Π^0_2-generated ideals are unwitnessable

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Abstract

We show that by assuming the Continuum Hypothesis it is possible to construct a Π^0_3 set $P \subseteq \mathbb{R}$ and a σ-ideal $\mathcal{I} \subseteq 2^{\mathbb{R}}$ such that $\mathcal{I}$ contains the Σ^0_3 sets which are the subsets of P, for every $A \in \mathcal{I}$ there is a Π^0_2 set $G \in \mathcal{I}$ satisfying $A \subseteq G$ (i.e. $\mathcal{I}$ is strongly Π^0_2-generated) but $P \notin \mathcal{I}$. This result is motivated by a question of Arnold Miller. We also prove in ZFC several covering results related to (strongly) Π^0_2-generated ideals.

1 Introduction

Let (X,τ) be a Polish space and let $\mathcal{I} \subseteq 2^X$ be a σ-ideal which is generated by its $\Pi^0_1(\tau)$ members, i.e. $A \in \mathcal{I}$ if and only if there are $\Pi^0_1(\tau)$ sets $F_i \in \mathcal{I}$ $(i<\omega)$ such that $A \subseteq \bigcup_{i<\omega} F_i$. This condition

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makes our task simple if we need to decide and to prove whether an analytic set A is in $\mathcal{I}$ or not *and A happens to be in $\mathcal{I}$;* then $A \in \mathcal{I}$ is witnessed by a countable union of properly chosen $\Pi^0_1(\tau)$ sets. But what if $A \notin \mathcal{I}$? Unlike the previous case this problem seems to be much more complex since it may really involve analytic sets. However, this case is also as simple as possible, as shown by the following result of S. Solecki ([9], Theorem 1 on page 1023).

Theorem 1. *(S. Solecki) Given a family of closed sets $\mathcal{I}$ and an analytic set A in some Polish space X, either A can be covered by the union of countably many members of $\mathcal{I}$ or A contains a nonempty Π^0_2 set G with the property that $F \cap G$ is meager in G for every $F \in \mathcal{I}$.*

Less formally this result says that if a σ-ideal $\mathcal{I}$ is generated by its Π^0_1 members the problem whether an analytic set belongs to $\mathcal{I}$ or not depends only on sets of low complexity, namely on Π^0_2 sets: $A \notin \mathcal{I}$ can be verified by a good Π^0_2 subset of A and Baire category. It is natural to ask whether this holds for every Borel class (see [8]).

Question 2. *(A. Miller) For $2 \leq \xi < \omega_1$ let $\mathcal{I}$ be a σ-ideal which is generated by its Π^0_ξ members. Is it true that for every analytic set $A \subseteq X$, either $A \in \mathcal{I}$ or there is a $\Pi^0_{\xi+1}$ set $B \subseteq A$ such that $B \notin \mathcal{I}$?*

Before all further speculations it is important to note that this question is already refuted by the following unpublished result of A. Kechris and M. Zelený. First we give a definition.

Definition 3. Let $\mathcal{I}$ be an ideal and $\mathcal{F} \subseteq \mathcal{I}$. We say that $\mathcal{I}$ *is strongly generated by $\mathcal{F}$* if for every $G \in \mathcal{I}$ there is an $F \in \mathcal{F}$ such that $G \subseteq F$.

Theorem 4. *(A. Kechris-M. Zelený) Assume $V = L$. Then there is an analytic set $A \subseteq 2^\omega$ and a σ-ideal $\mathcal{I}$ strongly generated by its $\Pi^0_2(\tau_{2^\omega})$ members such that $\mathcal{I}$ contains every Borel subset of A.*

That is the answer to Question 2 is consistently negative. Among other things, in this paper we aim to relax the condition $V = L$, as follows. We introduce a notation in advance.

Definition 5. Let $0 < \eta < \omega_1$. If (X, τ) is a Polish space and $P \subseteq X$, $\mathcal{S}^0_\eta(P)$ ($\mathcal{P}^0_\eta(P)$, resp.) denotes the collection of $\Sigma^0_\eta(\tau)$ ($\Pi^0_\eta(\tau)$, resp.) subsets of P.

In the next theorem and in the rest of the paper CH abbreviates the Continuum Hypothesis.

Theorem 6. *Let (X, τ) be a Polish space and $P \subseteq X$ be a Borel not $\Sigma^0_3(\tau)$ set. Let $\mathcal{A} \subseteq \mathcal{S}^0_3(P)$ be arbitrary of cardinality ω_1. Then there is a σ-ideal $\mathcal{I}$ such that*

1. *$\mathcal{I}$ is strongly generated by its $\Pi^0_2(\tau)$ members;*
2. *$\mathcal{A} \subseteq \mathcal{I}$;*
3. *$P \notin \mathcal{I}$.*

In particular, by assuming CH there is a σ-ideal $\mathcal{I}$ such that

1. *$\mathcal{I}$ is strongly generated by its $\Pi^0_2(\tau)$ members;*
2. *$\mathcal{S}^0_3(P) \subseteq \mathcal{I}$;*
3. *$P \notin \mathcal{I}$.*

Unfortunately this result *does not* show that Question 2 is consistently false. It only reproves a variant of Theorem 4 using only CH. Theorem 6 is sharp when P is $\Pi^0_3(\tau)$ in the sense that $\mathcal{I}$ contains all the Borel subsets of P which has lower Borel class than P. We also prove two ZFC results related to ideal generation. The first construct a Π^0_3 covering for Π^0_2-generated ideals and we think that this is the best one can obtain in ZFC.

Theorem 7. *Let (X,τ) be an uncountable Polish space and $P \subseteq X$ be a $\Pi^0_3(\tau)$ set which is not $\Sigma^0_3(\tau)$. Then there is a mapping $\Phi\colon \mathcal{S}^0_3(P) \to \mathcal{P}^0_3(P)$ such that $A \subseteq \Phi(A)$ and*

$$P \setminus \bigcup_{i<\omega} \Phi(A^i) \neq \emptyset \ (A, A^i \in \mathcal{S}^0_3(P)\ \ (i<\omega)).$$

The following easy theorem is the analogous result of Theorem 1 for strong generation.

Theorem 8. *Let (X,τ) be a Polish space and $A \subseteq X$ be an analytic set. If an ideal $\mathcal{I} \subseteq 2^X$ is strongly generated by its $\Pi^0_1(\tau)$ members then either $A \in \mathcal{I}$ or there is a $\Sigma^0_2(\tau)$ set $S \subseteq A$ so that $S \notin \mathcal{I}$.*

Theorem 6 also shows that the Theorem 7 and Theorem 8 are in some sense the best possible ZFC results. The nontrivial proofs are based on the theory of topologized Hurewicz test sets initiated in [6] and developed in [5]

Let us turn back to Question 2. The analogous question for strong generation is the following.

Question 9. *Let (X,τ) be a Polish space and let $0<\xi<\omega_1$. If a σ-ideal $\mathcal{I} \subseteq 2^X$ is strongly generated by its $\Pi^0_\xi(\tau)$ members and an* *analytic set $A \subseteq X$ is not in $\mathcal{I}$ does there exists a $\Sigma^0_{\xi+1}(\tau)$ set $B \subseteq A$ such that $B \notin \mathcal{I}$?*

Despite Theorem 4, in view of Theorem 6 it seems to be interesting to look for an alternative proof (without $V = L$) for the fact that for every $2 \leq \xi < \omega_1$ the answer to Question 2 and/or to Question 9 is consistently negative. Mostly because in the proof of Theorem 6, CH is used for having only ω_1 many Σ^0_3 sets to handle, so this result is more "descriptive" than "set theory".

In the end we remark that Theorem 1 has numerous important corollaries so it would be nice if an analogous statement was consistently true for the higher Borel classes. However, it seems to be open whether the answer to Question 2 and/or to Question 9 is consistently positive for every/some $2 \leq \xi < \omega_1$.

2 Preliminaries

Our terminology and notation follow [1]. As usual, $\Pi^0_\xi(\tau)$ ($\Sigma^0_\xi(\tau)$ resp.) ($0 < \xi < \omega$) stands for the ξ^{th} multiplicative (additive resp.) Borel class in the Polish space (X, τ), starting with $\Pi^0_1(\tau)$ = closed sets, $\Sigma^0_1(\tau)$ = open sets. A set is called *proper* $\Pi^0_\xi(\tau)$ if it is $\Pi^0_\xi(\tau)$ but not $\Sigma^0_\xi(\tau)$ ($0 < \xi < \omega_1$).

Let (C, τ_C) denote the Polish space 2^ω with its usual product topology. For two finite sequences $s, t \in \omega^{<\omega}$, we write $s \subseteq t$ ($s \subset t$, resp.) if t is an extension (a proper extension, resp.) of s. The length of s is denoted by $|s|$. If $s = (s_0 s_1 \dots s_{n-1})$ and $i < \omega$, then $s^\frown i$ stands for the sequence $(s_0 s_1 \dots s_{n-1} i)$.

