

Covering the Edges of a Graph by Three Odd Subgraphs

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ABSTRACT

We prove that any finite simple graph can be covered by three of its odd subgraphs, and we construct an infinite sequence of graphs where an edge-disjoint covering by three odd subgraphs is not possible. © ??? John Wiley & Sons, Inc.

Keywords: odd subgraph, edge covering

1. INTRODUCTION

A graph is called odd if the degree of its vertices is odd or zero. L. Pyber raises the problem of edge-covering with odd subgraphs in [9] as the counterpart of even-subgraph covering problems. He immediately shows there that the edges of every finite simple graph can be covered by at most four disjoint odd subgraphs, observes that an Euler graph with an odd number of vertices cannot be covered by two odd subgraphs and proves the following theorem.

Theorem 1. (Proposition 1.4 in [9]) The edge set of any finite simple graph on an even number of vertices can be partitioned into three edge-disjoint odd graphs.

Our first result reduces the number of necessary covering subgraphs to three; the second shows that edge-disjointness is lost for an infinite sequence of graphs.

Journal of Graph Theory Vol. , 1?? ()

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Theorem 2. Every finite simple graph can be covered by three odd subgraphs.

Example. There exists an infinite sequence of finite simple connected graphs not coverable by three edge-disjoint odd subgraphs.

These results can be related to the following. A. D. Scott [10] has shown that the edge set of any graph can be partitioned into $5k^2 \log k$ sets each of which is the edge set of a graph with all degrees congruent to 1 mod k . So Theorem 1, Theorem 2 and the Example together show the exact value of the number of covering graphs one needs for $k = 2$. The existence of odd and even factors in a graph is characterized by Tutte type conditions by A. Amahashi [1], C. P. Chen, M. Kano and C. Yuting [4], [12], J. Topp and J. Vestergaard [11] while structure theorems and results on the order of a maximum odd subgraph are given in the works of M. Kano, Gy. Y. Katona and H. Matsuda [6], [7], [8]. These references also consider odd graphs separately and among other things provide special conditions for the existence of various odd factors in odd graphs. So a covering by three odd subgraphs can mean that considering only odd graphs is not a serious loss of generality. The methods appearing in these references, such as eliminating odd forests or coforests, tree partitions, conditions on connectedness etc., are the main techniques we use here. Since our results remained unpublished for years, they may have become folklore. Also, in the meantime, partial results have been obtained independently by another work group [5].

In the next section we give the Example. We continue with covering trees with odd subgraphs, and after proving several seemingly disparate covering results we conclude with the proof of Theorem 2. Covering always means edge covering and we follow the terminology and notation of [2]; in particular $G = (V, E)$ denotes a finite simple undirected graph; if a graph G' is given, $V(G')$, $E(G')$ stand for the vertex set and edge set of G' . For $v \in V$, $d_G(v)$ denotes the degree of v in G and $\Gamma_G(v) = \{x \in V: vx \in E\}$ the neighborhood of v .

2. THE EXAMPLE

What we show first is that every graph which cannot be covered by three edge-disjoint odd subgraphs generates an infinite sequence of graphs with the same property. Since W_4 , the wheel of four spokes, cannot be covered by three disjoint subgraphs we have the sequence we need.

Let us suppose that a graph W is not coverable by three disjoint odd subgraphs. Let v_1v_2 be an edge of W . Take an even number of copies W_i ($i = 1, 2, \dots, 2k$) of W , add a new vertex v and substitute in each W_i the edge v_1v_2 with edges vv_1 and vv_2 . We claim that this graph cannot be covered by three disjoint odd subgraphs. Suppose that such a covering does exist. The vertex v is connected by two edges to each W_i . None of the covering classes can contain both members of such an edge-pair because by replacing vv_1 and vv_2 with v_1v_2 in that particular W_i and putting v_1v_2 into the class where the edge-

pair was without changing the partition of the other edges we would obtain a disjoint covering of $W_i = W$, a contradiction. The degree of v is even, so its edges are featured in two classes and both of these classes contain exactly one edge of each pair, that is $2k$ edges each; a contradiction. Therefore, the graph we construct above cannot be covered by three edge-disjoint odd subgraphs.

3. ODD FORESTS AND COFORESTS

We start with a lemma which allows us isolate forests and coforests which are odd at prescribed vertices.

