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A characterization of spaces l-equivalent to the unit interval ☆

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Abstract

Two topological spaces X and Y are said to be l-equivalent if there is a linear homeomorphism between $C_p(X)$ and $C_p(Y)$, the spaces of continuous functions on X and Y endowed with the pointwise convergence topology. We prove a necessary condition on a topological space to be l-equivalent to the n-dimensional cube. This, in particular, combined with a recent result of R. Górak, characterizes the topological spaces l-equivalent to the compact interval.

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1. Introduction

Let X be a topological space. We denote by $C_p(X)$ the linear space of continuous functions on X with the pointwise convergence topology. If for two topological spaces X and Y there is a linear (uniform) homeomorphism between $C_p(X)$ and $C_p(Y)$, then X and Y are said to be l-equivalent (u-equivalent).

It is natural to ask the question which topological properties are preserved by l-equivalence or u-equivalence, that is for which topological properties P is it true that if the spaces X and Y are l-equivalent (u-equivalent), then X has property P if and only if Y has property P (see [1, Chapter 0] for a detailed study). Among others, it has been proven

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by Arkhangel'skiĭ [2] that the class of compact metrizable spaces is preserved by linear equivalence, while Pestov [9] obtained in a very general setting that dimension is preserved by linear equivalence. The analogous results for uniform equivalence have been proven by Uspenskiĭ [10] and Gul'ko [7], while a much finer classification in case of certain types of zero-dimensional spaces has been carried out by Baars, de Groot, van Mill and Pelant (see [3] and [4]).

Motivated by a question of Arkhangel'skiĭ, in this paper we focus on the problem of characterizing the topological spaces l-equivalent to I^n, the n-dimensional cube for $n \geqslant 1$ (see [8] for a comprehensive overview on this question and other related problems and results). For uniform equivalence, this characterization was recently given by Górak [6, Main Theorem 1] as follows.

Theorem 1. *For every positive integer n, a space X is u-equivalent to I^n if and only if*

(1) *X is n-dimensional compact metrizable*;
(2) *every nonempty closed subset of X contains a relatively open nonempty subset which can be embedded into I^n.*

He also obtained the following result on linear equivalence [6, Theorem 4.2]. For the definition of the n-dimensional embedding derivative $X^{[i,n]}$ see Definition 5.

Theorem 2. *For a space X such that $X^{[i,n]} = X^{[i+1,n]}$ for some $n > 0$ and $i \in \mathbb{N}$ the following conditions are equivalent*:

(1) *X is l-equivalent to I^n*;
(2) *X is n-dimensional compact metrizable and $X^{[i,n]} = \emptyset$.*

This result inspired the conjecture [6, Hypothesis 4.3] that X is l-equivalent to I^n if and only if X is n-dimensional compact metrizable and $X^{[i,n]} = \emptyset$ for some $i \in \mathbb{N}$. Among others, we prove this conjecture for $n = 1$ (Corollary 21), or equivalently, with the formulation introduced above, we show the property of having empty ith one-dimensional embedding derivative for some finite i is preserved by l-equivalence (Corollary 22).

2. Preliminaries

From this point, ind indicates always small inductive dimension. Let X be a topological space. We denote by $C_p(X)$ and $C_p^\star(X)$ the linear space of continuous and bounded continuous functions on X with the pointwise convergence topology, while $C(X)$ stands for the Banach space of bounded continuous functions on X with the usual $\|\cdot\|_\infty$ norm.

Let X, Y be topological spaces, and consider a linear homeomorphism

$$H : C_p^\star(X) \to C_p^\star(Y).$$

Then the graph of H,

$$\Gamma(H) \subset C_p^\star(X) \times C_p^\star(Y)$$

is closed. Since the norm topology $C(X)$ is finer than the pointwise convergence topology $C_p(X)$, this implies that

$$\Gamma(H) \subset C(X) \times C(Y)$$

is also closed, so by the Closed Graph Theorem the operator

$$H : C(X) \to C(Y)$$

is bounded. By symmetry, we have that its inverse is also bounded, which makes possible the following definition.

Definition 3. For a given linear homeomorphism $H : C_p^\star(X) \to C_p^\star(Y)$ we set

$$L(H) = \max\bigl\{\|H\|, \bigl\|H^{-1}\bigr\|\bigr\}. \tag{1}$$

Now let $H : C_p(X) \to C_p(Y)$ be a linear homeomorphism. Since for every $y \in Y$ the linear functional

$$\begin{aligned}&\nu(y) : C_p(X) \to \mathbb{R},\\&\nu(y)f = (Hf)(y)\end{aligned}$$

is continuous, it can be identified with an element of

$$C_p(X)^\star = \left\{\sum_{x\in D} \alpha_x\delta_x\colon\ D \subset X,\ \operatorname{card} D < \infty,\ \alpha_x \in \mathbb{R}\right\},$$

where δ_x stands for the Dirac-measure at x. The coefficient of δ_x in $\nu(y)$ will be denoted by $\nu_x(y)$, that is

$$\nu(y) = \sum_{x\in X} \nu_x(y)\delta_x.$$

Moreover, since for every continuous function $f \in C_p(X)$ the image function $Hf \in C_p(Y)$ is also continuous, we have that $\nu : Y \to C_p(X)^\star$ is also continuous (where $C_p(X)^\star$ is always endowed with the weak* topology). The continuous mapping $\mu : X \to C_p(Y)^\star$ and coefficients $\mu_y(x)$ are defined analogously. The support of a measure $\mu(x)$ will be denoted by $\operatorname{supp}\mu(x)$.

