On a σ-ideal of compact sets

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Abstract

We recall from [10] a G_δ σ-ideal of compact subsets of 2^ω and prove that it is not Tukey reducible to the ideal $\mathcal{I}_{1/n} = \{H \subseteq \omega \colon \sum_{h \in H} 1/h < \infty\}$. This result answers a question of S. Solecki and S. Todorčević in the negative.

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1. Complexity of σ-ideals of compact sets

Let X be a Polish space and let $\mathcal{K}(X)$ denote the family of compact subsets of X. A subfamily $\mathcal{I} \subseteq \mathcal{K}(X)$ is a *σ-ideal of compact subsets of X* if $\mathcal{I}$ has the following properties:

1. for every $K, L \in \mathcal{K}(X)$, $L \subseteq K \in \mathcal{I}$ implies $L \in \mathcal{I}$;
2. if $K_i \in \mathcal{I}$ $(i < \omega)$ and $\bigcup_{i<\omega} K_i \in \mathcal{K}(X)$ then $\bigcup_{i<\omega} K_i \in \mathcal{I}$.

That is, $\mathcal{I}$ is the restriction of an ordinary σ-ideal on X to $\mathcal{K}(X)$.

Many important families of compact sets form σ-ideals. Examples include the families of countable compact sets, compact meager sets or compact Lebesgue null sets, which unarguably play an important role in topology and real analysis. However, there are many other σ-ideals of compact sets originating from various branches of mathematics. Classical examples are the sets of uniqueness, the sets of extended uniqueness or other families of thin sets of harmonic analysis, the smooth sets for

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Borel equivalence relations, and others. For more examples, the interested reader can consult the recent survey [9]; the deep and fruitful interplay between harmonic analysis and the descriptive set theory of σ-ideals of compact sets is studied in [3]-[7]. So a thorough understanding of the "complexity" of σ-ideals of compact sets is beneficial for many areas of mathematics.

Of course, we have to specify what we mean by "complexity". The notion of complexity we are concerned with in the present note is Tukey reducibility, i.e. the comparison of cofinal types. We recall from [10] a G_δ σ-ideal of compact sets which turns out to be the most complicated object of its kind. It was designed to provide a counterexample to a question related to the descriptive set theoretic complexity of G_δ σ-ideals of compact sets (see [10] for more details), but as we will prove in this note, it is not Tukey reducible to the ideal $\mathcal{I}_{1/n} = \{H \subseteq \omega \colon \sum_{h \in H} 1/h < \infty\}$. This answers [13, Question 1 p. 1909] in the negative. We recall the relevant definitions and results in the following section. We refer to [2] for basic notions of descriptive set theory.

1.1. Tukey reducibility

The notion of complexity we consider in this note is related to the cofinal type of directed partial orders. We recall the relevant definitions following [1].

Definition 1.1. Let $(D, \leq)$ and $(E, \leq)$ be directed partial orders. We say $(D, \leq)$ is *Tukey reducible* to $(E, \leq)$, $(D, \leq) \leq_T (E, \leq)$ in notation, if there is a mapping $f \colon D \to E$ such that the images of unbounded subsets of D under f are unbounded in E. We write $(D, \leq) <_T (E, \leq)$ if $(D, \leq) \leq_T (E, \leq)$ but $(E, \leq) \not\leq_T (D, \leq)$.

In the sequel we will consider only one partial order on a given set, so we do not write out $\leq$. It can be shown that $D \leq_T E$ if and only if there is a function $g \colon E \to D$ such that the images of cofinal subsets of E under g are cofinal in D; thus requiring the existence of such a g may serve as an equivalent definition of Tukey reducibility.

The relevance of the concept of Tukey reducibility comes from the fact that the existence or nonexistence of a Tukey reduction between two directed partial orders relates many structural properties of the partial orders. To illustrate this, observe that it is immediate, using Definition 1.1, that $D \leq_T E$ implies $\mathrm{add}(E) \leq \mathrm{add}(D)$; while by the other equivalent definition, $D \leq_T E$ implies $\mathrm{cf}(D) \leq \mathrm{cf}(E)$ (see e.g. [1, Theorem 1J p. 180]). For a detailed exposition of these ideas and a very thorough survey on the Tukey reducibility between almost all the "natural" directed partial orders we refer to [1] and the references therein.

