

On typical properties of Hilbert space operators

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Abstract

We study the typical behavior of bounded linear operators on infinite dimensional complex separable Hilbert spaces in the norm, strong-star, strong, weak polynomial and weak topologies. In particular, we investigate typical spectral properties, the problem of unitary equivalence of typical operators, and their embeddability into C_0 -semigroups. Our results provide information on the applicability of Baire category methods in the theory of Hilbert space operators.

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1. Introduction

Baire category is a fundamental tool in analysis because it provides a natural notion of smallness, i.e. meagerness, in every Baire topological space. However,

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there is an obvious limitation of this technique: only such properties can be studied using Baire category arguments which are non-trivial in a suitable Baire topology, i.e. which hold at least on a non-meager set, or in optimal cases on a co-meager set.

In this spirit, given a property Φ on the points of a Baire topological space X , we say that a typical point of X satisfies Φ if $\{x \in X : x \text{ satisfies } \Phi\}$ is a co-meager subset of X . The importance of this concept is clearly indicated by the great impact of the study of typical properties of continuous functions on the theory of real functions (for a primer, see e.g. [2]). In this paper we continue the investigations of the first author into the typical properties of contractive linear operators on infinite dimensional complex separable Hilbert spaces in the norm, strong-star, strong, weak polynomial and weak topologies (for the definitions, see Definition 2.1). Before summarizing our results, let us try to justify our settings.

We choose contractions as the underlying set of operators because it becomes a Baire space in all the five topologies we consider. By writing the set of bounded linear operators as a countable union of scaled copies of the set of contractions, suitable extensions of our result can be easily obtained. In our investigations of contractions, several other important classes of operators (e.g. isometries, positive self-adjoint operators, unitary operators) come into play, and we obtain information about typical properties in these subclasses, as well.

We work only in infinite dimensional separable Hilbert spaces because removing any of these assumptions invalidates most of our results. Infinite dimensionality guarantees that the families of operators under consideration are sufficiently rich. Separability is essential for our descriptive set theoretic arguments. The Hilbert space structure not only allows us to use a well-developed spectral theory, but also facilitates the construction of operators using orthogonal decomposition. It is of limited importance that our Hilbert spaces are over the complex field; analogous results hold in real Hilbert spaces, as well. We defer the further discussion of possible extensions of our work until Section 3.

We obtain the following results. In Section 2.1, we recall some results obtained by the first author about typical properties in the weak topology. In this topology, a typical contraction is unitary, it has maximal spectrum and empty point spectrum, it can be embedded into a C_0 -semigroup, and typical contractions are not unitarily equivalent. The importance of the weak topology in operator theory is an obvious motivation for our investigations.

In Section 2.2 we consider the weak polynomial topology. Our main observations are that the contractions endowed with this topology form a Baire space, where the set of unitary operators is a co-meager subset. Since on the set of unitary operators the weak and weak polynomial topologies coincide, we conclude that the typical

properties of contractions in the weak and weak polynomial topologies coincide. This part of our work is motivated by the increasing interest in this unusual topology.

Section 2.3 treats the strong topology. We show that a typical contraction is unitarily equivalent to the infinite dimensional backward unilateral shift operator. An analogous result for strongly continuous semigroups is obtained, as well. In particular, the point spectrum of a typical contraction is the open unit disk, typical contractions are unitarily equivalent and can be embedded into a C_0 -semigroup. Contrast these results with the behavior in the weak topology. Just as for the weak topology, our interest in the strong topology necessitates no clarification.

We study the strong-star topology in Section 2.4. Our main result gives that the theory of typical properties of contractions in the strong-star topology can be reduced to the theories of typical properties of unitary and positive self-adjoint operators in the strong topology. As a corollary, we obtain that a typical contraction has maximal spectrum and empty point spectrum, and two typical contractions are not unitarily equivalent. We included the strong-star topology in our research because it may play the most important role while extending our investigations into general Banach*-algebras.

Section 2.5 contains our results on the norm topology, the only non-separable topology we consider. We obtain that in this topology there is no non-trivial typical behavior. Intuitively, the reason for this phenomenon is that the norm topology is fine enough to allow for the coexistence of many different properties on non-meager sets.

To conclude this introduction we note that in research works studying contractions, it is customary to point out that contractions in general are hard to study. The theory of contractions as a whole is often contrasted to the theories of normal, self-adjoint or unitary operators where a satisfactory classification can be obtained, e.g. via spectral measures (see e.g. [14] and [20]). Our results provide an explanation of this intuitive observation. As we pointed out above, as far as Baire category methods are concerned, in the four separable topologies we consider, the theory of contractions is reduced to the theory of unitary operators (weak and weak polynomial topologies), to the theory of one shift operator (strong topology) or to the theories of unitary and positive self-adjoint operators (strong-star topology). Thus if the oversimplified pictures captured by these separable topologies are dissatisfactory for an analyst, then necessarily the very fine norm topology has to be used, in which case non-separability can be made responsible for being complicated.

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2. Typical properties in various topologies

As general references, see [12] for descriptive set theory, [20] for functional analysis, and [9] for semigroup theory. Recall that a topological space X is a *Baire space* if every non-empty open set in X is non-meager, or equivalently if the intersection of countably many dense open sets in X is dense (see e.g. [12, (8.1) Proposition and (8.2) Definition p. 41]). Polish spaces, i.e. separable complete metric spaces, are well-known examples of Baire spaces.

A set in a topological space is G_δ if it can be obtained as an intersection of countably many open sets. We will often use the observation that in every topological space, every dense G_δ set is co-meager.

Let X be a Baire space and let Φ be a property on the points of X . We say that Φ is a *typical property on X* , or that *a typical element of X satisfies Φ* if $\{x \in X : x \text{ satisfies } \Phi\}$ is a co-meager subset of X . Note that if Φ_n ($n \in \mathbb{N}$) are typical properties on X then a typical element of X satisfies all Φ_n ($n \in \mathbb{N}$) simultaneously.

In the sequel $(H, \|\cdot\|)$ always denotes an infinite dimensional complex separable Hilbert space. The scalar product on H is denoted by $\langle \cdot, \cdot \rangle$. For every $U \subseteq H$, $\text{span}\{U\}$ denotes the linear subspace of H generated by U , and $U^\perp = \{x \in H : \langle x, u \rangle = 0 \text{ (} u \in U)\}$. For every $U, V \subseteq H$, we write $U \perp V$ if $\langle u, v \rangle = 0$ ($u \in U, v \in V$). If $V \leq H$ is a subspace, we define $B_V = \{v \in V : \|v\| \leq 1\}$, $S_V = \{v \in V : \|v\| = 1\}$.

Let $B(H)$, $C(H)$, $U(H)$ and $P(H)$ denote the sets of bounded, contractive, unitary and contractive positive self-adjoint linear $H \rightarrow H$ operators. The identity operator is denoted by Id . For every $V \leq H$, the orthogonal projection onto V is denoted by Pr_V . For every $A \in B(H)$ the adjoint of A is denoted by A^* . For every $A \in B(H)$ and $U \subseteq H$ we set $A[U] = \{Ax : x \in U\}$, $\ker A = \{x \in H : Ax = 0\}$ and $\text{Ran } A = A[H]$. For every $A \in B(H)$, let $\mathcal{O}(A) = \{UAU^{-1} : U \in U(H)\}$.

For every $A \in B(H)$ the spectrum, the point spectrum, the continuous spectrum, and the residual spectrum of A is denoted by $\sigma(A)$, $P_\sigma(A)$, $C_\sigma(A)$, and $R_\sigma(A)$ (see e.g. [20, Definition p. 209]).

We recall the definitions and some elementary properties of the weak, weak polynomial, strong and strong-star topologies (see e.g. [20, Definition 3 p. 112], [15, Definition p. 1142], [20, Definition p. 69], and [21, Definition p. 220]).

Definition 2.1. Let $A, A_n \in B(H)$ ($n \in \mathbb{N}$) be arbitrary.

1. We say that $\{A_n : n \in \mathbb{N}\}$ converges to A weakly, $A = w\text{-}\lim_{n \in \mathbb{N}} A_n$ in notation, if for every $x, y \in H$, $\lim_{n \in \mathbb{N}} \langle A_n x, y \rangle = \langle Ax, y \rangle$. The topology corresponding to this notion of convergence is called the *weak topology*. Topological notions referring to the weak topology are preceded by w -.

2. We say that $\{A_n : n \in \mathbb{N}\}$ converges to A weakly polynomially, $A = pw\text{-}\lim_{n \in \mathbb{N}} A_n$ in notation, if for every $k \in \mathbb{N}$, $w\text{-}\lim_{n \in \mathbb{N}} A_n^k = A^k$. The topology corresponding to this notion of convergence is called the *weak polynomial topology*. Topological notions referring to the weak polynomial topology are preceded by *pw*-.
3. We say that $\{A_n : n \in \mathbb{N}\}$ converges to A strongly, $A = s\text{-}\lim_{n \in \mathbb{N}} A_n$ in notation, if for every $x \in H$, $\lim_{n \in \mathbb{N}} A_n x = Ax$. The topology corresponding to this notion of convergence is called the *strong topology*. Topological notions referring to the strong topology are preceded by *s*-.
4. We say that $\{A_n : n \in \mathbb{N}\}$ converges to A in the strong-star sense, $A = s^*\text{-}\lim_{n \in \mathbb{N}} A_n$ in notation, if $s\text{-}\lim_{n \in \mathbb{N}} A_n = A$ and $s\text{-}\lim_{n \in \mathbb{N}} A_n^* = A^*$. The topology corresponding to this notion of convergence is called the *strong-star topology*. Topological notions referring to the strong-star topology are preceded by *s**-.

Proposition 2.2. ([19, Section 2 p. 67], [21, Section 7.f p. 121]) Let $\{e_i : i \in \mathbb{N}\} \subseteq H$ be an orthonormal basis.

1. For every $A, B \in B(H)$, set

$$d_w(A, B) = \sum_{i, j \in \mathbb{N}} 2^{-i-j} |\langle Ae_i, e_j \rangle - \langle Be_i, e_j \rangle|.$$

Then d_w is a complete separable metric on $C(H)$ which generates the weak topology.

2. For every $A, B \in B(H)$, set

$$d_s(A, B) = \sum_{i \in \mathbb{N}} 2^{-i} \|Ae_i - Be_i\|.$$

Then d_s is a complete separable metric on $C(H)$ which generates the strong topology.

The strong-star topology on uniformly bounded sets is generated by the metric $d_{s^*}(A, B) = d_s(A, B) + d_s(A^*, B^*)$. It is easy to see that d_{s^*} is a complete metric on $C(H)$. We mention without proof that $C(H)$ endowed with the weak polynomial topology is *not* a Polish space; nevertheless, as we will see in Section 2.2, it is a Baire space.