Definition. Let G be a graph and H be an arbitrary subset of V . A subgraph F of G is called a *forest associated to H* (*coforest associated to H* resp.) if F is a forest ($G \setminus F$ is a forest resp.) and the vertices of F with odd degree are exactly the elements of H .

Lemma 1. Let G be a connected graph and H be an arbitrary even subset of V .

- (1) There exists a forest F associated to H ; moreover, if K is a subset of V disjoint to H and $G[V \setminus K]$ is connected, F can be chosen such that $V(F) \cap K = \emptyset$.
- (2) There exists a coforest C associated to H ; moreover, if we fix a vertex v which is not a cutvertex of G and an edge e on v , C can be chosen such that $d_{G \setminus C}(v) = 0$ (if $d_G(v)$ is odd and $v \in H$, or $d_G(v)$ is even and $v \notin H$) or $d_{G \setminus C}(v) = 1$ (if $d_G(v)$ is even and $v \in H$, or $d_G(v)$ is odd and $v \notin H$) with single edge e .

Proof. For (1), let $H = \{v_1, v_2, \dots, v_{2k}\}$ and let P_i be the paths connecting v_{2i-1} to v_{2i} for $i = 1, 2, \dots, k$. If K is non-empty, by the connectedness of $G[V \setminus K]$ these paths can be chosen avoiding K . We define F as the mod 2 sum of the paths P_i , i.e. for $e \in E$ we have $e \in E(F)$ if and only if e belongs to an odd number of the paths P_i . Then the only vertices with odd degree in F are the elements of H . If F is not a forest remove one-by-one the cycles from F ; this neither increases $V(F)$ nor changes the parity of degrees.

For (2), consider a forest F associated to H and add the cycles of $G \setminus F$ to F until only one forest remains; this defines C .

Now consider a v not cutvertex of G and an edge $e = vw$. Let $\{v_1, v_2, \dots, v_{2k}\} = \Gamma_{G \setminus C}(v)$ if $d_G(v)$ is odd and $v \in H$ or $d_G(v)$ is even and $v \notin H$; while if $d_G(v)$ is even and $v \in H$ or $d_G(v)$ is odd and $v \notin H$, let $\{v_1, v_2, \dots, v_{2k}\} = \Gamma_{G \setminus C}(v) \setminus \{w\}$ if $w \in \Gamma_{G \setminus C}(v)$ and $\{v_1, v_2, \dots, v_{2k}\} = \Gamma_{G \setminus C}(v) \cup \{w\}$ if $w \notin \Gamma_{G \setminus C}(v)$. Since v is not a cutvertex in G there exist P_i paths connecting v_{2i-1} to v_{2i} for $i = 1, 2, \dots, k$ avoiding v . We construct C' as the mod 2 sum of C and P_i, vv_i for $i = 1, 2, \dots, 2k$. Finally we add to C' the cycles of $G \setminus C'$ that might appear. The resulting graph fulfills the requirements. ■

Lemma 2. Let T be a tree, $v \in V(T)$ be fixed and suppose that a parity π for v is prescribed, which is even if $V(T) = \{v\}$.

- (1) Then T can be partitioned into two subtrees T_I, T_{II} such that T_I is an odd subgraph and in T_{II} all the vertices different from v have odd or zero degree and the parity of $d_{T_{II}}(v)$ is π .
- (2) If by removing a vertex w of a graph G we get a forest then G can be partitioned into two subgraphs where all the vertices different from w have odd or zero degree.

Proof. We prove (1) by induction on $|V(T)|$; for $|V(T)| = 1$ the statement clearly holds so suppose that the partition is possible for $|V(T)| < n$ and take a tree with $|V(T)| = n$. Pick a $v' \in \Gamma_T(v)$ and delete the edge $e = vv'$. Let T_1, T_2 denote the remaining trees, say $v \in V(T_1)$. We have $|V(T_1)|, |V(T_2)| < n$ so we can apply the induction hypothesis for T_1 with $v_1 = v$, prescribing π_1 even if and only if $V(T_1) = \{v\}$ or π is odd, and for T_2 with $v_2 = v', \pi_2$ even. We define T_I as

- (a) the union of class I of T_1 and class I of T_2 if $V(T_1) \neq \{v\}$;
- (b) the union of e and class II of T_2 if $V(T_1) = \{v\}$ and π is even;
- (c) class I of T_2 if $V(T_1) = \{v\}$ and π is odd;

while T_{II} is defined as

- (a) the union of e , class II of T_1 and class II of T_2 if $V(T_1) \neq \{v\}$;
- (b) class I of T_2 if $V(T_1) = \{v\}$ and π is even;
- (c) the union of e and class II of T_2 if $V(T_1) = \{v\}$ and π is odd.