In the present note we would like to focus on the participants of the following picture. The arrangement on the picture originates from [13].

$$
\begin{array}{ccccc}
 & & (\mathcal{P}(\omega), \subseteq^\star) & & \\
 & & \downarrow & & \\
 & \mathcal{Z}_0 & \in \{\text{analytic } P\text{-ideals}\} \leq_T & \mathcal{I}_{1/n} & \\
 \nearrow_{<_T} & & & & \\
1 <_T \omega <_T \omega^\omega & \vee_T & & \vee_T & \\
 \searrow_{<_T} & & & & \\
 & \mathcal{N} \cap \mathcal{K}(2^\omega) & \leq_T & \mathcal{M} \cap \mathcal{K}(2^\omega) & <_T \mathfrak{I}
\end{array}
$$

Tukey picture

We introduce the notation and give the partial orders on the sets which appear above.

- 1 denotes the one element set with the trivial order;
- ω stands for the first infinite ordinal with its usual well-order;
- the order on ω^ω, the set of all functions from ω to ω, is the (not necessarily strict) dominance at every coordinate;
- $\mathcal{N}$ denotes the σ-ideal of Lebesgue null subsets of 2^ω, ordered by inclusion;
- $\mathcal{M}$ stands for the σ-ideal of meager subsets of 2^ω, also ordered by inclusion;
- $\mathcal{Z}_0 = \{H \subseteq \omega\colon \lim_{n<\omega} |H \cap n|/n = 0\}$ is an ideal, and the order on $\mathcal{Z}_0$ is the inclusion;
- $\mathcal{I}_{1/n} = \{H \subseteq \omega\colon \sum_{h \in H} 1/h < \infty\}$ is also an ideal, and the order on $\mathcal{I}_{1/n}$ is the inclusion;
- $\mathfrak{I}$ is the G_δ σ-ideal of compact subsets of 2^ω which will be presented in Section 2, ordered by inclusion.

The Tukey picture summarizes the following results.

- The nontrivial parts of $1 <_T \omega <_T \omega^\omega <_T \mathcal{Z}_0 <_T \mathcal{I}_{1/n}$ can be found in [1, Proposition 3K p. 208], [1, Proposition p. 211] and [8, Theorem 7 p. 187].

- For $\omega^\omega <_T \mathcal{N} \cap \mathcal{K}(2^\omega) \leq_T \mathcal{M} \cap \mathcal{K}(2^\omega) <_T \mathcal{I}_{1/n}$ we refer to [1, Theorem 3B p. 198], [1, Corollary 3E p. 202] and [1, Proposition p. 211].

- The ideal $\mathcal{I}_{1/n}$ is Tukey maximal among all analytic P-ideals (see [8, Theorem 5 p. 181]). We remark that $\mathcal{I}_{1/n}$ is ω-Tukey equivalent with $\mathcal{N}$ (see [1, Theorem 2F p. 191] for the details), so roughly speaking, $\mathcal{I}_{1/n}$ and the σ-ideal of Lebesgue null sets have the same cofinal types. Before the appearance of $\mathfrak{I}$, $\mathcal{M} \cap \mathcal{K}(2^\omega)$ was Tukey maximal among all *known* analytic σ-ideals of compact sets.

- As a substantial improvement of earlier results, it was shown in [13, Corollary 6.4 p. 1895] that no "nontrivial" analytic P-ideal is Tukey below a σ-ideal of compact sets. Furthermore, we also have $\mathcal{N} \cap \mathcal{K}(2^\omega) <_T \mathcal{Z}_0$ (see [1, Proposition 3K p. 208]). These results motivate the arrangement that puts analytic P-ideals above σ-ideals of compact sets.

- On the top of our Tukey picture, $(\mathcal{P}(\omega), \subseteq^\star) \hookrightarrow \{\text{analytic } P\text{-ideals}\}$ refers to [8, Theorem 6 p. 183]. It states that the set of all subsets of ω partially ordered by essential inclusion can be embedded into the set of analytic P-ideals partially ordered by Tukey reducibility. Thus analytic P-ideals are rich in cofinal types.

