

Tamás Mátrai,* University of Toronto, 40 St George St., Toronto, ON,
Canada, M5S 2E4. email: matrait@renyi.hu

ON ℓ^p -LIKE EQUIVALENCE RELATIONS

Abstract

For $f: [0, 1] \rightarrow \mathbb{R}^+$, consider the relation \mathbf{E}_f on $[0, 1]^\omega$ defined by $(x_n)\mathbf{E}_f(y_n) \Leftrightarrow \sum_{n < \omega} f(|y_n - x_n|) < \infty$. We study the Borel reducibility of Borel equivalence relations of the form \mathbf{E}_f . Our results indicate that for every $1 \leq p < q < \infty$, the order \leq_B of Borel reducibility on the set of equivalence relations $\{\mathbf{E}: \mathbf{E}_{\text{Id}^p} \leq_B \mathbf{E} \leq_B \mathbf{E}_{\text{Id}^q}\}$ is more complicated than expected, e.g. consistently every linear order of cardinality continuum embeds into it.

1 Introduction

Let $f: [0, 1] \rightarrow \mathbb{R}^+$ be an arbitrary function and consider the relation \mathbf{E}_f on $[0, 1]^\omega$ defined by setting, for every $(x_n)_{n < \omega}, (y_n)_{n < \omega} \in [0, 1]^\omega$,

$$(x_n)\mathbf{E}_f(y_n) \Leftrightarrow \sum_{n < \omega} f(|y_n - x_n|) < \infty. \quad (1)$$

Several natural questions arise, e.g.

- (i) when is \mathbf{E}_f an equivalence relation?
- (ii) which equivalence relations can be obtained in the form \mathbf{E}_f ?
- (iii) for what $f, g: [0, 1] \rightarrow [0, 1]$ is \mathbf{E}_f Borel reducible to \mathbf{E}_g ?

Mathematical Reviews subject classification: Primary: 03E15; Secondary: 46A45

Key words: Borel equivalence relation, Borel reduction, ℓ^p space, Lipschitz embedding,

E_1 *The research was partially supported by the OTKA grants K 61600, K 49786 and K 72655. We thank Máté Vizer for the inspiring discussions. We also thank Longyun Ding and the referee for pointing out an error in an earlier version of this paper.

In the present paper we answer (i), we initiate a study of (ii) and we obtain various conditions for (iii).

The prototypes of equivalence relations of the form \mathbf{E}_f are induced by the Banach spaces ℓ^p ($1 \leq p < \infty$), i.e. they are defined by the functions $f = \text{Id}^p$ for $1 \leq p < \infty$, where $\text{Id}: [0, 1] \rightarrow [0, 1]$ is the identity function. The Borel reducibility among these equivalence relations is fully described by a classical result of R. Dougherty and G. Hjorth [3, Theorem 1.1 p. 1836 and Theorem 2.2 p. 1840] stating that for every $1 \leq p, q < \infty$,

$$\mathbf{E}_{\text{Id}^p} \leq_B \mathbf{E}_{\text{Id}^q} \Leftrightarrow p \leq q. \quad (2)$$

We note, however, that e.g. for the function $f(0) = 0$, $f(x) = 1$ ($0 < x \leq 1$) we have \mathbf{E}_f is the equivalence relation of eventual equality on $[0, 1]^\omega$, also denoted by E_1 in the literature; that is, the investigation of equivalence relations of the form \mathbf{E}_f concerns equivalence relations which are not necessarily reducible to \mathbf{E}_{Id^p} for some $1 \leq p < \infty$.

Our investigations were motivated by a question of S. Gao in [4] p. 74, asking whether for $1 \leq p < \infty$, \mathbf{E}_{Id^p} is the greatest lower bound of $\{\mathbf{E}_{\text{Id}^q} : p < q < \infty\}$; we note that formally the question in [4] p. 74 refers to equivalence relations on \mathbb{R}^ω , but as we will see later in Lemma 3, the two formulations are equivalent. We answer this question in the negative by showing, for fixed $1 \leq p < \infty$, that $\mathbf{E}_{\text{Id}^p} <_B \mathbf{E}_f <_B \mathbf{E}_{\text{Id}^q}$ for every $q > p$ whenever

$$\lim_{x \rightarrow +0} \frac{f(x)}{x^p} = 0 \text{ and } \lim_{x \rightarrow +0} \frac{f(x)}{x^q} = \infty \text{ (} p < q < \infty \text{)}, \quad (3)$$

and f satisfies some additional technical assumptions (see e.g. Corollary 27). However, toward this result we aim to carry out a general study of the relations \mathbf{E}_f and their Borel reducibility. To this end, in Section 2 we characterize the functions for which \mathbf{E}_f is an equivalence relation and, roughly speaking, we show that f is continuous if and only if $E_1 \not\leq_B \mathbf{E}_f$. In Section 3 and in Section 4 we prove general reducibility and nonreducibility results for equivalence relations of the form \mathbf{E}_f . The results of these sections heavily build on techniques developed in [3]. Finally, in Section 5 we conclude our investigations by applying the technical results of the previous sections to concrete functions; in particular, we answer the above mentioned question of S. Gao, and we show that for $1 \leq p < q < \infty$, every linear order which embeds into $(\mathcal{P}(\omega)/\text{fin}, \subset)$ also embeds into the set of equivalence relations $\{\mathbf{E}_f : \mathbf{E}_{\text{Id}^p} \leq_B \mathbf{E}_f \leq_B \mathbf{E}_{\text{Id}^q}\}$ ordered by $<_B$.

Our results produce just examples. We are far from giving a full description of the Borel equivalence relations of the form \mathbf{E}_f or a complete picture of the Borel reducibility relation among the \mathbf{E}_f s. In particular, it remains open

whether there are two functions f and g such that \mathbf{E}_f and \mathbf{E}_g are incomparable under \leq_B . Nevertheless, we have one qualitative observation. Conditions (2) and (3) may suggest that reducibility among the \mathbf{E}_f s is essentially governed by the growth order of the f s. However, this is far from being true. As we will see in Section 3, under mild additional assumptions on f we have e.g. $\mathbf{E}_{\text{Id}^p} \leq_B \mathbf{E}_f$ whenever $\lim_{x \rightarrow +0} f(x)/x^{p-\varepsilon} = 0$ for every $\varepsilon > 0$ (for the precise statement, see Theorem 15); this is in contrast with (3).

For basic terminology in descriptive set theory we refer to [5]. As above, if X and Y are Polish spaces, E and F are equivalence relations on X and Y , then we say E is *Borel reducible to F* , $E \leq_B F$ in notation, if there exists a Borel function $\vartheta: X \rightarrow Y$ satisfying

$$xE x' \Leftrightarrow \vartheta(x)F\vartheta(x').$$

We say E and F are *Borel equivalent* if $E \leq_B F$ and $F \leq_B E$, while we write $E <_B F$ if $E \leq_B F$ but $F \not\leq_B E$.

Depending on the context, $|\cdot|$ denotes the absolute value of a real number, the length of a sequence or the cardinality of a set; $\lfloor \cdot \rfloor$ and $\{\cdot\}$ stand for lower integer part and fractional part. We denote by \mathbb{Z} and \mathbb{R}^+ the set of integers and nonnegative reals.

2 Basic properties

Definition 1. Let $(G, +)$ be an Abelian group and let $H \subseteq G$ satisfy

(H_1) $0 \in H$;

(H_2) for every $x, y \in H$, $x - y \in H$ or $y - x \in H$;

(H_3) for every $x, y, z \in H$, $x - y \in H$ and $y - z \in H$ implies $x - z \in H$.

For every $x \in H \cup -H$, let $x^+ = x$ if $x \in H$ and $x^+ = -x$ if $x \in -H \setminus H$.

For every function $f: H \rightarrow \mathbb{R}^+$, we define the relation \mathbf{E}_f on H^ω by setting, for every $(x_n)_{n < \omega}, (y_n)_{n < \omega} \in H^\omega$,

$$(x_n)\mathbf{E}_f(y_n) \Leftrightarrow \sum_{n < \omega} f((y_n - x_n)^+) < \infty; \quad (4)$$

the definition is valid by (H_2).

We say $f: H \rightarrow \mathbb{R}^+$ is *even* if for every $x \in H \cap -H$, $f(x) = f(-x)$.

Observe that for $\tilde{f}: H \rightarrow [0, 1]$, $\tilde{f}(x) = \min\{f(x), 1\}$ ($x \in H$) we have $\mathbf{E}_{\tilde{f}} = \mathbf{E}_f$. So in the sequel we only consider bounded functions.

We start this section by characterizing the bounded functions $f: H \rightarrow \mathbb{R}^+$ for which \mathbf{E}_f is an equivalence relation. To avoid a meticulous bookkeeping of non-relevant constants, we will use the terminology “by (\star) , $A \lesssim B$ ” to abbreviate that “by property (\star) , there is a constant $C > 0$ depending on the parameters of (\star) such that $A \leq CB$ ”. The relations \gtrsim and \approx are defined analogously.

Proposition 2. *Let $f: H \rightarrow \mathbb{R}^+$ be a bounded even function. Let \mathbf{E}_f be the relation on H^ω defined by (4). Then \mathbf{E}_f is an equivalence relation if and only if the following conditions hold:*

$$(R_1) \quad f(0) = 0;$$

$$(R_2) \quad \text{there is a } C \geq 1 \text{ such that for every } x, y \in H \text{ with } x + y \in H,$$

$$(a) \quad f(x + y) \leq C(f(x) + f(y)),$$

$$(b) \quad f(x) \leq C(f(x + y) + f(y)).$$

Proof. Since f is even, \mathbf{E}_f is symmetric. It is obvious that (R_1) is equivalent to \mathbf{E}_f being reflexive, so it remains to show that (R_2) is equivalent to transitivity.

Suppose first (R_2) holds and let $(x_n)_{n < \omega}$, $(y_n)_{n < \omega}$, $(z_n)_{n < \omega} \in H^\omega$ such that $(x_n)\mathbf{E}_f(y_n)$ and $(y_n)\mathbf{E}_f(z_n)$. Let $n < \omega$ be fixed. Since the role of x_n and z_n is symmetric, by (H_2) we can assume $z_n - x_n \in H$. We distinguish several cases.

If $x_n - y_n \in H$ then by (H_3) , $z_n - y_n \in H$ so by (R_2b) using $(z_n - y_n) = (z_n - x_n) + (x_n - y_n)$,

$$f(z_n - x_n) \lesssim f(z_n - y_n) + f((y_n - x_n)^+).$$

If $y_n - x_n \in H$ then either $z_n - y_n \in H$, hence by (R_2a) , using $(z_n - x_n) = (z_n - y_n) + (y_n - x_n)$,

$$f(z_n - x_n) \lesssim f(z_n - y_n) + f(y_n - x_n);$$

or $y_n - z_n \in H$ hence by (R_2b) using $(y_n - z_n) + (z_n - x_n) = (y_n - x_n)$,

$$f(z_n - x_n) \lesssim f((z_n - y_n)^+) + f(y_n - x_n).$$

Thus

$$\sum_{n < \omega} f((z_n - x_n)^+) \lesssim \sum_{n < \omega} f((z_n - y_n)^+) + \sum_{n < \omega} f((y_n - x_n)^+) < \infty,$$

which gives $(x_n)\mathbf{E}_f(z_n)$; i.e. (R_2) implies transitivity.

To see the other direction, suppose first there is no $C \geq 1$ for which (R_2a) holds, i.e. for every $n < \omega$ there are $\xi_n, \eta_n \in H$ such that $\xi_n + \eta_n \in H$ and

$$f(\xi_n + \eta_n) > 2^n(f(\xi_n) + f(\eta_n)).$$

Set $k_n = \max\{1, \lfloor 1/f(\xi_n + \eta_n) \rfloor\}$; if $B \geq 1$ is an upper bound of f , we have

$$B \geq k_n f(\xi_n + \eta_n) \geq \frac{1}{2} \text{ and } B > 2^n k_n (f(\xi_n) + f(\eta_n)). \quad (5)$$

Let $(x_m)_{m < \omega} \in H^\omega$ be the sequence which, for every $n < \omega$, admits the value ξ_n with multiplicity k_n ; and define the sequence $(y_m)_{m < \omega} \in H^\omega$ to admit η_n exactly there where $(x_m)_{m < \omega}$ admits ξ_n ($n < \omega$). Then by (5),

$$\sum_{m < \omega} f(x_m) < 2B \text{ and } \sum_{m < \omega} f(y_m) < 2B,$$

i.e. if $\underline{0}$ denotes the constant zero sequence we have $\underline{0}\mathbf{E}_f(x_m)$ and $(x_m)\mathbf{E}_f(x_m + y_m)$. Also by (5),

$$\sum_{m < \omega} f(x_m + y_m) = \infty,$$

i.e. $\underline{0}\mathbf{E}_f(x_m + y_m)$, which shows transitivity fails.

Finally suppose there is no $C \geq 1$ for which (R_2b) holds, i.e. for every $n < \omega$ there are $\xi_n, \eta_n \in H$ such that

$$f(\xi_n) > 2^n(f(\xi_n + \eta_n) + f(\eta_n)).$$

Set $k_n = \max\{1, \lfloor 1/f(\xi_n) \rfloor\}$ and let $(x_m)_{m < \omega}, (y_m)_{m < \omega}$ be as above. Then $(y_m)\mathbf{E}_f\underline{0}$ and $\underline{0}\mathbf{E}_f(x_m + y_m)$ but $(y_m)\mathbf{E}_f(x_m + y_m)$. ■

If f is an arbitrary function, thus \mathbf{E}_f is not necessarily an equivalence relation, then one could consider the equivalence relation generated by \mathbf{E}_f . However, it is very hard to control the properties of this generated equivalence relation by the properties of f , in particular we do not know how to ensure \mathbf{E}_f is Borel. Therefore, from now on, we restrict our attention to such functions f for which \mathbf{E}_f is an equivalence relation.

Despite of the general setting of Definition 1 and Proposition 2, in the present paper we will work only with two special cases. At some point, we will set $G = H$ to be the circle group $\mathbb{S} = [0, 1)$ with mod 1 addition. Then (H_1) - (H_3) obviously hold, $x^+ = x$ ($x \in \mathbb{S}$), moreover (R_2a) and (R_2b) are equivalent. But mainly we will work with $G = \mathbb{R}$ and $H = [0, 1]$; then (H_1) - (H_3) hold and $x^+ = |x|$. Our reason for working with functions f defined on

$[0, 1]$ instead of \mathbb{R} is that on a smaller domain it is easier to define f such that it satisfies (R_1) and (R_2) . Next we show that for \mathbf{E}_{Id^p} , this change of domain makes no difference.