Note that the weak, weak polynomial, strong, strong-star, and norm topologies all refine the preceding topologies in this list. We also need the following.

Proposition 2.3. ([19, Remark 4.10 p. 84]) *On $U(H)$ the weak, weak polynomial, strong, and strong-star topologies coincide. With this topology, $U(H)$ is a Polish space.*

2.1. *The weak topology*

In the weak topology, the theories of typical properties of contractions and of unitary operators coincide.

Theorem 2.4. ([4, Theorem 2.2 p. 2]) *A w -typical contraction is unitary.*

Equivalently, Theorem 2.4 says that $U(H)$ is a w -co-meager subset of $C(H)$. By Proposition 2.3, $U(H)$ endowed with the weak topology is also a Polish space. Hence the notions related to Baire category make sense relative to $U(H)$, and a set $A \subseteq C(H)$ is w -co-meager in $C(H)$ if and only if $A \cap U(H)$ is w -co-meager in $U(H)$.

Unitary operators are well-understood. E.g. the theory of spectral measures allows a detailed description of the spectral properties and conjugacy classes of unitary operators. We refer to [20, Chapter XI.4 p. 306] and [14, Chapter 2 p. 17] for an introduction to spectral measures. The following proposition briefly summarizes how the spectral properties of w -typical contractions can be obtained from the theory of spectral measures. We set $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. For a measure μ on S^1 , $\text{supp } \mu$ denotes the closed support of μ . For measures μ, ν on S^1 , we write $\mu \perp \nu$ if μ and ν are mutually singular.

Proposition 2.5. *A w -typical contraction U satisfies $C_\sigma(U) = S^1$.*

Proof. Let U be a w -typical contraction. By Theorem 2.4, U is unitary. Let μ_U denote the maximal spectral type of U (see [14, Section 8.22 p. 55]), which is a Borel measure on S^1 . E.g. by the uniqueness of the measure class of μ_U , $\lambda \in \sigma(U)$ if and only if $\lambda \in \text{supp } \mu_U$, and $\lambda \in P_\sigma(U)$ if and only if λ is an atom of μ_U .

By [14, 8.25 Theorem (a) p. 56],

$$\{U \in U(H) : P_\sigma(U) = \emptyset\} = \{U \in U(H) : \mu_U \text{ is atomless}\}$$

is a w -co-meager set in $U(H)$. By a similar argument, using [14, 7.7 Corollary p. 46], we obtain that

$$\{U \in U(H) : \sigma(U) = S^1\} = \{U \in U(H) : \text{supp } \mu_U = S^1\}$$

is also a w -co-meager set in $U(H)$. Since $\sigma(U) = P_\sigma(U) \cup C_\sigma(U) \cup R_\sigma(U)$ and $R_\sigma(U) = \emptyset$ for every unitary operator, we conclude that a w -typical contraction U satisfies $C_\sigma(U) = S^1$, as required. ■

By analogous applications of spectral measures, one can isolate numerous additional w -typical properties of contractions (see e.g. [14, 8.25 Theorem p. 56]). We refer to e.g. [6, Section IV.3] for asymptotic properties of w -typical contractions, and mention that P. Zorin [22] showed recently that a w -typical contraction admits a fixed cyclic vector. Here we restrict ourselves to pointing out that despite the abundance of w -typical properties, typical contractions are not unitarily equivalent.

Proposition 2.6. *For every $U \in C(H)$, $\mathcal{O}(U)$ is w -meager in $C(H)$. In particular, for a w -typical pair of contractions $(U_1, U_2) \in C(H) \times C(H)$ we have that U_1 and U_2 are not unitarily equivalent.*

Proof. Let $U \in C(H)$ be arbitrary. By Theorem 2.4, the statement follows if $U \notin U(H)$. So for the rest of the argument, we can assume $U \in U(H)$. By the conjugacy invariance of the measure class of the maximal spectral type, for every $V \in U(H)$ we have that μ_U and $\mu_{VUV^{-1}}$ are mutually absolutely continuous. By [14, 8.25 Theorem (b) p. 56], for every measure ν on S^1 , the set $\{V \in U(H) : \mu_V \perp \nu\}$ is w -co-meager in $U(H)$. Thus $\mathcal{O}(U)$ is w -meager in $U(H)$ and so in $C(H)$, as well.

Finally consider the set

$$E = \{(U_1, U_2) \in C(H) \times C(H) : U_1, U_2 \text{ are unitarily equivalent}\}.$$

The set E is clearly analytic hence has the Baire property (see e.g. [12, (29.14) Corollary p. 229]). As we have seen above, for every $U_1 \in C(H)$ we have $\{U_2 \in C(H) : (U_1, U_2) \in E\}$ is w -meager in $C(H)$. So by the Kuratowski-Ulam theorem (see e.g. [12, (29.14) Corollary p. 229]) we get that E is w -meager in $C(H) \times C(H)$. This completes the proof. ■

2.2. The weak polynomial topology

The main result of this section is the following.

Proposition 2.7. *The set $C(H)$ endowed with the weak polynomial topology is a Baire space. Moreover, $U(H)$ is a pw -co-meager pw - G_δ subset of $C(H)$.*

This result immediately implies that the theories of pw -typical and w -typical properties of contractions coincide.

Corollary 2.8. *A set $\mathcal{C} \subseteq C(H)$ is pw -co-meager in $C(H)$ if and only if $\mathcal{C} \cap U(H)$ is w -co-meager in $U(H)$. In particular, a property Φ of contractions is pw -typical if and only if Φ is w -typical.*

Proof. By Proposition 2.7, $U(H)$ is a pw -co-meager subset of the Baire space $C(H)$. By Proposition 2.3, the weak and weak polynomial topologies coincide on $U(H)$ and in these topologies $U(H)$ is a Polish space. So the notions related to Baire category make sense relative to $U(H)$. By $U(H) \subseteq C(H)$ being pw -co-meager, a set $\mathcal{M} \subseteq U(H)$ is pw -meager in $U(H)$ if and only if $\mathcal{M} \subseteq C(H)$ is pw -meager in $C(H)$.

We obtained that $\mathcal{C} \subseteq C(H)$ is pw -co-meager in $C(H)$ if and only if $\mathcal{C} \cap U(H) \subseteq C(H)$ is pw -co-meager in $C(H)$. This is equivalent to $\mathcal{C} \cap U(H) \subseteq U(H)$ being pw -co-meager in $U(H)$, which is the same as $\mathcal{C} \cap U(H) \subseteq U(H)$ being w -co-meager in $U(H)$. Finally by Theorem 2.4, this is equivalent to $\mathcal{C} \subseteq C(H)$ being w -co-meager in $C(H)$. This completes the proof. ■

By Corollary 2.8, for the pw -typical properties of contractions one can refer to Section 2.1.

Proof of Proposition 2.7. By [4, Theorem 2.2], $U(H)$ is a w -co-meager w - G_δ subset of $C(H)$. Since the weak polynomial topology is finer than the weak topology, $U(H) \subseteq C(H)$ is pw - G_δ , as well. By [15, Theorem 1 p. 1142], $U(H)$ is pw -dense in $C(H)$, thus $U(H)$ is pw -co-meager in $C(H)$.

To prove that $C(H)$ is Baire in the weak polynomial topology, let $G_n \subseteq C(H)$ ($n \in \mathbb{N}$) be pw -dense pw -open sets, and let $G \subseteq C(H)$ be an arbitrary non-empty pw -open set; we have to show that $\bigcap_{n \in \mathbb{N}} G_n \cap G \neq \emptyset$. Since $U(H) \subseteq C(H)$ is pw -dense, we get that $G_n \cap U(H) \subseteq U(H)$ ($n \in \mathbb{N}$) are pw -dense pw -open sets in $U(H)$, and $G \cap U(H) \subseteq U(H)$ is a non-empty pw -open set in $U(H)$. By Proposition 2.3, the weak and weak polynomial topologies coincide on $U(H)$, hence $G_n \cap U(H) \subseteq U(H)$ ($n \in \mathbb{N}$) are w -dense w -open sets in $U(H)$, and $G \cap U(H) \subseteq U(H)$ is a non-empty w -open set in $U(H)$. Since $U(H)$ endowed with the weak topology is a Polish space, we get $\bigcap_{n \in \mathbb{N}} G_n \cap G \cap U(H) \neq \emptyset$, which completes the proof. ■

2.3. The strong topology

Probably the most surprising observation in the present paper is that some typical properties of contractions in the strong and weak topologies are completely different. While typical contractions in the weak topology are not unitarily equivalent, typical contractions in the strong topology are unitarily equivalent to an infinite dimensional backward unilateral shift operator, hence the investigation of s -typical properties of contractions is reduced to the study of one particular operator. We introduce this operator in the following definition. Since infinite dimensional complex separable Hilbert spaces are isometrically isomorphic, we can restrict ourselves to the study of a particular one.

Definition 2.9. Set $H = \ell^2(\mathbb{N} \times \mathbb{N})$ and denote the canonical orthonormal basis of H by $\{e_i(n) : i, n \in \mathbb{N}\}$. We define the *infinite dimensional backward unilateral shift operator* $S \in C(H)$ by $Se_0(n) = 0$ and $Se_{i+1}(n) = e_i(n)$ ($i, n \in \mathbb{N}$).

The main result of this section is the following. Recall $\mathcal{O}(A) = \{UAU^{-1} : U \in U(H)\}$.

Theorem 2.10. *The set $\mathcal{O}(S)$ is an s -co-meager subset of $C(H)$.*

By Theorem 2.10, a property of contractions is s -typical if and only if S has this property. In the following corollary, we recall only the properties of S we usually concern in this paper.

Corollary 2.11. *An s -typical contraction A satisfies that*

1. $P_\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, and for every $\lambda \in P_\sigma(A)$ we have $\dim \ker(\lambda \cdot \text{Id} - A) = \infty$;
2. $C_\sigma(A) = S^1$;
3. *A can be embedded into a strongly continuous semigroup.*

Moreover, typical contractions are unitarily equivalent.

Proof. Statements 1 and 2 follow from Theorem 2.10 and the results of [17], while 3 is a corollary of [4, Proposition 4.3]. ■

Our strategy to prove Theorem 2.10 is the following. The main observation is that for an s -typical contraction A , its *adjoint* A^* is an isometry. Then by the Wold decomposition theorem (see e.g. [18, Theorem 1.1 p. 3]), A is unitarily equivalent to a direct sum of unitary and backward unilateral shift operators, and the number of shifts in the direct sum depends on the dimension of $\ker A$. Since an s -typical contraction A is strongly stable, the unitary part is trivial. So we complete the proof by showing $\dim \ker A = \infty$.

2.3.1. Elementary observations

We collect here some elementary results we need later in our analysis.