- We will show $\mathcal{M} \cap \mathcal{K}(2^\omega) <_T \mathfrak{I}$ in Section 3. This gives a negative answer to [8, Question 3 p. 194]. We remark that independently of our result, a G_δ σ-ideal of compact sets, constructed in [12], was proved in [11] to be strictly above $\mathcal{M} \cap \mathcal{K}(2^\omega)$ in the Tukey order. Since, as far as nonreduction is concerned, instead of $\mathfrak{I} \not\leq_T \mathcal{M} \cap \mathcal{K}(2^\omega)$ we will prove the stronger $\mathfrak{I} \not\leq_T \mathcal{I}_{1/n}$, we also give a negative answer to [13, Question 1 p. 1909].

It would be nice to conclude this introductory section by recalling the open problems around Tukey reducibility of directed partial orders. However, we will not do this since so many problems are open, and because the area is in an initial stage of research that evolves too quickly. We only remark that it is open whether $\mathcal{N} \cap \mathcal{K}(2^\omega)$ is strictly below $\mathcal{M} \cap \mathcal{K}(2^\omega)$ in the Tukey order[1], and we refer to [1], [8], [11] and [13] for open problems.

[1] Recently we obtained $\mathcal{M} \cap \mathcal{K}(2^\omega) \not\leq_T \mathcal{N} \cap \mathcal{K}(2^\omega)$ by methods similar to the proof of Theorem 3.1 below.

2. The construction of $\mathfrak{I}$

In this section, we present the construction of $\mathfrak{I}$. We aim to construct the complement of our G_δ σ-ideal $\mathfrak{I}$. Technically, we perform this construction by coding perfect subsets of 2^ω using perfect trees on $2^{<\omega}$ the usual way (see e.g. [2, Section 2.B p. 7]).

We construct the perfect trees coding the perfect members of $\mathcal{K}(2^\omega) \setminus \mathfrak{I}$ by successively extending finite trees. At each step we extend our finite tree by attaching finite trees to its terminal nodes. The heart of the construction is to specify which particular finite trees can be used for the extension.

Intuitively, the complement of a σ-ideal should consist of "large" sets. Therefore our construction should reflect this expectation. As we will see, the perfect trees coding the perfect members of $\mathcal{K}(2^\omega) \setminus \mathfrak{I}$ will be large because every finite tree used in the extension procedure is either "dense" or "concentrated". We return to this heuristics after having presented the construction.

We recall some notation following [2]. For every $s, t \in 2^{<\omega}$, $|s|$ denotes the length of s and $s^\frown t$ stands for the sequence $s(0) \dots s(|s|-1)t(0) \dots t(|t|-1)$. If $T \subseteq 2^{<\omega}$ is a tree

(i) the maximal branches, or terminal nodes, of T are denoted by $\mathfrak{T}(T)$;

(ii) for $s \in 2^{<\omega}$, $T_s = \{t \in 2^{<\omega} \colon s^\frown t \in T\}$ and $s^\frown T = \{t \in 2^\omega \colon \exists u \in T \ (t \sqsubseteq s^\frown u)\}$;

(iii) $[T] = \{x \in 2^\omega \colon \forall n < \omega \ (x|_n \in T)\}$.

For every $n < \omega$, we identify 2^n with the maximal branches of the full binary tree $2^{<n}$, i.e. $2^n = \mathfrak{T}(2^{<n})$, indexed according to the lexicographic order. If $\sigma \colon 2^n \to 2$ is given, $T(\sigma)$ is the subtree of $2^{<n}$ generated by $\bigcup \sigma^{-1}(1)$.

For $2 \le n < \omega$ and $\sigma \colon 2^n \to 2$ we set (see (1) below)

$$g_l(\sigma) = \min\{i < 2^n \colon \sigma(i) = 1\},$$

$$g_r(\sigma) = 2^n - 1 - \max\{i < 2^n \colon \sigma(i) = 1\},$$

$$b(\sigma) = \max\{d \le 2^n \colon \forall i \in [g_l(\sigma), g_l(\sigma) + d) \ (\sigma(i) = 1)\},$$

and let $n(\sigma)$ denote the length of the longest sequence of consecutive 0s in $[2^{n-2}, 2^n - 2^{n-2} - 1]$.

$$n = 5: \qquad \sigma = \underbrace{000}_{g_l(\sigma)=3}\,\underbrace{1111}_{b(\sigma)=4}\,001\,\underbrace{00}_{n(\sigma)=2}\,11011101111011\,\underbrace{000000}_{g_r(\sigma)=6} \tag{1}$$

For every $2 \leq n < \omega$ set

$$\Sigma_n = \{\sigma \in 2^{2^n} : g_l(\sigma) \leq 2^{n-2},\ g_r(\sigma) \leq 2^{n-2} \text{ and } n(\sigma) \leq b(\sigma)\}; \tag{2}$$

e.g. the sequence of (1) is in Σ_5.