Lemma 3. *For $1 \leq p < \infty$, let ℓ^p denote the equivalence relation defined by (4) with $f: \mathbb{R} \rightarrow \mathbb{R}^+$, $f(x) = |x|^p$ ($x \in \mathbb{R}$). Then ℓ^p and $\ell^p|_{[0,1]^\omega \times [0,1]^\omega}$ are Borel equivalent.*

Proof. It is obvious that $\ell^p|_{[0,1]^\omega \times [0,1]^\omega} \leq_B \ell^p$. To see the other direction, for every $k \in \mathbb{Z}$ let $\rho_k: \mathbb{R} \rightarrow [0, 1]$,

$$\rho_k(x) = \begin{cases} 1, & \text{if } k < \lfloor x \rfloor; \\ \{x\}, & \text{if } k = \lfloor x \rfloor; \\ 0, & \text{if } k > \lfloor x \rfloor; \end{cases}$$

and set $\vartheta: \mathbb{R}^\omega \rightarrow [0, 1]^{\mathbb{Z} \times \omega}$,

$$\vartheta((x_n)_{n < \omega}) = (\rho_k(x_n))_{k \in \mathbb{Z}, n < \omega}.$$

For every $x, y \in \mathbb{R}$ with $|y - x| \leq 1$, we have $\rho_k(x) \neq \rho_k(y)$ only if $k = \lfloor x \rfloor$ or $k = \lfloor y \rfloor$; moreover

$$|y - x| = \sum_{k \in \mathbb{Z}} |\rho_k(y) - \rho_k(x)| \quad (x, y \in \mathbb{R}).$$

Thus

$$\sum_{k \in \mathbb{Z}} |\rho_k(y) - \rho_k(x)|^p \leq |y - x|^p \quad (x, y \in \mathbb{R});$$

and for $x, y \in \mathbb{R}$ with $|y - x| \leq 1$,

$$|y - x|^p \leq 2^p \sum_{k \in \mathbb{Z}} |\rho_k(y) - \rho_k(x)|^p.$$

Since $(x_n)\ell_p(y_n)$ implies $\lim_{n < \omega} |y_n - x_n| = 0$, after reindexing the coordinates of its range, ϑ reduces ℓ^p to $\ell^p|_{[0,1]^\omega \times [0,1]^\omega}$, as required. ■

As we have seen already in the introduction, \mathbf{E}_f may be an equivalence relation for a discontinuous f , e.g., for the function $f(0) = 0$, $f(x) = 1$ ($0 < x \leq 1$) we have \mathbf{E}_f is the equivalence relation of eventual equality on $[0, 1]^\omega$. Following the literature, we denote this equivalence relation by E_1 . In the remaining part of this section we show that f is continuous in zero if and only if $E_1 \not\leq_B \mathbf{E}_f$.

Theorem 4. *Let $f: [0, 1] \rightarrow \mathbb{R}^+$ be a bounded Borel function such that \mathbf{E}_f is an equivalence relation. Then f is continuous in zero if and only if $E_1 \not\leq_B \mathbf{E}_f$.*

Before proving Theorem 4 we show that up to Borel reducibility, requiring continuity in zero or continuity on the whole $[0, 1]$ is the same condition for \mathbf{E}_f .

Proposition 5. *Let $f: [0, 1] \rightarrow \mathbb{R}^+$ be a bounded function such that \mathbf{E}_f is an equivalence relation. If f is continuous in zero then there exists a continuous function $\tilde{f}: [0, 1] \rightarrow \mathbb{R}^+$ such that $\mathbf{E}_f = \mathbf{E}_{\tilde{f}}$.*

As a corollary of Theorem 4 and Proposition 5, we obtain the following surprising result.

Corollary 6. *Let $f, g: [0, 1] \rightarrow \mathbb{R}^+$ be bounded Borel functions such that \mathbf{E}_f and \mathbf{E}_g are equivalence relations. If g is continuous and $\mathbf{E}_f \leq_B \mathbf{E}_g$ then f is continuous in zero hence there is a continuous function $\tilde{f}: [0, 1] \rightarrow \mathbb{R}^+$ such that $\mathbf{E}_f = \mathbf{E}_{\tilde{f}}$.*

We start with the proof of Proposition 5.

Proof of Proposition 5. Let $C \geq 1$ be the constant of (R_2) . First we show that there exists an increasing function $\varepsilon: [0, 1] \rightarrow [0, 1]$ such that $\varepsilon(a) > 0$ for $a > 0$ and for every $x, y \in [0, 1]$,

$$|y - x| \leq \varepsilon(f(x)) \Rightarrow \frac{f(x)}{2C} \leq f(y) \leq 2Cf(x). \quad (6)$$

Set

$$\varepsilon(a) = \frac{1}{2} \sup \left\{ y \in [0, 1] : f(d) \leq \frac{a}{2C} \text{ for } 0 \leq d \leq y \right\};$$

then ε is increasing and since $f(0) = 0$ and f is continuous in zero, $\varepsilon(a) > 0$ for $a > 0$. We show (6). By (R_2a) ,

$$f(y) \leq C(f(x) + f(y - x)) \leq 2Cf(x) \quad (0 \leq y - x \leq \varepsilon(f(x)))$$

and

$$\frac{f(x)}{2C} \leq \frac{f(x)}{C} - f(x - y) \leq f(y) \quad (0 \leq x - y \leq \varepsilon(f(x)));$$

and by (R_2b) ,

$$\frac{f(x)}{2C} \leq \frac{f(x)}{C} - f(y - x) \leq f(y) \quad (0 \leq y - x \leq \varepsilon(f(x)))$$

and

$$f(y) \leq C(f(x) + f(x - y)) \leq 2Cf(x) \quad (0 \leq x - y \leq \varepsilon(f(x))),$$

as required.

As a corollary of (6), we get $U = \{x \in [0, 1]: f(x) > 0\}$ is an open set. Moreover, for every $a > 0$ the $\varepsilon(a)$ -neighborhood of $\{x \in [0, 1]: f(x) > a\}$ is contained in U , i.e. f is continuous at every point of $[0, 1] \setminus U = \{x \in [0, 1]: f(x) = 0\}$. For every $x \in U$, set

$$I_x = (x - \varepsilon(f(x)), x + \varepsilon(f(x))) \cap [0, 1].$$

Then $I_x \subseteq U$ and $\{I_x: x \in U\}$ is an open cover of U . Since the covering dimension of U is one, there is an open refinement $J_x \subseteq I_x$ ($x \in U$) such that $\{J_x: x \in U\}$ is an open cover of U of order at most two, i.e. for every $x \in U$, $|\{y \in U: x \in J_y\}| \leq 2$. So the function $\varphi: U \rightarrow 2^{\mathbb{R}}$,

$$\varphi(x) = \bigcup_{\substack{y \in U \\ x \in J_y}} \left[\frac{f(y)}{2C}, 2Cf(y) \right]$$

is closed convex valued and lower semicontinuous, hence Michael's Selection Theorem [9, Theorem 3.2 p. 364] can be applied to have a continuous function $\tilde{f}: U \rightarrow \mathbb{R}$ satisfying $\tilde{f}(x) \in \varphi(x)$ ($x \in U$). Since f is continuous at every point of $[0, 1] \setminus U$, \tilde{f} extends continuously to $[0, 1]$ with $\tilde{f}(x) = 0$ for $x \in [0, 1] \setminus U$.

For fixed $x \in [0, 1]$, $x \in J_y$ implies $x \in I_y$. So by (6),

$$f(x) \in \left[\frac{f(y)}{2C}, 2Cf(y) \right] \text{ hence } f(y) \in \left[\frac{f(x)}{2C}, 2Cf(x) \right].$$

Thus

$$\bigcup_{\substack{y \in U \\ x \in J_y}} \left[\frac{f(y)}{2C}, 2Cf(y) \right] \subseteq \left[\frac{f(x)}{4C^2}, 4C^2f(x) \right]$$

and so

$$\frac{f(x)}{4C^2} \leq \tilde{f}(x) \leq 4C^2f(x) \text{ (} x \in [0, 1]\text{)}.$$

Therefore $\mathbf{E}_f = \mathbf{E}_{\tilde{f}}$, as required. ■

We close this section with the proof of Theorem 4. We obtain the nonreducibility of E_1 to \mathbf{E}_f for a continuous f via [7, Theorem 4.1 p. 238], which says that E_1 is not reducible to any equivalence relation induced by a Polish group action. To this end, first we show that for continuous f , \mathbf{E}_f is essentially induced by a Polish group action. Recall that \mathbb{S} denotes the circle group $[0, 1)$ with mod 1 addition.

Lemma 7. *Let $f: [0, 1] \rightarrow \mathbb{R}^+$ be a continuous function such that \mathbf{E}_f is an equivalence relation. Then either f is identically zero or there is a continuous even function $\tilde{f}: \mathbb{S} \rightarrow \mathbb{R}^+$ such that $\tilde{f}(x) > 0$ for $x \neq 0$, $\mathbf{E}_{\tilde{f}}$ is an equivalence relation and $\mathbf{E}_f \leq_B \mathbf{E}_{\tilde{f}}$.*

Proof. Suppose f is not identically zero. We distinguish two cases. Suppose first $f(x) > 0$ for $x > 0$. Then set

$$\tilde{f}(x) = \begin{cases} f(2x), & \text{if } 0 \leq x < 1/2; \\ f(2-2x), & \text{if } 1/2 \leq x < 1. \end{cases}$$

It is obvious that \tilde{f} is continuous, even and $\tilde{f}(x) > 0$ for $x \neq 0$. We show that $\mathbf{E}_{\tilde{f}}$ is an equivalence relation by verifying the conditions of Proposition 2. We have (R_1) ; since (R_2a) implies (R_2b) , we prove only (R_2a) . Let C be the constant of (R_2) for f . If $x \in [1/4, 3/4]$ or $y \in [1/4, 3/4]$ then

$$\tilde{f}(x+y) \leq \frac{\max \tilde{f}}{\min \tilde{f}|_{[1/4, 3/4]}} (\tilde{f}(x) + \tilde{f}(y)).$$

If $x, y \in [0, 1/4]$ or $x, y \in [3/4, 1]$ then by (R_2a) for f , (R_2a) holds for \tilde{f} with C . Finally if exactly one of x and y is in $[0, 1/4]$ and $[3/4, 1]$ then by (R_2b) for f , (R_2a) holds for \tilde{f} with C .

Also, $\vartheta: [0, 1]^\omega \rightarrow \mathbb{S}^\omega$, $\vartheta((x_n)_{n < \omega}) = (x_n/2)_{n < \omega}$ is a reduction of \mathbf{E}_f to $\mathbf{E}_{\tilde{f}}$, so the proof of first case is complete.

In the second case, suppose $f(x) = 0$ for some $x \in (0, 1]$. By (R_2) , the nonempty set $\{x \in [0, 1]: f(x) = 0\}$ is closed under additions that are in $[0, 1]$. Hence by the continuity of f , $x^* = \inf\{x \in (0, 1]: f(x) = 0\}$ satisfies $x^* > 0$ and $f(x^*) = 0$.

Set

$$\tilde{f}(x) = \begin{cases} f(xx^*), & \text{if } 0 \leq x < 1/2; \\ f((1-x)x^*), & \text{if } 1/2 \leq x < 1. \end{cases}$$

It is obvious that \tilde{f} is continuous, even and $\tilde{f}(x) > 0$ for $x \neq 0$. Similarly to the previous case, we get $\mathbf{E}_{\tilde{f}}$ is an equivalence relation by distinguishing several cases. If $x \in [1/4, 3/4]$ or $y \in [1/4, 3/4]$ then

$$\tilde{f}(x+y) \leq \frac{\max \tilde{f}}{\min \tilde{f}|_{[1/4, 3/4]}} (\tilde{f}(x) + \tilde{f}(y)).$$

If $x, y \in [0, 1/4]$ then by (R_2a) for f , (R_2a) holds for \tilde{f} with C . If $x, y \in [3/4, 1]$ then again by (R_2a) for f ,

$$\tilde{f}(x+y) = f(2x^* - (x+y)x^*) \lesssim f(x^* - xx^*) + f(x^* - yx^*) = \tilde{f}(x) + \tilde{f}(y).$$

Finally if exactly one of x and y is in $[0, 1/4]$ and $[3/4, 1)$ then by (R_2b) for f , (R_2a) holds for \tilde{f} with C .

For every $x \in [0, 1]$, let $\langle x \rangle = x/x^* - \lfloor x/x^* \rfloor$. We show that $\vartheta: [0, 1]^\omega \rightarrow \mathbb{S}^\omega$, $\vartheta((x_n)_{n < \omega}) = (\langle x_n \rangle)_{n < \omega}$ is a reduction of \mathbf{E}_f to $\mathbf{E}_{\tilde{f}}$. For every $0 \leq x \leq y \leq 1$, with $k = \lfloor y/x^* \rfloor - \lfloor x/x^* \rfloor$ we have $\langle y \rangle - \langle x \rangle = y/x^* - x/x^* - k$, so

$$\tilde{f}(\langle y \rangle - \langle x \rangle) = \begin{cases} f(y - x - kx^*), & \text{if } 0 \leq y - x - kx^* < x^*/2; \\ f(-y + x + kx^*), & \text{if } 0 \leq -y + x + kx^* < x^*/2; \\ f(x^* - y + x + kx^*), & \text{if } x^*/2 \leq y - x - kx^* < x^*; \\ f(x^* + y - x - kx^*), & \text{if } x^*/2 \leq -y + x + kx^* < x^*. \end{cases}$$

For $l = k$ or $l = k \pm 1$, in any of the cases where applicable, by (R_2) we have

$$f(y - x) \lesssim f(y - x - lx^*) + f(lx^*), \quad f(y - x) \lesssim f(lx^*) + f(lx^* - y + x),$$

$$f(y - x - lx^*) \lesssim f(y - x) + f(lx^*), \quad f(lx^* - y + x) \lesssim f(lx^*) + f(y - x).$$

So $f(y - x) \approx \tilde{f}(\langle y \rangle - \langle x \rangle)$ follows from $f(lx^*) = 0$. This implies that ϑ is a reduction, so the proof is complete. ■

In the next lemma, for an \tilde{f} as in Lemma 7, we find a Polish group action inducing $\mathbf{E}_{\tilde{f}}$.

Definition 8. Let $f: H \rightarrow \mathbb{R}^+$ be an arbitrary function. For every $x = (x_n)_{n < \omega} \in H^\omega$ and $I \subseteq \omega$ we set

$$\|x\|_f = \sum_{n < \omega} f(x_n), \quad \|x|_I\|_f = \sum_{n \in I} f(x_n).$$

We define $\mathcal{N}_f = \{x \in H^\omega: \|x\|_f < \infty\}$.

Lemma 9. Let $f: \mathbb{S} \rightarrow \mathbb{R}^+$ be a continuous even function such that $f(x) > 0$ for $x \neq 0$ and \mathbf{E}_f is an equivalence relation.

1. There is a unique topology τ_f on \mathcal{N}_f such that for every $x \in \mathcal{N}_f$, the sets

$$B(x, \varepsilon) = \{y \in \mathcal{N}_f: \|y - x\|_f < \varepsilon\} \quad (\varepsilon > 0)$$

form a neighborhood base at x . This topology is regular, second countable and refines the topology inherited from \mathbb{S}^ω .