In this note we will notoriously refine Polish topologies by turning countably many closed sets into open sets. We do this as described in [1], that is the open sets of the ancient topology together with their portion on the members of our collection of closed sets serve as a subbase of the new, finer topology. We will use that the topology obtained in this way is also Polish.

Definition 10. Let (X, τ) be a Polish space, $\mathcal{P} = \{P_i : i < \omega\}$ be a countable collection of $\Pi^0_1(\tau)$ sets. Then $\tau[\mathcal{P}]$ denotes that Polish topology refining τ where each P_i $(i < \omega)$ is turned successively into an open set.

It is easy to see that the resulting finer topology $\tau[\mathcal{P}]$ is independent from the enumeration of $\mathcal{P}$. This will be clear shortly when we fix a base of $\tau[\mathcal{P}]$.

Definition 11. If the basic open sets $\mathcal{G}$ are fixed in the Polish space (X, τ) and $\mathcal{P}$ is a countable collection of $\Pi^0_1(\tau)$ subsets of X *the basic open sets* of $\tau[\mathcal{P}]$ are the sets of the form $G \cap F_0 \cap \dots \cap F_{n-1}$ or G with $G \in \mathcal{G}$, $F_i \in \mathcal{P}$ $(i < n)$; a basic $\tau[\mathcal{P}]$-open set is said to be *proper* if it is not τ-open.

Observe that the basic open sets defined on this way form a basis of $\tau[\mathcal{P}]$. From now on whenever a Polish space (X, τ) appears we assume that a countable basis comprised of basic τ-open sets is fixed with respect to the convention of Definition 11. We take X to be basic τ-open.

The closure of a set $A \subseteq (X, \tau)$ is denoted by $\mathrm{cl}_\tau(A)$. We will never have to fix a special compatible metric on our Polish spaces but we will condition on the diameter of sets. In this case diam_τ denotes the diameter in an arbitrary fixed metric generating τ. We assume that $\mathrm{diam}_\tau(X) \leq 1$.

We recall that a $\Pi^0_2(\tau)$ subset G of the Polish space (X, τ) is itself a Polish space with the restricted topology $\tau|_G$ (see e.g. [1], (3.11) Theorem). In particular, the notions related to category in the topology τ make sense relative to G.

We will use the following result in the special $\xi = 3$ case (see e.g. [3], page 433). Recall that for to sets $A_0, A_1 \subseteq X$ we say that $G \subseteq X$ *separates* A_0 *from* A_1 if we have $A_0 \subseteq G \subseteq X \setminus A_1$.

Theorem 12. *(A. Louveau, J. Saint Raymond) Let* $3 \leq \xi < \omega_1$ *and* (X, τ) *be a Polish space. If* $P_\xi \subseteq C$ *is proper* $\Pi^0_\xi(\tau_C)$ *and* $A_0, A_1 \subseteq X$ *is any pair of disjoint analytic sets, then either* A_0 *can be separated from* A_1 *by a* $\Sigma^0_\xi(\tau)$ *set or there is a continuous one-to-one map* $\varphi\colon (C, \tau_C) \to (X, \tau)$ *with* $\varphi(P_\xi) \subseteq A_0$ *and* $\varphi(C \setminus P_\xi) \subseteq A_1$.

3 The Π^0_3 set of Lusin

We fix one special $\Pi^0_3(\tau_C)$ set which will play multiple roles in the sequel. Since the method we apply had already been used by Lusin to build (probably the first) proper $\Pi^0_3(\tau_C)$ set, we denote it by P_L. This construction has already been used in [7] in a special case.

Let (X, τ) be an uncountable Polish space. For every finite sequence $s \in \omega^{<\omega}$, fix a nonempty perfect set $P_s \subseteq X$ with the following properties:

$$P_\emptyset = X; \tag{1}$$

$$P_{s^\frown i} \cap P_{s^\frown j} = \emptyset \ (s \in \omega^{<\omega},\ i < j < \omega); \tag{2}$$

$$P_s \subseteq P_t \text{ and } P_s \text{ is } \tau|_{P_t}\text{-nowhere dense in } (P_t, \tau|_{P_t}) \ (t \subset s \in \omega^{<\omega}); \tag{3}$$

$$\bigcup_{i<\omega} P_{s^\frown i} \text{ is } \tau|_{P_s}\text{-dense in } (P_s, \tau|_{P_s}) \ (s \in \omega^{<\omega}). \tag{4}$$

To have $P_{s^\frown i} \subseteq P_s$ $(i < \omega)$, one simply has to take a countable dense subset $D_s = \{d_1, d_2, \dots\} \subseteq P_s$ and cover successively every d_i with a perfect set $P_{s^\frown i}$ which is nowhere dense in $(P_s, \tau_C|_{P_s})$ and disjoint from $P_{s^\frown j}$ $(j < i)$. Then (1-4) obviously hold. Once this done, let

$$\mathcal{P}_n = \{P_s \colon s \in \omega^{<\omega},\ |s| < n\} \ (n \leq \omega), \tag{5}$$

$$\mathcal{P} = \{X \setminus P^i \colon i < \omega\}, \tag{6}$$

and let $P^\omega = \emptyset$,

$$P^i = \bigcup_{\substack{s \,\in\, \omega^{<\omega} \\ |s| \,=\, i}} P_s \ (i < \omega),\ P_L = \bigcap_{i<\omega} P^i = X \setminus \bigcup_{i<\omega} \left(X \setminus P^i\right). \tag{7}$$

We introduce two topologies related to this construction.

Definition 13. With the notation of (5-6) we set $\tau^<_{P_L} = \tau[\mathcal{P}_\omega]$ and $\tau_{P_L} = \tau^<_{P_L}[\mathcal{P}]$.

From now on if we *take a P_L in X* we assume that the above construction has been carried out such that (1-7) hold. Observe that $P^j \subseteq P^i$ $(i \leq j < \omega)$ by (3), P^i is $\tau[\mathcal{P}_i]$-dense in X by (4) and $\tau[\mathcal{P}_i]$-meager in X by (3). By (2) and (3), a basic $\tau^<_{P_L}$-open set is of the form $G \cap P_s$ and a basic τ_{P_L}-open set is of the form $G \cap P_s \cap (X \setminus P^i)$ where G is basic τ-open, $s \in \omega^{<\omega}$ and $i \leq \omega$. We will use this without further reference. We prove a claim on the relation of P_L and the topologies of Definition 13.

Claim 14.

1. *P_L is a $\Pi^0_3(\tau)$ set.*
2. *The topologies $\tau^<_{P_L}$ and τ_{P_L} are Polish and refine τ.*
3. *If G is basic τ_{P_L}-open and $G \cap P_L \neq \emptyset$ then G is in fact basic $\tau^<_{P_L}$-open.*
4. *The topologies $\tau_{P_L}|_{P_L}$ and $\tau^<_{P_L}|_{P_L}$ coincide.*
5. *If G is basic $\tau^<_{P_L}$-open and $G \cap X \setminus P^i \neq \emptyset$ then G is basic $\tau[\mathcal{P}_i]$-open.*
6. *The topologies $\tau_{P_L}|_{P_s \setminus P^{|s|+1}}$ and $\tau|_{P_s \setminus P^{|s|+1}}$ coincide $(s \in \omega^{<\omega})$.*
7. *P_L is a τ_{P_L}-nowhere dense $\Pi^0_1(\tau_{P_L})$ set.*
8. *P_L is a $\tau^<_{P_L}$-residual $\Pi^0_2(\tau^<_{P_L})$ set.*

Proof. We have 1 by (7). By Definition 13, the topologies $\tau^<_{P_L}$ and τ_{P_L} are Polish and refine τ, which proves 2.

Since $\tau_{P_L} = \tau^<_{P_L}[\mathcal{P}]$, proper basic τ_{P_L}-open sets do not intersect P_L, which shows 3. Form this we immediately get 4.

By definition, if a set is basic $\tau^<_{P_L}$-open then it is basic $\tau[\mathcal{P}_n]$-open for some $n < \omega$. But a basic $\tau[\mathcal{P}_{i+1}]$-open but not $\tau[\mathcal{P}_i]$-open set G' satisfies $G' \subseteq P^i$ $(i < \omega)$, so 5 follows.

For 6, fix an $s \in \omega^{<\omega}$ and take a basic τ_{P_L}-open set H so that $H \cap P_s \setminus P^{|s|+1} \neq \emptyset$. By definition, $H = G \cap P_t \cap (X \setminus P^i)$ for some basic τ-open set $G \subseteq X$, $t \in \omega^{<\omega}$ and $i \leq \omega$. From (2) and (3) we get that $|s| < i$ and $t \subseteq s$, hence

$$\big(G \cap P_t \cap (X \setminus P^i)\big) \cap \left(P_s \setminus P^{|s|+1}\right) = G \cap P_s \setminus P^{|s|+1}.$$

Thus the topologies $\tau_{P_L}|_{P_s \setminus P^{|s|+1}}$ and $\tau|_{P_s \setminus P^{|s|+1}}$ coincide.