We note that by (1), for the total variation of $\nu(y)$, $\mu(x)$ we have

$$\begin{aligned}&\bigl\|\nu(y)\bigr\| = \sum_{x\in X}\bigl|\nu_x(y)\bigr| \leqslant L(H), \quad \forall y \in Y,\\&\bigl\|\mu(x)\bigr\| = \sum_{y\in Y}\bigl|\mu_y(x)\bigr| \leqslant L(H), \quad \forall x \in X.\end{aligned}$$

We introduce the two different notions of embedding derivatives.

Definition 4. We say that a topological space X is *n-dimensional everywhere*, if for every open set $U \subset X$, $\operatorname{ind} U = n$.

Definition 5. Let X be a topological space. For an $n \in \mathbb{N}$, let

$$I_n(X) = \bigcup_{s \in S} U_s, \qquad J_n(X) = \bigcup_{t \in T} V_t,$$

where $\{U_s\colon s \in S\}$ is the family of all open subsets of X which can be embedded into I^n and $\{V_t\colon t \in T\}$ is the family of all open subsets of X which are at most $(n-1)$-dimensional.

With this notation, for an ordinal α, the *αth n-dimensional embedding derivative* $X^{[\alpha,n]}$ is defined on the following way:

$$X^{[0,n]} = X;$$
$$X^{[\alpha+1,n]} = X^{[\alpha,n]} \setminus I_n\big(X^{[\alpha,n]}\big);$$
$$X^{[\alpha,n]} = \bigcap_{\beta<\alpha} X^{[\beta,n]} \quad \text{if } \alpha \text{ is limit.}$$

Similarly, we define the *αth n-dimensional strong embedding derivative* $X^{[[\alpha,n]]}$ by

$$X^{[[0,n]]} = X \setminus J_n(X);$$
$$X^{[[\alpha+1,n]]} = X^{[[\alpha,n]]} \setminus \big(I_n\big(X^{[[\alpha,n]]}\big) \cup J_n\big(X^{[[\alpha,n]]} \setminus I_n\big(X^{[[\alpha,n]]}\big)\big)\big);$$
$$X^{[[\alpha,n]]} = \bigcap_{\beta<\alpha} X^{[[\beta,n]]} \setminus J_n\Bigg(\bigcap_{\beta<\alpha} X^{[[\beta,n]]}\Bigg) \quad \text{if } \alpha \text{ is limit.}$$

That is, in the case of the n-dimensional embedding derivative, one derivation step throws away every open set which can be embedded into the n-dimensional cube, while the n-dimensional strong embedding derivative throws away every lower dimensional open subset as well. In particular, as we shall see later, every n-dimensional strong embedding derivative is n-dimensional everywhere or empty (Lemma 13).

We note that every embedding or strong embedding derivative of a compact metric space is itself compact metric. We will show that for $n = 1$, $X^{[i,1]} = \emptyset$ for some $i \in \mathbb{N}$ if and only if $X^{[[j,1]]} = \emptyset$ for some $j \in \mathbb{N}$ (Lemma 15). On the other hand, for $n \geqslant 2$, any universal $(n-1)$-dimensional compact metric space shows that the analogous statement does not hold. This is the reason why our result cannot be applied for the characterization of spaces l-equivalent to two or higher dimensional cubes.

With these notions our main result can be stated as follows.

Theorem 6. *Let X be a topological space. If X is l-equivalent to the n-dimensional cube, then $X^{[[m,n]]} = \emptyset$ for some $m \in \mathbb{N}$.*

For $n = 1$ and one-dimensional compact metrizable X, this condition is sufficient.

We shall prove the first part in Section 4 in a quantitative form (Theorem 18 and Corollary 19), while one-dimensional case will be proven in Section 5 (Corollary 21).

From now on we consider only *compact* topological spaces, for which the function spaces $C_p^\star(X)$ and $C_p(X)$ coincide. To avoid the use of multiple stars, we will use the notation $C_p(X)$ even if we work with the notions introduced for $C_p^\star(X)$.

3. Basic properties

We will use the following classical results of dimension theory. For the proofs, see, e.g., [5, Corollary 1.8.11, p. 76] [5, Theorem 1.11.7, p. 126] and [5, Theorem 1.5.3, p. 42].