2. With τ_f , $(\mathcal{N}_f, +)$ is a Polish group. The natural action of \mathcal{N}_f on \mathbb{S}^ω is continuous, and the equivalence relation induced by this action is \mathbf{E}_f .

Proof. For 1, we show that for every $x \in \mathcal{N}_f$, $\varepsilon > 0$ and $y \in B(x, \varepsilon)$ there is a $\delta > 0$ such that $B(y, \delta) \subseteq B(x, \varepsilon)$; once this done, the first part of the statement follows from elementary topology (see e.g. [2]). Let $C \geq 1$ be the constant of (R_2) , fix $x \in \mathcal{N}_f$, $\varepsilon > 0$ and $y \in B(x, \varepsilon)$. Let $n < \omega$ be such that

$$\|(y - x)|_{\omega \setminus n}\|_f < \frac{\varepsilon - \|y - x\|_f}{3C}.$$

Let $\delta > 0$ satisfy $\delta < (\varepsilon - \|y - x\|_f)/(3C)$, and such that for every $i < n$ and $z_i \in [0, 1]$, $f(z_i - y_i) < \delta$ implies

$$|f(z_i - x_i) - f(y_i - x_i)| < \frac{\varepsilon - \|y - x\|_f}{3n};$$

such a δ exists by the continuity of f and by $f(x) > 0$ for $x \neq 0$.

Let $z \in B(y, \delta)$; then by (R_2) ,

$$\begin{aligned} \|z - x\|_f &= \|(z - x)|_n\|_f + \|(z - x)|_{\omega \setminus n}\|_f < \\ &\|(y - x)|_n\|_f + n \frac{\varepsilon - \|y - x\|_f}{3n} + C(\|(z - y)|_{\omega \setminus n}\|_f + \|(y - x)|_{\omega \setminus n}\|_f) < \\ &\|y - x\|_f + \frac{\varepsilon - \|y - x\|_f}{3} + \frac{\varepsilon - \|y - x\|_f}{3} + \frac{\varepsilon - \|y - x\|_f}{3} = \varepsilon, \end{aligned}$$

as required.

Since $f(x) > 0$ for $x \neq 0$, τ_f refines the topology inherited from \mathbb{S}^ω . The countable set of eventually zero rational sequences shows separability and hence second countability. To see regularity, let $F \subseteq (\mathcal{N}_f, \tau_f)$ be a closed set and take $x \notin F$. Then $\gamma = \inf\{\|y - x\|_f : y \in F\} > 0$. By (R_2) , $B(x, \gamma/(2C)) \cap B(F, \gamma/(2C)) = \emptyset$, as required.

For 2, first we show $(\mathcal{N}_f, +)$ is a topological group. Let $x, y \in \mathcal{N}_f$ and $\gamma > 0$. By (R_2) , $B(x, \gamma/2C) + B(y, \gamma/2C) \subseteq B(x + y, \gamma)$, so addition is continuous. The continuity of the inverse operation is obvious, so the statement follows.

Next we show (\mathcal{N}_f, τ_f) is strong Choquet (for the definition and notation see [5, Section 8.D p. 44]). The closed balls $\bar{B}(x, \varepsilon) = \{y \in \mathcal{N}_f : \|y - x\|_f \leq \varepsilon\}$ are closed in \mathbb{S}^ω , thus every $\|\cdot\|_f$ -Cauchy sequence is convergent in \mathcal{N}_f . If player I plays $(x_n, U_n)_{n < \omega}$, a winning strategy for player II is to choose $V_n = B(x_n, \gamma_n)$ such that $\bar{B}(x_n, \gamma_n) \subseteq U_n$ and $\gamma_n \leq 1/2^n$ ($n < \omega$). So (\mathcal{N}_f, τ_f) is strong Choquet, hence Polish by Choquet's Theorem (see e.g. [5, (8.18) Theorem p. 45]).

The continuity of the action of \mathcal{N}_f on \mathbb{S}^ω follows from the fact that τ_f refines the topology inherited from \mathbb{S}^ω . It is obvious that the equivalence relation induced by this action is \mathbf{E}_f , so the proof complete. ■

Definition 10. For a topological space X and $G \subseteq X$, we set

$$V(G) = \bigcup \{U \subseteq X : U \text{ is open, } G \cap U \text{ is comeager in } U\}.$$

The next lemma is a folklore result on the existence of a perfect set with special distance set.

Lemma 11. *Let $G \subseteq [0, 1]$ be a Borel set such that zero is adherent to $V(G)$. Then there exists a nonempty perfect set $P \subseteq [0, 1]$ such that*

$$\{|y - x| : x, y \in P\} \subseteq G \cup \{0\}. \quad (7)$$

Proof. By passing to a subset, we can assume that G is a comeager G_δ subset of $V(G)$. Set $\tilde{G} = G \cup (-G) \cup \{0\}$ and let $d_{\tilde{G}}$ be the metric on \tilde{G} for which $(\tilde{G}, d_{\tilde{G}})$ is a Polish space with the topology inherited from $[-1, 1]$ (see e.g. [5, (3.11) Theorem p. 17]). We construct inductively a sequence $(x_n)_{n < \omega} \subseteq [0, 1]$ with the following properties:

1. for every $n < \omega$, $x_{n+1} < x_n/2$;
2. for every $s \in \{-1, 0, +1\}^{<\omega}$, $\sum_{i < |s|} s(i)x_i \in \tilde{G}$;
3. for every $s \in \{-1, 0, +1\}^{<\omega} \setminus \{\emptyset\}$, $d_{\tilde{G}}(\sum_{i < |s|-1} s(i)x_i, \sum_{i < |s|} s(i)x_i) \leq 1/2^{|s|}$.

Let $x_0 \in G$ be arbitrary. Let $0 < n < \omega$ and suppose x_i ($i < n$) are defined such that 2 and 3 hold for every $s \in \{-1, 0, +1\}^{\leq n}$. By 2, if $s \in \{-1, 0, +1\}^n$ and $\sum_{i < n} s(i)x_i \neq 0$ then \tilde{G} is comeager in a neighborhood of $\sum_{i < n} s(i)x_i$. Since zero is adherent to $V(G)$, by the Baire Category Theorem we can pick $x_n \in G$ sufficiently close to zero such that 1 holds; and for every $s \in \{-1, 0, +1\}^{n+1}$ with $\sum_{i < n} s(i)x_i \neq 0$ we have $\sum_{i < n+1} s(i)x_i \in \tilde{G}$, hence by $x_n \in G$, $\sum_{i < n+1} s(i)x_i \in \tilde{G}$ for every $s \in \{-1, 0, +1\}^{n+1}$; and in addition 3 holds. This completes the inductive step.

We show

$$P = \left\{ \sum_{n < \omega} \sigma(n)x_n : \sigma \in 2^\omega \right\}$$

fulfills the requirements. By 3, for every $\sigma \in \{-1, 0, +1\}^\omega$, $(\sum_{i < n} \sigma(i)x_i)_{n < \omega}$ is a Cauchy sequence in \tilde{G} , so $\sum_{n < \omega} \sigma(n)x_n \in \tilde{G}$. In particular $P \subseteq G \cup \{0\}$.

Let $x, x' \in P$, $x = \sum_{n < \omega} \sigma(n)x_n$ and $x' = \sum_{n < \omega} \sigma'(n)x_n$ with $\sigma, \sigma' \in 2^\omega$, $\sigma \neq \sigma'$; say for the first $n < \omega$ with $\sigma(n) \neq \sigma'(n)$ we have $\sigma(n) = 0$, $\sigma'(n) = 1$. Then for $\delta \in \{-1, 0, +1\}^\omega$, $\delta(n) = \sigma'(n) - \sigma(n)$ ($n < \omega$) we have

$$|x' - x| = x' - x = \sum_{n < \omega} \delta(n)x_n \in \tilde{G},$$

moreover by 1, $x' - x > 0$ i.e. $x' - x \in G$. Thus P is a nonempty perfect set and satisfies (7), which completes the proof. ■

The last lemma points out a property of an f discontinuous in zero.

Lemma 12. *Let $f: [0, 1] \rightarrow \mathbb{R}^+$ be a bounded Borel function such that \mathbf{E}_f is an equivalence relation. If f is not continuous in zero then there exists an $a > 0$ such that $G = \{x \in [0, 1]: f(x) > a\}$ satisfies the condition of Lemma 11, i.e. zero is adherent to $V(G)$.*

Proof. Let C be the constant of (R_2) . Since f is not continuous in zero, there exists an $a > 0$ such that zero is adherent to $\{x \in [0, 1]: f(x) > 2Ca\}$. If for every $x \in [0, 1]$ with $f(x) > 2Ca$, x is adherent to

$$\bigcup \{U \subseteq (x, 1): U \text{ is open, } \{y \in U: f(y) > a\} \text{ is comeager in } U\}$$

then the statement follows. If not, by f being Borel, there is an $x \in [0, 1]$ with $f(x) > 2Ca$ and a $\delta > 0$ such that

$$Y = \{y \in (x, x + \delta): f(y) \leq a\}$$

is comeager in $(x, x + \delta)$. Since $f(x) > 2Ca$, by (R_2b) we have $f(y - x) > a$ whenever $y \in Y$. Hence $\{x \in [0, 1]: f(x) > a\}$ is comeager in $(0, \delta)$, which finishes the proof. ■

Proof of Theorem 4. Suppose first f is not continuous in zero. By Lemma 11 and Lemma 12, there is an $a > 0$ and a nonempty perfect set $P \subseteq [0, 1]$ such that $f(|y - x|) > a$ for every $x, y \in P$, $x \neq y$. Thus \mathbf{E}_f restricted to P^ω is E_1 .

Suppose now that f is continuous in zero. By Proposition 5, we can assume f is continuous on $[0, 1]$. If $f \equiv 0$ then $E_1 \not\leq_B \mathbf{E}_f$ is obvious. Else by Lemma 7 and Lemma 9, $\mathbf{E}_f \leq_B \mathbf{E}_{\tilde{f}}$ where $\mathbf{E}_{\tilde{f}}$ is induced by a Polish group action. Hence $E_1 \not\leq_B \mathbf{E}_{\tilde{f}}$ by [7, Theorem 4.1 p. 238] and [6]; in particular $E_1 \not\leq_B \mathbf{E}_f$. This completes the proof. ■

3 Reducibility results

In the remaining part of the paper, in most cases, we restrict our attention to equivalence relations \mathbf{E}_f where $f: [0, 1] \rightarrow \mathbb{R}^+$ is a *continuous* function. As we have seen in Proposition 5, requiring continuity on $[0, 1]$ and continuity in zero for f are equivalent, and by Theorem 4, for Borel f it is the necessary and sufficient condition to have $E_1 \not\leq_B \mathbf{E}_f$. This assumption is acceptable to

us since we aim to study equivalence relations \mathbf{E}_f for which $\mathbf{E}_f \leq_B \mathbf{E}_{\text{Id}^q}$ for some $1 \leq q < \infty$.

The main restriction, in addition to $(R_1) + (R_2)$, we impose in the sequel on the function f is formulated in the following definition.

Definition 13. Let (R, \leq) be an ordered set and $f: R \rightarrow \mathbb{R}^+$ be a function. We say f is *essentially increasing* if for some $C \geq 1$, $\forall x, y \in R$ ($x \leq y \Rightarrow f(x) \leq Cf(y)$). Similarly, f is *essentially decreasing* if for some $C \geq 1$, $\forall x, y \in R$ ($x \leq y \Rightarrow Cf(x) \geq f(y)$).

Lemma 14. *With the notation of Definition 13, f is essentially increasing (resp. essentially decreasing) if and only if there is an increasing (resp. decreasing) function \tilde{f} such that $\tilde{f} \approx f$.*

Proof. If f is essentially increasing, set $\tilde{f}: R \rightarrow \mathbb{R}^+$, $\tilde{f}(x) = \sup\{f(y) : y \leq x\}$. Then \tilde{f} is increasing and $f(x) \leq \tilde{f}(x) \leq Cf(x)$ ($x \in R$). If f is essentially decreasing, let $\tilde{f}(x) = \inf\{f(y) : y \leq x\}$; then, as above, $\tilde{f} \approx f$. The other directions are obvious, so the proof is complete. ■

We remark that for $R = [0, 1]$ or $R = (0, 1]$, by its definition above, \tilde{f} is continuous if f is so. By $\tilde{f} \approx f$, \tilde{f} and f have the same asymptotic behavior in 0.

In this section we prove the following two theorems.

Theorem 15. *Let $1 \leq \alpha < \infty$ and let $\psi: (0, 1] \rightarrow (0, +\infty)$ be an essentially decreasing continuous function such that $\text{Id}^\alpha \psi$ is bounded and for every $\delta > 0$, $\liminf_{x \rightarrow +0} x^\delta \psi(x) = 0$. Set $g(x) = x^\alpha \psi(x)$ for $0 < x \leq 1$ and $g(0) = 0$. Suppose \mathbf{E}_g is an equivalence relation. Then $\mathbf{E}_{\text{Id}^\alpha} \leq_B \mathbf{E}_g$.*

Theorem 16. *Let $f, g: [0, 1] \rightarrow \mathbb{R}^+$ be continuous essentially increasing functions such that \mathbf{E}_f and \mathbf{E}_g are equivalence relations. Suppose there exists a function $\kappa: \{1/2^i : i < \omega\} \rightarrow [0, 1]$ satisfying the recursion*

$$f(1) = g(\kappa(1)), \quad f(1/2^n) = \sum_{i=0}^n g(\kappa(1/2^i)/2^{n-i}) \quad (0 < n < \omega) \quad (8)$$

such that for some $L \geq 1$,

$$\sum_{i=n}^{\infty} g(\kappa(1/2^i)) \leq L \sum_{i=0}^n g(\kappa(1/2^i)/2^{n-i}) \quad (n < \omega) \quad (9)$$

and

$$\kappa(1/2^n) \leq L \cdot \max\{\kappa(1/2^i)/2^{n-i} : i < n\} \quad (n < \omega). \quad (10)$$

Then $\mathbf{E}_f \leq_B \mathbf{E}_g$.

Theorem 15 illustrates, e.g. by choosing $\psi(x) = 1 - \log(x)$ ($0 < x \leq 1$), that reducibility among the \mathbf{E}_f s is *not* characterized by the growth order of the f s. Theorem 16 is a stronger version of [3, Theorem 1.1 p. 1836], but we admit that our improvement is of technical nature. However, in Section 5 it will allow us to show the reducibility among \mathbf{E}_f s for new families of f s.

These results neither give a complete description of the reducibility between the equivalence relations \mathbf{E}_f nor are optimal. Nevertheless, we note that in Theorem 15, Id^α cannot be replaced by an arbitrary “nice” function: as we will see, e.g. $\mathbf{E}_{\text{Id}^\alpha} <_B \mathbf{E}_{\text{Id}^\alpha/(1-\log)}$. Also, the condition ψ is decreasing cannot be left out: e.g. we need the techniques of Theorem 16 in order to treat the $\psi(x) = x$ case, i.e. to show $\mathbf{E}_{\text{Id}^\alpha} \leq_B \mathbf{E}_{\text{Id}^{\alpha+1}}$. We comment on the optimality of Theorem 16 after its proof.