For 7, P_L is $\Pi^0_1(\tau_{P_L})$ by (6) and (7). So it remains to show that P_L does not contain any nonempty basic τ_{P_L}-open set. Suppose that $G \subseteq P_L$ and G is nonempty basic τ_{P_L}-open. Then by 3, G is basic $\tau^<_{P_L}$-open hence basic $\tau[\mathcal{P}_n]$-open for some $n < \omega$. But P^n is $\tau[\mathcal{P}_n]$-meager, so $G \not\subseteq P^n$ which contradicts $G \subseteq P_L \subseteq P^n$.

For 8, it is enough to show that P^i is $\tau^<_{P_L}$-dense $(i < \omega)$; being $\tau^<_{P_L}$-open this implies that each P^i and hence P_L is $\tau^<_{P_L}$-residual. By (3) and (4) we have that P^i is $\tau[\mathcal{P}_i]$ dense in X and contains P^j $(i \leq j < \omega)$. So if G is a basic $\tau^<_{P_L}$-open set then either G is $\tau[\mathcal{P}_i]$-open and so $P^i \cap G \neq \emptyset$, or G is not $\tau[\mathcal{P}_i]$-open hence $G \subseteq P^i$. This proves the statement.■

The next theorem contains the striking property of P_L.

Theorem 15. *Let (X, τ) be a Polish space, $(P_s)_{s \in \omega^{<\omega}}$ be a family of $\Pi^0_1(\tau)$ subsets of X satisfying (1-4). Let $\mathcal{P}_n$, P^n $(n \leq \omega)$ and P_L be as in (5-7). Then for every $\Pi^0_2(\tau)$ set $A \subseteq X$ and basic τ_{P_L}-open set G*

with $G \cap P_L \neq \emptyset$, whenever $A \cap P_L$ is $\tau_{P_L}|_{P_L}$-residual in $G \cap P_L$ then A is τ_{P_L}-residual in G.

Proof. Let $A \subseteq X$ be $\Pi^0_2(\tau)$, G be a basic τ_{P_L}-open set with $G \cap P_L \neq \emptyset$ and suppose that $A \cap P_L$ is $\tau_{P_L}|_{P_L}$-residual in $G \cap P_L$. Observe that since A is $\Pi^0_2(\tau)$, ($\Pi^0_2(\tau[\mathcal{P}_n])$ $(n \leq \omega)$, $\Pi^0_2(\tau_{P_L})$, resp.), the notions "meager" and "nowhere dense" coincide for A in the topology τ, ($\tau[\mathcal{P}_n]$ $(n \leq \omega)$, τ_{P_L}, resp.).

By Claim 14.3, G is actually $\tau^<_{P_L}$-open. Since by Claim 14.8, P_L is $\tau^<_{P_L}$-residual in X and by Claim 14.4 the topologies $\tau_{P_L}|_{P_L}$ and $\tau^<_{P_L}|_{P_L}$ coincide, we have that A is $\tau^<_{P_L}$-residual in G. Suppose that A is not τ_{P_L}-residual in G; that is we have a nonempty basic τ_{P_L}-open set $G' \subseteq G$ such that $A \cap G'$ is τ_{P_L}-meager thus τ_{P_L}-nowhere dense in G'. So by passing to a nonempty proper basic τ_{P_L}-open subset we can assume that $A \cap G' = \emptyset$. Since $P^j \subseteq P^i$ $(i \leq j < \omega)$, we have $G' = G'' \cap (X \setminus P^n)$ for some $n < \omega$ where $G'' \subseteq G$ is basic $\tau^<_{P_L}$-open. Since $G'' \cap (X \setminus P^n) \neq \emptyset$, from Claim 14.5 we get that G'' is basic $\tau[\mathcal{P}_n]$-open. Now $G'' \subseteq G$ implies that A is $\tau^<_{P_L}$-residual hence $\tau^<_{P_L}$-dense in G''. Since $\tau[\mathcal{P}_n]$ is coarser than $\tau^<_{P_L}$, A is $\tau[\mathcal{P}_n]$-dense so $\tau[\mathcal{P}_n]$-residual in G''. Thus A and $X \setminus P^n$ are both $\tau[\mathcal{P}_n]$-residual in G'' which yields $A \cap G'' \cap (X \setminus P^n) \neq \emptyset$, a contradiction which completes the proof.■

The following theorem describes important corollaries of Theorem 15, and in fact the first four statements can be considered as its reformulation. The fifth statement points out an obvious fact.

Corollary 16. *Let $G \subseteq X$ be basic τ_{P_L}-open with $G \cap P \neq \emptyset$, or equivalently let G be basic $\tau^<_{P_L}$-open.*

1. *If* $A \subseteq X$ *is* $\Pi^0_2(\tau)$ *and* $\tau^<_{P_L}$*-residual in* G *then* A *is* τ_{P_L}*-residual in* G*.*

2. *If* $A \subseteq X$ *is* $\Sigma^0_3(\tau)$ *and of* $\tau^<_{P_L}$*-second category in* G *then* A *is of* τ_{P_L}*-second category in* G*.*

3. *If* $A \subseteq X$ *is* $\Sigma^0_2(\tau)$ *and of* τ_{P_L}*-second category in* G *then* A *is of* $\tau^<_{P_L}$*-second category in* G*.*

4. *If* $A \subseteq X$ *is* $\Pi^0_3(\tau)$ *and* τ_{P_L}*-residual in* G *then* A *is* $\tau^<_{P_L}$*-residual in* G*.*

5. P *is a proper* $\Pi^0_3(\tau)$ *set.*

Proof. For 1, let A be $\Pi^0_2(\tau)$ and $\tau^<_{P_L}$-residual in G. By Claim 14.8, P_L is a $\tau^<_{P_L}$-residual $\Pi^0_2(\tau^<_{P_L})$ set so $A \cap G \cap P_L \neq \emptyset$ and $A \cap G \cap P_L$ is $\tau^<_{P_L}|_{P_L}$-residual in $G \cap P_L$. By Claim 14.4 the topologies $\tau^<_{P_L}|_{P_L}$ and $\tau_{P_L}|_{P_L}$ coincide, so we have that $A \cap G \cap P_L$ is $\tau_{P_L}|_{P_L}$-residual in $G \cap P_L$. By Theorem 15 we conclude that A is τ_{P_L}-residual in G, as required.

For 2, let $A = \bigcup_{i<\omega} A_i$ with $\Pi^0_2(\tau)$ set A_i $(i < \omega)$. As in the proof of 1, if A is of $\tau^<_{P_L}$-second category in G then there exists an $i < \omega$ and a basic $\tau^<_P$-open set $G' \subseteq G$ with $G' \cap P_L \neq \emptyset$ such that $A_i \cap P_L$ is $\tau^<_{P_L}|_{P_L}$-residual in $G' \cap P_L$. Since by Claim 14.4 the topologies $\tau^<_{P_L}|_{P_L}$ and $\tau_{P_L}|_{P_L}$ coincide, we have that $A_i \cap P_L$ is $\tau_{P_L}|_{P_L}$-residual in $G' \cap P_L$. Thus G' is a basic τ_{P_L}-open set with $G' \cap P_L \neq \emptyset$, Theorem 15 applies and gives that A_i is τ_{P_L}-residual in G'. Thus A is of τ_{P_L}-second category in G, as required.

Statements 3 and 5 follow from 1 and 2 by taking complements.

Finally suppose that P_L is $\Sigma^0_3(\tau)$. Since P_L is $\tau^<_{P_L}$-residual in X by Claim 14.8, from 2 we get that P_L is of τ_{P_L}-second category in X. But P_L is τ_{P_L}-nowhere dense by Claim 14.7. This contradiction completes the proof.■

We prove here another, a bit more technical lemma but of the same flavor as Corollary 16. We fix a notation in advance.

Definition 17. *For every $A \subseteq X$ set*

$$Z^{X,P_L}_s(A) = cl_\tau(A \cap P_s) \ (s \in \omega^{<\omega})$$

and

$$Z^{X,P_L}(A) = P_L \cup \bigcup_{s \in \omega^{<\omega}} Z^{X,P_L}_s(A) \setminus P^{|s|+1}.$$

Lemma 18. *Let $A \subseteq X$ be a τ_{P_L}-meager $\Pi^0_2(\tau)$ set. Then*

1. *$Z^{X,P_L}_s(A)$ is $\tau|_{P_s}$-nowhere dense in P_s ($s \in \omega^{<\omega}$);*
2. *$X \setminus Z^{X,P_L}(A)$ is $\tau^<_{P_L}$-dense in X.*

Proof. For 1, suppose that $Z^{X,P_L}_s(A)$ is not $\tau|_{P_s}$-nowhere dense in P_s for some $s \in \omega^{<\omega}$. Then A is $\tau|_{P_s}$-dense in $G \cap P_s$ for some τ-open set G satisfying $G \cap P_s \neq \emptyset$. Since "residual" and "dense" coincide for $\Pi^0_2(\tau)$ sets, A is $\tau|_{P_s}$-residual in $G \cap P_s$. Now the $\Sigma^0_2(\tau)$ set $P^{|s|+1}$ is $\tau|_{P_s}$-meager in $G \cap P_s$ so we have that A is $\tau|_{P_s}$-residual in $G \cap P_s \setminus P^{|s|+1}$. By Claim 14.6 the topologies $\tau_{P_L}|_{P_s \setminus P^{|s|+1}}$ and $\tau|_{P_s \setminus P^{|s|+1}}$ coincide. So A is τ_{P_L}-residual in the τ_{P_L}-open $G \cap P_s \setminus P^{|s|+1}$, a contradiction.

