued analogue of the result of Malý or Theorem 1.2 does not hold even in finite dimensional spaces. On the other hand, he also proved ([6], Theorem 20.) that if $\det f'$ is non-vanishing then the graph of $\det f'$ is connected.

3. Let X be an infinite dimensional Banach space with separable dual, and let $M \subset X^\star$ be an analytic set satisfying some extra arcwise connectivity conditions. M. Fabian, Ondřej Kalenda and Jan Kolář in [2] have proved that such a set M can be obtained as a range of a continuously differentiable bump. This implies that the range of a derivative does not have to be simply connected or locally connected.

Analogous finite and infinite dimensional results can be found in [3] or in [4].

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