We start with a technical lemma.

Lemma 17. *Let $f, g: [0, 1] \rightarrow \mathbb{R}^+$ be continuous functions such that $\mathbf{E}_f, \mathbf{E}_g$ are equivalence relations. Suppose there exists $K > 0$ and $I \in [\omega]^\omega$ such that for every $n \in I$ there is a mapping $\vartheta_n: \{i/n: 0 \leq i \leq n\} \rightarrow [0, 1]^\omega$ satisfying*

$$\frac{1}{K}f((j-i)/n) \leq \|\vartheta_n(j/n) - \vartheta_n(i/n)\|_g \leq Kf((j-i)/n) \quad (0 \leq i < j \leq n). \quad (11)$$

Then $\mathbf{E}_f \leq_B \mathbf{E}_g$.

Proof. For $x \in [0, 1]$ and $0 < n < \omega$ set $[x]_n = \max\{i/n: i/n \leq x, 0 \leq i \leq n\}$. Since f is uniformly continuous on $[0, 1]$, for every $k < \omega$ there is an $n_k \in I$ such that $|f(x) - f([x]_{n_k})| \leq 1/2^k$ ($x \in [0, 1]$). We show that $\vartheta: [0, 1]^\omega \rightarrow [0, 1]^\omega$,

$$\vartheta((x_k)_{k < \omega}) = (\vartheta_{n_k}([x_k]_{n_k}))_{k < \omega},$$

after reindexing the coordinates of the range, is a Borel reduction of \mathbf{E}_f to \mathbf{E}_g .

Let $(x_k)_{k < \omega}, (y_k)_{k < \omega} \in [0, 1]^\omega$. We have

$$|[y_k - x_k]_{n_k}| \leq |[y_k]_{n_k} - [x_k]_{n_k}| \leq |[y_k - x_k]_{n_k}| + 1/n_k \quad (k < \omega).$$

So by the choice of n_k , $|f(|y_k - x_k|) - f([y_k - x_k]_{n_k})| \leq 1/2^k$ and $|f([y_k - x_k]_{n_k}) - f([y_k]_{n_k} - [x_k]_{n_k})| \leq 1/2^k$, thus

$$|f(|y_k - x_k|) - f([y_k]_{n_k} - [x_k]_{n_k})| \leq 2/2^k \quad (k < \omega).$$

By (11), $\|\vartheta_{n_k}([y_k]_{n_k}) - \vartheta_{n_k}([x_k]_{n_k})\|_g \approx f([y_k]_{n_k} - [x_k]_{n_k})$ ($k < \omega$), so the statement follows. ■

Proof of Theorem 15. For some $B \geq 1$, let $x^\alpha \psi(x) \leq B$ ($0 < x \leq 1$). We find a $K > 0$ such that for every $0 < n < \omega$ there exist $M < \omega$ and $0 < \mu \leq 1$ such that for every $0 \leq i < j \leq n$,

$$\frac{1}{K} \left(\frac{j-i}{n} \right)^\alpha \leq M \left(\frac{j-i}{n} \mu \right)^\alpha \psi \left(\frac{j-i}{n} \mu \right) \leq K \left(\frac{j-i}{n} \right)^\alpha. \quad (12)$$

Once this done, the conditions of Lemma 17 are satisfied by the mapping $\vartheta_n: \{i/n: 0 \leq i \leq n\} \rightarrow [0, 1]^\omega$,

$$\vartheta_n(i/n) = (\underbrace{i\mu/n, \dots, i\mu/n}_M, 0, \dots).$$

Observe that (12) is equivalent to

$$1/K \leq M\mu^\alpha \psi((j-i)\mu/n) \leq K \quad (0 \leq i < j \leq n).$$

Since ψ is essentially decreasing, it is enough to have $1/2 \leq M\mu^\alpha \psi(\mu)$ and $M\mu^\alpha \psi(\mu/n) \leq 2B$. We will find a $0 < \mu \leq 1$ satisfying $\psi(\mu/n) \leq 2\psi(\mu)$. Then by choosing M to be minimal such that $1/2 \leq M\mu^\alpha \psi(\mu)$, by $\mu^\alpha \psi(\mu) \leq B$ and $B \geq 1$ we have $M\mu^\alpha \psi(\mu/n) \leq 2M\mu^\alpha \psi(\mu) \leq 2B$, so we fulfilled the requirements.

Suppose such a μ does not exist, i.e. $\psi(\mu/n) > 2\psi(\mu)$ ($0 < \mu \leq 1$). Then for every $k < \omega$ and $\mu \in [1/n, 1]$, $\psi(n^{-k}\mu) \geq 2^k \psi(\mu)$. We have $x = n^{-k}\mu$ runs over $(0, 1]$ as (k, μ) runs over $\omega \times [1/n, 1]$. So since ψ is essentially decreasing, with $\delta = \log(2)/\log(n)$ we have $\psi(x)x^\delta \gtrsim 1/n^\delta \psi(1) > 0$ ($0 < x \leq 1$). This contradicts $\liminf_{x \rightarrow +0} x^\delta \psi(x) = 0$, so the proof is complete. ■

Proof of Theorem 16. Let $n < \omega$ be fixed. For $0 < l \leq 2^n$ let $r(l) \leq n$, $s(l) < \omega$ be such that $l/2^n = s(l)/2^{r(l)}$ and $s(l)$ is odd. With $\text{Pr}_l x$ standing for the l^{th} coordinate of $x \in [0, 1]^{2^n}$, for every $0 \leq i \leq 2^n$ we define $\vartheta(i/2^n)$ by

$$\text{Pr}_l \vartheta(i/2^n) = \left(1 - 2^{r(l)} \left| \frac{i}{2^n} - \frac{l}{2^n} \right| \right) \kappa(1/2^{r(l)}) \text{ if} \\ l > 0 \text{ and } \left| \frac{i}{2^n} - \frac{l}{2^n} \right| \leq 1/2^{r(l)}, \quad (13)$$

else let $\text{Pr}_l \vartheta(i/2^n) = 0$. We show (11) holds for $\vartheta_{2^n} = \vartheta$.

Let $0 \leq i < j \leq 2^n$ be arbitrary. Let $m \leq n$ be minimal such that for some $e < 2^m$ we have

$$\frac{i}{2^n} \leq \frac{e}{2^m} < \frac{(e+1)}{2^m} \leq \frac{j}{2^n}.$$

We distinguish several cases.

Suppose first $i/2^n = e/2^m$ and $j/2^n = (e+1)/2^m$. For every $k \leq m$ there is exactly one l with $r(l) = k$ such that

$$|e/2^m - l/2^n| \leq 1/2^k \text{ and } |(e+1)/2^m - l/2^n| \leq 1/2^k;$$

and for this l , by (13),

$$|\text{Pr}_l(\vartheta((e+1)/2^m) - \vartheta(e/2^m))| = \kappa(1/2^k)/2^{m-k}.$$

All the other coordinates of $\vartheta((e+1)/2^m)$ and $\vartheta(e/2^m)$ are zero so by (8),

$$\|\vartheta((e+1)/2^m) - \vartheta(e/2^m)\|_g = \sum_{k=0}^m g(\kappa(1/2^k)/2^{m-k}) = f(1/2^m), \quad (14)$$

i.e. (11) holds with $K = 1$.

Next suppose e is even, $i/2^n = e/2^m$ and $(e+1)/2^m < j/2^n$; then we have $m \geq 1$. Observe that by the choice of m we have $j/2^n < (e+2)/2^m$. For every $k < m$ there is exactly one l with $r(l) = k$ such that

$$|e/2^m - l/2^n| \leq 1/2^k \text{ and } |j/2^n - l/2^n| \leq 1/2^k;$$

and for this l , $l/2^n \notin (e/2^m, j/2^n)$. So by (13),

$$\frac{\kappa(1/2^k)}{2^{m-k}} \leq |\text{Pr}_l(\vartheta(j/2^n) - \vartheta(e/2^m))| \leq 2 \frac{\kappa(1/2^k)}{2^{m-k}} \quad (0 \leq k < m).$$

Since e is even, $\vartheta(e/2^m)$ has no other nonzero coordinates. For every $m \leq k \leq n$ there is exactly one l with $r(l) = k$ such that $|j/2^n - l/2^n| \leq 1/2^k$, and for this l , $\text{Pr}_l(\vartheta(j/2^n)) \leq \kappa(1/2^k)$. Since g is essentially increasing, we have

$$\begin{aligned} \sum_{k=0}^{m-1} g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right) &\lesssim \|\vartheta(j/2^n) - \vartheta(e/2^m)\|_g \lesssim \\ &\sum_{k=0}^{m-1} g\left(2 \frac{\kappa(1/2^k)}{2^{m-k}}\right) + \sum_{k=m}^n g(\kappa(1/2^k)). \end{aligned} \quad (15)$$

By (R_2) ,

$$g\left(2 \frac{\kappa(1/2^k)}{2^{m-k}}\right) \lesssim g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right) \quad (0 \leq k < m). \quad (16)$$

By (8) and since f is essentially increasing,

$$f\left(\frac{j}{2^n} - \frac{e}{2^m}\right) \lesssim f\left(\frac{1}{2^{m-1}}\right) = \sum_{k=0}^{m-1} g\left(\frac{\kappa(1/2^k)}{2^{m-1-k}}\right),$$

so by (15),

$$f\left(\frac{j}{2^n} - \frac{e}{2^m}\right) \lesssim \|\vartheta(j/2^n) - \vartheta(e/2^m)\|_g. \quad (17)$$

By (9) and (16), the right hand side of (15) is $\lesssim \sum_{k=0}^m g(\kappa(1/2^k)/2^{m-k})$, so since f is essentially increasing,

$$\begin{aligned} \|\vartheta(j/2^n) - \vartheta(e/2^m)\|_g &\lesssim \\ &\sum_{k=0}^m g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right) = f(1/2^m) \lesssim f\left(\frac{j}{2^n} - \frac{e}{2^m}\right). \end{aligned} \quad (18)$$

The case $e+1$ is even, $i/2^n < e/2^m$ and $j/2^n = (e+1)/2^m$ can be treated by an analogous argument.

Suppose now e is even, $i/2^n < e/2^m$ and $(e+1)/2^m \leq j/2^n$; then we have $m \geq 2$. By (R_2) , (18), and also by (14) if $j/2^n = (e+1)/2^m$,

$$\begin{aligned} \|\vartheta(j/2^n) - \vartheta(i/2^n)\|_g &\lesssim \\ \|\vartheta(j/2^n) - \vartheta(e/2^m)\|_g + \|\vartheta(e/2^m) - \vartheta(i/2^n)\|_g &\lesssim \\ \left(f\left(\frac{j}{2^n} - \frac{e}{2^m}\right) + f\left(\frac{e}{2^m} - \frac{i}{2^n}\right)\right) &\lesssim f\left(\frac{j-i}{2^n}\right). \end{aligned}$$

To have a lower bound, observe that for every $k < m-1$ there is exactly one l with $r(l) = k$ such that

$$|i/2^n - l/2^n| \leq 1/2^k \text{ and } |j/2^n - l/2^n| \leq 1/2^k.$$

For this l , $l/2^n \notin (i/2^n, j/2^n)$ if $l/2^n \neq e/2^m$, i.e. if $k \neq k_0 = r(2^{n-m}e)$. So by (13),

$$\frac{\kappa(1/2^k)}{2^{m-k}} \leq |\text{Pr}_l(\vartheta(j/2^n) - \vartheta(i/2^n))| \quad (k < m-1, k \neq k_0).$$

By (10), (R_2) and since g is essentially increasing,

$$g\left(\frac{\kappa(1/2^{k_0})}{2^{m-k_0}}\right) \lesssim \max_{i < k_0} g\left(\frac{\kappa(1/2^i)}{2^{m-i}}\right).$$

So by (R_2) and since g is essentially increasing,

$$\begin{aligned} f(1/2^{m-2}) &= \sum_{k=0}^{m-2} g\left(\frac{\kappa(1/2^k)}{2^{m-2-k}}\right) \lesssim \sum_{k=0}^{m-2} g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right) \lesssim \\ &\sum \left\{ g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right) : k < m-1, k \neq k_0 \right\} \lesssim \|\vartheta(j/2^n) - \vartheta(i/2^n)\|_g. \end{aligned}$$

By the choice of m we have $(j - i)/2^n < 4/2^m$. So since f is essentially increasing, $f((j - i)/2^n) \lesssim f(4/2^m) = f(1/2^{m-2})$; thus $f((j - i)/2^n) \lesssim \|\vartheta(j/2^n) - \vartheta(i/2^n)\|_g$.

The case $e+1$ is even, $i/2^n \leq e/2^m$ and $(e+1)/2^m < j/2^n$ follows similarly, so the proof is complete. ■

The assumptions of Theorem 16 are not necessary, they merely make possible to imitate the construction in the proof of [3, Theorem 1.1 p. 1836]. We note however that the problem of characterizing whether $\{i/2^n : 0 \leq i \leq 2^n\}$ endowed with the $\|\cdot\|_f$ -distance Lipschitz embeds into $[0, 1]^\omega$ endowed with the $\|\cdot\|_g$ -distance is very hard even if the distances $\|\cdot\|_f$ and $\|\cdot\|_g$ can be related to norms (see e.g. [8] and the references therein). So it is unlikely that there is a simple characterization of reducibility among \mathbf{E}_f s using the approach of Lemma 17.

4 Nonreducibility results

In this section we improve [3, Theorem 2.2 p. 1840] in order to obtain nonreducibility results for a wider class of \mathbf{E}_f s, as follows.

Theorem 18. *Let $1 \leq \alpha < \infty$ and let $\varphi, \psi: [0, 1] \rightarrow [0, +\infty)$ be continuous functions. Set $f = \text{Id}^\alpha \varphi$, $g(x) = \text{Id}^\alpha \psi$ and suppose that f, g are bounded and \mathbf{E}_f and \mathbf{E}_g are equivalence relations. Suppose $\psi(x) > 0$ ($x > 0$), and*

(A₁) *there exist $\varepsilon > 0$, $M < \omega$ such that for every $n > M$ and $x, y \in [0, 1]$,*

$$\varphi(x) \leq \varepsilon \varphi(y) \varphi(1/2^n) \Rightarrow x \leq \frac{y}{2^{n+1}};$$

(A₂) $\lim_{n \rightarrow \infty} \psi(1/2^n)/\varphi(1/2^n) = 0$.

Then $\mathbf{E}_g \not\leq_B \mathbf{E}_f$.

Observe that $\varphi \equiv 1$, $\psi = \text{Id}^\beta$ ($0 < \beta < \infty$) satisfy the assumptions of Theorem 18, so it generalizes [3, Theorem 2.2 p. 1840].