For 2, let G be a basic $\tau^<_{P_L}$-open set, say $G = G' \cap P_s$ where G' is basic τ-open and $s \in \omega^{<\omega}$. We show that $G \cap X \setminus Z^{X,P_L}(A) \neq \emptyset$. By

(2) and (3) we have

$$P_s \cap Z^{X,P_L}(A) \subseteq Z_s^{X,P_L}(A) \cup P^{|s|+1},$$

thus

$$G \cap X \setminus Z^{X,P_L}(A) =$$
$$= G' \cap P_s \setminus (P_s \cap Z^{X,P_L}(A)) \supseteq G' \cap P_s \setminus (Z_s^{X,P_L}(A) \cup P^{|s|+1}).$$

Since $Z_s^{X,P_L}(A)$ is $\tau|_{P_s}$-nowhere dense in P_s by 1, and $P^{|s|+1}$ is $\tau|_{P_s}$-meager in P_s by (3), $G' \cap P_s \setminus (Z_s^{X,P_L}(A) \cup P^{|s|+1}) \neq \emptyset$. Hence $G \cap X \setminus Z^{X,P_L}(A) \neq \emptyset$, which completes the proof.■

3.1 Constructible coverings

Our next goal is to show that if (X, τ) is a Polish space and $P \subseteq X$ is a proper $\Pi_3^0(\tau)$ set then the σ-ideal generated by the $\Sigma_3^0(\tau)$ subsets of P can be covered by a σ-ideal $\mathcal{I}$ strongly generated by some $\Pi_3^0(\tau)$ subsets of P such that $P \notin \mathcal{I}$. This is surprising because as we will see later for $X = C$ there are $\Sigma_3^0(\tau_C)$ sets contained by P_L which can be covered only by $\tau_{P_L}|_{P_L}$-residual $\Pi_3^0(\tau_C)$ subsets of P_L. As we mentioned in the introduction, we think that this is the best covering result for Σ_3^0-generated ideals which can be obtained in ZFC.

Proof of Theorem 7. First we construct $\Phi = \Phi_L$ for $(X, \tau) = (C, \tau_C)$ and $P = P_L$. For every $A \in \mathcal{S}_3^0(P_L)$ fix a presentation $A = \bigcup_{j<\omega} A_j$ where A_j is $\Pi_2^0(\tau_C)$ $(j < \omega)$. Set

$$\Phi_s(A) = \bigcup_{j<|s|} \mathrm{cl}_{\tau_C}(A_j \cap P_s) \ (s \in \omega^{<\omega}),$$

$$\Phi_n(A) = \bigcup_{|s|=n} \Phi_s(A) \ (n < \omega),\ \Phi_L(A) = \bigcap_{m<\omega} \bigcup_{m \le n < \omega} \Phi_n(A).$$

It is clear that $\Phi_L(A)$ is $\Pi^0_3(\tau_C)$; $\Phi_n(A) \subseteq P^n \subseteq P^m$ $(m \le n < \omega)$ shows $\Phi_L(A) \subseteq P_L$. Since $A \subseteq P_L \subseteq P^n$ implies $A_j \subseteq \Phi_n(A)$ for $j < n < \omega$, we also have $A \subseteq \Phi_L(A)$. It remains to show that if $A^i \in \mathcal{S}^0_3(P_L)$ with its fixed presentation $A^i = \bigcup_{j<\omega} A^i_j$ $(i < \omega)$ then we can find a point in $P_L \setminus \bigcup_{i<\omega} \Phi_L(A^i)$. We do this by constructing inductively a sequence $s_n \in \omega^{<\omega}$, $|s_n| = n$ $(n < \omega)$ and a basic τ_C-open set G_n $(n < \omega)$ such that

$$s_n \subset s_{n+1} \ (n < \omega); \tag{8}$$

$$G_n \cap P_{s_n} \neq \emptyset \ (n < \omega); \tag{9}$$

$$\mathrm{cl}_{\tau_C}(G_{n+1}) \subseteq G_n \ (n < \omega); \tag{10}$$

$$G_{n+1} \cap P_{s_{n+1}} \subseteq C \setminus \bigcup_{i \le n} \Phi_n\left(A^i\right) \ (n < \omega). \tag{11}$$

Then by (8), (9) and (10) we have

$$\bigcap_{n<\omega} G_n \cap P_{s_n} \neq \emptyset \text{ and } \bigcap_{n<\omega} G_n \cap P_{s_n} \subseteq P_L,$$

so (11) gives

$$\bigcap_{n<\omega} G_n \cap P_{s_n} \subseteq P_L \setminus \bigcup_{i<\omega} \Phi_L(A^i).$$

It remains to make the construction. Set $s_0 = \emptyset$, $G_0 = C$. Suppose that s_n, G_n are already defined; we find our s_{n+1}, G_{n+1}. By (2), $\Phi_n(A^i) \cap P_{s_n} = \Phi_{s_n}(A^i)$ $(i < \omega)$. First we obtain a basic τ_C-open set $G \subseteq G_n$ such that $G \cap P_{s_n} \neq \emptyset$ and $G \cap P_{s_n} \cap \Phi_{s_n}(A^i) = \emptyset$ $(i \le n)$. For this we show that $\Phi_{s_n}(A^i)$ $(i \le n)$ is $\tau_C|_{P_{s_n}}$-nowhere dense in P_{s_n}.

We have $\Phi_{s_n}(A^i) = \bigcup_{j<|s|} \mathrm{cl}_{\tau_C}(A^i_j \cap P_{s_n})$ $(i \le n)$. Since P_L is τ_{P_L}-meager by Claim 14.7 and $A^i_j \subseteq P_L$, A^i_j is a τ_{P_L}-meager $\Pi^0_2(\tau_C)$ set

$(i \leq n,\ j < |s_n|)$. Hence by Lemma 18.1, $\mathrm{cl}_{\tau_C}(A^i_j \cap P_{s_n})$ $(i \leq n,\ j < |s_n|)$ is $\tau_C|_{P_{s_n}}$-nowhere dense in P_{s_n}. So $\Phi_{s_n}(A^i)$ $(i \leq n)$ is indeed $\tau_C|_{P_{s_n}}$-nowhere dense in P_{s_n}.

We obtained that there is a basic τ_C-open set $G \subseteq G_n$ which satisfies $G \cap P_{s_n} \neq \emptyset$ and $G \cap P_{s_n} \subseteq C \setminus \bigcup_{i \leq n} \Phi_n(A^i)$. We can pass to a basic τ_C-open subset $G_{n+1} \subseteq \mathrm{cl}_{\tau_C}(G_{n+1}) \subseteq G$ such that $G_{n+1} \cap P_{s_n} \neq \emptyset$; then we have (10). By (4) we can find an $s_n \subset s_{n+1} \in \omega^{<\omega}$, $|s_{n+1}| = n+1$ with $P_{s_{n+1}} \cap G_{n+1} \neq \emptyset$, thus (8), (9) and (11) hold. This choice completes the inductive step and the proof of the special case.

If (X, τ) and P are arbitrary, by Theorem 12 we can take a continuous one-to-one map $\varphi\colon (C, \tau_C) \to (X, \tau)$ such that $\varphi^{-1}(P) = P_L$. For $A \in \mathcal{S}^0_3(P)$ let

$$\Phi(A) = (P \setminus \varphi(P_L)) \cup \varphi(\Phi_L(\varphi^{-1}(A))).$$

Since homeomorphism preserves the Borel class of sets, this definition makes sense and fulfills the requirements.■

One could say that Theorem 7 is trivial if the $\Sigma^0_3(\tau_C)$ subsets of P_L (which are all $\tau_{P_L}|_{P_L}$-meager by Corollary 16.2, Claim 14.4 and Claim 14.7) could be covered by a $\tau_{P_L}|_{P_L}$-meager $\Pi^0_3(\tau_C)$ subset of P_L. Then a category argument would give that

$$P_L \setminus \bigcup_{i<\omega} \Phi(A^i) \neq \emptyset \ (A, A^i \in \mathcal{S}^0_3(P)\ (i < \omega)).$$

However, this is not the case. The construction in the following claim has already been used by S. Solecki in [9] to prove Theorem 1.