We define our σ-ideal $\mathfrak{I}$ as follows. Consider the following inductive construction of a sequence of finite trees $T^n \subseteq 2^{<\omega}$ ($n < \omega$). Set $T^0 = \{\emptyset\}$; let $n < \omega$ and suppose that T^n is already defined. For every $t \in \mathfrak{T}(T^n)$ take an arbitrary $2 \leq m(t) < \omega$ and pick an arbitrary sequence $\sigma_t \in \Sigma_{m(t)}$. We define T^{n+1} by extending T^n at every $t \in \mathfrak{T}(T^n)$ by $T(\sigma_t)$, that is

$$T^{n+1} = \{u \in 2^{<\omega} : u \sqsubseteq t^\frown s,\ t \in \mathfrak{T}(T^n), s \in T(\sigma_t)\}.$$

Such a sequence $(T^n)_{n<\omega}$ is called *admissible*. A tree $T \subseteq 2^{<\omega}$ is *admissible* if $T = \bigcup_{n<\omega} T^n$ for some admissible sequence $(T^n)_{n<\omega}$. For every $s \in 2^{<\omega}$ set

$$\mathcal{P}_s = \{[s^\frown T] : T \text{ is admissible}\}. \tag{3}$$

For every $\mathcal{P} \subseteq \mathcal{K}(2^\omega)$ let

$$\mathcal{P}^\uparrow = \{A \in \mathcal{K}(2^\omega) : \exists P \in \mathcal{P}\ (P \subseteq A)\}. \tag{4}$$

We define

$$\mathfrak{I} = \mathcal{K}(2^\omega) \setminus \bigcup_{s \in 2^{<\omega}} \mathcal{P}_s^\uparrow. \tag{5}$$

We close this section by finishing the heuristic qualitative analysis above. As we have seen, those finite trees can be used in the extension procedure which have their coding sequence σ in Σ_n for some $2 \leq n < \omega$. Fix such an n and let $\sigma \in \Sigma_n$. If $b(\sigma)$ is small compared to 2^n then so is $n(\sigma)$. So in $[2^{n-2}, 2^n - 2^{n-2} - 1]$, σ often has 1s, which can be interpreted as a sort of density for $T(\sigma)$. On the other hand, if $b(\sigma)$ is of the order of 2^n, then the first block of 1s in σ is long, which translates to a concentration of branches in $T(\sigma)$. Therefore the extending finite trees are large because they are either dense or concentrated.

Observe that $\mathcal{M} \cap \mathcal{K}(2^\omega)$ and $\mathcal{N} \cap \mathcal{K}(2^\omega)$ can be obtained by using the construction scheme presented above, as follows. We get $\mathcal{M} \cap \mathcal{K}(2^\omega)$ if for every $n < \omega$, Σ_n contains only the constant 1 sequence; i.e. the only admissible tree is the complete binary tree $2^{<\omega}$. Then it is clear that

$$\mathcal{M} \cap \mathcal{K}(2^\omega) = \mathcal{K}(2^\omega) \setminus \bigcup_{s \in 2^{<\omega}} \{[s^\frown 2^{<\omega}]\}^\uparrow.$$

Consequently, requiring only concentration leads to $\mathcal{M} \cap \mathcal{K}(2^\omega)$. Similarly, we get $\mathcal{N} \cap \mathcal{K}(2^\omega)$ by putting appropriate conditions on the density of 1s in the sequences of the Σ_ns and on the heights of the extending trees. So in a sense, $\mathfrak{I}$ is a combination of the measure and category ideals.

3. $\mathfrak{J}$ on the Tukey picture

In this section we prove the following result.

Theorem 3.1. *With the notation of Section 1.1,*

1. $\mathfrak{J} \not\leq_T \mathcal{I}_{1/n}$;
2. $\mathcal{M} \cap \mathcal{K}(2^\omega) <_T \mathfrak{J}$.

It is the nonreduction result which requires a more involved argument. It is based on the observation that the following property is preserved under Tukey reducibility.