Theorem 7. *A compact subset of I^n is n-dimensional if and only if it has nonempty interior.*

Theorem 8. *Every zero-dimensional separable metric space can be embedded into I^1.*

Theorem 9. *Let X be a separable metric space. If X is a countable union of n-dimensional closed subspaces then X is n-dimensional.*

In the following lemmas we prove some properties of continuous mapping into $C_p(Y)^\star$. A detailed study of the techniques and the proof of our first lemma can be found in [3, Lemma 1.2.7, p. 21].

Lemma 10. *Let X, Y be metric spaces and let*

$$\mu : X \to C_p(Y)^\star$$

be a continuous function. Then for every open set $B \subset Y$ and for every $n \in \mathbb{N}$, the set

$$\bigl\{x \in X\colon \operatorname{card}\bigl(\operatorname{supp}\mu(x) \cap B\bigr) \geqslant n\bigr\}$$

is open.

Lemma 11. *Let X, Y be metric spaces, $s \in \mathbb{N}$ and let*

$$\mu : X \to C_p(Y)^\star$$

be a continuous function with $\operatorname{card}\operatorname{supp}\mu(x) = s$, $\forall x \in X$. *Then for every $x \in X$ there exist an open neighborhood $N_x \subset X$ of x and continuous functions*

$$\mu_1, \mu_2, \ldots, \mu_s : N_x \to Y,$$
$$\alpha_1, \alpha_2, \ldots, \alpha_s : N_x \to \mathbb{R}$$

such that

$$\mu(x) = \sum_{i=1}^{s} \alpha_i(x)\delta_{\mu_i(x)}.$$

The functions μ_i, $1 \leqslant i \leqslant s$ will be called the support functions *of μ.*

Proof. Let $x \in X$ be arbitrary and let $\operatorname{supp}\mu(x) = \{y_1, \ldots, y_s\}$. We can choose pairwise disjoint open sets $B_x^1, \ldots, B_x^s$ such that $y_i \in B_x^i$, $1 \leqslant i \leqslant s$. By Lemma 10, there is an open set $x \in N_x \subset X$ such that $\operatorname{card}(\operatorname{supp}\mu(z) \cap B_x^i) = 1$ for every $z \in N_x$, $1 \leqslant i \leqslant s$. Let

$$\mu_i(z) = \operatorname{supp}\mu(z) \cap B_x^i, \qquad \alpha_i(z) = \mu_{\mu_i(z)}(z)$$

for $z \in N_x$, $1 \leqslant i \leqslant s$. Again by Lemma 10, the functions μ_i are continuous on N_x, and it is easy to see that the continuity of μ implies the continuity of the coefficients α_i, which proves the lemma. □

Lemma 12. *Let X be a compact metric space and let X' be its compact subset. Let*

$$\mu : X \to C_p\big(I^n\big)^\star, \qquad \nu : I^n \to C_p(X)^\star$$

be continuous functions. Suppose that for an open set $U \subset X$, a number $l \in \mathbb{N}$, a sequence of numbers $s_1, s_2, \ldots, s_l \in \mathbb{N}$ and sequence of pairwise disjoint open balls $B_1, \ldots, B_l \subset I^n$ we have that the set

$$U_1 = \big\{x \in X' \cap U\colon\ \operatorname{card}\operatorname{supp}\mu(x) = l,\ \operatorname{card}\big(\operatorname{supp}\mu(x) \cap B_i\big) = 1,\\ \operatorname{card}\big(\operatorname{supp}\nu\big(\operatorname{supp}\mu(x) \cap B_i\big)\big) = s_i,\ 1 \leqslant i \leqslant l\big\}$$

is of second category in X'. Then U_1 has nonempty interior in X'.

Proof. From Lemma 10 we have that the sets

$$\big\{x \in X' \cap U\colon\ \operatorname{card}\big(\operatorname{supp}\mu(x) \cap B_i\big) \geqslant 1\big\}, \quad 1 \leqslant i \leqslant l,$$

are open in X', that is the set

$$U_{1,0} = \big\{x \in X' \cap U\colon\ \operatorname{card}\operatorname{supp}\mu(x) = l,\ \operatorname{card}\big(\operatorname{supp}\mu(x) \cap B_i\big) = 1,\ 1 \leqslant i \leqslant l\big\}$$

is the intersection of a finite number of relatively open and closed sets in X'. Again by Lemma 10, the sets

$$\big\{x \in U_{1,0}\colon\ \operatorname{card}\big(\operatorname{supp}\nu\big(\operatorname{supp}\mu(x) \cap B_i\big)\big) \geqslant s_i\big\}, \quad 1 \leqslant i \leqslant l,$$

are open in $U_{1,0}$, so U_1 is the intersection of a finite number of relatively open and closed sets in $U_{1,0}$, and thus in X', as well. Hence, since X' is compact metric, U_1 is of second category in X' if and only if it has nonempty interior in X'. This completes the proof. □

Finally we prove three lemmas on the embedding derivatives. The first is an obvious corollary of Theorem 9.