Claim 19. *There is a* $\Sigma^0_3(\tau_C)$ *set* $A \subseteq P_L$ *such that if* B *is* $\Pi^0_3(\tau_C)$ *and* $A \subseteq B$ *then* $B \cap P_L$ *is* $\tau_{P_L}|_{P_L}$*-residual in* P_L*.*

Proof. In this proof $B(x,\varepsilon)$ denotes the τ_C-open ball centered at $x \in C$ with radius $\varepsilon > 0$. Fix a $v \in \omega^{<\omega} \setminus \{\emptyset\}$ and a basic τ_C-open set V satisfying $V \cap P_v \neq \emptyset$. We define an injection $\varphi\colon \omega^{<\omega} \to \omega^{<\omega}$, a map $j\colon \omega^{<\omega} \to \omega$ and basic τ_C-open sets $(U_t)_{t\in\omega^{<\omega}}$ with the following properties.

$$U_\emptyset = V,\ \varphi(\emptyset) = v; \tag{12}$$

$$s \subseteq t \implies \varphi(s) \subseteq \varphi(t); \tag{13}$$

$$U_s \subseteq \mathrm{cl}_{\tau_C}(U_s) \subseteq U_t \setminus P_{\varphi(t)^\frown j(t)}\ (s,t \in \omega^{<\omega},\ t \subset s); \tag{14}$$

$$U_s \cap U_t = \emptyset\ (s,t \in \omega^{<\omega},\ |t| = |s|); \tag{15}$$

$$\mathrm{diam}_{\tau_C}(U_s) \leq 2^{-|s|}\ (s \in \omega^{<\omega}); \tag{16}$$

$$U_s \cap P_{\varphi(s)^\frown j(s)} \neq \emptyset\ (s \in \omega^{<\omega}); \tag{17}$$

$$\forall x \in U_s \cap P_{\varphi(s)^\frown j(s)}\ \forall \varepsilon > 0$$
$$\exists i < \omega\,(U_{s^\frown i} \subseteq B(x,\varepsilon))\ \ (s \in \omega^{<\omega}). \tag{18}$$

Suppose that the construction is done. Let

$$A_{v,V} = \bigcap_{n<\omega} \bigcup_{s\in\omega^{<\omega},\ |s|=n} U_s.$$

This set is $\Pi^0_2(\tau_C)$, we show that $A_{v,V} \subseteq P_L$. By (15) we have

$$A_{v,V} = \bigcup_{\sigma\in\omega^\omega} \bigcap_{n<\omega} U_{\sigma|_n}.$$

By (16), (17) and (3) we have

$$\bigcap_{n<\omega} U_{\sigma|_n} = \bigcap_{n<\omega} \left(U_{\sigma|_n} \cap P_{\varphi(\sigma|_n)}\right) \in P_L\ (\sigma \in \omega^\omega)$$

so indeed $A_{v,V} \subseteq P_L$.

Next we show that $U_s \cap P_{\varphi(s)^\frown j(s)} \subseteq \mathrm{cl}_{\tau_C}(U_s \cap A_{v,V})$ $(s \in \omega^{<\omega})$. Let $x \in U_s \cap P_{\varphi(s)^\frown j(s)}$. By (18) there is a sequence $(i_l)_{l<\omega} \subseteq \omega$ such that $U_{s^\frown i_l} \subseteq B(x, 1/l)$. By (14) and (16) we have $A_{v,V} \cap U_s \neq \emptyset$ $(s \in \omega^{<\omega})$, in particular $A_{v,V} \cap U_{s^\frown i_l} \neq \emptyset$ $(l < \omega)$ so indeed $x \in \mathrm{cl}_{\tau_C}(U_s \cap A_{v,V})$.

Finally we show that every $\Sigma^0_2(\tau_C)$ set $H \subseteq C$ for which $A_{v,V} \subseteq H$ is of $\tau^<_{P_L}$-second category in $V \cap P_v$. Let $H = \bigcup_{i<\omega} H_i$ where H_i is $\Pi^0_1(\tau_C)$ $(i < \omega)$. By the Baire Category Theorem in $A_{v,V}$, there is an $i < \omega$ such that $H_i \cap A_{v,V}$ is of $\tau_C|_{A_{v,V}}$-second category in $A_{v,V}$, say H_i contains a basic $\tau_C|_{A_{v,V}}$-open set $G \cap A_{v,V}$ where G is basic τ_C-open. By (16) there is an $s \in \omega^{<\omega}$ such that $U_s \subseteq G$. By (17) we have $U_s \cap P_{\varphi(s)^\frown j(s)} \neq \emptyset$. As we have seen above,

$$U_s \cap P_{\varphi(s)^\frown j(s)} \subseteq \mathrm{cl}_{\tau_C}(U_s \cap A_{v,V}) \subseteq \mathrm{cl}_{\tau_C}(G \cap A_{v,V}) \subseteq H_i.$$

Since $P_{\varphi(s)^\frown j(s)} \subseteq P_v$ by (12) and (13), H_i and thus H is indeed of $\tau^<_{P_L}$-second category in $V \cap P_v$.

To have A, fix an enumeration $\{(v_i, V_i) \colon i < \omega\}$ of the pairs (v, V) where $v \in \omega^{<\omega} \setminus \{\emptyset\}$, V is basic τ_C-open and $V \cap P_v \neq \emptyset$. Set $A = \bigcup_{i<\omega} A_{v_i,V_i}$. If B is a $\Pi^0_3(\tau_C)$ set containing A, say $B = \bigcap_{i<\omega} B_i$ with B_i $\Sigma^0_2(\tau_C)$ then as we have shown above, B_i $(i < \omega)$ is of $\tau^<_{P_L}$-second category in every nonempty $\tau^<_{P_L}$-open set. So B_i $(i < \omega)$ and hence B is $\tau^<_{P_L}$-residual. Then by Claim 14.8, $B \cap P_L$ is $\tau^<_{P_L}|_{P_L}$ residual in P_L, by Claim 14.4 the topologies $\tau^<_{P_L}|_{P_L}$ and $\tau_{P_L}|_{P_L}$ coincide, so $B \cap P_L$ is $\tau_{P_L}|_{P_L}$-residual in P_L, as stated.

So it remains to make the construction; we do this recursively. Set $U_\emptyset = V$, $\varphi(\emptyset) = v$ and let $j(\emptyset) < \omega$ be such that $V \cap P_{v^\frown j(\emptyset)} \neq \emptyset$; such a $j(\emptyset)$ exists by (4). So we have (12-17) for $s=\emptyset$. Suppose that we already have U_s, $\varphi(s)$ and $j(s)$ for $s \in \omega^{<\omega}$, $|s| \leq n$ such that (13-17) hold for $|s|, |t| \leq n$ and (18) holds for $|s|<n$. For every $s \in \omega^{<\omega}$ with $|s|=n$, using (3), we can fix a countable set

$$D_s=\{d_s(i) \colon i<\omega\} \subseteq U_s \cap P_{\varphi(s)} \setminus P_{\varphi(s)^\frown j(s)}$$

with the property that $\mathrm{cl}_{\tau_C}(D)=D \cup \mathrm{cl}_{\tau_C}(U_s \cap P_{\varphi(s)^\frown j(s)})$. Then we can find a basic τ_C-open neighborhood $U_{s^\frown i}$ of $d_s(i)$ such that

$$U_{s^\frown i} \cap P_{\varphi(s)} \neq \emptyset,\ \mathrm{cl}_{\tau_C}(U_{s^\frown i}) \subseteq U_s \setminus P_{\varphi(s)^\frown j(s)},$$
$$U_{s^\frown i} \cap U_{s^\frown j}=\emptyset\ (i,j<\omega,\ i \neq j)$$

and $\mathrm{diam}_{\tau_C}(U_{s^\frown i}) \leq 2^{n+1+i}$ $(i<\omega)$. Then (14-16) hold for $|s| \leq n+1$ and (18) holds for $|s| \leq n$. Define $\varphi(s) \subset \varphi(s^\frown i)$ to have $P_{\varphi(s^\frown i)} \cap U_{s^\frown i} \neq \emptyset$ $(i<\omega)$; this is possible by (4). Then (13) holds for $|s|, |t| \leq n+1$. Again by (4), we can have $j(s^\frown i)<\omega$ such that $P_{\varphi(s^\frown i)^\frown j(s^\frown i)} \cap U_{s^\frown i} \neq \emptyset$ $(i<\omega)$. Then also (17) holds for $|s| \leq n+1$. This completes the recursive step and the proof.■

To close this section we prove Theorem 8.