Definition 3.2. Let $(P, \leq)$ be a partially ordered set. We say $(P, \leq)$ has the $\mathcal{D}$-*property* if there is a function $D \colon \omega^{<\omega} \to 2^P$ such that

(i) $D(\emptyset) = P$;

(ii) for every $a \in \omega^{<\omega}$, $D(a) = \bigcup_{n<\omega} D(a^\frown n)$;

(iii) if $\varphi \colon 2^{<\omega} \to \omega^{<\omega}$ is strictly increasing then for every $d \colon 2^{<\omega} \to P$ satisfying $d(s) \in D(\varphi(s))$ $(s \in 2^{<\omega})$ we have $\{d(s) \colon s \in 2^{<\omega}\}$ is bounded.

Proposition 3.3. *If $(P, \leq)$ and $(Q, \leq)$ are directed partial orders, $(Q, \leq)$ has the $\mathcal{D}$-property and $P \leq_T Q$ then $(P, \leq)$ has the $\mathcal{D}$-property, as well.*

Proof. Let $f \colon P \to Q$ be a Tukey map. Let $D_Q \colon \omega^{<\omega} \to 2^Q$ witness the $\mathcal{D}$-property of Q. Set $D_P \colon \omega^{<\omega} \to 2^P$, $D_P(a) = f^{-1}(D_Q(a))$ $(a \in \omega^{<\omega})$. Then (i) holds, and by

$$D_P(a) = f^{-1}(D_Q(a)) = f^{-1}\left(\bigcup_{n<\omega} D_Q(a^\frown n)\right) = \bigcup_{n<\omega} f^{-1}(D_Q(a^\frown n)) = \bigcup_{n<\omega} D_P(a^\frown n),$$

D_P satisfies (ii). For (iii), let φ and d_P be as prescribed. Then $d_Q \colon 2^{<\omega} \to Q$, $d_Q(s) = f(d_P(s))$ $(s \in 2^{<\omega})$ satisfies $d_Q(s) \in D_Q(\varphi(s))$ $(s \in 2^{<\omega})$ hence $\{d_Q(s) \colon s \in 2^{<\omega}\}$ is bounded. Since f is a Tukey map,

$$\{d_P(s) \colon s \in 2^{<\omega}\} \subseteq \{f^{-1}(d_Q(s)) \colon s \in 2^{<\omega}\}$$

is also bounded, as required. This completes the proof.■

First we show that $\mathcal{I}_{1/n}$ has the $\mathcal{D}$-property. For every $H \subseteq \omega$, we define $\|H\| = \sum_{h \in H} 1/h$; thus $\mathcal{I}_{1/n} = \{H \subseteq \omega \colon \|H\| < \infty\}$.

Lemma 3.4. *$\mathcal{I}_{1/n}$ has the $\mathcal{D}$-property.*

Proof. Define $h\colon \omega^{<\omega} \to [\omega]^{<\omega}$ such that $h(\emptyset) = \emptyset$ and for every $a \in \omega^{<\omega}$,

$$\{h(a^\frown n)\colon n < \omega\} = \{h(a) \cup B\colon B \in [\omega]^{<\omega}\}. \tag{6}$$

For every $a \in \omega^{<\omega}$ we define $D(a) \subseteq \mathcal{I}_{1/n}$ by induction on $|a|$, as follows. Set $D(\emptyset) = \mathcal{I}_{1/n}$. Let $a \in \omega^{<\omega}$ be such that $D(a)$ is already defined. For every $n < \omega$ set

$$D(a^\frown n) = \{H \in D(a)\colon h(a^\frown n) \subseteq H,\ \|H \setminus h(a^\frown n)\| \leq 1/3^{|a|+1}\}.$$

Then (i) holds. For (ii) observe that by (6), for every $H \in D(a)$ there is an $n < \omega$ such that $h(a^\frown n) \subseteq H$ and $\|H \setminus h(a^\frown n)\| \leq 1/3^{|a|+1}$, so the statement follows.