This partition clearly satisfies the requirements.

For (2), let $\Gamma_G(w) = \{w_1, w_2, \dots, w_k\}$ and consider the graph G' obtained from $G[V \setminus \{w\}]$ by adding new vertices v_1, v_2, \dots, v_k and edges $w_1v_1, w_2v_2, \dots, w_kv_k$. Clearly, G' is a forest; so by (1), G' can be partitioned into two odd subgraphs. Partition G by putting the edges ww_i into the class of w_iv_i while the other edges of G go into the same class as in G' . All the vertices different from w will have odd degree which completes the proof. ■

4. COVERING BY THREE ODD SUBGRAPHS

Toward the proof of Theorem 2 first we cover several special graphs with three odd subgraphs respectively. Hopefully this makes transparent the several cases we have to distinguish in the end.

We continue using [2], specially Chapter III §2 on block-cutvertex decomposition. If R is an endblock of G on a cutvertex s then we call $D = R[V(R) \setminus \{s\}]$ the *pre-endblock* in R and s is the cutvertex corresponding to D . If G is 2-connected then G is the only block, endblock and pre-endblock in G . From now on we assume that G is connected.

Lemma 3. Suppose that $|V|$ is odd. If there is a cutvertex s of G so that for an endblock R on s we have $d_R(v)$ odd for $v \in V(R)$, then G can be covered by three disjoint odd subgraphs so that $E(R)$ intersects only one of them.

Proof. By Lemma 1.2 we have a coforest C in G associated to $H = V \setminus \{s\}$. Since $(G \setminus C) \cap R$ forms a forest with at most one vertex, namely s , with odd degree it is empty. We partition $G \setminus C = T_I \cup T_{II}$ by Lemma 2.1 prescribing π even for s . Put $G_I = C \setminus R$, $G_{II} = T_{II} \cup R$ and $G_{III} = T_I$; the fact $d_R(s)$ is odd implies that these are odd subgraphs so the proof is complete. ■

Lemma 4. Suppose that $|V|$ is odd. If we have an edge vw so that v is not a cutvertex in G , w is not a cutvertex in $G' = G[V \setminus \{v\}]$ and $d_G(v)$, $d_G(w)$ are even then G can be covered by three disjoint odd subgraphs.

Proof. Since $d_G(w)$ is even, we can apply Lemma 1.2 in G' to have a coforest C associated to $H = V(G')$ containing all the edges on w in G' . By Lemma 2.2, there is a partition $G \setminus \{C, vw\} = T_I \cup T_{II}$ so that only $d_{T_I}(v)$ and $d_{T_{II}}(v)$ can be even; since $d_G(v)$ is even exactly one of $d_{T_I}(v)$, $d_{T_{II}}(v)$ is even, say $d_{T_{II}}(v)$ is so. Let $G_I = C$, $G_{II} = T_{II} \cup \{vw\}$ and $G_{III} = T_I$; this covering fulfills the requirements. ■

Lemma 5. Suppose that $|V|$ is odd. If we have edges $vw, wq \in E$ so that $vq \notin E$, v is not a cutvertex in G , q is not a cutvertex in $G' = G[V \setminus \{v\}]$, $d_G(v)$ is even and $d_G(w)$ is odd then G can be covered by three odd subgraphs.

Proof. Since q is not a cutvertex in G' , by Lemma 1.2 there is a coforest C associated to $V(G')$ in G' with fixed vertex and edge q , $e = wq$. Partition $T = G \setminus \{C, vw, wq\}$ by Lemma 2.2; since $d_T(v)$ is odd for exactly one class of this partition, say T_{II} , we have $d_{T_{II}}(v)$ even. Let $G_I = C$. Set $G_{II} = T_{II} \cup \{vw, wq\}$, $G_{III} = T_I$ if $d_{T_{II}}(w) > 0$; while if $d_{T_{II}}(w) = 0$ we set $G_{II} = T_{II} \cup \{vw\}$, $G_{III} = T_I \cup \{wq\} \setminus G_I$.