Proof of Theorem 8. Suppose that $A \notin \mathcal{I}$. Let $S \subseteq A$ be a $\tau|_A$-dense countable set. Then $S \subseteq X$ is $\Sigma_2^0(\tau)$ and $A \subseteq \mathrm{cl}_\tau(S)$ shows $\mathrm{cl}_\tau(S) \notin \mathcal{I}$. Since $\mathcal{I}$ is strongly generated by its $\Pi_1^0(\tau)$ members this implies $S \notin \mathcal{I}$ and the proof is complete.■

3.2 An unwitnessable ideal

By assuming the Continuum Hypothesis we can cover the $\Sigma^0_3(\tau_C)$ subsets of P_L in C by $\Pi^0_2(\tau_C)$ sets such that P_L is not in the σ-ideal generated by the covering sets. Combining Theorem 15 with the previous example we see that in this case the covering $\Pi^0_2(\tau_C)$ sets cannot be the subsets of P_L: a $\Pi^0_2(\tau_C)$ subset of P_L should be $\tau_{P_L}|_{P_L}$-meager in P_L while we have constructed a $\Sigma^0_3(\tau_C)$ subset of P_L which can be covered only by $\tau_{P_L}|_{P_L}$-residual $\Pi^0_3(\tau_C)$ subsets of P_L. For this covering by $\Pi^0_2(\tau)$ sets, Theorem 12 is too "imprecise" in the sense that it does not care about the presentation of $\Pi^0_\xi(\tau)$ sets. We prove an alternative for $\xi = 3$.

Lemma 20. *Let (X,τ) be a Polish space, $(P_s)_{s\in\omega^{<\omega}}$ be a family of $\Pi^0_1(\tau)$ subsets of X satisfying (1-4). Let $\mathcal{P}_n$, P^n $(n<\omega)$ and P_L be as in (5). Let A be a fixed τ_{P_L}-meager $\Pi^0_2(\tau)$ set. If $x^\star \in X \setminus Z^{X,P_L}(A)$ and $U \subseteq X$ is basic τ-open with $x^\star \in U$ then there is a τ-compact set $F \subseteq X$ such that $x^\star \in F \subseteq U \setminus A$ and $(F\cap P_s)_{s\in\omega^{<\omega}}$ satisfies (1-4) in the Polish space $(F, \tau|_F)$.*

Proof. Let $\mathcal{G} = \{G_n : n<\omega\}$ be an enumeration of the sets of the form $G \cap P_s$ $(s\in\omega^{<\omega}, G$ basic τ-open). Let $r \in \omega^{<\omega}$ be such that $x^\star \in P_r \setminus P^{|r|+1}$. We define recursively a tree $T \subseteq \omega^{<\omega}$ and $\mathcal{T} = \{(V_s, X_s) : s \in T\}$ such that $T \cap \omega^n$ $(n<\omega)$ is finite, $V_s \subseteq X$ $(s\in T)$ is a basic τ-open set, $X_s = \{x_{s^\frown i} : i<\omega,\ s^\frown i \in T\} \subseteq X \setminus P_L$ $(s \in T)$ is a finite set, and with $V^n = \bigcup_{s\in T,|s|=n} V_s$, $X^n = \bigcup_{s\in T,|s|=n} X_s$ $(n<\omega)$

the following hold:

$$x^\star \in X^n \subseteq X^{n+1} \ (n < \omega); \tag{19}$$

$$\mathrm{cl}_\tau(V_{s^\frown i}) \subseteq V_s \subseteq U \ (i < \omega,\ s, s^\frown i \in T); \tag{20}$$

$$V_s \cap V_t = \emptyset \ (s, t \in T,\ |s| = |t|); \tag{21}$$

$$\mathrm{diam}_\tau(V_s) < 2^{-|s|} \ (s \in T); \tag{22}$$

$$V_{s^\frown i} \cap X_s = \{x_{s^\frown i}\} \ (i < \omega,\ s^\frown i \in T); \tag{23}$$

$$\mathrm{cl}_\tau(G_n) \cap X^n = \emptyset \text{ implies } G_n \cap V^{n+1} = \emptyset \ (n < \omega); \tag{24}$$

if $i \leq |r|$ is maximal such that $x_s \in P_{r|_i}$ then

$$V_s \cap \mathrm{cl}_\tau \left(P_{r|_i} \cap A\right) = \emptyset \ (s \in T); \tag{25}$$

if $x_s \in P_t$ for some $s \in T, t \in \omega^{<\omega}$ then

$$X_s \cap V_s \cap (P_{t|_m} \setminus P^{m+1}) \neq \emptyset \ (m < |t|); \tag{26}$$

if $x_s \in P_t$ for some $s \in T,\ t \in \omega^{<\omega}$ then

$$X_s \cap V_s \cap \bigcup_{i<\omega} P_{t^\frown i} \neq \emptyset. \tag{27}$$

Suppose that the construction is done, we show that

$$F = \bigcap_{n<\omega} V^n = \mathrm{cl}_\tau \left(\bigcup_{n<\omega} X^n \right)$$

fulfills the requirements; here equality follows from (20), (22) and (23). By (20), (21) and (22), F is compact, (19) and (20) imply that $x^\star \in F \subseteq U$.

We show that $F \cap A = \emptyset$. If $x \in F$ then by (21) and (22) there is a unique $\sigma \in \omega^\omega$ such that $x = \bigcap_{n<\omega} V_{\sigma|_n}$. Let $i \leq |r|$ be maximal for which $x_{\sigma|_n} \in P_{r|_i}$ for infinitely many $n < \omega$. By (23) we have $\lim_{n\to\infty} x_{\sigma|_n} = x$ so $x \in P_{r|_i}$. Fix an $n < \omega$ such that i is maximal for which $x_{\sigma|_n} \in P_{r|_i}$. Then by (25), $V_{\sigma|_n} \cap \mathrm{cl}_\tau(P_{r|_i} \cap A) = \emptyset$, in particular $x \in P_{r|_i} \setminus A$, as stated.

Now we prove that (1-4) holds for $(F \cap P_s)_{s \in \omega^{<\omega}}$ in the Polish space $(F, \tau|_F)$. Since (1) and (2) are automatic we only have to check (3) and (4).

First we show that $F \cap P_s = \mathrm{cl}_\tau(\bigcup_{n<\omega} X^n \cap P_s)$. It is clear that $F \cap P_s \supseteq \mathrm{cl}_\tau(\bigcup_{n<\omega} X^n \cap P_s)$. For the reverse containment it is enough to show that whenever $G \cap F \cap P_s \neq \emptyset$ for some basic τ-open set G then $G \cap X^n \cap P_s \neq \emptyset$ for some $n < \omega$; so let $G \cap F \cap P_s \neq \emptyset$. By regularity we can find another basic τ-open set $G' \subseteq \mathrm{cl}_\tau(G') \subseteq G$ such that $G' \cap F \cap P_s \neq \emptyset$. Let $G_n = G' \cap P_s$. Since $G_n \cap V^{n+1} \neq \emptyset$, by (24) we have $\mathrm{cl}_\tau(G_n) \cap X^n \neq \emptyset$ hence $G \cap X^n \cap P_s \neq \emptyset$, which proves the statement.

For (3), let $s \subset t$ and suppose that $G \cap F \cap P_t \neq \emptyset$ for some basic τ-open set G. We have to show that $G \cap F \cap P_s \setminus P_t \neq \emptyset$. By the preceding, $x_u \in G \cap P_t$ for some $u \in T$, so by (19), (20), (22) and (23), $x_u = x_v \in V_v \subseteq G$ for some $u \subseteq v \in T$. Then by (26) for $m = |s| < |t|$ we get $X_v \cap V_v \cap (P_{t|_m} \setminus P^{m+1}) \neq \emptyset$. Since $P_t \subseteq P^{m+1}$, $X_v \subseteq F$ and $V_v \subseteq G$, we have $G \cap F \cap (P_s \setminus P_t) \neq \emptyset$, as required.

For (4), suppose that $G \cap F \cap P_t \neq \emptyset$ for some basic τ-open set G and $t \in \omega^{<\omega}$. We have to find some $i < \omega$ such that $G \cap F \cap P_{t^\frown i} \neq \emptyset$. Let $x_u \in G \cap P_t$; as above, we have a $u \subseteq v \in T$ with $x_u = x_v \in V_v \subseteq G$. Then by (27), $X_v \cap V_v \cap \bigcup_{i<\omega} P_{t^\frown i} \neq \emptyset$, so from $V_v \subseteq G$ and $X_v \subseteq F$ we have $G \cap F \cap \bigcup_{i<\omega} P_{t^\frown i} \neq \emptyset$ and we are done.

We do the construction recursively such that after the N^{th} step of the recursion (19-25) hold for $n \leq N$ while (26) and (27) hold for $x_s \in X^{N-1}$. Put $\emptyset \in T$, set $x_\emptyset = x^\star$ and let $x^\star \in V_\emptyset \subseteq U$ be a basic τ-open set such that $V_\emptyset \cap \mathrm{cl}_\tau(P_r \cap A) = \emptyset$; this choice is possible since $x^\star \notin Z^{X,P_L}(A)$, which means that $x^\star \notin \mathrm{cl}_\tau(P_r \cap A)$. So $x_\emptyset, V_\emptyset$ meet the requirements.