It remains to see (iii). Let φ and d be as in the statement. For every $s \in 2^{<\omega} \setminus \{\emptyset\}$ and $i \in 2$, $d(s) \in D(\varphi(s))$ and $d(s^\frown i) \in D(\varphi(s^\frown i)) \subseteq D(\varphi(s))$ so $d(s^\frown i) \in D(\varphi(s))$. Hence $h(\varphi(s)) \subseteq d(s)$ and $h(\varphi(s)) \subseteq d(s^\frown i)$, moreover $\|d(s^\frown i) \setminus h(\varphi(s))\| \leq 1/3^{|\varphi(s)|}$. This shows $\|d(s^\frown i) \setminus d(s)\| \leq 1/3^{|\varphi(s)|} \leq 1/3^{|s|}$ for every $s \in 2^{<\omega} \setminus \{\emptyset\}$ and $i \in 2$. So for $M = \bigcup_{s \in 2^{<\omega}} d(s)$,

$$\|M\| \leq \|d(\emptyset)\| + \|d(0)\| + \|d(1)\| + \sum_{n<\omega} 2^{n+1}/3^n < \infty;$$

i.e. $\{d(s)\colon s \in 2^{<\omega}\}$ is bounded. This completes the proof.■

Next we show that $\mathfrak{I}$ does not have the $\mathcal{D}$-property. We introduce some notation in advance. For every $s \in 2^{<\omega}$ we set $N_s = \{x \in 2^\omega\colon s \sqsubseteq x\}$. For every $i \in \omega$ and $S \subseteq 2^i$, we define $N(S) = \bigcup_{s \in S} N_s$ and

$$U(S) = \{K \in \mathcal{K}(2^\omega)\colon K \subseteq N_S,\ K \cap N_s \neq \emptyset\ (s \in S)\}.$$

Lemma 3.5. *$\mathfrak{I}$ does not have the $\mathcal{D}$-property.*

Proof. In the sequel the abbreviation s.c.e. stands for "of second category everywhere". Suppose $D\colon \omega^{<\omega} \to 2^{\mathfrak{I}}$ satisfies (i)-(ii); we show that (iii) fails, i.e. we construct a strictly increasing $\varphi\colon 2^{<\omega} \to \omega^{<\omega}$ such that for some $d\colon 2^{<\omega} \to \mathfrak{I}$ satisfying $d(s) \in D(\varphi(s))$ $(s \in 2^{<\omega})$ we have $\{d(s)\colon s \in 2^{<\omega}\}$ is unbounded.

Before discussing technical details, let us summarize the strategy of the proof. By the definition of $\mathfrak{I}$ in (5) of Section 2, $\{d(s)\colon s \in 2^{<\omega}\}$ is unbounded if and only if the there is an $r \in 2^{<\omega}$ and a $P \in \mathcal{P}_r$ such that $P \subseteq \mathrm{cl}_{2^\omega}(\bigcup_{s \in 2^{<\omega}} d(s))$.

Of course, P will be defined as $P = [r^\frown T]$ for an admissible tree $T \subseteq 2^{<\omega}$. We obtain T as a union of an admissible sequence $(T^n)_{n<\omega}$. For every $n < \omega$, the extension of T^n to T^{n+1} will be made in two steps. The first step will guarantee that at each terminal node $t \in \mathfrak{T}(T^n)$, T_t^{n+1} is a tree $T(\sigma_t)$ with $g_l(\sigma_t) \le 2^{m(t)-2}$, $g_r(\sigma_t) \le 2^{m(t)-2}$. The second extension step will be responsible for $n(\sigma_t) \le b(\sigma_t)$. We turn to the details.

In addition to T^n $(n < \omega)$ and $\sigma_t \in \Sigma_{m(t)}$ $(t \in \mathfrak{T}(T^n),\ n < \omega)$, for every $s \in 2^{<\omega}$ we define $\varphi(s) \in \omega^{<\omega}$, $k(s) < \omega$ and $Z_s \subseteq 2^{k(s)}$ by induction on $|s|$ such that

(I) $(T^n)_{n<\omega}$ is an admissible sequence;

(II) $\mathfrak{T}(r^\frown T^n) \subseteq \bigcup_{s\in 2^n} Z_s$ $(n < \omega)$;

(III) $D(\varphi(s))$ is s.c.e. in $U(Z_s)$.

Let first $s = \emptyset$. By (i) and (ii), $\mathfrak{I} = \bigcup_{n<\omega} D(n)$. Thus there are $k(\emptyset) < \omega$ and $Z_\emptyset \subseteq 2^{k(\emptyset)}$ such that for some $n_\emptyset < \omega$, $D(n_\emptyset)$ is s.c.e. in $U(Z_\emptyset)$. We set $\varphi(\emptyset) = n_\emptyset$, let $r \in Z_\emptyset$ be arbitrary and let $T^0 = \{\emptyset\}$. Then (II) and (III) hold for $n = 0$.