The proof of [3, Theorem 2.2 p. 1840] has two fundamental constituents. The first idea is to pass to a subspace $X \subseteq [0, 1]^\omega$ where a hypothetic Borel reduction ϑ of \mathbf{E}_g to \mathbf{E}_f is modular, i.e. for $x \in X$, $\vartheta(x)$ consists of finite blocks, each of which depends only on a single coordinate of x . This technique can be adopted without any difficulty. The second tool is an excessive use of the fact that for $f = \text{Id}^p$, $f^{-1}(\|\cdot\|_f)$ is a norm, which does not follow from the assumptions of Theorem 18. We get around this difficulty by exploiting that φ is a perturbation when compared to Id^α .

Proof of Theorem 18. Suppose $\vartheta: [0, 1]^\omega \rightarrow [0, 1]^\omega$ is a Borel reduction of \mathbf{E}_g to \mathbf{E}_f . With $Z_k = \{i/2^k: 0 \leq i \leq 2^k\}$, set $Z = \prod_{k < \omega} Z_k$; then ϑ is a Borel reduction of $\mathbf{E}_g|_{Z \times Z}$ to \mathbf{E}_f . For every finite sequence $t \in \prod_{i < |t|} Z_i$, let $N_t = \{z \in Z: z(i) = t(i) \ (i < |t|)\}$. We import several lemmas from [3].

Lemma 19. ([3, Claim (i) p. 1840]) *For any $j, k < \omega$ there exist $l < \omega$, a finite sequence $s^* \in \prod_{i < |s^*|} Z_{k+i}$, and a comeager set $D \subseteq Z$ such that for all $x, \hat{x} \in D$, if we have $x = r \frown s^* \frown y$ and $\hat{x} = \hat{r} \frown s^* \frown y$ for some $r, \hat{r} \in [0, 1]^k$ and $y \in [0, 1]^\omega$, then $\|(\vartheta(x) - \vartheta(\hat{x}))|_{\omega \setminus l}\|_f < 2^{-j}$.*

Proof. For every $l < \omega$, we define $F_l: Z \rightarrow \mathbb{R}$ by

$$F_l(x) = \max\{\|(\vartheta(z) - \vartheta(\hat{z}))|_{\omega \setminus l}\|_f: \\ z, \hat{z} \in Z, z(i) = \hat{z}(i) = x(i) \ (k \leq i < \omega)\}.$$

For fixed $x \in Z$, there are only finitely many $z, \hat{z} \in Z$ satisfying $z(i) = \hat{z}(i) = x(i)$ ($k \leq i < \omega$). For each such pair we have $\|z - \hat{z}\|_g < \infty$, hence $\|\vartheta(z) - \vartheta(\hat{z})\|_f < \infty$, in particular $\lim_{l \rightarrow \infty} \|(\vartheta(z) - \vartheta(\hat{z}))|_{\omega \setminus l}\|_f = 0$. So $F_l(x) < \infty$ for all $l < \omega$ and $\lim_{l \rightarrow \infty} F_l(x) = 0$ ($x \in Z$). Therefore, by the Baire Category Theorem, there exists an $l < \omega$ such that $\{x \in Z: F_l(x) < 2^{-j}\}$ is not meager. By f being Borel, this set has the property of Baire, so there is a nonempty open set O on which it is relatively comeager.

We can assume $O = N_t$ for some finite sequence $t \in \prod_{i < |t|} Z_i$, and we can also assume $|t| \geq k$. Let $t = r^* \frown s^*$ where $|r^*| = k$. But $F_l(x)$ does not depend on the first k coordinates of x , so $\{x \in Z: F_l(x) < 2^{-j}\}$ is also relatively comeager in $N_{r^* \frown s^*}$ for all $r \in \prod_{i < k} Z_i$. Let D be a comeager set such that $F_l(x) < 2^{-j}$ whenever $x \in D \cap N_{r^* \frown s^*}$ for any r of length k . Now the conclusion of the claim follows from the definition of F_l . ■

By [5, (8.38) Theorem p. 52] there is a dense G_δ set $C \subseteq Z$ such that $\vartheta|_C$ is continuous.

Lemma 20. ([3, Claim (ii) p. 1841]) *For any $j, k, l < \omega$ there is a finite sequence $s^{**} \in \prod_{i < |s^{**}|} Z_{k+i}$ such that for all $x, \hat{x} \in C$, if we have $x = r \frown s^{**} \frown y$ and $\hat{x} = \hat{r} \frown s^{**} \frown y$ for some $r \in [0, 1]^k$ and $y, \hat{y} \in [0, 1]^\omega$, then $\|(\vartheta(x) - \vartheta(\hat{x}))|_l\|_f < 2^{-j}$.*

*Furthermore, if G is a given dense open subset of Z , then s^{**} can be chosen such that $N_{r \frown s^{**}} \subseteq G$ for all $r \in \prod_{i < k} Z_i$.*

Proof. There are only finitely many $r \in \prod_{i < k} Z_i$; enumerate them as r_0, r_1, \dots, r_{M-1} . We construct s^{**} by successive extensions.

Let $t_0 = \emptyset$. Let $m < M$ and suppose that we have the finite sequence $t_m \in \prod_{i < |t_m|} Z_{k+i}$. The basic open set $N_{r_m \frown t_m}$ meets the comeager set C , so

we can pick $w \in C \cap N_{r_m} \widehat{t}_m$. Since ϑ is continuous on C and f is continuous, we can pass to a smaller open neighborhood O of w such that for all $x, \hat{x} \in C \cap O$, $\|\vartheta(x) - \vartheta(\hat{x})\|_f < 2^{-j}$. We can assume $O = N_{r_m} \widehat{t}'_m$ for some extension t'_m of t_m . Since G is dense open, we can further extend t'_m to get t_{m+1} such that $N_{r_m} \widehat{t}_{m+1} \subseteq G$. Once the sequences t_m ($m \leq M$) are constructed, $s^{**} = t_M$ fulfills the requirements. ■

Lemma 21. [3, Claim (iii) p. 1842] *There exist strictly increasing sequences $(b_i)_{i < \omega}, (l_i)_{i < \omega} \subseteq \omega$ and functions $f_i: Z_{b_i} \rightarrow [0, 1]^{l_{j+1} - l_j}$ such that $b_0 = l_0 = 0$, for $Z' = \prod_{i < \omega} Z_{b_i}$ and $\vartheta': Z' \rightarrow [0, 1]^\omega$, $\vartheta'(x) = f_0(x_0) \frown \dots \frown f_i(x_i) \frown \dots$ we have*

$$\|x - \hat{x}\|_g < \infty \Leftrightarrow \|\vartheta'(x) - \vartheta'(\hat{x})\|_f < \infty. \quad (19)$$

Proof. We construct the sequences $(b_i)_{i < \omega}, (l_i)_{i < \omega} \subseteq \omega$, finite sequences s_i ($i < \omega$) and dense open sets D_i^j ($i, j < \omega$) by induction, as follows.

We have $b_0 = l_0 = 0$. Let $j < \omega$ and suppose that we have b_j, l_j and $D_i^{j'}$ for every $i < \omega$ and $j' < j$. We apply Lemma 19 for j and $k = b_j + 1$ to get $l_{j+1} = l < \omega$, a finite sequence $s_j^* \in \prod_{i < |s_j^*|} Z_{b_j+1+i}$ and a comeager set $D^j \subseteq Z$ satisfying the conclusions of Lemma 19. We can assume $l_{j+1} > l_j$ and $D^j \subseteq C$. Let $(D_i^j)_{i < \omega}$ be a decreasing sequence of dense open subsets of Z such that $\bigcap_{i < \omega} D_i^j \subseteq D^j$. We apply Lemma 20 for $j, k = b_j + 1 + |s_j^*|, l = l_{j+1}$, and $G = \bigcap_{j' < j} D_j^{j'}$ to get s_j^{**} as in Lemma 20. We set $s_j = s_j^* \frown s_j^{**}$ and $b_{j+1} = b_j + 1 + |s_j|$.

Let $Z' = \prod_{i < \omega} Z_{b_i}$ and set $h: Z' \rightarrow Z$,

$$h(x) = x_0 \frown s_0 \frown x_1 \frown s_1 \frown \dots \frown x_i \frown s_i \frown \dots$$

For every $i < \omega$, we define $f_i: Z_{b_i} \rightarrow [0, 1]^{l_{j+1} - l_j}$ by

$$f_i(a) = \vartheta(h(\underbrace{0 \frown \dots \frown 0}_i \frown a \frown 0 \frown 0 \frown \dots))|_{l_{j+1} \setminus l_j}; \quad (20)$$

and we set $\vartheta': Z' \rightarrow [0, 1]^\omega$, $\vartheta'(x) = f_0(x_0) \frown \dots \frown f_i(x_i) \frown \dots$

It remains to prove (19). To see this, it is enough to prove $\|\vartheta'(x) - \vartheta(h(x))\|_f < \infty$ for every $x \in Z'$ since then for every $x, \hat{x} \in Z'$, by (R_2) ,

$$\begin{aligned} \|\vartheta'(x) - \vartheta'(\hat{x})\|_f < \infty &\iff \|\vartheta(h(x)) - \vartheta(h(\hat{x}))\|_f < \infty \iff \\ &\|h(x) - h(\hat{x})\|_g < \infty \iff \|x - \hat{x}\|_g < \infty. \end{aligned}$$

Let $x \in Z'$ be arbitrary; for every $j < \omega$ we define $e_j, e'_j \in Z'$ by setting

$$\text{Pr}_i e_j = \begin{cases} x_i, & \text{if } i = j; \\ 0, & \text{if } i \in \omega \setminus \{j\}; \end{cases}, \quad \text{Pr}_i e'_j = \begin{cases} x_i, & \text{if } i \leq j; \\ 0, & \text{if } j < i < \omega. \end{cases}$$

Since $h(x)$ and $h(e'_j)$ agree on all coordinates below b_{j+1} , by the definition of s_j^{**} ,

$$\|(\vartheta(h(x)) - \vartheta(h(e'_j)))|_{l_{j+1}}\|_f < 2^{-j} \quad (j < \omega).$$

On the other hand, for $j > 0$, $h(e'_j)$ and $h(e_j)$ agree on all coordinates above b_{j-1} , so by the definition of s_{j-1}^* ,

$$\|(\vartheta(h(e'_j)) - \vartheta(h(e_j)))|_{\omega \setminus l_j}\|_f < 2^{-j+1} \quad (0 < j < \omega). \quad (21)$$

Moreover, (21) holds for $j = 0$, as well. Then by (R_2) ,

$$\begin{aligned} \|(\vartheta'(x) - \vartheta(h(x)))|_{l_{j+1} \setminus l_j}\|_f &= \|(\vartheta(h(e_j)) - \vartheta(h(x)))|_{l_{j+1} \setminus l_j}\|_f \lesssim \\ &\|(\vartheta(h(e_j)) - \vartheta(h(e'_j)))|_{\omega \setminus l_j}\|_f + \|(\vartheta(h(e'_j)) - \vartheta(h(x)))|_{l_{j+1}}\|_f \leq 3 \cdot 2^{-j}. \end{aligned}$$

Therefore

$$\|(\vartheta'(x) - \vartheta(h(x)))\|_f = \sum_{j < \omega} \|(\vartheta'(x) - \vartheta(h(x)))|_{l_{j+1} \setminus l_j}\|_f \leq \sum_{j < \omega} 3 \cdot 2^{-j} < \infty,$$

as required. ■

Lemma 22. [3, Claim (iv) p. 1843] *There exist $c > 0$ and $N < \omega$ such that with the notation of (20), for every $i > N$, $\|f_i(1) - f_i(0)\|_f > c$.*

Proof. If not, then we can find a strictly increasing sequence $(j_m)_{m < \omega} \subseteq \omega$ such that $\|f_{j_m}(1) - f_{j_m}(0)\|_f \leq 2^{-m}$ ($m < \omega$). Let \hat{x} be the constant 0 sequence, and let x be the sequence which is 1 at each coordinate j_m ($m < \omega$) and 0 at all other coordinates. Then $\|x - \hat{x}\|_g = \infty$ but

$$\begin{aligned} \|\vartheta'(x) - \vartheta'(\hat{x})\|_f &= \sum_{j < \omega} \|f_j(x(j)) - f_j(\hat{x}(j))\|_f = \\ &\sum_{m < \omega} \|f_{j_m}(1) - f_{j_m}(0)\|_f \leq \sum_{m < \omega} 2^{-m} < \infty, \end{aligned}$$

contradicting (19). ■

Lemma 23. *Let $c > 0$, $N < \omega$ be as in Lemma 22. For every $0 < D < \omega$ there exists $N_D > \max\{N, D\}$ such that for every $i \geq N_D$ there is a $0 \leq k < 2^{b_{N_D}}$ with*

$$\|f_i((k+1)/2^{b_{N_D}}) - f_i(k/2^{b_{N_D}})\|_f \geq Dg(1/2^{b_{N_D}}). \quad (22)$$

Proof. Let $\varepsilon > 0$ and $M < \omega$ be as in the assumptions of Theorem 18. Fix $0 < D < \omega$; by (A_2) there exists $N_D > \max\{M, N, D\}$ such that with $n = 2^{b_{N_D}}$, $2D/c < \varepsilon\varphi(1/n)/\psi(1/n)$. Fix $i \geq N_D$, set $l = l_{i+1} - l_i$ and

$$\gamma_j = |\text{Pr}_j(f_i(1) - f_i(0))| \quad (j < l).$$

For every $x = (x_j)_{j < l} \in [-1, 1]^l$ set

$$\|x\|_\Delta = \left(\sum_{j < l} |x_j|^\alpha \varphi(\gamma_j) \right)^{1/\alpha};$$

then $\|\cdot\|_\Delta$ satisfies the triangle inequality on $[0, 1]^l$. Since $\|f_i(1) - f_i(0)\|_f = \|f_i(1) - f_i(0)\|_\Delta^\alpha = \sum_{j < l} \gamma_j^\alpha \varphi(\gamma_j)$, by the triangle inequality there is a $0 \leq k < n$ such that

$$\|f_i((k+1)/n) - f_i(k/n)\|_\Delta \geq \frac{1}{n} \|f_i(1) - f_i(0)\|_f^{1/\alpha}.$$

With such a k , set

$$\delta_j = |\text{Pr}_j(f_i((k+1)/n) - f_i(k/n))| \quad (j < l);$$

i.e. we have

$$\sum_{j < l} \delta_j^\alpha \varphi(\gamma_j) \geq \frac{1}{n^\alpha} \|f_i(1) - f_i(0)\|_f. \quad (23)$$

Set $J = \{j < l: \varphi(\gamma_j) \leq \varphi(\delta_j)c/(2D\psi(1/n))\}$. Then

$$\sum_{j < l} \delta_j^\alpha \varphi(\gamma_j) \leq \sum_{j \in J} \delta_j^\alpha \varphi(\delta_j) \frac{c}{2D\psi(1/n)} + \sum_{j \notin J} \delta_j^\alpha \varphi(\gamma_j). \quad (24)$$