Suppose that $T \cap \omega^N$ is defined, we have X^{N-1} and V^N such that (19-25) hold for $n \leq N$, (26) and (27) hold for $x_s \in X^{N-1}$. We extend T up to level $N+1$ and define X^N, V^{N+1} such that (19-25) hold for $n \leq N+1$, (26) and (27) hold for $x_s \in X^N$ as follows. Let $s \in T$, $|s| = N$ be arbitrary; since $x_s \notin P_L$, to satisfy (26) and (27) for x_s we have to take only finitely many points. We show that we can pick each of these points x in V_s such that

$$\text{if } i \leq |r| \text{ is maximal for which } x \in P_{r|_i} \text{ then } x \notin \mathrm{cl}_\tau(P_{r|_i} \cap A). \quad (28)$$

Let $x_s \in P_t$ for some $t \in \omega^{<\omega}$. For (26) let $m < |t|$, and let $i \leq |r|$ be maximal such that $P_{t|_m} \subseteq P_{r|_i}$. We distinguish two cases. If $P_{t|_m} = P_{r|_i}$ then by Lemma 18.1, $\mathrm{cl}_\tau(P_{t|_m} \cap A)$ is $\tau|_{P_{t|_m}}$-nowhere dense in $P_{t|_m}$ so by (3) we can pick a point in $V_s \cap P_{t|_m} \setminus (P^{m+1} \cup \mathrm{cl}_\tau(P_{t|_m} \cap A))$. If $P_{t|_m} \subset P_{r|_i}$ then by (2) for every $y \in P_{t|_m}$, i is the maximal for which $y \in P_{r|_i}$. Since $x_s \in P_{t|_m}$, we have $V_s \cap \mathrm{cl}_\tau(P_{r|_i} \cap A) = \emptyset$ by (25). By (3) we can pick a point x in $V_s \cap P_{t|_m} \setminus P^{m+1}$, such that $x \notin \mathrm{cl}_\tau(P_{r|_i} \cap A)$ follows from $x \in V_s$.

For (27), if $x_s \in P^{|t|+1}$ then $x_s \in X_s$ shows (27). So suppose that $x_s \notin P^{|t|+1}$. By (4) we have a $j < \omega$ such that $V_s \cap P_{t^\frown j} \neq \emptyset$. By Lemma 18.1, $\mathrm{cl}_\tau(P_{t^\frown j} \cap A)$ is $\tau|_{P_{t^\frown j}}$-nowhere dense in $P_{t^\frown j}$ so we can pick a point x in $V_s \cap P_{t^\frown j} \setminus \mathrm{cl}_\tau(P_{t^\frown j} \cap A)$. Let $i \leq |r|$ be maximal for which $x \in P_{r|_i}$. If $r|_i = t^\frown j$ then we have $x \notin \mathrm{cl}_\tau(P_{r|_i} \cap A)$. If $r|_i \neq t^\frown j$ then $P_t \subseteq P_{r|_i}$. We show that for x_s, as well, i is the maximal such that $x_s \in P_{r|_i}$. If $P_t = P_{r|_i}$ then this follows from $x \notin P^{|t|+1}$ while if $P_t \subset P_{r|_i}$ then we are done by the maximality of i for x. Thus by (25), $V_s \cap \mathrm{cl}_\tau(P_{r|_i} \cap A) = \emptyset$ so $x \in V_s$ gives $x \notin \mathrm{cl}_\tau(P_{r|_i} \cap A)$.

Index x_s and the new points with $s^\frown i$ $(i < n_s)$ for some $n_s < \omega$, put $s^\frown i \in T$ $(i < n_s)$ and take pairwise disjoint basic τ-open sets $x_{s^\frown i} \in V_{s^\frown i} \subseteq \mathrm{cl}_\tau(V_{s^\frown i}) \subseteq V_s$ such that (21), (22), (24) and (25) hold; observe that (25) can be satisfied by (28). Since (19), (20), (23), (26) and (27) hold, this completes the recursive step and the proof.■

Instead of covering $\Sigma^0_3(\tau)$ sets by $\Pi^0_2(\tau)$ sets we define $\Sigma^0_2(\tau)$ sets avoiding them. The sequence of $\Sigma^0_2(\tau)$ sets constructed for this must be specially nested.

Definition 21. Let (X, τ) be a Polish space and consider P_L in X. Let $0 < \zeta < \omega_1$. Set $\mathcal{B}^0 = \{X\}$ while for $0 < \alpha < \zeta$, let $\mathcal{B}^\alpha = \{B^\alpha_i \colon i < \omega\}$ be a collection of pairwise disjoint $\Pi^0_1(\tau)$ subsets of X. Set $B^\alpha = \bigcup_{i<\omega} B^\alpha_i$ $(\alpha < \zeta)$, $B_\zeta = \bigcap_{\alpha<\zeta} B^\alpha$. Let $\tau[\alpha] = \tau[\bigcup_{\vartheta<\alpha} \mathcal{B}^\vartheta]$, $\tau^<_{P_L}[\alpha] = \tau^<_{P_L}[\bigcup_{\vartheta<\alpha} \mathcal{B}^\vartheta]$ and $\tau_{P_L}[\alpha] = \tau_{P_L}[\bigcup_{\vartheta<\alpha} \mathcal{B}^\vartheta]$ $(\alpha \leq \zeta)$. We say that $(\mathcal{B}^\alpha)_{\alpha<\zeta}$ is P_L-*nested* if

1. $B^\beta \subseteq B^\alpha$ $(\alpha \leq \beta < \zeta)$;
2. B^α is $\tau^<_{P_L}[\alpha]$-dense in X $(\alpha < \zeta)$;
3. B^α_i is $\tau[\alpha]$-compact $(\alpha < \zeta,\ i < \omega)$;
4. $(P_s \cap B_\alpha)_{s \in \omega^{<\omega}}$ satisfies (1-4) in the Polish space $(B_\alpha, \tau[\alpha]|_{B_\alpha})$ for every $\alpha \leq \zeta$ if ζ is successor and for $\alpha < \zeta$ if ζ is limit.

Since a compact Polish topology on a base set has no nontrivial compact Polish refinement, Definition 21.3 says in particular that on B_i^α the topologies $\tau[\beta]$ $(\beta \leq \alpha)$ coincide. In the sequel we use this property without further reference. Next we show that Definition 21.4 holds for limit $\alpha = \zeta$, as well.

Lemma 22. *Let $(\mathcal{B}^\alpha)_{\alpha<\zeta}$ be a P_L-nested sequence for some $\zeta < \omega_1$ and suppose that $A \subseteq B_\zeta$ is $\Pi^0_2(\tau[\zeta]|_{B_\zeta})$ and $\tau_{P_L}[\zeta]|_{B_\zeta}$-meager. Then*

1. *B_ζ is $\tau^<_{P_L}[\zeta]$-residual in X;*
2. *if ζ is a limit ordinal then $(P_s \cap B_\zeta)_{s\in\omega^{<\omega}}$ satisfies (1-4) in the Polish space $(B_\zeta, \tau[\zeta]|_{B_\zeta})$;*
3. *$B_\zeta \setminus Z^{B_\zeta, P_L\cap B_\zeta}(A)$ is $\tau^<_{P_L}[\zeta]|_{B_\zeta}$-dense in B_ζ;*
4. *$P_L \cap B_\zeta \neq \emptyset$.*

Proof. By Definition 21.1 and 2, B^α $(\alpha < \zeta)$ is $\tau^<_{P_L}[\alpha]$-dense and $\tau^<_{P_L}[\zeta]$-open $(\alpha < \zeta)$. Since a $\tau^<_{P_L}[\zeta]$-open but not $\tau^<_{P_L}[\alpha]$-open set is contained in B^α $(\alpha < \zeta)$, B^α $(\alpha < \zeta)$ is also $\tau^<_{P_L}[\zeta]$-dense. Hence B^α $(\alpha < \zeta)$ and so B_ζ is $\tau^<_{P_L}[\zeta]$-residual in X, which is 1.

For 2, we have (1) and (2) automatically. To have (3), let $t \subset s$ and suppose that $G \cap P_s \neq \emptyset$ for some basic $\tau[\zeta]|_{B_\zeta}$-open set. That is, $G = G' \cap B^\alpha_n \cap B_\zeta$ for some basic τ-open set G', $\alpha < \zeta$ and $n < \omega$. But (3) holds in B^α_n, in particular $U = B^\alpha_n \cap G' \cap P_t \setminus P_s \neq \emptyset$. Now U is a $\tau^<_{P_L}[\zeta]$-open set, so given that B_ζ is $\tau^<_{P_L}[\zeta]$-residual by 1, we have $U \cap B_\zeta \neq \emptyset$, as required.