Lemma 2.12. *Let $x, y \in S_H$ satisfy $x \neq -y$. Set $\alpha = (2 + 2 \cdot \text{Re}\langle x, y \rangle)^{-1/2}$. Then $\|\alpha(x + y)\| = 1$.*

Proof. We have $\|\alpha(x+y)\|^2 = \alpha^2(\|x\|^2 + \|y\|^2 + 2 \cdot \operatorname{Re}\langle x, y \rangle) = \alpha^2(2 + 2 \cdot \operatorname{Re}\langle x, y \rangle) = 1$, as required. ■

Lemma 2.13. *Let $A \in C(H)$ and let $(b_n)_{n \in \mathbb{N}} \subseteq S_H$, $z \in S_H$ satisfy $\lim_{n \in \mathbb{N}} Ab_n = z$. Then $(b_n)_{n \in \mathbb{N}}$ is convergent.*

Proof. It is enough to prove that for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for every $n, m \geq N$ we have $\|b_n - b_m\|^2 \leq \varepsilon$. So let $\varepsilon > 0$ be arbitrary. For every n, m sufficiently large we have $b_n \neq -b_m$, so $\alpha_{n,m} = (2 + 2 \cdot \operatorname{Re}\langle b_n, b_m \rangle)^{-1/2}$ is defined. By Lemma 2.12 we have $\|\alpha_{n,m}(b_n + b_m)\| = 1$. So since A is a contraction,

$$\begin{aligned} \alpha_{n,m} \cdot (2 - \|Ab_n - z\| - \|Ab_m - z\|) &\leq \\ \alpha_{n,m} \cdot \|2 \cdot z + (Ab_n - z) + (Ab_m - z)\| &= \\ \alpha_{n,m} \cdot \|Ab_n + Ab_m\| &= \|\alpha_{n,m} \cdot A(b_n + b_m)\| \leq 1. \end{aligned} \quad (1)$$

Let $N \in \mathbb{N}$ be such that for every $n \geq N$ we have $\|Ab_n - z\| \leq \varepsilon/8$. Then by (1), for every $n, m \geq N$ we get $\alpha_{n,m} \leq 1/(2 - 2\varepsilon/8)$, i.e. $2 + 2 \cdot \operatorname{Re}\langle b_n, b_m \rangle \geq 4 \cdot (1 - \varepsilon/8)^2$. Thus

$$\|b_n - b_m\|^2 = 2 - 2 \cdot \operatorname{Re}\langle b_n, b_m \rangle \leq 4 - 4 \cdot (1 - \varepsilon/8)^2 \leq \varepsilon,$$

as required. ■

The following lemma will be helpful to show that the kernel of a typical contraction is infinite dimensional.

Lemma 2.14. *Let $n \in \mathbb{N} \setminus \{0\}$ and let $\{e_i : i < n\} \subseteq H$ be an orthonormal family. Let $\{f_i : i < n\} \subseteq H$ satisfy $\|f_i - e_i\| < 1/n$ ($i < n$). Then $\{f_i : i < n\}$ are linearly independent.*

Proof. Let $\alpha_i \in \mathbb{C}$ ($i < n$) be arbitrary satisfying $\sum_{i < n} |\alpha_i| > 0$. We have

$$\begin{aligned} \left\| \sum_{i < n} \alpha_i e_i - \sum_{i < n} \alpha_i f_i \right\|^2 &\leq \left(\sum_{i < n} |\alpha_i| \|e_i - f_i\| \right)^2 < \\ &\left(\frac{1}{n} \sum_{i < n} |\alpha_i| \right)^2 \leq \frac{1}{n} \sum_{i < n} |\alpha_i|^2 = \frac{1}{n} \left\| \sum_{i < n} \alpha_i e_i \right\|^2. \end{aligned}$$

So by $\|\sum_{i < n} \alpha_i e_i\| > 0$ we have $\|\sum_{i < n} \alpha_i f_i\| > 0$, as required. ■

Next we point out a trivial sufficient condition for the direct sum of contractions to be a contraction.

Lemma 2.15. *Let $V_0, V_1 \leq H$ be subspaces satisfying $V_0 \perp V_1$. Let $A_i: V_i \rightarrow H$ ($i < 2$) be contractive linear operators such that $A_0[V_0] \perp A_1[V_1]$. Then $A: \text{span}\{V_0, V_1\} \rightarrow H$,*

$$A(\alpha v_0 + \beta v_1) = \alpha A_0 v_0 + \beta A_1 v_1 \quad (\alpha, \beta \in \mathbb{C})$$

is also contractive.

Proof. Let $v \in \text{span}\{V_0, V_1\}$, $\|v\| = 1$ be arbitrary. Then there are $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ and $v_i \in S_{V_i}$ ($i < 2$) such that $v = \alpha v_0 + \beta v_1$. We have

$$\begin{aligned} \|Av\|^2 &= \|A(\alpha v_0 + \beta v_1)\|^2 = \|\alpha A_0 v_0 + \beta A_1 v_1\|^2 = \\ &|\alpha|^2 \|A_0 v_0\|^2 + |\beta|^2 \|A_1 v_1\|^2 \leq |\alpha|^2 + |\beta|^2 = 1, \end{aligned}$$

so the proof is complete. ■

Finally we point out that strongly stable contractions form a s -co-meager s - G_δ subset of $C(H)$.

Definition 2.16. A contraction A is *strongly stable* if $s\text{-}\lim_{n \in \mathbb{N}} A^n = 0$, i.e., for every $x \in S_H$ and $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $\|A^n x\| < \varepsilon$. The set of strongly stable contractions is denoted by \mathcal{S} .

Lemma 2.17. *The set of strongly stable contractions is an s -co-meager s - G_δ subset of $C(H)$.*

Proof. By $A = \lim_{n \in \mathbb{N}} (1 - 2^{-n})A$ ($A \in C(H)$), the set of contractions A satisfying $\|A\| < 1$ is a norm dense and hence an s -dense subset of $C(H)$. Since every such operator is strongly stable, it remains to show that \mathcal{S} is s - G_δ . To this end, let $\{x_i: i \in \mathbb{N}\}$ be a dense subset of S_H . Note that for each $x \in H$ and $A \in \mathcal{S}$, the sequence $\|A^n x\|$ ($n \in \mathbb{N}$) is monotonically decreasing, so $\lim_{n \in \mathbb{N}} \|A^n x\| = 0$ is equivalent to $\inf_{n \in \mathbb{N}} \|A^n x\| = 0$. Thus

$$\mathcal{S} = \bigcap_{i, j \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \{A \in C(H): \|A^n x_i\| < 2^{-j}\}, \quad (2)$$

which completes the proof. ■

We remark that Lemma 2.17 holds in every separable Banach space. It is interesting to note the difference to the weak operator topology in which the set of weakly stable contractions is w -meager in $C(H)$ (see [7, Theorem 4.3]).

2.3.2. Mapping properties of s -typical contractions

The purpose of this section is to prove the following.

Theorem 2.18. *Let \mathcal{G} denote the set of contractive operators A satisfying the following properties:*

1. *for every $y \in S_H$ there exists an $x \in S_H$ such that $Ax = y$, i.e., $S_H \subseteq A[S_H]$;*
2. $\dim \text{Ker } A = \infty$.

Then \mathcal{G} is an s -co-meager subset of $C(H)$.

To prove Theorem 2.18, we need a geometric lemma saying that every contraction defined on a finite dimensional subspace of H can be extended to a contraction which is surjective in a very strong sense.

Lemma 2.19. *Let $V \leq H$ be a finite dimensional subspace and let $A: V \rightarrow H$ be a contractive linear operator. Let $W = A[V]$ and let $Y \leq H$ be an arbitrary subspace satisfying $Y \perp W$. Then for every $X \leq H$ satisfying $X \perp V$ and $\dim X = \dim W + \dim Y$ there exists a contractive linear operator $\tilde{A}: \text{span}\{V, X\} \rightarrow H$ such that $\tilde{A}|_V = A$ and $\tilde{A}[B_{\text{span}\{V, X\}}] = B_{\text{span}\{W, Y\}}$.*

Proof. We handle first the special $Y = \{0\}$ case. Let $\dim V = n$ and $\dim W = m$. The Gram-Schmidt orthogonalization theorem states that there exists an orthonormal base $\{v_i: i < n\} \subseteq V$ such that $\{Av_i: i < n\} \subseteq W$ are pairwise orthogonal, and

$$(i) \quad \|Av_0\| = \max\{\|Av\|: v \in S_V\};$$

$$(ii) \quad \text{for every } j < n - 1, \quad \|Av_{j+1}\| = \max\{\|Av\|: v \in S_V \cap \text{span}\{v_i: i \leq j\}^\perp\}.$$

By (i) and (ii) we have $\|Av_i\| > 0$ if and only if $i < m$. Set $w_i = Av_i/\|Av_i\|$ ($i < m$).

Let $X \leq H$ satisfy $X \perp V$ and $\dim X = m$. Fix an orthonormal base $\{x_i: i < m\}$ in X and define $\tilde{A}: \text{span}\{V, X\} \rightarrow H$ such that $\tilde{A}|_V = A$ and $\tilde{A}x_i = \sqrt{1 - \|Av_i\|^2}w_i$ ($i < m$).

First we show \tilde{A} is a contraction. Let $u \in \text{span}\{V, X\}$ satisfy $\|u\| = 1$. Then $u = \sum_{i < n} \alpha_i v_i + \sum_{i < m} \beta_i x_i$ where $\alpha_i \in \mathbb{C}$ ($i < n$), $\beta_i \in \mathbb{C}$ ($i < m$) satisfy $\sum_{i < n} |\alpha_i|^2 +$

$\sum_{i < m} |\beta_i|^2 = 1$. Then

$$\begin{aligned} \|\tilde{A}u\|^2 &= \left\| \tilde{A} \left(\sum_{i < n} \alpha_i v_i + \sum_{i < m} \beta_i x_i \right) \right\|^2 = \\ &= \left\| \sum_{i < m} (\alpha_i \|Av_i\| w_i + \beta_i \sqrt{1 - \|Av_i\|^2} w_i) \right\|^2 \leq \sum_{i < m} (|\alpha_i| \|Av_i\| + |\beta_i| \sqrt{1 - \|Av_i\|^2})^2 \end{aligned}$$

By the Cauchy-Schwarz inequality, for every $0 \leq p, q, r \leq 1$ we have $(pr + q\sqrt{1 - r^2})^2 \leq p^2 + q^2$. So $\|\tilde{A}u\| \leq \sum_{i < m} |\alpha_i|^2 + |\beta_i|^2 \leq 1$, as required.