Lemma 13. *For any compact metric space X and ordinal α, $X^{[[\alpha,n]]}$ is empty or n-dimensional everywhere.*

Lemma 14. *Let X be a compact metric space such that $X^{[[m,n]]} \neq \emptyset$ for some $m \in \mathbb{N}$, and every nonempty closed subset of X contains a nonempty relatively open subset which can be embedded into I^n.*

Then there is a sequence $X_1, X_2, \ldots, X_m$ of subsets of X with the following properties:

(1) *X_i is locally compact and n-dimensional everywhere, $i = 1, \ldots m$;*
(2) *$X_j \subset \operatorname{cl} X_i$, $1 \leqslant i < j \leqslant m$;*
(3) *$X_i \cap \operatorname{cl} X_j = \emptyset$, $1 \leqslant i < j \leqslant m$.*

Proof. Let

$$X_i = I_n\big(X^{[[i-1,n]]}\big), \quad i = 1, \ldots, m.$$

Since $X^{[[i-1,n]]} \subset X$ is nonempty and closed, we have that X_i is a dense open subset of $X^{[[i-1,n]]}$, so specially it is nonempty, locally compact and Lemma 13 implies that it is n-dimensional everywhere. Moreover, whenever $1 \leqslant i < j \leqslant m$,

$$X_j = I_n\bigl(X^{[[j-1,n]]}\bigr) \subset X^{[[j-1,n]]} \subset X^{[[i,n]]} = \operatorname{cl} X_i, \tag{2}$$

which proves property (1) and (2).

As (2) shows, $X^{[[i,n]]}$, a compact subset of $X^{[[i-1,n]]}$, contains X_j whenever $1 \leqslant i < j \leqslant m$, so we have

$$X_i \cap \operatorname{cl} X_j \subset X_i \cap X^{[[i,n]]} = I_n\bigl(X^{[[i-1,n]]}\bigr) \cap X^{[[i,n]]} = \emptyset, \quad 1 \leqslant i < j \leqslant m.$$

This proves property (3) and completes the proof. □

Lemma 15. *For every compact metric space X and $m \in \mathbb{N}$,*

$$X^{[2m+1,1]} \subset X^{[[m,1]]} \subset X^{[m,1]}.$$

Proof. Before starting the proof, we note that if X, Y are arbitrary topological spaces, then $X \subset Y$ implies $X \setminus I_1(X) \subset Y \setminus I_1(Y)$, since if a set $U \subset Y$ can be embedded into I^1 then $U \cap X$ can be embedded into I^1, as well. From Theorem 8 we also know that for every compact metric space X we have $J_1(X) \subset I_1(X)$.

It is obvious that $X^{[[m,1]]} \subset X^{[m,1]}$ for every $m \in \mathbb{N}$, so we prove only the first inclusion, by induction on m.

Let now $m = 0$. Then

$$X^{[1,1]} = X \setminus I_1(X) \subset X \setminus J_1(X) = X^{[[0,1]]},$$

as stated.

If we already know $X^{[2m-1,1]} \subset X^{[[m-1,1]]}$ for some $m \in \mathbb{N}$, then

$$X^{[2m,1]} = X^{[2m-1,1]} \setminus I_1\bigl(X^{[2m-1,1]}\bigr) \subset X^{[[m-1,1]]} \setminus I_1\bigl(X^{[[m-1,1]]}\bigr),$$

hence

$$\begin{aligned} X^{[2m+1,1]} &= X^{[2m,1]} \setminus I_1\bigl(X^{[2m,1]}\bigr) \\ &\subset \bigl(X^{[[m-1,1]]} \setminus I_1\bigl(X^{[[m-1,1]]}\bigr)\bigr) \setminus I_1\bigl(X^{[[m-1,1]]} \setminus I_1\bigl(X^{[[m-1,1]]}\bigr)\bigr). \end{aligned} \tag{3}$$

Since

$$J_1\bigl(X^{[[m-1,1]]} \setminus I_1\bigl(X^{[[m-1,1]]}\bigr)\bigr) \subset I_1\bigl(X^{[[m-1,1]]} \setminus I_1\bigl(X^{[[m-1,1]]}\bigr)\bigr),$$

from (3) we get

$$\begin{aligned} X^{[2m+1,1]} &\subset \bigl(X^{[[m-1,1]]} \setminus I_1\bigl(X^{[[m-1,1]]}\bigr)\bigr) \setminus J_1\bigl(X^{[[m-1,1]]} \setminus I_1\bigl(X^{[[m-1,1]]}\bigr)\bigr) \\ &= X^{[[m,1]]}. \end{aligned}$$

This proves the induction step and finishes the proof. □

4. Spaces not l-equivalent to the n-dimensional cube

In this section we prove Theorem 18, a quantitative version of the first part of Theorem 6. The proof, in spirit, is very similar to the proof of Theorem 2.5 in [4].