Suppose that for an $n < \omega$, for every $s \in 2^n$, $k(s) < \omega$, $Z_s \subseteq 2^{k(s)}$, $\varphi(s)$ and T^n are defined such that (II) and (III) hold. Take an arbitrary $s \in 2^n$. By (ii), $D(\varphi(s)) = \bigcup_{n<\omega} D(\varphi(s)^\frown n)$. By (III), $D(\varphi(s))$ is s.c.e. in $U(Z_s)$ hence there are $n_{s^\frown 0} < \omega$, $k^+(s) \in \omega \setminus k(s)$ and $Z_s^+ \subseteq 2^{k^+(s)}$ such that we have

(a$_0$) for every $t \in Z_s \cap \mathfrak{T}(r^\frown T^n)$ there are $z, z' \in Z_s^+$ with $t^\frown(00) \sqsubseteq z$, $t^\frown(11) \sqsubseteq z'$;

(b$_0$) $D(\varphi(s)^\frown n_{s^\frown 0})$ is s.c.e. in $U(Z_s^+)$.

For every $t \in Z_s \cap \mathfrak{T}(r^\frown T^n)$, we denote by z_t the leftmost $z \in Z_s^+$ satisfying $t^\frown(00) \sqsubseteq z$.

Similarly, there are $n_{s^\frown 1} < \omega$, $k(s^\frown 1) \in \omega \setminus (k^+(s) + 2)$ and $Z_{s^\frown 1} \subseteq 2^{k(s^\frown 1)}$ such that

(a$_1$) for every $t \in Z_s \cap \mathfrak{T}(r^\frown T^n)$ and $t' \in 2^{k^+(s)+1-k(s)}$ with $z_t \le_{\text{lex}} t^\frown t'$, there is a $z \in Z_{s^\frown 1}$ with $t^\frown t' \sqsubseteq z$;

(b$_1$) $D(\varphi(s)^\frown n_{s^\frown 1})$ is s.c.e. in $U(Z_{s^\frown 1})$.

We set $\varphi(s^\frown i) = \varphi(s)^\frown n_{s^\frown i}$ $(i < 2)$, $k(s^\frown 0) = k(s^\frown 1)$ and

$$Z_{s^\frown 0} = \{j \in 2^{k(s^\frown 0)} \colon \exists i \in Z_s^+ \ (i \subseteq j)\}. \tag{7}$$

For every $t \in \mathfrak{T}(T^n)$, say $r^\frown t \in Z_s$, let $m(t) = k(s^\frown 1) - k(s)$ and define $\sigma_t \colon 2^{m(t)} \to 2$ by

$$\sigma_t(i) = 1 \Leftrightarrow z_t \le_{\text{lex}} r^\frown t^\frown i \text{ and } r^\frown t^\frown i \in Z_{s^\frown 0} \cup Z_{s^\frown 1}. \tag{8}$$

This completes the inductive step of the construction.

By (8) we have (II) for $n+1$. Since $U(Z_{s^\frown 0}) \subseteq U(Z_s^+)$ by (7), (b_0) and (b_1) imply (III). It remains to see (I). For this, all we have to show is that for every $t \in \mathfrak{T}(T^n)$, $\sigma_t \in \Sigma_{m(t)}$.

Fix such a t, say $r^\frown t \in Z_s$. By (a_0) and (7), $r^\frown t^\frown(00) \sqsubseteq z_t$ and there is $z' \in Z_{s^\frown 0}$ such that $r^\frown t^\frown(11) \sqsubseteq z'$. So by (8), there are $j, j' \leq 2^{m(t)-2}$ such that $\sigma_t(j) = \sigma_t(2^{m(t)} - 1 - j') = 1$. Hence $g_l(\sigma_t) \leq 2^{m(t)-2}$, $g_r(\sigma_t) \leq 2^{m(t)-2}$.

By (7) and (8), we have $b(\sigma_t) \geq 2^{k(s^\frown 1)-k^+(s)}$. By ($a_1$), $n(\sigma_t) < 2 \cdot 2^{k(s^\frown 1)-k^+(s)-1}$. So we have $\sigma_t \in \Sigma_{m(t)}$, as required.