We have G_I odd. Since $d_{T_{II}}(v)$ is even and $d_T(q) = 0$, G_{II} is also odd. The graph T_I is also odd so to have G_{III} odd we have to show that if $d_{T_{II}}(w) = 0$ and $wq \notin E(G_I)$ then $d_{T_I}(w) = d_{T_I}(q) = 0$; $d_{T_I}(q) = 0$ follows from $d_T(q) = 0$. Now $d_G(w)$ odd and $wq \notin E(G_I) = E(C)$ imply that $d_T(w)$ is even hence if $d_{T_{II}}(w) = 0$ then $d_{T_I}(w) = d_T(w) = 0$. The proof is complete. ■

Lemma 6. Suppose that $|V|$ is odd. If there is a set $D \subseteq V$ and vertices $w \in D$, $v \in V$ so that $G[V \setminus (D \cup \{v\})]$ is connected, $d_G(v)$ is even, $d_G(x)$ is odd for every $x \in D$, $D \subseteq \Gamma_G(v)$ and $\Gamma_G(w) \subseteq D \cup \{v\}$ then G can be covered by three odd subgraphs.

Proof. Since $G[V \setminus (D \cup \{v\})]$ is connected, $|V|$ is odd and $d_G(v)$ is even, by Lemma 1.1 there is a forest T associated to $H = \{x \in V: d_G(x) \text{ is even}\} \setminus \{v\}$ avoiding $K = D \cup \{v\}$. By Lemma 2.1 we can cover T with two disjoint odd subgraphs T_I and T_{II} . Set

$$G_I = G \setminus (T \cup \{wx, vx: x \in \Gamma_G(w)\}),$$

$$G_{II} = T_{II} \cup \{wx: x \in \Gamma_G(w)\}, \text{ and}$$

$$G_{III} = T_I \cup \{vx: x \in (\Gamma_G(w) \setminus \{v\}) \cup \{w\}\}.$$

We have $d_{G_I}(x)$ odd for $x \notin K$ by the definition of T ; for $x \in D$, $d_{G_I}(x)$ is odd because $d_G(x)$ is odd while $d_{G_I}(v)$ is odd because $d_G(w)$ is odd and $d_G(v)$ is even. The graph G_{II} is odd because T_{II} is odd, avoids $D \cup \{v\}$ which contains $w, \Gamma_G(w)$ and $d_G(w)$ is odd. Finally G_{III} is odd because T_I is odd, avoids $D \cup \{v\}$ which contains $w, \Gamma_G(w)$ and $d_G(w)$ hence $|(\Gamma_G(w) \setminus \{v\}) \cup \{w\}|$ is odd; so we have a right covering. ■

Proof of Theorem 2. We prove the statement by induction on $|V|$. For $|V| \leq 3$ the statement is trivial; suppose that the theorem holds for $|V| < n$ and let G be a graph with $|V| = n$. We can assume that G is connected.

If n is even take by Lemma 1.1 a forest T associated to $H = \{x \in V: d_G(x) \text{ is even}\}$ and cover T using Lemma 2.1 with two disjoint odd subgraphs T_I and T_{II} . Set $G_I = G \setminus T$, $G_{II} = T_{II}$ and $G_{III} = T_{III}$, these are odd subgraphs which cover G .

From now on suppose that $|V|$ is odd. First we show that it is enough to prove the statement if there is a $v \in V$ so that

$$\begin{aligned} d_G(v) \text{ is even, } G' = G[V \setminus \{v\}] \text{ is connected,} \\ W = \{w \in \Gamma_G(v): w \text{ is in a pre-endblock of } G'\} \neq \emptyset. \end{aligned} \quad (1)$$

If G is 2-connected let $v \in V$ be arbitrary with $d_G(v)$ even; then G' is connected and $W \neq \emptyset$ so (1) holds. If G is not 2-connected let R be an endblock on a cutvertex s . If $d_R(x)$ is odd for every $x \in V(R)$ then Lemma 3 implies that G can be covered by three odd subgraphs. If $d_R(x)$ is odd for every $x \in V(R) \setminus \{s\}$ but $d_R(s)$ is even then cover $G \setminus R$ by the induction hypothesis with three odd subgraphs $\tilde{G}_I, \tilde{G}_{II}$ and \tilde{G}_{III} , say $d_{\tilde{G}_I}(s) > 0$, and set $G_I = \tilde{G}_I \cup R$, $G_{II} = \tilde{G}_{II}$, $G_{III} = \tilde{G}_{III}$. This is a right covering of G . Finally if $d_G(v)$ is even for some $v \in V(R) \setminus \{s\}$ then (1) again holds.