First we prove a lemma which is essential for the recursive step in the proof of the theorem, and which fails to work if we replace strong embedding derivative with embedding derivative.

Lemma 16. *Let X be a compact metric space and let X' be its compact subset which is n-dimensional everywhere. Let*

$$\mu : X \to C_p\big(I^n\big)^\star, \qquad \nu : I^n \to C_p(X)^\star$$

be continuous functions. Then for every $\varepsilon > 0$, every open set $U \subset X$ with $U \cap X' \neq \emptyset$ there is an open set $V \subset \operatorname{cl} V \subset U$, integers $0 \leqslant k \leqslant l$, a sequence of numbers $s_1, s_2, \ldots, s_l \in \mathbb{N}$ and a sequence of open sets $G_1, G_2, \ldots, G_l \subset I^n$ such that

(1) $V \cap X' \neq \emptyset$;
(2) $G_i \cap G_j = \emptyset$, $1 \leqslant i < j \leqslant l$;
(3) $\operatorname{card}\operatorname{supp}\mu(x) = l$ *for every* $x \in V \cap X'$;
(4) $\operatorname{card}(\operatorname{supp}\mu(x) \cap G_i) = 1$ *for every* $x \in V \cap X'$, $1 \leqslant i \leqslant l$;
(5) $\operatorname{card}(\operatorname{supp}\nu(\operatorname{supp}\mu(x) \cap G_i)) = s_i$ *for every* $x \in V \cap X'$, $1 \leqslant i \leqslant l$;
(6) *for every* $x \in V \cap X'$ *and* $y \in \operatorname{supp}\mu(x)$, $\operatorname{supp}\nu(y) \cap V \neq \emptyset$ *implies that* $y \in G_i$ *for some* $1 \leqslant i \leqslant k$ *and* $\operatorname{supp}\nu(y) \cap V = \{x\}$;
(7) *for every* $1 \leqslant i \leqslant k$ *and* $y \in G_i$, $\operatorname{supp}\nu(y) \cap V \neq \emptyset$ *implies*
$$\operatorname{supp}\nu(y) \cap V \subset X';$$
(8) *for every* $1 \leqslant i \leqslant k$, $u \in V$ *and* $u' \in V \cap X'$,
$$\sum_{y \in \operatorname{supp}\mu(u) \cap G_i} \big|\mu_y(u)\big| \geqslant \big|\mu_{\operatorname{supp}\mu(u') \cap G_i}(u')\big| - \frac{\varepsilon}{k};$$
(9) *for every* $1 \leqslant i \leqslant k$ *and nonempty open set* $W \subset V \cap X'$,
$$\bigcup_{x \in W} \operatorname{supp}\mu(x) \cap G_i$$
is n-dimensional.

Proof. Let $S \subset I^n$ be a dense countable set and consider the following subsets of $X' \cap U$. For a given $l \in \mathbb{N}$, sequence of numbers $s_1, s_2, \ldots, s_l \in \mathbb{N}$ and sequence of pairwise disjoint open balls $B_1, \ldots, B_l \subset I^n$ centered at S having rational radius let

$$U^{B_1,\ldots,B_l}_{s_1,s_2,\ldots,s_l} = \big\{x \in X' \cap U\colon\ \operatorname{card}\operatorname{supp}\mu(x) = l,\ \operatorname{card}\big(\operatorname{supp}\mu(x) \cap B_i\big) = 1,\\ \operatorname{card}\big(\operatorname{supp}\nu\big(\operatorname{supp}\mu(x) \cap B_i\big)\big) = s_i,\ 1 \leqslant i \leqslant l\big\}.$$

Since $\{U^{B_1,\ldots,B_l}_{s_1,s_2,\ldots,s_l}\}$ is a countable cover of the locally compact metric space $X' \cap U$, some

$$U_1 = U^{B_1,\ldots,B_l}_{s_1,s_2,\ldots,s_l}$$

is of second category. By Lemma 12 we have that U_1 has nonempty interior in X'.

We show that by passing to open subsets of U_1, we can find an open set $U_2 \subset X'$ such that

(1) for every fixed i, either

$$D_i = \bigcup_{x \in U_2} \operatorname{supp} \mu(x) \cap B_i$$

is at most $(n-1)$-dimensional, or for every nonempty open set $W \subset U_2$,

$$\bigcup_{x \in W} \operatorname{supp} \mu(x) \cap B_i$$

is n-dimensional;

(2) μ on U_2 and ν on the corresponding D_i, $1 \leqslant i \leqslant l$ are of the form as in the conclusion of Lemma 11;

(3) for every fixed i, the function $\operatorname{card}(\operatorname{supp} \nu(y) \cap X')$ is constant on D_i;

(4) for every fixed i and support function ν_i^j, $1 \leqslant j \leqslant s_i$ of $\nu|_{D_i}$, either

$$\nu_i^j\bigl(\operatorname{supp} \mu(x) \cap D_i\bigr) \cap \operatorname{cl} U_2 = \emptyset, \quad \forall x \in U_2 \quad \text{or}$$

$$\nu_i^j\bigl(\operatorname{supp} \mu(x) \cap D_i\bigr) \cap U_2 = \{x\}, \quad \forall x \in U_2.$$

Note that the sets D_i change as U_2 decreases.