Next we show that for every $y \in B_W$ there is an $x \in B_{\text{span}\{V, X\}}$ such that $\tilde{A}x = y$. Let $y = \sum_{i < m} \beta_i w_i$ where $\beta_i \in \mathbb{C}$ ($i < m$) satisfy $\sum_{i < m} |\beta_i|^2 \leq 1$. Let

$$x = \sum_{i < m} \beta_i (\|Av_i\| v_i + \sqrt{1 - \|Av_i\|^2} x_i).$$

Then

$$\|x\| = \sum_{i < m} \left(\beta_i^2 \|Av_i\|^2 + \beta_i^2 \sqrt{1 - \|Av_i\|^2}^2 \right) \leq 1,$$

and

$$\begin{aligned} \tilde{A}x &= \sum_{i < m} \beta_i (\|Av_i\| Av_i + \sqrt{1 - \|Av_i\|^2} \tilde{A}x_i) = \\ &= \sum_{i < m} \beta_i (\|Av_i\| \|Av_i\| w_i + \sqrt{1 - \|Av_i\|^2} \sqrt{1 - \|Av_i\|^2} w_i) = y, \end{aligned}$$

as required. This completes the proof of the special $Y = \{0\}$ case.

In the general case write $X = X_0 \oplus X_1$ where $X_0 \perp X_1$ and $\dim X_0 = \dim W$, $\dim X_1 = \dim Y$. By the special case above, there is a contraction $\tilde{A}: \text{span}\{V, X_0\} \rightarrow H$ such that $\tilde{A}|_V = A$ and $\tilde{A}[B_{\text{span}\{V, X_0\}}] = B_W$. Extend further \tilde{A} by setting $\tilde{A}|_{X_1}: X_1 \rightarrow Y$ be any isometric isomorphism. By Lemma 2.15, \tilde{A} is a contraction which clearly satisfies $\tilde{A}[B_{\text{span}\{V, X\}}] = B_{\text{span}\{W, Y\}}$. This completes the proof. ■

From Lemma 2.19, we immediately get the following.

Proposition 2.20. *The set of contractive operators A such that for every $y \in S_H$ there exists an $x \in S_H$ such that $Ax = y$ is an s -co-meager subset of $C(H)$.*

Proof. Let

$$\mathcal{M} = \{A \in C(H) : \forall \varepsilon > 0, y \in S_H \exists x \in S_H (\|y - Ax\| < \varepsilon)\}.$$

First we show that \mathcal{M} is an s -dense s - G_δ subset of $C(H)$.

Fix $y \in S_H$ and $\varepsilon > 0$. The set

$$C(y, \varepsilon) = \{A \in C(H) : \exists x \in S_H (\|y - Ax\| < \varepsilon)\}$$

is s -open. We show that it is s -dense.

Let $U \subseteq C(H)$ be any non-empty s -open set. By passing to a subset, we can assume $U = \{A \in C(H) : \|y_i - Ax_i\| < \varepsilon_i \ (i \in I)\}$ where $x_i, y_i \in H$, $\varepsilon_i > 0 \ (i \in I)$ and I is finite. Let $V = \text{span}\{x_i : i \in I\}$ and take an arbitrary $A \in U$. By restricting A to V we can assume $A|_{V^\perp} = 0$.

Let $Y \leq H$ be an at most one dimensional subspace such that $Y \perp W$ and $y \in \text{span}\{W, Y\}$. Let $X \leq H$ be a $\dim W + \dim Y$ dimensional subspace such that $X \perp V$. By Lemma 2.19, there exists a contraction $\tilde{A} : \text{span}\{V, X\} \rightarrow H$ such that $\tilde{A}|_V = A|_V$ and $\tilde{A}[B_{\text{span}\{V, X\}}] = B_{\text{span}\{W, Y\}}$. In particular, there is an $x \in B_{\text{span}\{V, X\}}$ such that $\tilde{A}x = y$. Since \tilde{A} is a contraction and $\|y\| = 1$, we get $x \in S_H$. Extend further \tilde{A} by setting $\tilde{A}|_{\text{span}\{V, X\}^\perp} = 0$. Then $\tilde{A} \in U \cap C(y, \varepsilon)$; i.e. we concluded that $C(y, \varepsilon)$ is s -dense.

Let $D \subseteq S_H$ be a countable dense set. We have

$$\mathcal{M} = \bigcap \{C(y, 2^{-n}) : y \in D, n \in \mathbb{N}\},$$

so by the Baire category theorem, \mathcal{M} is an s -dense s - G_δ subset of $C(H)$.

It now suffices to show that every $A \in \mathcal{M}$ satisfies $S_H \subseteq A[S_H]$. To this end, let $A \in \mathcal{M}$ and $z \in S_H$ be arbitrary. By the definition of \mathcal{M} , there is a sequence $(b_n)_{n \in \mathbb{N}} \subseteq S_H$ such that $\lim_{n \in \mathbb{N}} Ab_n = z$. By Proposition 2.13, $(b_n)_{n \in \mathbb{N}}$ is convergent, say $\lim_{n \in \mathbb{N}} b_n = x$. Then $Ax = z$, which completes the proof. ■

To prove that a typical contraction has infinite dimensional kernel, we need a lemma showing that a typical contraction approximates the zero operator on arbitrarily large finite dimensional subspaces.

Lemma 2.21. *The set of contractive operators A such that for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $Z \leq H$ with $\dim Z \geq n$ and $\|A|_Z\| < \varepsilon$ is an s -dense s - G_δ subset of $C(H)$.*

Proof. For every $n \in \mathbb{N}$ and $\varepsilon > 0$, the set

$$C(n, \varepsilon) = \{A \in C(H) : \exists Z \leq H (\dim Z \geq n, \|A|_Z\| < \varepsilon)\}$$

is s -open. We show that $C(n, \varepsilon)$ ($n \in \mathbb{N}, \varepsilon > 0$) are s -dense.

Fix arbitrary $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $U \subseteq C(H)$ be any non-empty s -open set. By passing to a subset, we can assume $U = \{A \in C(H) : \|y_i - Ax_i\| < \varepsilon_i \ (i \in I)\}$ where $x_i, y_i \in H$, $\varepsilon_i > 0$ ($i \in I$) and I is finite. Let $A \in U$ be arbitrary. Let $V = \text{span}\{x_i : i \in I\}$, and define $B \in C(H)$ by $B|_V = A|_V$, $B|_{V^\perp} = 0$. Since $\dim V^\perp = \infty$, we obtained $B \in C(n, \varepsilon) \cap U$; i.e. we concluded $C(n, \varepsilon)$ is s -dense.

The set of contractive operators A such that for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $Z \leq H$ with $\dim Z \geq n$ and $\|A|_Z\| < \varepsilon$ is

$$\bigcap \{C(n, 2^{-m}) : n, m \in \mathbb{N}\}.$$

So by the Baire category theorem, this is an s -dense s - G_δ subset of $C(H)$, which completes the proof. ■

We are ready to prove the second part of Theorem 2.18.

Proposition 2.22. *The set of contractions A which satisfy $\dim \ker A = \infty$ is an s -co-meager subset of $C(H)$.*

Proof. By Lemma 2.21 and Proposition 2.20, the set of contractions A which satisfy that

1. for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $Z \leq H$ with $\dim Z \geq n$ and $\|A|_Z\| < \varepsilon/n$;
2. for every $y \in S_H$ there exists an $x \in S_H$ such that $Ax = y$;

is an s -co-meager subset of $C(H)$. We show that every member A of this set satisfies $\dim \ker A = \infty$; this will complete the proof.

Fix an arbitrary $n \in \mathbb{N} \setminus \{0\}$. Set $\varepsilon = 1$ and let $Z \leq H$ satisfy 1 for this n and ε . Let $\{e_i : i < n\} \subseteq Z$ be an orthonormal family. We find $f_i \in H$ ($i < n$) such that $\|f_i - e_i\| < 1/n$ ($i < n$) and $Af_i = 0$. Then by Lemma 2.14, $\{f_i : i < n\}$ are linearly independent. Then $\dim \ker A \geq n$, and since n was arbitrary, we concluded $\dim \ker A = \infty$.

Fix $i < n$; we define f_i as follows. If $Ae_i = 0$, set $f_i = e_i$. Else observe that by 2, there exists $x_i \in S_H$ with $Ax_i = Ae_i/\|Ae_i\|$. Define now

$$f_i = e_i - \|Ae_i\|x_i.$$

Then $\|f_i - e_i\| = \|Ae_i\| < 1/n$ and we have $Af_i = Ae_i - \|Ae_i\|Ax_i = 0$, so the construction is complete. ■

Proof of Theorem 2.18. The statement follows from Proposition 2.20 and Proposition 2.22. ■

2.3.3. Unitary equivalence of s -typical contractions to a shift

Operators in the set \mathcal{G} introduced in Theorem 2.18 have the following properties.

Proposition 2.23. *Let $A \in \mathcal{G}$. Then $AA^* = \text{Id}$ and A^*A is the projection onto $\text{Ran } A^*$, which is an infinite dimensional and infinite co-dimensional subspace of H . In particular, A^* is an isometry, hence A is a co-isometry and in addition, A is an isometry on $(\text{Ker } A)^\perp$.*

Proof. Let $\{e_i: i \in \mathbb{N}\}$ be an orthonormal basis of H . By the definition of \mathcal{G} , for every $i \in \mathbb{N}$ there is an $a_i \in S_H$ such that $Aa_i = e_i$. Note that for every $i \in \mathbb{N}$, $1 = \langle e_i, e_i \rangle = \langle Aa_i, Aa_i \rangle = \langle A^*Aa_i, a_i \rangle$. By A, A^* being contractions, this is possible only if $A^*e_i = A^*Aa_i = a_i$ ($i \in \mathbb{N}$), thus $AA^*e_i = Aa_i = e_i$ ($i \in \mathbb{N}$). This proves that $AA^* = \text{Id}$, $\text{Ran } A^* = \text{span}\{A^*e_i: i \in \mathbb{N}\} = \text{span}\{a_i: i \in \mathbb{N}\}$, and that A^*A is the projection onto $\text{Ran } A^*$. Again by $A, A^* \in C(H)$, this implies that A^* is an isometry, and A is an isometry on $\text{Ran } A^*$. By the definition of \mathcal{G} , $\text{Ran } A^* = (\text{Ker } A)^\perp$ is infinite dimensional and infinite co-dimensional. This completes the proof. ■

Proof of Theorem 2.10. By Lemma 2.17 and Theorem 2.18, it is enough to show that every $A \in \mathcal{S} \cap \mathcal{G}$ is unitarily equivalent to the operator S of Definition 2.9. By Proposition 2.23, A is a co-isometry. So by the Wold decomposition theorem (see e.g. [18, Theorem 1.1 p. 3]), we have $H = H_u \oplus H_s$ such that $A|_{H_u}$ is unitary and $A|_{H_s}$ is unitarily equivalent to the backward unilateral shift operator on $l^2(\mathbb{N}, \text{ker } A)$, i.e. the Hilbert space of square summable $\mathbb{N} \rightarrow \text{ker } A$ functions. By $A \in \mathcal{S}$ we have $H_u = \{0\}$. By Theorem 2.18.2, $\dim \text{ker } A = \infty$ hence $l^2(\mathbb{N}, \text{ker } A)$ is isometrically isomorphic to $\ell^2(\mathbb{N} \times \mathbb{N})$. This completes the proof. ■

2.3.4. The continuous case

In this section we show that a typical strongly continuous contraction semigroup on H is unitarily equivalent to an infinite dimensional backward unilateral shift semigroup.