By the choice of N_D , $2D/c < \varepsilon\varphi(1/n)/\psi(1/n)$. So for $j \notin J$, $\varphi(\delta_j) < \varepsilon\varphi(\gamma_j)\varphi(1/n)$. This, by (A_1) and by $b_{N_D} \geq N_D > M$, implies $\delta_j \leq \gamma_j/(2n)$ ($j \notin J$). Hence

$$\sum_{j \notin J} \delta_j^\alpha \varphi(\gamma_j) \leq \frac{1}{(2n)^\alpha} \sum_{j \notin J} \gamma_j^\alpha \varphi(\gamma_j) = 2^{-\alpha} \frac{\|f_i(1) - f_i(0)\|_f}{n^\alpha}.$$

So by (23) and (24),

$$\sum_{j \in J} \delta_j^\alpha \varphi(\delta_j) \frac{c}{2D\psi(1/n)} \geq (1 - 2^{-\alpha}) \frac{\|f_i(1) - f_i(0)\|_f}{n^\alpha}$$

which implies

$$\|f_i((k+1)/n) - f_i(k/n)\|_f = \sum_{j < l} \delta_j^\alpha \varphi(\delta_j) \geq D \frac{\psi(1/n)}{n^\alpha} = Dg(1/n),$$

as required. ■

For every $0 < D < \omega$ let N_D be as in Lemma 23. Since $g(0) = 0$ and g is continuous, by reassigning N_D we can assume $g(1/2^{b_{N_D}}) \leq 1/D^2$ ($0 < D < \omega$). Let $I_D \subseteq \omega \setminus N_D$ ($0 < D < \omega$) be pairwise disjoint sets such that $1/D \leq |I_D|Dg(1/2^{b_{N_D}}) < 2/D$. For every $0 < D < \omega$ and $i \in I_D$ pick a $0 \leq k_{i,D} < 2^{b_{N_D}}$ satisfying (22). Define $x, \hat{x} \in Z'$ by $x(i) = k_{i,D}/2^{b_{N_D}}$, $\hat{x}(i) = (k_{i,D} + 1)/2^{b_{N_D}}$ ($i \in I_D$, $0 < D < \omega$), else $x(i) = \hat{x}(i) = 0$. Then

$$\begin{aligned} \|\vartheta'(\hat{x}) - \vartheta'(x)\|_f &= \\ & \sum_{0 < D < \omega} \sum_{i \in I_D} \|f_i((k_{i,D} + 1)/2^{b_{N_D}}) - f_i(k_{i,D}/2^{b_{N_D}})\|_f \geq \\ & \sum_{0 < D < \omega} D|I_D|g(1/2^{b_{N_D}}) = \infty \end{aligned}$$

while

$$\|\hat{x} - x\|_g = \sum_{0 < D < \omega} |I_D|g(1/2^{b_{N_D}}) < \sum_{0 < D < \omega} \frac{2}{D^2} < \infty;$$

i.e. $x \mathbf{E}_g \hat{x}$ but $\vartheta'(x) \not\mathbf{E}_f \vartheta'(\hat{x})$. This contradiction completes the proof. ■

5 Applications

In this section we construct several families of functions for which our reducibility and nonreducibility results can be applied. Let $1 \leq \alpha \leq \beta < \infty$, let $\varphi: (0, 1] \rightarrow \mathbb{R}$, $\psi: [0, 1] \rightarrow \mathbb{R}$ be continuous functions and set $f = \text{Id}^\alpha \varphi$, $f(0) = 0$ and $g = \text{Id}^\beta \psi$.

5.1 Definition of φ from ψ and κ

In order to facilitate the checking of the conditions of Theorem 16, we may use the following approach. Instead of defining κ from φ and ψ , we may define φ from ψ and κ . To this end we set $\kappa(1/2^n) = \mu(n)/2^{n\alpha/\beta}$ ($n < \omega$) where μ will be specified later. We assume $\mu(0) = \varphi(1) = \psi(1) = 1$. Then (8), (9) and (10) read as

$$\varphi\left(\frac{1}{2^n}\right) = \sum_{i=0}^n 2^{(\alpha-\beta)(n-i)} \mu(i)^\beta \psi\left(\frac{2^{(1-\alpha/\beta)i} \mu(i)}{2^n}\right) \quad (n < \omega), \quad (25)$$

$$\sum_{i=n}^{\infty} \frac{1}{2^{i\alpha}} \mu(i)^\beta \psi\left(\frac{\mu(i)}{2^{i\alpha/\beta}}\right) \leq L \sum_{i=0}^n 2^{i(\beta-\alpha)} \frac{\mu(i)^\beta}{2^{n\beta}} \psi\left(\frac{2^{(1-\alpha/\beta)i} \mu(i)}{2^n}\right), \quad (26)$$

$$\mu(n) \leq L \cdot \max_{i < n} \mu(i) \frac{2^{(n-i)\alpha/\beta}}{2^{n-i}} \quad (n < \omega). \quad (27)$$

Given μ and ψ , we can define $\varphi(1/2^n)$ ($n < \omega$) by (25) and then extend φ to $(0, 1]$ to be a continuous function which is affine on $[1/2^{n+1}, 1/2^n]$ ($n < \omega$). When we say below “we define φ from μ , α , β and ψ ”, we mean this definition.

We show that for a φ defined this way, if there exist $\varepsilon > 0$, $M < \omega$ such that for $n > M$,

$$\varphi(1/2^i) \leq \varepsilon \varphi(1/2^j) \varphi(1/2^n) \Rightarrow i \geq j + n + 3 \quad (i, j < \omega) \quad (28)$$

then (A_1) of Theorem 18 holds. Let $x, y \in (0, 1]$, say $1/2^{i+1} < x \leq 1/2^i$ and $1/2^{j+1} < y \leq 1/2^j$. We have

$$\varphi(x) \in [\varphi(1/2^{i+1}), \varphi(1/2^i)], \quad \varphi(y) \in [\varphi(1/2^{j+1}), \varphi(1/2^j)],$$

thus $\varphi(x) \leq \varepsilon \varphi(y) \varphi(1/2^n)$ implies

$$\min\{\varphi(1/2^{i+1}), \varphi(1/2^i)\} \leq \varepsilon \max\{\varphi(1/2^{j+1}), \varphi(1/2^j)\} \varphi(1/2^n).$$

So by (28), for $n > M$ we have $i \geq n + j + 2$, which implies $x \leq y/2^{n+1}$, as required.

5.2 Explicit examples

We introduce a family of functions for which our theorems can be applied and whose growth order is easy to calibrate. For $n < \omega$, let $t_n: (0, 1] \rightarrow \mathbb{R}$,

$$t_n(x) = \underbrace{1 + \log(1 + \dots \log(1 - \log(x)) \dots)}_n \quad (0 < x \leq 1).$$

For $\eta \in [0, 1)^{<\omega}$ we define $l_\eta: (0, 1] \rightarrow \mathbb{R}$, $l_\eta(x) = \prod_{i < |\eta|} t_i^{\eta_i}$ ($0 < x \leq 1$); e.g.,

$$l_\emptyset(x) = 1, \quad l_{(\eta_0)}(x) = (1 - \log(x))^{\eta_0}, \\ l_{(\eta_0 \eta_1)}(x) = (1 - \log(x))^{\eta_0} (1 + \log(1 - \log(x)))^{\eta_1}, \text{ etc.}$$

Let $<_{\text{lex}}$ denote the lexicographic order. We summarize some elementary properties of the functions l_η , which will be used in the sequel.

Lemma 24. *For every $\eta, \eta' \in [0, 1)^{<\omega}$ with $\eta <_{\text{lex}} \eta'$, $1 \leq \alpha < \infty$ and $\delta > 0$,*

- (a) $1 \leq l_\eta(xy) \leq l_\eta(x)l_\eta(y)$ ($0 < x, y \leq 1$);
- (b) $l_\eta \circ \text{Id}^\delta \approx l_\eta$ and $l_\eta \lesssim \text{Id}^{-\delta}$;
- (c) $l_\eta(1/2^{n+1}) - l_\eta(1/2^n) \leq 1$ for every $n < \omega$ sufficiently large;
- (d) l_η is continuous and strictly decreasing, moreover if $\eta <_{\text{lex}} \eta'$ then $l_\eta/l_{\eta'}$ is strictly increasing in a neighborhood of 0, so by $l_\eta(x)/l_{\eta'}(x) > 0$ ($x > 0$), $l_\eta/l_{\eta'}$ is essentially increasing and $\lim_{x \rightarrow +0} l_\eta(x)/l_{\eta'}(x) = 0$;
- (e) $\text{Id}^\delta l_\eta$ is bounded and $\lim_{x \rightarrow +0} x^\delta l_\eta(x) = 0$;
- (f) $f(x) = x^\delta l_\eta(x)$ ($0 < x < 1$), $f(0) = 0$ is continuous, strictly increasing in a neighborhood of 0, so by $f(x) > 0$ ($x > 0$), f is essentially increasing;
- (g) $f(x) = x^\alpha l_\eta(x)$ ($0 < x < 1$), $f(0) = 0$ is continuous, satisfies (R_1) and (R_2) hence \mathbf{E}_f is an equivalence relation;
- (h) $f(x) = x^\alpha/l_\eta(x)$ ($0 < x < 1$), $f(0) = 0$ is continuous and strictly increasing, satisfies (R_1) and (R_2) hence \mathbf{E}_f is an equivalence relation;
- (i) $\varphi = 1/l_\eta$ satisfies (A_1) of Theorem 18.

Proof. It is enough to prove (a) for t_n ($n < \omega$). We do this by induction on n . For $n = 0$, the statement follows from

$$1 \leq 1 - \log(xy) = 1 - \log(x) - \log(y) \leq 1 - \log(x) - \log(y) + \log(x)\log(y) = (1 - \log(x))(1 - \log(y)).$$

Let now $n > 1$; then $t_n = 1 + \log t_{n-1}$, hence $1 \leq t_n$. By the inductive hypothesis,

$$t_n(xy) = 1 + \log t_{n-1}(xy) \leq 1 + \log t_{n-1}(x) + \log t_{n-1}(y) \leq (1 + \log t_{n-1}(x))(1 + t_{n-1}(y)) = t_n(x)t_n(y),$$

as required.

Similarly, it is enough to show (b) for t_n ($n < \omega$); we use induction on n . For $n = 0$, the first statement follows from $1 - \log(x^\delta) = 1 - \delta \log(x)$ ($0 < x \leq 1$), while $t_0 \lesssim \text{Id}^{-\delta}$ is elementary analysis. Let now $n > 1$; we have $t_n = 1 + \log t_{n-1}$. By the inductive hypothesis and $t_{n-1} \geq 1$, $1 + \log(t_{n-1} \circ \text{Id}^\delta) \approx 1 + \log t_{n-1}$, so the first statement follows. Also by the inductive hypothesis, $1 + \log t_{n-1} \lesssim 1 - \delta \log \lesssim \text{Id}^{-\delta}$, so the proof is complete.

We show $(l_\eta(1/2^{n+1}) - l_\eta(1/2^n))_{n < \omega}$ is a null sequence; then (c) follows. By elementary analysis, for every $\delta \in [0, 1)$ and $m < \omega$, $(t_m^\delta(1/2^{n+1}) -$

$t_m^\delta(1/2^n)_{n < \omega}$ is a null sequence. Since l_η is a finite product of t_m^δ s, the statement follows.

Statements (d), (e) and (f) are elementary analysis. For (g), (R_1) is immediate; (R_2a) follows from $(x+y)^\alpha \lesssim x^\alpha + y^\alpha$ ($0 \leq x, y \leq 1$) and l_η being decreasing; while (R_2b) follows from Id^α/l_η being essentially increasing.

Consider now (h). Since l_η is strictly decreasing, Id^α/l_η is strictly increasing. So (R_1) is immediate and (R_2b) holds. To see (R_2a) , observe that by (a), for $0 < v/2 \leq u \leq v \leq 1$ we have

$$l_\eta(u) \leq l_\eta(v/2) \leq l_\eta(1/2)l_\eta(v).$$

So for $0 < x, y \leq 1$,

$$(x+y)^\alpha/l_\eta(x+y) \lesssim l_\eta(1/2)(x^\alpha/l_\eta(x) + y^\alpha/l_\eta(y)),$$

as required.

It remains to prove (i). It is enough to show that for every $n < \omega$,

$$l_\eta(x) \geq l_\eta(1/2)l_\eta(y)l_\eta(1/2^n) \Rightarrow x \leq \frac{y}{2^{n+1}} \quad (i, j < \omega).$$

By (a), $l_\eta(y/2^{n+1}) \leq l_\eta(1/2)l_\eta(y)l_\eta(1/2^n)$, so since l_η is decreasing, the statement follows. ■

Corollary 25. *Let $1 \leq \alpha < \infty$ and let $\eta, \eta' \in [0, 1)^{<\omega}$ satisfy $\eta <_{lex} \eta'$.*

1. *The functions $\psi = l_\eta$, $g(x) = x^\alpha l_\eta(x)$ ($0 < x \leq 1$), $g(0) = 0$ satisfy the conditions of Theorem 15.*
2. *The functions $\varphi(x) = 1/l_\eta(x)$, $\psi(x) = 1/l_{\eta'}(x)$ ($0 < x \leq 1$), $\varphi(0) = \psi(0) = 0$ and $f = \text{Id}^\alpha/l_\eta$, $g = \text{Id}^\alpha/l_{\eta'}$ satisfy the conditions of Theorem 18.*

Proof. Statement 1 follows from (d), (e) and (g) of Lemma 24. For 2, \mathbf{E}_f and \mathbf{E}_g are equivalence relations by (h) of Lemma 24; while (A_1) and (A_2) follow from (i) and (d) of Lemma 24. This completes the proof. ■

5.3 The counterintuitive case

In this section we present an example illustrating that the comparison of the growth order of functions does not decide Borel reducibility. Let $\alpha = \beta$ and $\psi \equiv 1$. Then (25) turns to $\varphi(1/2^n) = \sum_{i=0}^n \mu(i)^\alpha$, i.e.

$$\mu(n)^\alpha = \varphi(1/2^n) - \varphi(1/2^{n-1}) \quad (0 < n < \omega); \quad (29)$$

(26) reads as

$$\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \mu(n+i)^\alpha \leq L\varphi(1/2^n); \quad (30)$$

and (27) means

$$\mu(n) \leq L \cdot \max_{i < n} \mu(i). \quad (31)$$

Since $\mu(n)^\alpha \leq \varphi(1/2^n)$, (30) holds if

$$\sum_{i=0}^{\infty} 1/2^{i\alpha} \varphi(1/2^{n+i}) \leq L\varphi(1/2^n). \quad (32)$$