To have property (1), it is enough to reduce U_2 whenever

$$\bigcup_{x \in W} \operatorname{supp} \mu(x) \cap B_i$$

is at most $(n-1)$-dimensional for some nonempty open set $W \subset U_2$ and $1 \leqslant i \leqslant l$.

It is clear that we can get property (2) by a successive Lemma 11.

Now we obtain property (3). Let

$$d_i = \min_{y \in D_i}\bigl(\operatorname{card} \operatorname{supp} \nu(y) \cap X'\bigr).$$

By property (2), the set

$$\bigl\{y \in D_i\colon \operatorname{card}(\operatorname{supp} \nu(y) \cap X') = d_i\bigr\}$$

is nonempty and open in D_i. So by passing, for every $1 \leqslant i \leqslant l$ successively, to a nonempty open subset we have that $\operatorname{card}(\operatorname{supp} \nu(y) \cap X')$ is constant on D_i, $1 \leqslant i \leqslant l$.

Finally we show how to get property (4). If

$$x_0 \neq \nu_i^j\bigl(\operatorname{supp} \mu(x_0) \cap D_i\bigr) \cap U_2$$

for some $x_0 \in U_2$, then by property (2),

$$N_{x_0} \cap \bigcup_{x \in N_{x_0}} \nu_i^j\bigl(\operatorname{supp} \mu(x_0) \cap D_i\bigr) \cap \operatorname{cl} U_2 = \emptyset$$

holds for a sufficiently small open neighborhood $N_{x_0} \subset U_2$ of x_0. By passing to such a neighborhood whenever it is possible, we obtain property (4).

We can suppose that $D_1, \dots, D_k$ are those sets which are n-dimensional.

We choose an open set $V_1 \subset X$ with $\operatorname{cl} V_1 \subset U$ such that $V_1 \cap X' \neq \emptyset$, $\operatorname{cl} V_1 \cap X' \subset U_2$. Let

$$W_i = \bigcup_{x \in (\operatorname{cl} V_1) \cap X'} \operatorname{supp} \mu(x) \cap B_i, \quad 1 \leqslant i \leqslant l.$$

By the property (2), these sets are continuous images of a compact set, so they are compact. For the same reason, for every $i = 1, \dots, l$, the sets

$$Z_i = \bigcup_{y \in W_i} \operatorname{supp} \nu(y)$$

are the union of s_i compact sets,

$$Z_i^j = \bigcup_{y \in W_i} \nu_i^j(y), \quad 1 \leqslant j \leqslant s_i,$$

respectively. As a weakening reformulation of property (4) of U_2, we have that for every $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant s_i$ either $Z_i^j = (\operatorname{cl} V_1) \cap X'$ or $Z_i^j \cap \operatorname{cl} U_2 = \emptyset$. Since the sets Z_i^j and $\operatorname{cl} U_2$ are all compact, we can find, by separating all the nonintersecting Z_i^j sets from $\operatorname{cl} U_2$, an open set $V_2 \subset \operatorname{cl} V_2 \subset V_1$ in X such that $V_2 \cap X' \neq \emptyset$ and $V_2 \cap Z_i^j = V_2 \cap Z_i = V_2 \cap X'$ for every $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant s_i$. Now let

$$G_j = \operatorname{int} \bigcup_{x \in (\operatorname{cl} V_2) \cap X'} \operatorname{supp} \mu(x) \cap B_j$$

for $1 \leqslant j \leqslant k$ and $G_j = B_j$ for $k + 1 \leqslant j \leqslant l$. We define the open set $V \subset V_2$ by

$$V = \{x \in V_2\colon \operatorname{supp} \mu(x) \cap G_j \neq \emptyset,\ 1 \leqslant j \leqslant k\} \tag{4}$$

if $k \neq 0$, while for $k = 0$ we set $V = V_2$.

First we show that the sets G_j and $V \cap X'$ are nonempty. It is clear for $k + 1 \leqslant j \leqslant l$. For $1 \leqslant j \leqslant k$, we know from Theorem 7 that the set

$$\Bigl(\bigcup_{x \in (\operatorname{cl} V_2) \cap X'} \operatorname{supp} \mu(x) \cap B_j\Bigr) \setminus G_j,$$

as a compact subset of I^n with empty interior, is at most $(n-1)$-dimensional, so by and property (1) and (2) of U_2 we have that G_j is dense in

$$\bigcup_{x \in (\operatorname{cl} V_2) \cap X'} \operatorname{supp} \mu(x) \cap B_j.$$

So again by property (1) and (2) of U_2, the sets

$$\{x \in V_2 \cap X'\colon \operatorname{supp} \mu(x) \cap G_j \neq \emptyset\}$$

are also dense open in $V_2 \cap X'$, so we have that $V \cap X' \neq \emptyset$. For $k = 0$ the statements are obvious, which proves conclusion (1).