Let $C^c(H)$ denote the set of contractive C_0 -semigroups on H . Here we endow $C^c(H)$ with the topology induced by the uniform strong convergence on compact time intervals, i.e., by the metric

$$d_s^c(T(\cdot), S(\cdot)) = \sum_{j,n \in \mathbb{N}} 2^{-(j+n)} \sup_{t \in [0,n]} \|T(t)e_j - S(t)e_j\|,$$

where $\{e_j: j \in \mathbb{N}\}$ is an orthonormal basis of H . With respect to this topology, $C^c(H)$ is a Polish space. With an abuse of notation, topological notions referring to this topology are also preceded by s -.

We again restrict ourselves without loss of generality to a particular infinite dimensional complex separable Hilbert space and introduce on it the infinite dimensional backward unilateral shift semigroup.

Definition 2.24. Set $H = L^2(\mathbb{R}_+^2)$. We define the *infinite dimensional backward unilateral shift semigroup* $S(\cdot) \in C^c(H)$ by

$$[S(t)f](s, w) = f(s + t, w) \quad (t, s, w \in \mathbb{R}_+, f \in H).$$

We also set $\mathcal{O}(S(\cdot)) = \{(U^{-1}S(t)U)_{t \in \mathbb{R}_+} : U \in U(H)\} \subseteq C^c(H)$.

The following shows that an s -typical contraction semigroup is unitarily equivalent to $S(\cdot)$.

Theorem 2.25. *The set $\mathcal{O}(S(\cdot))$ is an s -co-meager subset of $C^c(H)$.*

Note that for every $t > 0$, $S(t)$ is unitarily equivalent to the backward unilateral shift operator S of Definition 2.9. So by Theorem 2.25, s -typical contractive C_0 -semigroups $T(\cdot)$ satisfy that for every $t > 0$, the operator $T(t)$ has the same properties as S . Moreover, s -typical contraction semigroups are unitarily equivalent.

To prove Theorem 2.25, we need to introduce the following not so well-known concept from semigroup theory.

Definition 2.26. Let $T(\cdot)$ be a C_0 -semigroup on H . Let the generator A of $T(\cdot)$ satisfy $1 \in \rho(A)$. Then the operator $V \in B(H)$ defined by

$$V = (A + \text{Id})(A - \text{Id})^{-1} = \text{Id} - 2R(1, A)$$

is called the *cogenerator* of $T(\cdot)$.

The cogenerator is a bounded operator which determines the semigroup uniquely. Moreover, it shares many properties of $T(\cdot)$ such as being contractive, unitary, self-adjoint, normal, isometric, strongly stable etc. (see [18, Section III.8-9] for details). We will use the following (see e.g. [18, Theorem III.8.1]).

Lemma 2.27. *With the notation of Definition 2.26, V is the cogenerator of a contraction semigroup if and only if V is contractive and satisfies $1 \notin P_\sigma(V)$.*

Our key observation is the following.

Lemma 2.28. *Set*

$$\mathcal{V} = \{V \in C(H) : 1 \notin P_\sigma(V)\},$$

endowed with the strong topology. Define $J: C^c(H) \rightarrow \mathcal{V}$ by $J(T(\cdot)) = V$ where V is the cogenerator of $T(\cdot)$ ($T(\cdot) \in C^c(H)$). Then J is a homeomorphism.

Proof. The statement follows from Lemma 2.27 and the First Trotter–Kato Theorem (see e.g. [9, Theorem III.4.8]). ■

Proof of Theorem 2.25. By Corollary 2.11, \mathcal{V} is s -co-meager in $C(H)$. So with the notation of (2) and Theorem 2.18, by Lemma 2.17 and Theorem 2.18 we have that $\mathcal{S} \cap \mathcal{G} \cap \mathcal{V}$ is s -co-meager in \mathcal{V} . Hence the set $J^{-1}(\mathcal{S} \cap \mathcal{G} \cap \mathcal{V})$ is s -co-meager in $C^c(H)$. So it is enough to show that $J^{-1}(\mathcal{S} \cap \mathcal{G} \cap \mathcal{V}) \subseteq \mathcal{O}(S(\cdot))$.

Let $T(\cdot) \in J^{-1}(\mathcal{S} \cap \mathcal{G} \cap \mathcal{V})$ be arbitrary. Let V denote the cogenerator of $T(\cdot)$; then $V \in \mathcal{S} \cap \mathcal{G} \cap \mathcal{V}$. By Proposition 2.23, V is a strongly stable co-isometry. Since the cogenerator of $T^*(\cdot)$ is V^* , and since the semigroup and the cogenerator share strong stability and the isometric property (see [18, Theorem III.9.1]), $T(\cdot)$ is a strongly stable co-isometric contraction semigroup. By Wold’s decomposition for semigroups (see [18, Theorem III.9.3]), $T(\cdot)$ is unitarily equivalent to the backward unilateral shift semigroup on $L^2(\mathbb{R}_+, Y)$, where $Y = (\text{Ran } V^*)^\perp$. Since $(\text{Ran } V^*)^\perp = \text{Ker } V$ is infinite dimensional, the statement follows. ■

2.4. The strong-star topology

As we have seen in the previous section, the theory of s -typical properties of contractions is reduced to the study of one particular operator. The reason behind this phenomenon is that s -convergence does not control the adjoint, i.e. the function $A \mapsto A^*$ is not s -continuous. A straightforward remedy to this problem is to refine the strong topology such that taking adjoint becomes a continuous operation. This naturally leads to the investigation of the strong-star topology.

As one may expect, the structure of an s^* -typical contraction is more complicated than the structure of an s -typical contraction. We show that the theory of s^* -typical properties of contractions can be reduced to the theories of typical properties of unitary and positive self-adjoint operators in the better understood strong topology.

Theorem 2.29. *There exist an s^* -co-meager s^* - G_δ set $\mathcal{H} \subseteq C(H)$ and an s -co-meager s - G_δ set $\mathcal{P} \subseteq P(H)$ such that the function $\Psi: U(H) \times \mathcal{P} \rightarrow \mathcal{H}$, $\Psi(U, P) = U \cdot P$ is a homeomorphism, where $U(H)$ and \mathcal{P} are endowed with the strong topology and \mathcal{H} is endowed with the strong-star topology.*

Moreover, if $(\psi_0, \psi_1): \mathcal{H} \rightarrow U(H) \times \mathcal{P}$ denotes the inverse of Ψ , then for every $A \in \mathcal{H}$ and $U \in U(H)$ we have $UAU^{-1} \in \mathcal{H}$ and $\psi_i(UAU^{-1}) = U\psi_i(A)U^{-1}$ ($i < 2$).

Note that Theorem 2.29 immediately implies the following. Recall $\mathcal{O}(A) = \{UAU^{-1} : U \in U(H)\}$.

Corollary 2.30. *For every $A \in C(H)$, $\mathcal{O}(A)$ is an s^* -meager subset of $C(H)$. In particular, s^* -typical contractions are not unitarily equivalent.*

Proof. We follow the notation of Theorem 2.29. Let $A \in C(H)$ be arbitrary. If $A \notin \mathcal{H}$ then by the unitary invariance of \mathcal{H} we have $\mathcal{O}(A) \subseteq C(H) \setminus \mathcal{H}$, i.e. $\mathcal{O}(A)$ is s^* -meager. So we can assume $A \in \mathcal{H}$. We have

$$\begin{aligned} \Psi^{-1}(\mathcal{O}(A)) &= \{(U\psi_0(A)U^{-1}, U\psi_1(A)U^{-1}) : U \in U(H)\} \subseteq \\ &\{U\psi_0(A)U^{-1} : U \in U(H)\} \times \{U\psi_1(A)U^{-1} : U \in U(H)\} = \mathcal{O}(\psi_0(A)) \times \mathcal{O}(\psi_1(A)). \end{aligned}$$

By Proposition 2.6, $\mathcal{O}(\psi_0(A)) \subseteq U(H)$ is s -meager. Hence $\Psi^{-1}(\mathcal{O}(A))$ is $s \times s$ -meager in $U(H) \times \mathcal{P}$. By Ψ being a homeomorphism, we obtain that $\mathcal{O}(A)$ is s^* -meager in \mathcal{H} and so in $C(H)$, as well. The further corollary that unitarily equivalent pairs are $s^* \times s^*$ -meager in $C(H) \times C(H)$ follows as in the proof of Proposition 2.6. ■

Similarly to our approach in the previous section, for the proof of Theorem 2.29 we need to describe the mapping properties of s^* -typical contractions. These investigations will also help us to determine the s^* -typical spectral properties of contractions.

2.4.1. Mapping properties of s^* -typical contractions

We prove the following.

Proposition 2.31. *The set*

$$\mathcal{T} = \{A \in C(H) : \forall \lambda \in \mathbb{C} \text{ (Ran } (A - \lambda \cdot I) \text{ is dense in } H)\}$$

is s^ -co-meager and s^* - G_δ in $C(H)$.*

We need the following elementary property of the backward unilateral shift operator. It follows from the fact that its adjoint, the unilateral shift operator, on ℓ^2 has empty point spectrum (see e.g. [16]).

Lemma 2.32. *Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal base in H . Define $D \in C(H)$ by $De_0 = 0$, $De_{i+1} = e_i$ ($i \in \mathbb{N}$). Then for every $\lambda \in \mathbb{C}$, $\text{Ran}(D - \lambda \cdot \text{Id})$ is dense in H .*

Proof of Proposition 2.31. First we show that \mathcal{T} is s^* -dense in $C(H)$. Let $U \subseteq C(H)$ be a non-empty s^* -open set. Then there exist $\{x_i: i < n\} \subseteq S_H$, $\varepsilon > 0$ and $A \in C(H)$ such that for every $B \in C(H)$, $\|Bx_i - Ax_i\| \leq \varepsilon$ ($i < n$) and $\|B^*x_i - A^*x_i\| \leq 2\varepsilon$ ($i < n$) imply $B \in U$. It is enough to find a $B \in U$ such that for every $\lambda \in \mathbb{C}$, $\text{Ran}(B - \lambda \cdot \text{Id})$ is dense in H .