Now suppose (1). Pre-endblocks contain no cutvertices; so if $d_G(w)$ is even for some $w \in W$ then we are in the situation of Lemma 4 hence G can be covered by three odd subgraphs.

If $d_G(w)$ is odd for every $w \in W$, let $D_w = \{x \in \Gamma_G(w): x \text{ is not a cutvertex of } G'\}$ ($w \in W$). If for some $w \in W$ and $q \in D_w$ we have $vq \notin E$ then Lemma 5 applies and G can be covered by three odd subgraphs.

If $vx \in E$ for every $x \in D_w$ ($w \in W$) then fix some $w_0 \in W$ and let $G[D]$ be the pre-endblock of G' containing w_0 . We show that $D \subseteq \Gamma_G(v)$; let $x \in D$. Since pre-endblocks are connected it is enough to show that if $xy \in E$ for some $y \in \Gamma_G(v) \cap D$ then $x \in \Gamma_G(v)$. But $y \in \Gamma_G(v)$ implies $y \in W$ hence $x \in D_y$ and so $x \in \Gamma_G(v)$. To summarize, $G[V \setminus (D \cup \{v\})]$ is connected, $D \subseteq \Gamma_G(v)$ hence $D \subseteq W$ so $d_G(x)$ ($x \in D$) is odd. Thus if $\Gamma_G(w) \subseteq D \cup \{v\}$ for some $w \in W$ then we are in the situation of Lemma 6 and G can be covered by three odd subgraphs.

If not then $D \neq V(G')$, that is G' is not 2-connected, and for the cutvertex c corresponding to $G[D]$ we have $D \subseteq \Gamma_G(c)$. Since $G[D]$ is an odd connected graph $|D|$ is even. By the induction hypothesis, $G[V \setminus D]$ can be covered by three odd subgraphs $\tilde{G}_I, \tilde{G}_{II}, \tilde{G}_{III}$. We have that $d_G(v)$ is even and so $d_{G[V \setminus D]}(v)$ is also even. If $d_{G[V \setminus D]}(v)$ is nonzero,

then the degree of v is nonzero in at least two of the covering graphs, say \tilde{G}_I and \tilde{G}_{II} . If $d_{\tilde{G}_I}(c) > 0$ then let $G_I = \tilde{G}_I \cup G[D \cup \{v, c\}]$, $G_{II} = \tilde{G}_{II}$ and $G_{III} = \tilde{G}_{III}$, if $d_{\tilde{G}_{II}}(c) > 0$ let $G_I = \tilde{G}_I$, $G_{II} = \tilde{G}_{II} \cup G[D \cup \{v, c\}]$ and $G_{III} = \tilde{G}_{III}$; else let $G_I = \tilde{G}_I \cup \{xv: x \in D\}$, $G_{II} = \tilde{G}_{II} \cup D$ and $G_{III} = \tilde{G}_{III} \cup \{xc: x \in D\}$. This yields a right covering.

If $d_{G[V \setminus D]}(v) = 0$ then let $w, w' \in D$, $w \neq w'$. By Lemma 3, $G \setminus \{vw, wc\}$ can be covered by three odd subgraphs $\tilde{G}_I, \tilde{G}_{II}, \tilde{G}_{III}$ so that $R = G[D \cup \{c, v\}] \setminus \{vw, wc\}$ belongs to one class, say \tilde{G}_I . Let $G_I = \tilde{G}_I$, $G_{II} = \tilde{G}_{II} \cup \{vw\}$ and $G_{III} = \tilde{G}_{III} \cup \{wc\}$ if $d_{\tilde{G}_{III}}(c) = 0$ while $G_{III} = \tilde{G}_{III} \cup \{wc, w'c\}$ if $d_{\tilde{G}_{III}}(c) > 0$. This yields a right covering and completes the proof. ■

ACKNOWLEDGMENTS

I thank Gábor Rudolf for his helpful remarks. I also thank László Pyber for his advices and guidance in the domain of covering problems.

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