To see (4) fix an $s \in \omega^{<\omega}$ and a basic $\tau[\zeta]|_{B_\zeta}$-open set G with $G \cap P_s \neq \emptyset$. As before, we have that $G = G' \cap B^\alpha_n \cap B_\zeta$ for some basic τ-open set G', $\alpha < \zeta$ and $n < \omega$. Now (4) holds in B^α_n hence $B^\alpha_n \cap G' \cap \bigcup_{i<\omega} P_{s^\frown i} \neq \emptyset$, say $U = B^\alpha_n \cap G' \cap P_{s^\frown i} \neq \emptyset$ for some $i < \omega$. Again, U is a $\tau^<_{P_L}[\zeta]$-open set, so given that B_ζ is $\tau^<_{P_L}[\zeta]$-residual by 1, we have $U \cap B_\zeta \neq \emptyset$, which proves the statement.

We have 3 since for ζ limit by 2, for ζ successor by Definition 21.4, Lemma 18.2 holds in the Polish space $(B_\zeta, \tau[\zeta]|_{B_\zeta})$.

Finally by 2 for ζ limit and by Definition 21.4 for ζ successor, Theorem 15.8 applies and we get that $P_L \cap B_\zeta$ is $\tau^<_{P_L}[\zeta]|_{B_\zeta}$-residual in B_ζ; in particular, $P_L \cap B_\zeta \neq \emptyset$. This shows 4 and completes the proof.■

Now we have everything to construct our unwitnessable ideal.

Proof of Theorem 6. If CH is assumed set $\mathcal{A} = \mathcal{S}^0_3(P_L)$. First we prove the special case $(X, \tau) = (C, \tau_C)$, $P = P_L$. For every $A \in \mathcal{A}$ fix a presentation $A = \bigcup_{i<\omega} A(i)$ where $A(i) \subseteq P_L$ is $\Pi^0_2(\tau_C)$. Let $\{A_\alpha \colon \alpha < \omega_1\}$ be an enumeration of $\{A(i) \colon A \in \mathcal{A},\ i < \omega\} \subseteq \mathcal{P}^0_2(P_L)$ such that $A_0 = \emptyset$. We shall construct a P_L-nested sequence $(\mathcal{B}^\alpha)_{\alpha<\omega_1}$ such that

$$A_\alpha \cap B^\alpha = \emptyset \ \ (\alpha < \omega_1)\,. \tag{29}$$

Once this done set

$$\mathcal{I}_{P_L} = \{G \subseteq C \colon \exists \alpha < \omega_1 \,(B_\alpha \cap G = \emptyset)\},$$

or, since $B_{\alpha+1} \subseteq B^\alpha \subseteq B_\alpha$ $(\alpha < \omega_1)$ by Definition 21.1, equivalently

$$\mathcal{I}_{P_L} = \{G \subseteq C \colon \exists \alpha < \omega_1 \,(B^\alpha \cap G = \emptyset)\}.$$

By Definition 21, $\mathcal{I}_{P_L}$ is a σ-ideal. Also by Definition 21.1, $\mathcal{I}_{P_L}$ is strongly generated by its $\Pi^0_2(\tau_C)$ members. By (29) it contains $\mathcal{A}$. Finally Lemma 22.4 implies that $P_L \notin \mathcal{I}$, as required.

It remains to make the construction. We proceed by induction; to start with, set $\mathcal{B}^0 = \{B^0_0\}$ with $B^0_0 = C$. Suppose that $\mathcal{B}^\alpha$ is defined for $\alpha < \zeta$ such that the sequence $(\mathcal{B}^\alpha)_{\alpha<\zeta}$ is P_L-nested and (29) hold. By Definition 21.4 if ζ is a successor and by Lemma 22.2 if ζ is limit, $\{P_L \cap B_\zeta, \tau_{P_L}[\zeta]|_{B_\zeta}\}$ is a topological Hurewicz test pair; in particular, $P_L \cap B_\zeta$ is $\tau_{P_L}[\zeta]|_{B_\zeta}$ meager in B_ζ. Since $A_\zeta \cap B_\zeta \subseteq P_L \cap B_\zeta$, $A_\zeta \cap B_\zeta$ is a $\tau_{P_L}[\zeta]|_{B_\zeta}$-meager $\Pi^0_2(\tau[\zeta]|_{B_\zeta})$ set. So by Lemma 22.3 we can fix a countable set

$$Y = \{x_n : n < \omega\} \subseteq B_\zeta \setminus Z^{B_\zeta, P_L \cap B_\zeta}(A_\zeta \cap B_\zeta)$$

such that Y is $\tau^<_{P_L}[\zeta]|_{B_\zeta}$-dense in the Polish space $(B_\zeta, \tau^<_{P_L}[\zeta]|_{B_\zeta})$. We define recursively B^ζ_n $(n < \omega)$ such that $\bigcup_{i<n} B^\zeta_i \neq B_\zeta$ $(n < \omega)$, $(\mathcal{B}^\alpha)_{\alpha<\zeta+1}$ is P_L-nested and (29) hold.

Suppose that we already have $B^\zeta_i \subseteq B_\zeta$ for $i < n$. Let $m < \omega$ be minimal with $x_m \notin \bigcup_{i<n} B^\zeta_i$ and take a basic $\tau_C[\zeta]|_{B_\zeta}$-open set $U \subseteq B_\zeta$ such that $x^\star = x_m \in U \subseteq B_\zeta \setminus \bigcup_{i<n} B^\zeta_i$ and $U \cup \bigcup_{i<n} B^\zeta_i \neq B_\zeta$. By Definition 21.4 if ζ is successor and by Lemma 22.2 if ζ is limit, we can apply Lemma 20 in the polish space $(X, \tau) = (B_\zeta, \tau_C[\zeta]|_{B_\zeta})$ for $x^\star$, U, $(P_s \cap B_\zeta)_{s\in\omega^{<\omega}}$ and $A_\zeta \cap B_\zeta$. Let $B^\zeta_n \subseteq B_\zeta$ be the resulting $\tau_C[\zeta]|_{B_\zeta}$-compact set. This defines recursively $\mathcal{B}^\zeta$.

We have $B^\zeta \cap A_\zeta = \emptyset$ so we have to check the conditions of Definition 21; 1 and 3 are obvious while 2 follows from $Y \subseteq B^\zeta$ using Lemma 22.1. If ζ is limit, 4 follows for $\alpha = \zeta$ from Lemma 22.2. If ζ is a successor then we have 4 for $\alpha = \xi$ by the induction hypothesis. Now we check 4 for $\alpha = \xi + 1$; (1) and (2) are automatic. To have (3), let $t \subset s$ and suppose that $G \cap P_s \neq \emptyset$ for some basic $\tau_C[\zeta + 1]|_{B_{\zeta+1}}$-open set G. Since B^ζ is $\tau^{<}_{P_L}[\zeta + 1]|_{B_{\zeta+1}}$-open and $\tau^{<}_{P_L}[\zeta + 1]|_{B_{\zeta+1}}$-dense, we can assume that G is of the form $G' \cap B^\zeta_n$ for some basic τ_C-open set G and $n < \omega$. But (3) holds in B^ζ_n, in particular $B^\zeta_n \cap G' \cap P_t \setminus P_s \neq \emptyset$, as required.

To see (4) fix an $s \in \omega^{<\omega}$ and a basic $\tau_C[\zeta + 1]|_{B_{\zeta+1}}$-open set G with $G \cap P_s \neq \emptyset$; as before, we can assume that $G = G' \cap B^\zeta_n$ for some basic τ_C-open set G' and $n < \omega$. Now (4) holds in B^ζ_n hence $B^\zeta_n \cap G' \cap \bigcup_{i<\omega} P_{s^\frown i} \neq \emptyset$ which proves the statement. This completes the recursive step and the proof of the special case.

For an arbitrary uncountable Polish space (X, τ) and Borel not $\Sigma^0_3(\tau)$ set P take a continuous one-to-one map $\varphi\colon (C, \tau_C) \to (X, \tau)$ such that $\varphi^{-1}(P) = P_L$. Construct $\mathcal{I}_{P_L}$ for the family

$$\mathcal{A}^{-1} = \{\varphi^{-1}(A)\colon A \in \mathcal{A}\} \subseteq \mathcal{S}^0_3(P_L).$$

Let

$$\mathcal{I} = \left\{G \subseteq X\colon \exists G' \in \mathcal{I}_{P_L}(G \cap \varphi(C) \subseteq \varphi(G'))\right\}.$$

Since homeomorphism preserves the Borel class of sets and $X \setminus \varphi(C)$ is τ-open hence $\Pi^0_2(\tau)$, this σ-ideal is strongly generated by its $\Pi^0_2(\tau)$ members. If $A \subseteq P$ is in $\mathcal{A}$ then $\varphi^{-1}(A) \subseteq P_L$ is in $\mathcal{A}^{-1}$ so $\varphi^{-1}(A) \in \mathcal{I}_{P_L}$ and hence $A \in \mathcal{I}$. This shows $\mathcal{A} \subseteq \mathcal{I}$. Since $P_L \notin \mathcal{I}_{P_L}$ we have $P \notin \mathcal{I}$, which completes the proof.■

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