Set $V = \text{span}\{x_i, Ax_i, A^*x_i: i < n\}$. Let $Q: V \rightarrow V$ be defined by $Q = (1 - \varepsilon) \cdot \text{Pr}_V A|_V$. Let $\dim V = m$ and let $\{x_i(0): i < m\}$ be an orthonormal base in V . Let $\{x_i(j): i < m, j \in \mathbb{N} \setminus \{0\}\}$ be an orthonormal base in V^\perp . Define $T: V^\perp \rightarrow H$ by $Tx_i(j) = x_i(j - 1)$ ($i < m, j \in \mathbb{N} \setminus \{0\}$). Set $B = Q \oplus \varepsilon \cdot T$; we show that B fulfills the requirements.

We have $\|B\| \leq \|B|_V\| + \|B|_{V^\perp}\| = \|Q\| + \varepsilon \cdot \|T\| \leq 1 - \varepsilon + \varepsilon = 1$, so $B \in C(H)$. For every $\lambda \in \mathbb{C}$,

$$\text{Ran}(B - \lambda \cdot \text{Id}) \supseteq \text{Ran}(B|_{V^\perp} - \lambda \cdot \text{Id}|_{V^\perp}) \supseteq \text{Ran}(T - \lambda \cdot \text{Id}|_{V^\perp}).$$

Since T is an m -fold sum of the operator D of Lemma 2.32, $\text{Ran}(T - \lambda \cdot \text{Id}|_{V^\perp})$ is dense in H , as required.

For every $i < n$,

$$\|Bx_i - Ax_i\| = \|Qx_i - Ax_i\| = \|Qx_i - \text{Pr}_V A|_V x_i\| \leq \varepsilon,$$

$$\begin{aligned} \|B^*x_i - A^*x_i\| &= \|Q^*x_i + \varepsilon \cdot T^*x_i - A^*x_i\| \leq \\ &\varepsilon \cdot \|T^*x_i\| + \|Q^*x_i - A^*x_i\| \leq \varepsilon + \|Q^*x_i - \text{Pr}_V A^*|_V x_i\| \leq 2\varepsilon, \end{aligned}$$

i.e. $B \in U$, as required.

It remains to show that \mathcal{T} is s^* - G_δ in $C(H)$. To this end, for every $y \in S_H$, $\delta > 0$ and $L \geq 0$ set

$$R(y, \delta, L) = \{A \in C(H): \exists \lambda \in \mathbb{C} (|\lambda| \leq L, \text{dist}(\text{Ran}(A - \lambda \cdot I), y) \geq \delta)\}. \quad (3)$$

It is clear that $R(y, \delta, 1)$ ($y \in S_H, \delta > 0$) are s -closed, and so s^* -closed, as well. Let $Y \subseteq S_H$ be a dense countable set. Since $\mathcal{T} = C(H) \setminus \bigcup\{R(y, 2^{-n}, 1): y \in Y, n \in \mathbb{N}\}$, the statement follows. ■

For technical reasons, we state the following corollary of Proposition 2.31.

Corollary 2.33. *The set*

$$\mathcal{E} = \{A \in C(H): \text{Ran } A \text{ is dense in } H\}$$

is s^ -co-meager and s^* - G_δ in $C(H)$.*

Proof. By Proposition 2.31, \mathcal{E} is s^* -co-meager in $C(H)$. With the notation of (3), $\mathcal{E} = C(H) \setminus \bigcup\{R(y, 2^{-n}, 0) : y \in Y, n \in \mathbb{N}\}$, so the statement follows. ■

We also need a similar result for positive self-adjoint operators.

Proposition 2.34. *The set*

$$\mathcal{P} = \{P \in P(H) : \text{Ran } P \text{ is dense in } H\}$$

is s -co-meager and s - G_δ in $P(H)$.

Proof. First we show that \mathcal{P} is s -dense in $P(H)$. To this end, let $U \subseteq P(H)$ be a non-empty s -open set. Then there exist $\{x_i : i < n\} \subseteq S_H$, $\varepsilon > 0$ and $A \in P(H)$ such that for every $B \in P(H)$, $\|Bx_i - Ax_i\| \leq \varepsilon$ ($i < n$) implies $B \in U$. We find a $B \in U$ such that $\text{Ran } B$ is dense in H .

Set $V = \text{span}\{x_i, Ax_i : i < n\}$. Let $Q : V \rightarrow V$ be an invertible contractive positive self-adjoint operator such that $\|Q - \text{Pr}_V A|_V\| \leq \varepsilon$; such a Q exists, e.g. any $Q = (1-\delta) \cdot \text{Pr}_V A|_V + \delta' \cdot \text{Id}|_V$ fulfills the requirements for suitable $0 < \delta, \delta' \leq \varepsilon/2$. Set $B = Q \oplus \text{Id}|_{V^\perp}$. Then B is a positive self-adjoint operator, $\|B|_V\| \leq 1$, $\|B|_{V^\perp}\| = 1$ and B is invertible, hence $B \in P(H)$ and $\text{Ran } B$ is dense in H . For every $i < n$,

$$\|Bx_i - Ax_i\| = \|Qx_i - Ax_i\| = \|Qx_i - \text{Pr}_V A|_V x_i\| \leq \varepsilon,$$

i.e. $B \in U$, as required.

It remains to show that \mathcal{P} is s - G_δ . Observe that for every $y \in S_H$ and $\delta > 0$, the set $R(y, \delta) = \{A \in P(H) : \text{dist}(\text{Ran } A, y) \geq \delta\}$ is s -closed. Let $Y \subseteq S_H$ be a dense countable set, then

$$\mathcal{P} = P(H) \setminus \bigcup\{R(y, 2^{-n}) : y \in Y, n \in \mathbb{N}\},$$

which completes the proof. ■

2.4.2. Spectral properties of s^* -typical contractions

Our main result in this section is the following.

Proposition 2.35. *An s^* -typical contraction A satisfies $C_\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.*

We start with a lemma.

Lemma 2.36. *The set*

$$\mathcal{S} = \{A \in C(H) : \forall \lambda \in \mathbb{C}, |\lambda| \leq 1 (\lambda \in \sigma(A))\}$$

is s^ -co-meager and s^* - G_δ in $C(H)$.*

Proof. First we show that \mathcal{S} is s^* -dense in $C(H)$. Let $U \subseteq C(H)$ be a non-empty s^* -open set. Then there exist $\{x_i: i < n\} \subseteq S_H$, $\varepsilon > 0$ and $A \in C(H)$ such that for every $B \in C(H)$, $\|Bx_i - Ax_i\| < \varepsilon$ ($i < n$) and $\|B^*x_i - A^*x_i\| < \varepsilon$ ($i < n$) imply $B \in U$. It is enough to find a $B \in U$ such that for every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, $\lambda \in \sigma(B)$.

Set $V = \text{span}\{x_i, Ax_i, A^*x_i: i < n\}$, and let $Q: V \rightarrow V$ be defined by $Q = \text{Pr}_V A|_V$. Let $T \in C(V^\perp)$ be arbitrary satisfying $\lambda \in \sigma(T)$ ($\lambda \in \mathbb{C}, |\lambda| \leq 1$), say the backward unilateral shift operator, and set $B = Q \oplus T$. We show that B fulfills the requirements.

Since $\sigma(T) \subseteq \sigma(B)$, we have $\lambda \in \sigma(B)$ ($\lambda \in \mathbb{C}, |\lambda| \leq 1$). For every $i < n$,

$$\|Bx_i - Ax_i\| = \|Qx_i - Ax_i\| = \|\text{Pr}_V A|_V x_i - Ax_i\| = 0 < \varepsilon,$$

$$\|B^*x_i - A^*x_i\| = \|Q^*x_i - A^*x_i\| = \|\text{Pr}_V A^*|_V x_i - A^*x_i\| = 0 < \varepsilon,$$

i.e. $B \in U$, as required.

It remains to show that \mathcal{S} is s^*G_δ in $C(H)$. For every $\delta > 0$, set

$$S(\delta) = \{A \in C(H): \exists \lambda \in \mathbb{C} (|\lambda| \leq 1 \text{ and } \|(A - \lambda \cdot I)x\|, \|(A^* - \bar{\lambda} \cdot I)x\| \geq \delta (x \in S_H))\}.$$

It is clear that $S(\delta)$ ($\delta > 0$) are s^* -closed. So the proof will be complete if we show $\mathcal{S} = C(H) \setminus \bigcup_{n \in \mathbb{N}} S(2^{-n})$.

Let $A \in C(H)$ be arbitrary. If $\lambda \notin \sigma(A)$ for a $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ then $(A - \lambda \cdot I)^{-1}$ and $(A^* - \bar{\lambda} \cdot I)^{-1}$ exist so $A \in S(\delta)$ for some sufficiently small $\delta > 0$. This proves $C(H) \setminus \mathcal{S} \subseteq \bigcup_{n \in \mathbb{N}} S(2^{-n})$.

Similarly, if $A \in S(\delta)$ for some $\delta > 0$, say $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ witnesses this, then $\lambda \notin P_\sigma(A) \cup C_\sigma(A)$ and $\bar{\lambda} \notin P_\sigma(A^*)$. By $\text{Ker}(A^* - \bar{\lambda} \cdot I) = \text{Ran}(A - \lambda \cdot I)^\perp$, $\bar{\lambda} \notin P_\sigma(A^*)$ implies $\lambda \notin R_\sigma(A)$. So we obtained $\lambda \notin \sigma(A)$, i.e. $A \notin \mathcal{S}$. This proves $\bigcup_{n \in \mathbb{N}} S(2^{-n}) \subseteq C(H) \setminus \mathcal{S}$ and finishes the proof. ■

Proof of Proposition 2.35. The map $A \mapsto A^*$ is a s^* -homeomorphism of $C(H)$. So by Proposition 2.31, a typical contraction A satisfies that for every $\lambda \in \mathbb{C}$, $\text{Ran}(A - \lambda \cdot I)$ and $\text{Ran}(A^* - \lambda \cdot I)$ are dense in H . Since $\text{Ker}(A - \lambda \cdot I) = \text{Ran}(A^* - \bar{\lambda} \cdot I)^\perp$, we get $\text{Ker}(A - \lambda \cdot I) = \emptyset$ ($\lambda \in \mathbb{C}$). Hence $P_\sigma(A) = P_\sigma(A^*) = \emptyset$ and $R_\sigma(A) = R_\sigma(A^*) = \emptyset$. So $C_\sigma(A) = \{\lambda \in \mathbb{C}: |\lambda| \leq 1\}$ is an immediate corollary of Lemma 2.36. ■

2.4.3. Proof of Theorem 2.29

With the notation of Corollary 2.33, set

$$\mathcal{H} = \mathcal{E} \cap \{A^*: A \in \mathcal{E}\} = \{A \in C(H): \text{Ran } A \text{ and } \text{Ran } A^* \text{ are dense in } H\}.$$

By Corollary 2.33, \mathcal{H} is an s^* -co-meager s^* - G_δ set in $C(H)$. Define $\psi_1: \mathcal{H} \rightarrow P(H)$ by $\psi_1(A) = (A^*A)^{1/2}$ ($A \in \mathcal{H}$). For every $A \in C(H)$, $(A^*A)^{1/2}$ is a positive self-adjoint contraction so the definition makes sense. Moreover, for $A \in \mathcal{H}$ we have $\text{Ran}(A^*A)^{1/2} = \text{Ran}(A^*A)$ is dense in H , so with the notation of Proposition 2.34, ψ_1 maps into \mathcal{P} . Note that the mapping $A \mapsto A^*A$ is s^* -continuous. We need that ψ_1 is s^* -continuous. Since we couldn't find a reference, we outline a proof of the following.