Corollary 26. *Let $\varphi: (0, 1] \rightarrow (0, +\infty)$ be an essentially decreasing continuous function such that $\text{Id}^\alpha \varphi$ is essentially increasing, $\mathbf{E}_{\text{Id}^\alpha \varphi}$ is an equivalence relation, for every $\delta > 0$, $\liminf_{x \rightarrow +0} x^\delta \varphi(x) = 0$ and (32) holds. Define μ by (29) and suppose (31) holds. Then $\mathbf{E}_{\text{Id}^\alpha}$ and $\mathbf{E}_{\text{Id}^\alpha \varphi}$ are Borel equivalent.*

Proof. By Theorem 15, $\mathbf{E}_{\text{Id}^\alpha} \leq_B \mathbf{E}_{\text{Id}^\alpha \varphi}$. By Lemma 14, we can assume in addition that φ is decreasing. Then the definition of μ in (29) is valid. So by Theorem 16, $\mathbf{E}_{\text{Id}^\alpha \varphi} \leq_B \mathbf{E}_{\text{Id}^\alpha}$. ■

We show that (32) holds if for some $\varepsilon > 0$, $\text{Id}^{\alpha-\varepsilon} \varphi$ is essentially increasing. Then

$$1/2^{(n+i)(\alpha-\varepsilon)} \varphi(1/2^{n+i}) \lesssim 1/2^{n(\alpha-\varepsilon)} \varphi(1/2^n) \quad (i < \omega),$$

i.e. $1/2^{i\alpha} \varphi(1/2^{n+i}) \lesssim 1/2^{i\varepsilon} \varphi(1/2^n)$ ($i < \omega$), so the statement follows. In particular, by Corollary 25.1 and by (d), (f) and (g) of Lemma 24, $\varphi = l_\eta$ fulfills these requirements for every $\eta \in [0, 1)^{<\omega}$. By (c) of Lemma 24 and by $\mu(0) = 1$, (31) also holds for $\varphi = l_\eta$ ($\eta \in [0, 1)^{<\omega}$). That is, $\mathbf{E}_{\text{Id}^\alpha l_\eta}$ and $\mathbf{E}_{\text{Id}^\alpha}$ are Borel equivalent. We will see below in (36) that for every $\eta \in [0, 1)^{<\omega}$, $\mathbf{E}_{\text{Id}^\alpha} <_B \mathbf{E}_{\text{Id}^\alpha / l_\eta}$. So the comparison of the growth order of functions does not decide Borel reducibility.

5.4 The $\alpha < \beta$ case

Since the previous and following sections contain the analysis of the reducibility of $\mathbf{E}_{\text{Id}^\beta}$ to $\mathbf{E}_{\text{Id}^\beta \psi}$, in the $\alpha < \beta$ case we assume $\psi \equiv 1$. Then (25) and (26) turn to

$$\varphi\left(\frac{1}{2^n}\right) = \sum_{i=0}^n 2^{(\alpha-\beta)(n-i)} \mu(i)^\beta \quad (n < \omega), \quad (33)$$

$$\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \mu(n+i)^\beta \leq L \sum_{i=0}^n 2^{(\alpha-\beta)(n-i)} \mu(i)^\beta. \quad (34)$$

To satisfy (33), we have to define

$$\mu(n)^\beta = \varphi(1/2^n) - \varphi(1/2^{n-1})/2^{\beta-\alpha} \quad (0 < n < \omega), \quad (35)$$

and then (34) follows from (32).

Corollary 27. *Let $1 \leq \alpha < \beta < \infty$. Suppose $\varphi: [0, 1] \rightarrow \mathbb{R}^+$ is continuous, essentially increasing, $\varphi/\text{Id}^{\beta-\alpha}$ is essentially decreasing, $\mathbf{E}_{\text{Id}^\alpha \varphi}$ is an equivalence relation and for the μ defined by (35), (27) holds. Then $\mathbf{E}_{\text{Id}^\alpha \varphi} \leq_B \mathbf{E}_{\text{Id}^\beta}$.*

Proof. By Lemma 14, we can assume $\varphi/\text{Id}^{\beta-\alpha}$ is decreasing, so that (35) is valid; while φ being essentially increasing implies (32). So $\mathbf{E}_{\text{Id}^\alpha \varphi} \leq_B \mathbf{E}_{\text{Id}^\beta}$ follows from Theorem 16. ■

The assumptions of Corollary 27 are affordable:

- if φ is essentially decreasing, Corollary 26 gives the Borel equivalence of $\mathbf{E}_{\text{Id}^\alpha \varphi}$ and $\mathbf{E}_{\text{Id}^\beta}$ under suitable assumptions;
- in order to not to be in the counterintuitive case, we may assume that $\varphi/\text{Id}^{\beta-\alpha-\delta}$ is decreasing for some $\delta > 0$, so by Corollary 27, $\mathbf{E}_{\text{Id}^\alpha \varphi} \leq_B \mathbf{E}_{\text{Id}^{\beta-\delta}} <_B \mathbf{E}_{\text{Id}^\beta}$;

So Corollary 27 indicates that in the $\alpha < \beta$ case growth order decides Borel reducibility. Moreover, in the next section we will see that in order to guarantee $\mathbf{E}_{\text{Id}^\alpha \varphi} \leq_B \mathbf{E}_{\text{Id}^\beta}$ by growth order estimates, we need $\text{Id}^\beta/(\text{Id}^\alpha \varphi)$ to be bounded; the assumptions of Corollary 27 reflect this constraint.

Finally we check that for every $\eta \in [0, 1)^{<\omega}$, the function $\varphi(0) = 0$, $\varphi(x) = 1/l_\eta(x)$ ($0 < x \leq 1$) satisfies the assumptions of Corollary 27. By $\lim_{n \rightarrow \infty} l_\eta(1/2^{n+1})/l_\eta(1/2^n) = 1$ we have

$$\mu(n+1) \lesssim 1/l_\eta^{1/\beta}(1/2^{n+1}) \lesssim 1/l_\eta^{1/\beta}(1/2^n) \lesssim \mu(n) \quad (n < \omega),$$

i.e. (27) holds. The other assumptions follow from (d), (f) and (h) of Lemma 24. So

$$\mathbf{E}_{\text{Id}^\alpha/l_\eta} \leq_B \mathbf{E}_{\text{Id}^\gamma} <_B \mathbf{E}_{\text{Id}^\beta} \quad (\eta \in [0, 1)^{<\omega}, 1 \leq \alpha < \gamma < \beta < \infty). \quad (36)$$

5.5 The $\alpha = \beta$ case

This is the most interesting case for us. Now (25), (26) and (27) turn to

$$\varphi\left(\frac{1}{2^n}\right) = \sum_{i=0}^n \mu(i)^\alpha \psi\left(\frac{\mu(i)}{2^n}\right) \quad (n < \omega), \quad (37)$$

$$\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \mu(n+i)^\alpha \psi\left(\frac{\mu(n+i)}{2^{n+i}}\right) \leq L \sum_{i=0}^n \mu(i)^\alpha \psi\left(\frac{\mu(i)}{2^n}\right) \quad (n < \omega), \quad (38)$$

$$\mu(n) \leq L \cdot \max_{i < n} \mu(i) \quad (n < \omega). \quad (39)$$

We obtain a sufficient condition for (38) and (39).

Lemma 28. *Assume ψ is essentially increasing, $\psi(x) > 0$ for $x > 0$ and $\mu(n) \leq 1$ for every $n < \omega$ sufficiently large. Then (38) and (39) hold.*

Proof. Since ψ is essentially increasing, for every n sufficiently large we have $\psi(\mu(n+i)/2^{n+i}) \lesssim \psi(1/2^n)$ ($0 \leq i < \omega$). Hence

$$\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \mu(n+i)^\alpha \psi\left(\frac{\mu(n+i)}{2^{n+i}}\right) \lesssim \frac{1}{(1-1/2^\alpha)} \psi(1/2^n) \quad (n < \omega), \quad (40)$$

thus by $\mu(0) = 1$, (38) follows. Also by $\mu(0) = 1$ we have (39), so the proof is complete. ■

5.5.1 The question of S. Gao

In this section, in the spirit of (3), we give the negative answer to the question of S. Gao mentioned in the introduction.

Corollary 29. *Let $1 \leq \alpha < \infty$ be arbitrary. Let $\mu: \omega \rightarrow [0, \infty)$ be such that $\mu(0) = 1$. Let $\psi: [0, 1] \rightarrow [0, \infty)$ be a continuous essentially increasing function such that $\psi(1) = 1$, (38) holds and there is a $K > 0$ for which*

$$\frac{1}{K} \psi(1/2^n) \leq \psi\left(\frac{\mu(i)}{2^n}\right) \leq K \psi(1/2^n) \quad (0 \leq i \leq n < \omega). \quad (41)$$

Set $\sigma_{\mu^\alpha}(n) = \sum_{i=1}^n \mu^\alpha(i)$ ($n < \omega$). Suppose $((1 + \sigma_{\mu^\alpha}(n)) \psi(1/2^n))_{n < \omega}$ is essentially decreasing and (39) holds.

Define φ from μ , α and ψ . Set $f(x) = x^\alpha \varphi(x)$ ($0 < x \leq 1$), $f(0) = 0$ and $g = \text{Id}^\alpha \psi$ and suppose \mathbf{E}_f and \mathbf{E}_g are equivalence relations. Then $\mathbf{E}_f \leq_B \mathbf{E}_g$.

If, in addition, φ satisfies A_1 of Theorem 18, or equivalently φ satisfies (28), and $\lim_{n \rightarrow \infty} \sigma_{\mu^\alpha}(n) = \infty$, then $\mathbf{E}_g \not\leq_B \mathbf{E}_f$.

Proof. By (41), from (37) we get

$$\frac{1}{K} (1 + \sigma_{\mu^\alpha}(n)) \psi\left(\frac{1}{2^n}\right) \leq \varphi\left(\frac{1}{2^n}\right) \leq K (1 + \sigma_{\mu^\alpha}(n)) \psi\left(\frac{1}{2^n}\right). \quad (42)$$

Since $((1 + \sigma_{\mu^\alpha}(n))\psi(1/2^n))_{n < \omega}$ is essentially decreasing, φ is essentially increasing. So by Theorem 16, $\mathbf{E}_f \leq_B \mathbf{E}_g$.

Moreover, if φ satisfies A_1 of Theorem 18, which follows e.g. if φ satisfies (28), then since $\lim_{n \rightarrow \infty} \sigma_{\mu^\alpha}(n) = \infty$ implies (A_2) of Theorem 18, $\mathbf{E}_g \not\leq_B \mathbf{E}_f$. This completes the proof. ■

Many natural functions satisfy the conditions of Corollary 29 for both φ and ψ , in particular the functions $1/l_\eta$. By Lemma 3, the following result gives the negative answer to the question of S. Gao.

Corollary 30. *For every $1 \leq \alpha < \beta < \infty$ and $\eta, \eta' \in [0, 1)^{<\omega}$ with $\eta <_{lex} \eta'$,*

$$\mathbf{E}_{\text{Id}^\alpha} <_B \mathbf{E}_{\text{Id}^\alpha/l_\eta} <_B \mathbf{E}_{\text{Id}^\alpha/l_{\eta'}} <_B \mathbf{E}_{\text{Id}^\beta}.$$

Proof. By Lemma 24 (h), $\mathbf{E}_{\text{Id}^\alpha/l_\eta}$ is an equivalence relation, and in (36) we obtained $\mathbf{E}_{\text{Id}^\alpha/l_\eta} <_B \mathbf{E}_{\text{Id}^\beta}$. By Lemma 24 (d), $(l_{\eta'}(1/2^n)/l_\eta(1/2^n))_{n < \omega}$ is strictly increasing for n sufficiently large. Thus there is a function $\mu: \omega \rightarrow \mathbb{R}^+$ such that $\mu(0) = 1$, and for every $n < \omega$ sufficiently large,

$$\mu^\alpha(n) = l_{\eta'}(1/2^n)/l_\eta(1/2^n) - l_{\eta'}(1/2^{n-1})/l_\eta(1/2^{n-1}).$$

Let $\psi = 1/l_{\eta'}$ ($0 < x \leq 1$), $\psi(0) = 0$ and define φ from μ , α and ψ . We check the conditions of Corollary 29.

First we show that for every $\varepsilon > 0$,

$$2^{-n\varepsilon} \leq \mu^\alpha(n) \leq 1 \tag{43}$$

holds for n sufficiently large. By Lemma 24 (a), (c) and (d), for every n sufficiently large,

$$\begin{aligned} \mu^\alpha(n) &= \frac{l_{\eta'}(1/2^n)}{l_\eta(1/2^n)} - \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^{n-1})} = \\ &= \frac{l_{\eta'}(1/2^n) - l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^n)} + \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^n)} - \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^{n-1})} \leq \\ &= \frac{l_{\eta'}(1/2^n) - l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^n)} \leq \frac{1}{l_\eta(1/2^n)} \leq 1. \end{aligned}$$

For the lower bound, take an $m > |\eta'|$ and consider t_m . By Lemma 24 (d), $l_{\eta'}(1/2^n)/(l_\eta(1/2^n)t_m(1/2^n))$ is still strictly increasing for n sufficiently large. So for n sufficiently large,

$$\mu^\alpha(n) = \frac{l_{\eta'}(1/2^n)}{l_\eta(1/2^n)} - \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^{n-1})} \geq \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^{n-1})} \frac{t_m(1/2^n) - t_m(1/2^{n-1})}{t_m(1/2^{n-1})}.$$

It is elementary analysis that $t_m(1/2^n) - t_m(1/2^{n-1}) \geq 1/n^2$ for n sufficiently large, so the statement follows.

By Lemma 24 (d), ψ is continuous, essentially increasing and $\psi(1) = 1$. Lemma 28 gives (38) and (39). Also, (41) follows from Lemma 24 (b) using that $2^{-n/2} \leq \mu^\alpha(n) \leq 2^{n/2}$ holds for n sufficiently large.

We have

$$(1 + \sigma_{\mu^\alpha}(n)) \psi(1/2^n) \approx 1/l_\eta(1/2^n) \quad (n < \omega),$$

so $((1 + \sigma_{\mu^\alpha}(n)) \psi(1/2^n))_{n < \omega}$ is essentially decreasing.

By (42),

$$\varphi\left(\frac{1}{2^n}\right) \approx (1 + \sigma_{\mu^\alpha}(n)) \psi\left(\frac{1}{2^n}\right) \approx l_{\eta'}(1/2^n)/l_\eta(1/2^n) \psi\left(\frac{1}{2^n}\right) \approx 1/l_\eta(1/2^n),$$

so by Corollary 29, $\mathbf{E}_{\text{Id}^\alpha/l_\eta} \leq_B \mathbf{E}_{\text{Id}^\alpha/l_{\eta'}}$.