By the continuity of μ, we can suppose in addition that conclusion (8) holds for V.

Conclusion (2), (3) and (5) hold since even the balls $B_1, \ldots, B_l$ has been chosen to satisfy them.

U_1 was defined to satisfy conclusion (4) for the balls $B_1, \ldots, B_l$, so conclusion (4) holds for $k+1 \leqslant i \leqslant l$, while $\operatorname{card}(\operatorname{supp}\mu(x) \cap G_i) \leqslant 1$ holds whenever $x \in V \cap X'$, $1 \leqslant i \leqslant k$. Taking into consideration (4), the definition of V guarantees that conclusion (4) holds.

To prove conclusion (6), let $x \in V \cap X'$, $y \in \operatorname{supp}\mu(x) \cap G_i$ and suppose that $\operatorname{supp}\nu(y) \cap V \neq \emptyset$. Since $V \subset V_2$, we have

$$\begin{aligned}\operatorname{supp}\nu(y) \cap V &\subset \operatorname{supp}\nu(y) \cap V_2\\ &= \operatorname{supp}\nu(y) \cap V_2 \cap X' \subset \operatorname{supp}\nu(y) \cap U_2 \neq \emptyset,\end{aligned}$$

so from property (4) of U_2 we have $\operatorname{supp}\nu(y) \cap V = \{x\}$. Again by property (4) of U_2,

$$\operatorname{supp}\nu\bigl(\operatorname{supp}\mu(x) \cap G_i\bigr) \cap V = \{x\}$$

holds for every $x \in V \cap X'$, that is the function

$$f : V \cap X' \to \bigcup_{x \in V \cap X'} \operatorname{supp}\mu(x) \cap G_i, \qquad f(x) = \operatorname{supp}\mu(x) \cap G_i$$

has a continuous inverse

$$g : \bigcup_{x \in V \cap X'} \operatorname{supp}\mu(x) \cap G_i \to V \cap X', \qquad g(y) = \operatorname{supp}\nu(y) \cap V.$$

Since $V \cap X'$ is n-dimensional and homeomorphism preserves dimension, we get from the definition of k that $i \leqslant k$, $y \in \operatorname{supp}\mu(x) \cap G_i$ and $\operatorname{supp}\nu(y) \cap V = \{x\}$. This proves conclusion (6).

By the definition of G_i, for every $1 \leqslant i \leqslant k$ and $y \in G_i$ there exists an $x \in (\operatorname{cl} V_2) \cap X'$ for which $y = \operatorname{supp}\mu(x) \cap G_i$. Since $V \subset V_2$, $\operatorname{supp}\nu(y) \cap V \neq \emptyset$ implies $\operatorname{supp}\nu(y) \cap V \subset X'$, which proves conclusions (7).

Conclusion (9) follows immediately from the definition of k, so the proof is complete. □

Corollary 17. *Let X, X', U, V, k, l and $G_1, \ldots G_l$ be as in Lemma* 16. *Then for any open set $B \subset I^n$ we can choose V and G_j, $1 \leqslant j \leqslant k$ such that for every fixed j either $G_j \subset B$ or $G_j \cap B = \emptyset$.*

Proof. Let $B_0 = B$, $B_1 = I^n \setminus \operatorname{cl} B$. For a $\sigma \in \{0,1\}^k$, let

$$C_\sigma = \bigl\{x \in V \cap X' \colon \operatorname{supp}\mu(x) \cap G_j \in B_{\sigma(j)},\ 1 \leqslant j \leqslant k\bigr\}.$$

The sets C_σ are open in X'. Since by Theorem 7 the boundary of B is at most $(n-1)$-dimensional, from conclusion (9) of Lemma 16 we have that

$$\bigcup_{\sigma \in \{0,1\}^k} C_\sigma$$

is dense in $V \cap X'$, in particular $C_\sigma \cap X'$ is nonempty for some $\sigma \in \{0,1\}^k$. We apply again Lemma 16 for an open set $U \subset V$ with $U \cap X' \neq \emptyset$, $U \cap X' \subset C_\sigma$. The resulting sets V and G_j, $1 \leqslant j \leqslant k$ obviously can be chosen to fulfill the requirements. □

Theorem 18. *Let X be an n-dimensional compact metric space, and let*

$$H : C_p(X) \to C_p\big(I^n\big)$$

be a linear homeomorphism. If $X^{[[m,n]]} \neq \emptyset$ for some $m \in \mathbb{N}$, then $L(H) \geqslant \sqrt{m}$.