Lemma 2.37. *The function $\cdot^{1/2}: P(H) \rightarrow P(H)$, $A \mapsto A^{1/2}$ ($A \in P(H)$) is s -continuous.*

Proof. Let $A \in P(H)$ be arbitrary; note that $\sigma(A) \subseteq [0, 1]$. By [20, Theorem XI.6.1 p. 313], [20, Theorem XI.4.1 p. 307], [20, Proposition XI.5.2 p. 310] and [20, Theorem XI.8.1 p. 319], there is a s -left-continuous function $F: [0, 1] \rightarrow P(H)$ such that F is projection-valued, $F(0) = 0$, $F(1) = \text{Id}$, $F(\vartheta)F(\vartheta') = F(\min\{\vartheta, \vartheta'\})$ ($\vartheta, \vartheta' \in [0, 1]$), and for every continuous function $f: [0, 1] \rightarrow \mathbb{R}$, $\int_0^1 f(\vartheta)dF(\vartheta)$ exists as the s -limit of Riemann-Stieltjes sums and defines a bounded linear operator, denoted by $f(A)$, satisfying $\|f(A)\| \leq \|f\|_\infty$. Moreover, by [20, Example XI.12.1 p. 338] and [20, Theorem XI.12.3 p. 343], this defines a functional calculus compatible with power series expansion, in particular $A^{1/2}A^{1/2} = A$.

Let $\{A_n: n \in \mathbb{N}\} \subseteq P(H)$ satisfy $s\text{-}\lim_{n \in \mathbb{N}} A_n = A$. Let $x \in S_H$ and $\varepsilon > 0$ be arbitrary. Let $p: [0, 1] \rightarrow \mathbb{R}$ be a polynomial satisfying $\max_{x \in [0, 1]} |p(x) - x^{1/2}| < \varepsilon$. Since $s\text{-}\lim_{n \in \mathbb{N}} A_n^k = A^k$ ($k \in \mathbb{N}$), we have $\lim_{n \in \mathbb{N}} p(A_n)x = p(A)x$. As we observed above, $\|A^{1/2} - p(A)\|, \|A_n^{1/2} - p(A_n)\| < \varepsilon$ ($n \in \mathbb{N}$). This implies $\limsup_{n \in \mathbb{N}} \|A^{1/2}x - A_n^{1/2}x\| < 2\varepsilon$; since ε was arbitrary, the statement follows. ■

Corollary 2.38. *The function $\psi_1: \mathcal{H} \rightarrow \mathcal{P}$ is s^* -continuous.*

Proof. Since the strong and strong-star topologies coincide on $P(H)$, the statement follows. ■

Let $A \in \mathcal{H}$ be arbitrary. As we observed above, $\psi_1(A)$ is a positive self-adjoint operator with dense range. Hence $\psi_1(A)^{-1}$ is a closed densely defined positive self-adjoint operator. Consider the densely defined operator $A \cdot \psi_1(A)^{-1}$. It has a dense range, and for every $x \in \text{Ran } \psi_1(A)$ we have

$$\begin{aligned} \langle A \cdot \psi_1(A)^{-1}x, A \cdot \psi_1(A)^{-1}x \rangle &= \langle A^*A \cdot \psi_1(A)^{-1}x, \psi_1(A)^{-1}x \rangle = \\ &= \langle \psi_1(A)^{-1} \cdot A^*A \cdot \psi_1(A)^{-1}x, x \rangle = \langle x, x \rangle. \end{aligned}$$

Hence $A \cdot \psi_1(A)^{-1}$ is a densely defined isometry with dense range, i.e. it is closable and its closure is an unitary operator. Let $\psi_0(A) \in U(H)$ denote the closure of $A \cdot \psi_1(A)^{-1}$. Note that by definition, $A = \psi_0(A) \cdot \psi_1(A)$.

Corollary 2.39. *Every $A \in \mathcal{H}$ can be written as $A = \psi_0(A) \cdot \psi_1(A)$ where $\psi_0(A) \in U(H)$ and $\psi_1(A) \in \mathcal{P}$. This decomposition is unique and unitary invariant, i.e. for every $A \in \mathcal{H}$ and $U \in U(H)$ we have $\psi_i(UAU^{-1}) = U\psi_i(A)U^{-1}$ ($i < 2$).*

Proof. To see uniqueness, let $U, V \in U(H)$ and $P, Q \in \mathcal{P}$ be arbitrary. Then $U \cdot P = V \cdot Q$ implies $P^2 = P^*U^*UP = Q^*V^*VQ = Q^2$, so $P = Q$ by the uniqueness of positive square root. By P, Q having dense range, we get $U = V$.

If $A \in \mathcal{H}$ and $U \in U(H)$ then $UAU^{-1} \in \mathcal{H}$ and $UAU^{-1} = U\psi_0(A)U^{-1} \cdot U\psi_1(A)U^{-1}$. Hence by the uniqueness of this decomposition, we have $\psi_0(UAU^{-1}) = U\psi_0(A)U^{-1}$ and $\psi_1(UAU^{-1}) = U\psi_1(A)U^{-1}$. ■

Recall $\Psi: U(H) \times P(H) \rightarrow \mathcal{H}$, $\Psi(U, P) = U \cdot P$, which is an s^* -continuous function. By Corollary 2.39, Ψ is injective. It is obvious that (ψ_0, ψ_1) is the inverse of Ψ , so the following completes the proof of Theorem 2.29.

Lemma 2.40. *The function ψ_0 is s^* -continuous.*

Proof. Let $\{A_n: n \in \mathbb{N}\} \subseteq \mathcal{H}$ and $A \in \mathcal{H}$ be such that $s^*\text{-}\lim_{n \in \mathbb{N}} A_n = A$. For every $y \in \text{Ran } \psi_1(A)$, say $y = \psi_1(A)x$, we have

$$\begin{aligned} \|\psi_0(A_n)y - \psi_0(A)y\| &= \|\psi_0(A_n)\psi_1(A)x - \psi_0(A)\psi_1(A)x\| \leq \\ &\|\psi_0(A_n)\psi_1(A)x - \psi_0(A_n)\psi_1(A_n)x\| + \|\psi_0(A_n)\psi_1(A_n)x - \psi_0(A)\psi_1(A)x\| \leq \\ &\|\psi_0(A_n)\| \cdot \|\psi_1(A)x - \psi_1(A_n)x\| + \|A_nx - Ax\|. \end{aligned}$$

Since ψ_1 is s^* -continuous, the statement follows from $\text{Ran } \psi_1(A) \subseteq H$ being dense. ■

2.5. The norm topology

In the norm topology it is not possible to give a simple description of the spectral properties of typical operators. An intuitive explanation may be that the norm topology is non-separable, hence several different properties can coexist on non-meager sets.

In this section, every topological notion refers to the norm topology. We prove the following.

Theorem 2.41. *Let $\lambda \in \mathbb{C}$ be arbitrary. Then the following sets of operators have non-empty interior.*

1. $\{A \in B(H): \lambda \notin \sigma(A)\};$
2. $\{A \in B(H): \lambda \in R_\sigma(A)\};$
3. $\{A \in B(H): \lambda \in P_\sigma(A)\}.$

In particular, the following sets have non-empty interior.

4. $\{A \in B(H): \text{Ran}(A - \lambda \cdot I) = H\};$
5. $\{A \in B(H): \text{Ran}(A - \lambda \cdot I) \text{ is not dense in } H\}.$

On the other hand, the following sets of operators are nowhere dense.

6. $\{A \in B(H): \lambda \in C_\sigma(A)\};$
7. $\{A \in B(H): \text{Ran}(A - \lambda \cdot I) \text{ is dense in } H \text{ but not equal to } H\}.$

We need some terminology in advance.

Definition 2.42. Let $A \in B(H)$ and $\lambda \in \mathbb{C}$ be arbitrary. We say $\lambda \in \sigma(A)$ is *stable* if there is an $\varepsilon > 0$ such that for every $D \in B(H)$ with $\|D\| < \varepsilon$ we have $\lambda \in \sigma(A + D)$. Similarly, we say $\lambda \in P_\sigma(A)$ is *stable* if there is an $\varepsilon > 0$ such that for every $D \in B(H)$ with $\|D\| < \varepsilon$ we have $\lambda \in P_\sigma(A + D)$.

With this terminology, [10, Theorem 2 p. 912] can be reformulated as follows.

Proposition 2.43. *Let $A \in B(H)$ and $\lambda \in \mathbb{C}$ be arbitrary. Then the following are equivalent.*

1. $\lambda \in \sigma(A)$ is stable;
2. $\dim \text{Ker}(A - \lambda \cdot I) \neq \dim \text{Ran}(A - \lambda \cdot I)^\perp$ and there is a $\varepsilon > 0$ such that for every $x \in \text{Ker}(A - \lambda \cdot I)^\perp$ we have $\|(A - \lambda \cdot I)x\| \geq \varepsilon \cdot \|x\|.$

We prove a similar result for the point spectrum.

Proposition 2.44. *Let $A \in B(H)$ and $\lambda \in \mathbb{C}$ be arbitrary. Then the following are equivalent.*

1. $\lambda \in P_\sigma(A)$ is stable;
2. $\dim \text{Ran}(A - \lambda \cdot I)^\perp < \dim \text{Ker}(A - \lambda \cdot I)$ and there is a $\varepsilon > 0$ such that for every $x \in \text{Ker}(A - \lambda \cdot I)^\perp$ we have $\|(A - \lambda \cdot I)x\| \geq \varepsilon \cdot \|x\|$.

Proof. By linearity, we can assume $\lambda = 0$. Suppose first $0 \in P_\sigma(A)$ is stable. Then $0 \in \sigma(A)$ is stable so by Proposition 2.43, $\dim \text{Ker}(A) \neq \dim \text{Ran}(A)^\perp$ and there is a $\varepsilon > 0$ such that for every $x \in \text{Ker}(A)^\perp$ we have $\|Ax\| \geq \varepsilon \cdot \|x\|$. If $\dim \text{Ker}(A) < \dim \text{Ran}(A)^\perp$ then for every $\varepsilon > 0$ there is a $D \in B(H)$ such that $\|D\| < \varepsilon$, $D|_{\text{Ker}(A)^\perp} = 0$ and for every $x \in \text{Ker}(A) \setminus \{0\}$, $Dx \in \text{Ran}(A)^\perp \setminus \{0\}$. Then for every $x \in H \setminus \{0\}$ we have $(A + D)x \neq 0$, hence $0 \notin P_\sigma(A + D)$. This contradicts the assumption that $0 \in P_\sigma(A)$ is stable. Thus $\dim \text{Ran}(A)^\perp < \dim \text{Ker}(A)$, as required.