By Lemma 24 (d), $\lim_{x \rightarrow +0} l_{\eta'}(x)/l_\eta(x) = \infty$, i.e. $\lim_{n \rightarrow \infty} \sigma_{\mu^\alpha}(n) = \infty$. By Lemma 24 (i), $1/l_\eta$ satisfies A_1 of Theorem 18, so again by Corollary 29, $\mathbf{E}_{\text{Id}^\alpha/l_{\eta'}} \not\leq_B \mathbf{E}_{\text{Id}^\alpha/l_\eta}$. The $\eta = \emptyset$ special case gives $\mathbf{E}_{\text{Id}^\alpha} <_B \mathbf{E}_{\text{Id}^\alpha/l_\eta}$, so the proof is complete. ■

5.5.2 Embedding long linear orders

In this section we show that every linear order which can be embedded into $(\mathcal{P}(\omega)/\text{fin}, \subset)$ also embeds into the set of Borel equivalence relations \mathbf{E}_f satisfying $\mathbf{E}_{\text{Id}^\alpha} \leq_B \mathbf{E}_f \leq_B \mathbf{E}_{\text{Id}^\alpha/(1-\log)}$ ordered by $<_B$. We refer to [1] for results on embedding ordered sets into $(\mathcal{P}(\omega)/\text{fin}, \subset)$, here we only remark that it is consistent with ZFC, e.g. under the Continuum Hypothesis, that every ordered set of size continuum embeds into $(\mathcal{P}(\omega)/\text{fin}, \subset)$.

Corollary 31. *Let $1 \leq \alpha < \infty$ be fixed. There is a mapping $\mathcal{F}: \mathcal{P}(\omega)/\text{fin} \rightarrow C[0, 1]$ such that for every $U, V \in \mathcal{P}(\omega)/\text{fin}$, $\mathbf{E}_{\mathcal{F}(U)}$ is an equivalence relation satisfying $\mathbf{E}_{\text{Id}^\alpha} \leq_B \mathbf{E}_{\mathcal{F}(U)} \leq_B \mathbf{E}_{\text{Id}^\alpha/(1-\log)}$ and $U \subset V \Rightarrow \mathbf{E}_{\mathcal{F}(V)} <_B \mathbf{E}_{\mathcal{F}(U)}$.*

Proof. Let $\gamma = 17/16$. For every $U \in \mathcal{P}(\omega)$ set

$$\mu_U(0) = 1, \quad \mu_U^\alpha(n) = \gamma^{|U \cap [1+\log(n)]|} \quad (0 < n < \omega).$$

Let $\psi_0(x) = 1/(1 - \log(x))^2$ ($0 < x \leq 1$), $\psi_0(0) = 0$. For every $U \in \mathcal{P}(\omega)$ we define φ_U from μ_U , α and ψ_0 , and we set $\mathcal{F}(U) = \text{Id}^\alpha \varphi_U$.

First we show that for every $U \in \mathcal{P}(\omega)$, μ_U and ψ_0 satisfy (41). By definition, $1 \leq \mu_U^\alpha(n) \leq \gamma^{1+\log(n)} \leq \gamma n$ ($0 < n < \omega$), so (41) follows.

Next we show that for every $U \in \mathcal{P}(\omega)$, φ_U is essentially increasing. Since (41) holds, by (42) it is enough to show that $((1 + \sigma_{\mu_U^\alpha}(n)) \psi_0(1/2^n))_{n < \omega}$ is

essentially decreasing. We have $\psi_0(1/2^n) \approx 1/n^2$ ($0 < n < \omega$). Let $0 < n < m < \omega$ be fixed, say $m = \rho n$ for some $\rho > 1$. If $\mu_U^\alpha(n) = \gamma^k$, then $\sigma_{\mu_U^\alpha}(n) \geq n\gamma^{k-1}/2$, and

$$\sigma_{\mu_U^\alpha}(m) \leq \sigma_{\mu_U^\alpha}(n) + (m-n)\gamma^{k+1+\log(m)-\log(n)} \leq \sigma_{\mu_U^\alpha}(n) + (\rho-1)n\gamma^{k+1+\log(\rho)}.$$

Hence

$$\begin{aligned} \frac{\sigma_{\mu_U^\alpha}(m)}{m^2} &\leq \frac{\sigma_{\mu_U^\alpha}(n) + (\rho-1)n\gamma^{k+1+\log(\rho)}}{(\rho n)^2} \leq \\ &\frac{\sigma_{\mu_U^\alpha}(n)}{n^2} + \frac{\gamma^{k+1+\log(\rho)}}{\rho n} \leq \frac{\sigma_{\mu_U^\alpha}(n)}{n^2} \left(1 + \frac{2\gamma^{2+\log(\rho)}}{\rho}\right) \leq 9\frac{\sigma_{\mu_U^\alpha}(n)}{n^2}. \end{aligned}$$

This shows $((1 + \sigma_{\mu_U^\alpha}(n))\psi_0(1/2^n))_{n < \omega}$ is essentially decreasing.

Next we check that for every $U \in \mathcal{P}(\omega)$, $\mathbf{E}_{\mathcal{F}(U)}$ is an equivalence relation. By definition, (R_1) holds; (R_2a) holds for $\text{Id}^\alpha \psi_0$ with $C = 8\alpha$, so since φ_U/ψ_0 is decreasing, (R_2a) holds for $\text{Id}^\alpha \varphi_U$, as well. Finally (R_2b) follows from $\text{Id}^\alpha \varphi_U$ is essentially increasing.

Our task is to prove that if $U, V \in \mathcal{P}(\omega)$ satisfy $U \subseteq^* V$, $|V \setminus U| = \infty$ then $\mathbf{E}_{\mathcal{F}(V)} <_B \mathbf{E}_{\mathcal{F}(U)}$. Observe that if $U, U' \in \mathcal{P}(\omega)$ differ only by a finite set then $\mathcal{F}(U) \approx \mathcal{F}(U')$ hence $\mathbf{E}_{\mathcal{F}(U)} = \mathbf{E}_{\mathcal{F}(U')}$. So we can assume $U \subseteq V$, $0 \in V \setminus U$.

Our strategy is to show that $\varphi = \varphi_V$ can be obtained from $\psi = \varphi_U$ as in (37) with a μ satisfying the assumptions of Corollary 29. Set $\mu(0) = 1$,

$$\mu^\alpha(n+1) = \frac{1 + \sigma_{\mu_U^\alpha}(n+1)}{1 + \sigma_{\mu_U^\alpha}(n+1)} - \frac{1 + \sigma_{\mu_U^\alpha}(n)}{1 + \sigma_{\mu_U^\alpha}(n)} \quad (n < \omega).$$

Later on will prove

$$\frac{\gamma-1}{(n+2)^3} \leq \mu^\alpha(n) \leq 1 \quad (n < \omega); \quad (44)$$

now we assume (44) and verify the conditions of Corollary 29.

We have φ_U is continuous and $\varphi_U(0) = 1$. As we have seen above, φ_U is essentially increasing. By $\mu \leq 1$, Lemma 28 gives (38) and (39). By (44),

$$\varphi_U \left(\frac{1}{2^{n+3\lfloor \log(n+2) \rfloor + 7}} \right) \lesssim \varphi_U \left(\frac{\mu(i)}{2^n} \right) \lesssim \varphi_U \left(\frac{1}{2^n} \right) \quad (0 \leq i \leq n < \omega),$$

so (41) follows from

$$\begin{aligned} \varphi_U(1/2^n) &\approx (1 + \sigma_{\mu_U^\alpha}(n))\psi_0(1/2^n) \approx \\ &(1 + \sigma_{\mu_U^\alpha}(n + 3\lfloor \log(n+2) \rfloor + 7))\psi_0(1/2^{n+3\lfloor \log(n+2) \rfloor + 7}) \approx \\ &\varphi_U \left(1/2^{n+3\lfloor \log(n+2) \rfloor + 7} \right). \end{aligned}$$

Let φ be defined from μ , α and φ_U . We have

$$1 + \sigma_{\mu^\alpha}(n) = (1 + \sigma_{\mu_V^\alpha}(n))/(1 + \sigma_{\mu_V^\alpha}(n)) \quad (n < \omega),$$

so by (42),

$$\begin{aligned} \varphi\left(\frac{1}{2^n}\right) &\approx \frac{1 + \sigma_{\mu_V^\alpha}(n)}{1 + \sigma_{\mu_V^\alpha}(n)} \varphi_U(n) \approx \\ &\frac{1 + \sigma_{\mu_V^\alpha}(n)}{1 + \sigma_{\mu_V^\alpha}(n)} (1 + \sigma_{\mu_V^\alpha}(n)) \psi_0\left(\frac{1}{2^n}\right) = \varphi_V\left(\frac{1}{2^n}\right). \end{aligned}$$

Thus $\mathbf{E}_{\mathcal{F}(V)} = \mathbf{E}_{\text{Id}^\alpha \varphi}$; and $((1 + \sigma_{\mu^\alpha}(n)) \psi(1/2^n))_{n < \omega}$ is essentially decreasing. So by Corollary 29, $\mathbf{E}_{\mathcal{F}(V)} \leq_B \mathbf{E}_{\mathcal{F}(U)}$.

Observe that ψ_0 satisfies (28) with $M = 0$ and $\varepsilon = 1/8$. Since $(1 + \sigma_{\mu_V^\alpha}(n))_{n < \omega}$ is increasing, $\varphi_U(1/2^n) \approx (1 + \sigma_{\mu_V^\alpha}(n)) \psi_0(1/2^n)$ ($n < \omega$) also satisfies (28) with the same M and a smaller ε . Thus φ_U satisfies A_1 of Theorem 18.

Since $|U \cap \log(n)| + k \leq |V \cap \log(n)|$ implies $\gamma^k \mu_V^\alpha(n) \leq \mu_V^\alpha(n)$, we have

$$\lim_{n \rightarrow \infty} (1 + \sigma_{\mu_V^\alpha}(n))/(1 + \sigma_{\mu_V^\alpha}(n)) = 0$$

hence $\lim_{n \rightarrow \infty} \sigma_{\mu^\alpha}(n) = \infty$. So again by Corollary 29, $\mathbf{E}_{\mathcal{F}(U)} \leq_B \mathbf{E}_{\mathcal{F}(V)}$. For $U = \emptyset$ and $V = \omega$, $\varphi_U \approx 1/(1 - \log)$ and $\varphi_V \approx 1$, so $\mathbf{E}_{\text{Id}^\alpha} \leq_B \mathbf{E}_{\mathcal{F}(U)} \leq_B \mathbf{E}_{\text{Id}^\alpha/(1-\log)}$ ($U \in \mathcal{P}(\omega)$).

It remains to prove (44). For $n = 1$, $\mu^\alpha(1) = (1 + \gamma)/2 - 1$; for $n = 2$, $\mu^\alpha(2) = (1 + 2\gamma)/3 - (1 + \gamma)/2$. So (44) holds for $n = 1, 2$. Let $n \geq 2$; then

$$1 + \sigma_{\mu_V^\alpha}(n) = 1 + \gamma + a_n, \quad 1 + \sigma_{\mu_V^\alpha}(n) = 2 + b_n$$

and

$$1 + \sigma_{\mu_V^\alpha}(n+1) = 1 + \gamma + a_n + \gamma^c, \quad 1 + \sigma_{\mu_V^\alpha}(n+1) = 2 + b_n + \gamma^d,$$

where $c \geq d+1$ and $\gamma \leq a_n/b_n \leq \gamma^{c-d}$ ($1 < n < \omega$). Then for every $2 \leq n < \omega$,

$$\begin{aligned} \frac{1 + \sigma_{\mu_V^\alpha}(n+1)}{1 + \sigma_{\mu_V^\alpha}(n)} - \frac{1 + \sigma_{\mu_V^\alpha}(n)}{1 + \sigma_{\mu_V^\alpha}(n)} &= \frac{1 + \gamma + a_n + \gamma^c}{2 + b_n + \gamma^d} - \frac{1 + \gamma + a_n}{2 + b_n} = \\ \frac{\gamma^c - \gamma^d \frac{1 + \gamma + a_n}{2 + b_n}}{2 + b_n + \gamma^d} &= \frac{\gamma^c - \gamma^d \left(\frac{a_n}{b_n} - \frac{2a_n - (1 + \gamma)}{2 + b_n} \right)}{2 + b_n + \gamma^d} \geq \\ \frac{\gamma^c - \gamma^d \left(\gamma^{c-d} - \frac{2\gamma - (1 + \gamma)}{2 + b_n} \right)}{2 + b_n + \gamma^d} &= \gamma^d \frac{\gamma - 1}{(2 + b_n)(2 + b_n + \gamma^d)}. \quad (45) \end{aligned}$$

We have $d \leq \lfloor \log(n) \rfloor$, so $b_n \leq n\gamma^d \leq n^2$ ($2 \leq n < \omega$). So (45) can be estimated from below by

$$\frac{\gamma - 1}{(2/\gamma^d + b_n/\gamma^d)(2 + b_n + \gamma^d)} \geq \frac{\gamma - 1}{(2/\gamma^d + n)(2 + n^2 + n)} \geq \frac{\gamma - 1}{(n + 2)^3},$$

as stated.

For the upper bound, as we have seen in (45), it is enough to show

$$\gamma^c - \gamma^d \frac{1 + \gamma + a_n}{2 + b_n} \leq 2 + b_n + \gamma^d \quad (2 \leq n < \omega).$$

Since $c \leq 1 + \log(n + 1)$ and $n - 2 \leq b_n$,

$$\gamma^c \leq \gamma(n + 1)^{\log(\gamma)} \leq n \leq 2 + b_n \quad (2 \leq n < \omega)$$

for our γ . This completes the proof. ■

References

- [1] A. Bella, A. Dow, K. P. Hart, M. Hrušák, J. van Mill, P. Ursino, *Embeddings into $\mathcal{P}(\mathbb{N})/\text{fin}$ and extension of automorphisms*, *Fund. Math.* **174** (2002), 271–284.
- [2] E. Čech, *Topological spaces*, Interscience Publishers John Wiley & Sons, London-New York-Sydney (1966).
- [3] R. Dougherty, G. Hjorth, *Reducibility and nonreducibility between ℓ^p equivalence relations*, *Trans. Amer. Math. Soc.* **351** (1999), no. 5, 1835–1844.
- [4] S. Gao, *Equivalence relations and classical Banach spaces*, *Mathematical logic in Asia*, World Sci. Publ., Hackensack, NJ, (2006), 70–89.
- [5] A. S. Kechris, *Classical Descriptive Set Theory*, *Graduate Texts in Mathematics* **156**, Springer-Verlag (1994).
- [6] A. S. Kechris, *Countable sections for locally compact group actions*, *Ergodic Theory Dynam. Systems* **12** (1992), no. 2, 283–295.
- [7] A. S. Kechris, A. Louveau, *The classification of hypersmooth Borel equivalence relations*, *J. Amer. Math. Soc.* **10** (1997), no. 1, 215–242.
- [8] J. Matoušek, *On embedding trees into uniformly convex Banach spaces*, *Israel J. Math.* **114** (1999), 221–237.
- [9] E. Michael, *Continuous selections I*, *Ann. of Math. (2)* **63** (1956), no. 2, 361–382.