Let $d\colon 2^{<\omega} \to P$ be arbitrary satisfying $d(s) \in D(\varphi(s)) \cap U(Z_s)$ ($s \in 2^{<\omega}$); this is possible by (III). Then by (II), $P = [r^\frown T]$ satisfies $P \subseteq \mathrm{cl}_{2^\omega}(\bigcup_{s \in 2^{<\omega}} d(s))$. Since $P \notin \mathfrak{I}$ by (I), we have $\{d(s)\colon s \in 2^{<\omega}\}$ unbounded, as required.■

Proof of Theorem 3.1. Statement 1 follows from Proposition 3.3, Lemma 3.4 and Lemma 3.5. By $\mathcal{M} \cap \mathcal{K}(2^\omega) <_T \mathcal{I}_{1/n}$, $\mathfrak{I} \not\leq_T \mathcal{M} \cap \mathcal{K}(2^\omega)$ follows. So for statement 2 it remains to show $\mathcal{M} \cap \mathcal{K}(2^\omega) \leq_T \mathfrak{I}$.

We define an admissible tree S, as follows. Let $S^0 = \{\emptyset\}$. Let $n < \omega$ and suppose that S^n is already defined. For every $t \in \mathfrak{T}(S^n)$, let $m(t) = 2$, $\sigma_t = \{(01), (10)\}$; then $\sigma_t \in \Sigma_2$. We define S^{n+1} by extending S^n at every $t \in \mathfrak{T}(S^n)$ by σ_t. Let $S = \bigcup_{n<\omega} S^n$ and $P = [S]$. We show $\mathfrak{I} \cap \mathcal{K}(P)$ coincides with the σ-ideal of compact meager subsets of P (see the proof of [11, Theorem 3.1] for a similar argument).

Observe that there is no $4 \leq m < \omega$ and $\sigma \in \Sigma_m$ such that $S \cap 2^{\leq m} = T(\sigma)$. Thus its defining sequence $(S^n)_{n<\omega}$ is the unique decomposition of S into the union of an admissible sequence of finite trees, and no proper subtree of S is admissible. Thus no proper compact subset of P is in $\mathcal{P}_\emptyset = \{[T]\colon T \text{ is admissible}\}$. Therefore, by the definition of $\mathfrak{I}$ in (5) and by the self-similarity of $[S]$, $K \in \mathcal{K}(P)$ is not in $\mathfrak{I}$ if and only if K has nonempty interior relative to P. This completes the proof.■

References

[1] D. H. Fremlin, *The Partially Ordered Sets of Measure Theory and Tukeys Ordering*, Note Mat., Vol. XI (1991), 177–214.

[2] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics 156 (Springer-Verlag, 1994).

[3] A. S. Kechris, *Hereditary properties of the class of closed sets of uniqueness for trigonometric series*, Israel J. Math. 73 (1991), no. 2, 189–198.

[4] A. S. Kechris, *The descriptive set theory of σ-ideals of compact sets*, Logic Colloq. '88 (Padova, 1988), 117–138, Stud. Logic Found. Math., 127, North-Holland, Amsterdam, 1989.

[5] A. S. Kechris, A. Louveau, *Descriptive Set Theory and Harmonic Analysis*, J. Symb. Logic, Vol. 57, No. 2 (1992), 413–441.

[6] A. S. Kechris, A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, London Math. Soc. Lecture Note Series 128, Cambridge University Press.

[7] A. S. Kechris, A. Louveau, W. H. Woodin, *The structure of σ-ideals of compact sets*, Trans. Amer. Math. Soc. 301 (1987), no. 1, 263–288.

[8] A. Louveau, B. Veličković, *Analytic Ideals and Cofinal Types*, Ann. Pure Appl. Logic 99, No. 1-3 (1999), 171–195.

[9] E. Matheron, M. Zelený, *Descriptive set theory of families of small sets*, Bull. Symb. Logic 13 (2007), No. 4, 482–537.

[10] T. Mátrai, *Kenilworth*, Proc. Amer. Math. Soc. 137, No. 3 (2009), 1115–1125.

[11] J. T. Moore, S. Solecki, *A G_δ Ideal of Compact Sets Strictly Above the Nowhere Dense Ideal in the Tukey Order*, Ann. Pure Appl. Logic 156, No. 2-3 (2008), 270–273.

[12] S. Solecki, *G_δ Ideals of Compact Sets*, J. Eur. Math. Soc., to appear.

[13] S. Solecki, S. Todorčević. *Cofinal Types of Topological Directed Orders*, Ann. Inst. Fourier (Grenoble) 54, No. 6 (2004), 1877–1911.