To see the converse, suppose the conditions of statement 2 hold. Then we have $\dim \text{Ker}(A) > 0$ hence $0 \in P_\sigma(A)$. We also have that $A: \text{Ker}(A)^\perp \rightarrow \text{Ran}(A)$ is a bounded invertible operator, in particular $\text{Ran}(A)$ is a closed co-finite dimensional subspace of H . Let $k = 1 + \dim \text{Ran}(A)^\perp$.

Let $D \in B(H)$ be arbitrary with $\|D\| < \varepsilon/2k^2$. For every $x \in \text{Ker}(A)$ consider the following inductive definition of a sequence $\{x_n: n \in \mathbb{N}\} \subseteq H$. Set $x_0 = x$. Let $n \in \mathbb{N}$ be arbitrary and suppose that $x_n \in H$ is defined. Write $Dx_n = u_n + v_n$ where $u_n \in \text{Ran}(A)$ and $v_n \in \text{Ran}(A)^\perp$. Set $x_{n+1} = A^{-1}u_n$. This completes the inductive step of the definition of $\{x_n: n \in \mathbb{N}\}$.

We set $\xi(x) = \sum_{n \in \mathbb{N}} (-1)^n x_n$, $\rho(x) = \sum_{n \in \mathbb{N}} (-1)^n v_n$. Note that $\|x_{n+1}\| \leq \|u_n\|/\varepsilon \leq \|Dx_n\|/\varepsilon < \|x_n\|/2k^2$, hence $\|x_n\| < \|x\|/(2k^2)^n$ ($n \in \mathbb{N}$) so the definitions make sense. Moreover, the functions ξ and ρ are linear, $\rho(x) \in \text{Ran}(A)^\perp$ and $\|\xi(x) - x\| < 1/k$. So by Lemma 2.14, if $\{x(i): i < k\}$ is an orthonormal system in $\text{Ker}(A)$ then $\{\xi(x(i)): i < k\}$ are linearly independent.

Observe that

$$\begin{aligned} (A + D)\xi(x) &= (A + D) \sum_{n \in \mathbb{N}} (-1)^n x_n = \sum_{n \in \mathbb{N}} (-1)^n (Ax_n + Dx_n) = \\ &= Ax_0 + \sum_{n \in \mathbb{N} \setminus \{0\}} (-1)^n Ax_n + \sum_{n \in \mathbb{N}} (-1)^n (u_n + v_n) = \sum_{n \in \mathbb{N}} (-1)^n v_n = \rho(x). \end{aligned}$$

By $\dim \text{Ker}(A) \geq k$, there is an orthonormal system $\{x(i): i < k\}$ in $\text{Ker}(A)$. Since $\dim \text{Ran}(A)^\perp < k$, there is an $x \in \text{span}\{x(i): i < k\} \setminus \{0\}$ such that $\rho(x) = 0$. Then $(A + D)\xi(x) = 0$ shows $0 \in P_\sigma(A + D)$, so the proof is complete. ■

Proof of Theorem 2.41. By linearity, it is enough to consider the $\lambda = 0$ case. The set of invertible operators shows statements 1 and 4. By Proposition 2.44, a neighborhood of the backward unilateral shift operator shows statement 3.

The set of statement 7 contains the set of statement 6 so it is sufficient to prove statement 7. Let $A \in B(H)$ be arbitrary satisfying $\text{Ran}(A)$ is dense in H but not equal to H . Then A cannot be invertible. Moreover, note that condition $\|Ax\| \geq \varepsilon \cdot \|x\|$ ($x \in \text{Ker}(A)^\perp$) in Proposition 2.43.2 would imply $\text{Ran}(A)$ is closed. Since $\text{Ran}(A)$ cannot be closed, A cannot satisfy Proposition 2.43.2. So the set of statement 7 is contained in the boundary of the set of invertible operators hence it is nowhere dense.

By Proposition 2.43, a neighborhood of the unilateral shift operator is contained in $\{A \in B(H) : 0 \in \sigma(A) \setminus P_\sigma(A)\}$. So by statement 6, statements 2 and 5 follow. ■

Observe that the backward unilateral shift operator D of Lemma 2.32 satisfies the conditions of statement 2 of Proposition 2.44 for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, hence every such $\lambda \in P_\sigma(D)$ is stable. So by taking direct sums of scaled and translated copies of D we can obtain arbitrarily complicated stable spectra. We also obtain the following.

Corollary 2.45. *The set of embeddable operators and the set of non-embeddable operators have non-empty interior.*

Proof. By [5, Theorem 2.1 p. 452], each operator in a small neighborhood of Id is embeddable. To show a non-empty open set of non-embeddable operators, we aim to use [5, Theorem 3.1 p. 454]. Recall the backward unilateral shift operator D of Lemma 2.32. By Proposition 2.44, each operator A in a small neighborhood of D satisfies $0 \in P_\sigma(A)$. So we get A is non-embeddable if we show $\dim \text{Ker}(A) \leq 1$.

Note that if $\|A - D\| < 1/\sqrt{2}$ then for every $x, y \in \text{Ker}(A) \cap S_H$ we have $\|Dx\|, \|Dy\| < 1/\sqrt{2}$. This implies $\|x - D^*Dx\|, \|y - D^*Dy\| > 1/\sqrt{2}$, and since D^*D is an orthogonal projection, we have

$$\begin{aligned} |\langle x, y \rangle| &= |\langle x - D^*Dx, y - D^*Dy \rangle + \langle D^*Dx, D^*Dy \rangle| \geq \\ &|\langle x - D^*Dx, y - D^*Dy \rangle| - |\langle D^*Dx, D^*Dy \rangle| > 0. \end{aligned}$$

Thus $\dim \text{Ker}(A) \leq 1$, which completes the proof. ■

The results of this section provide no information about the size of the spectrum of a typical operator. We state two questions related to this in Problem 3.2. Here we only point out the following.

Proposition 2.46. *The following sets are dense.*

1. $\{A \in B(H) : P_\sigma(A) \neq \emptyset\}$;

2. $\{A \in B(H) : R_\sigma(A) \neq \emptyset\}$.

Proof. Since $A \mapsto A^*$ ($A \in B(H)$) is a homeomorphism of $B(H)$ and $P_\sigma(A^*) = R_\sigma(A)$ ($A \in B(H)$), it suffices to show 1.

Let $B \in B(H)$ be arbitrary. Since the approximative point spectrum of B is nonempty, there exists a $\lambda \in \sigma(B)$ and a sequence $\{x_n : n \in \mathbb{N}\} \subseteq S_H$ such that $\lim_{n \in \mathbb{N}} \|Bx_n - \lambda x_n\| = 0$. For every $n \in \mathbb{N}$, let P_n denote the orthogonal projection onto $\text{span}\{x_n\}$, and set $A_n = B(\text{Id} - P_n) + \lambda \cdot P_n$. We have $A_n x_n = \lambda x_n$ and $\|A_n - B\| = \|\lambda \cdot P_n - B P_n\| = \|\lambda x_n - B x_n\|$ ($n \in \mathbb{N}$). Hence $P_\sigma(A_n) \neq \emptyset$ ($n \in \mathbb{N}$) and $\lim_{n \in \mathbb{N}} \|A_n - B\| = 0$. This completes the proof. ■

3. An outlook to general Banach spaces

The proofs in the previous section could convince the reader that an attempt to extend our investigations to tackle the typical properties of contractions on arbitrary Banach spaces could encounter considerable technical difficulties. In addition, such an endeavor has to face some problems of more fundamental nature. Recent developments in the theory of Banach spaces resulted in numerous spaces exhibiting surprising functional analytic properties. On the famous Banach space of S. A. Argyros and R. G. Haydon, every bounded linear operator is of the form $\lambda \cdot \text{Id} + K$ where $\lambda \in \mathbb{C}$ and K is a compact operator (see [1]). Spaces were constructed with only trivial isometries (see e.g. [3] and the references therein), moreover even a renorming of a Banach space may result in an arbitrary isometry group (see [11] and the references therein). The relevance of isometries comes from the important role played by unitary operators in the theory developed in the previous section.

Observe that renorming does not change the topology of the underlying Banach space, so the five topologies we consider on operators remain unchanged, as well. Instead, renorming affects the size of various classes of operators. So in a sense, the study of typical properties of operators in various topologies is more related to the geometry of the underlying Banach space than to the topology it carries. Therefore, reasonable extensions of our investigations should be pursued in Banach spaces where the geometry of the space is of special significance.

The most obvious such spaces are the L^p spaces ($1 \leq p \leq \infty$). The functional analysis of these spaces is well-developed, isometric operators are characterized (see e.g. [13]). Nevertheless, we expect that none of our main results extends to arbitrary L^p spaces.

Another promising extension of the theory of typical behavior of operators could tackle Banach*-algebras. Developing our results to that generality could separate

typical properties which are of operator theoretic nature from typical properties exploiting the geometry of the underlying space.

Finally let us propose some concrete problems which stem from the results of the previous section.

Problem 3.1. *We say $A, B \in B(H)$ are similar if there is an invertible operator $T \in B(H)$ such that $A = TBT^{-1}$.*

1. *Is it true that w -typical contractions are similar?*
2. *Is it true that s^* -typical contractions are similar?*

Problem 3.2. *Determine the s -typical properties of self-adjoint and positive self-adjoint operators.*

Problem 3.3. *Is a s^* -typical contraction embeddable into a strongly continuous semigroup?*

Problem 3.4.

1. *Is the set $\{A \in B(H) : P_\sigma(A) \neq \emptyset\}$ co-meager in the norm topology?*
2. *Is the set $\{A \in B(H) : C_\sigma(A) = \emptyset\}$ co-meager in the norm topology?*

It would be instructive to examine whether suitable analogues of our results hold for strongly continuous semigroups instead of single operators. Let $C^c(H)$ denote the set of contractive C_0 -semigroups. Given any metric d on $C(H)$, one can endow $C^c(H)$ with the topology generated by uniform d -convergence on compact time intervals. This topology is induced by the metric

$$d_c(T(\cdot), S(\cdot)) = \sum_{n \in \mathbb{N}} 2^{-n} \sup_{t \in [0, n]} d(T(t), S(t)).$$

Problem 3.5. *Under which of these topologies is $C_c(H)$ a Baire space? What are the typical properties of contractive C_0 -semigroups in these topologies?*

Recall that the strong topology case was treated in Section 2.3.4. Some results related to Problem 3.5 for the weak topology can be found in [4, Section 4], [7] and [8].

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