Proof. Fix an $\eta > 0$. From Theorem 1 we know that every nonempty closed subset of X contains a relatively open nonempty subset which can be embedded into I^n, so we can apply Lemma 14 for X. Let $X_1, X_2, \ldots, X_m$ be the resulting sequence. We will define recursively a sequence of nonempty open sets

$$V_1 \subset \cdots \subset V_s \subset \cdots \subset V_m \subset X \tag{5}$$

and a sequence of finite families of open sets

$$\big\{G_i^j \subset I^n \colon 1 \leqslant i \leqslant q_j\big\}, \quad 1 \leqslant j \leqslant m,$$

such that

$$V_j \cap X_{m-j+1} \neq \emptyset, \quad 1 \leqslant j \leqslant m, \tag{6}$$

$$G_i^j \cap G_{i'}^{j'} = \emptyset \tag{7}$$

whenever $(i, j) \neq (i', j')$, and for every $1 \leqslant t \leqslant j \leqslant m$ and $x \in V_j$,

$$\sum_{i=1}^{q_t} \sum_{y \in G_i^t} \big|\mu_y(x)\big| \geqslant \frac{1}{L(H)} - \frac{\eta}{2^t}. \tag{8}$$

Once this done, using (7) and (8) we have

$$\big\|\mu(x)\big\| \geqslant \sum_{t=1}^{m} \sum_{i=1}^{q_t} \sum_{y \in G_i^t} \big|\mu_y(x)\big| \geqslant \frac{m}{L(H)} - \sum_{t=1}^{m} \frac{\eta}{2^t}$$

for every $x \in V_m$, so

$$\frac{m}{L(H)} - \eta \leqslant \big\|\mu(x)\big\| \leqslant L(H),$$

which, by taking $\eta \to 0$, implies $L(H) \geqslant \sqrt{m}$ and completes the proof.

Let $N_1 \subset X_m$ be open with compact closure. Now we apply Lemma 16 for $X' = \operatorname{cl} N_1$, $U = U_1 = X$ and $\varepsilon = \eta/2$. Let V_1 denote the resulting open set, k_1, l_1 be the resulting integers and $G_1, G_2, \ldots, G_{l_1}$ denote the resulting sequence of open sets. We set $q_1 = k_1$, $G_j^1 = G_j$, $1 \leqslant j \leqslant q_1$, and let

$$G^1 = \bigcup_{i=1}^{q_1} G_i^1.$$

Then (5) and (7) holds for $s = 1$ and $j = j' = 1$, $1 \leqslant i, i' \leqslant q_1$.

By conclusion (1) of Lemma 16 we have $V_1 \cap \operatorname{cl} N_1 \neq \emptyset$, so (6) holds for $j = 1$ and we can have an $x_1 \in V_1 \cap \operatorname{cl} N_1$. We show that

$$\sum_{y \in G^1} \big|\mu_y(x_1)\big| \geqslant \frac{1}{L(H)}. \tag{9}$$

By the Urysohn lemma, there is a function $f \in C_p(X)$ with

$$\|f\|_\infty = f(x_1) = 1$$

such that $\operatorname{supp} f \subset V_1$. From conclusion (6) of Lemma 16 we have

$$1 = f(x_1) = \sum_{y \in I^n} \mu_y(x_1)(Hf)(y) = \sum_{y \in G^1} \mu_y(x_1)(Hf)(y), \tag{10}$$

so from $\|Hf\|_\infty \leqslant L(H)$ we have (9).

Let $x \in V_1$ be arbitrary. Conclusion (8) of Lemma 16 for $u = x$, $u' = x_1$ implies that

$$\sum_{y \in G_i^1} \left|\mu_y(x)\right| \geqslant \left|\mu_{[\operatorname{supp}\mu(x_1)] \cap G_i^1}(x_1)\right| - \frac{\eta}{2q_1}, \quad 1 \leqslant i \leqslant q_1. \tag{11}$$

Thus from (9) we get that

$$\sum_{i=1}^{q_1} \sum_{y \in G_i^1} \left|\mu_y(x)\right| \geqslant \frac{1}{L(H)} - \frac{\eta}{2},$$

that is (8) holds for $t = 1$ whenever $x \in V_1$.

Suppose that $V_1, \ldots, V_{s-1}$ is already defined with the corresponding open sets G_i^j, $1 \leqslant i \leqslant q_j$, $1 \leqslant j \leqslant s-1$ satisfying (5), (6), (7) and (8). We define V_s and $G_1^s, \ldots, G_{q_s}^S$ with the required properties.

Since V_{s-1} is open, from conclusion (2) of Lemma 14 and from (6) with $j = s-1$ we have that

$$\operatorname{cl} X_{m-s+1} \cap V_{s-1} \neq \emptyset,$$

thus also

$$X_{m-s+1} \cap V_{s-1} \neq \emptyset.$$

So by conclusion (1) of Lemma 14 we can find an open subset N_s of the set $V_{s-1} \cap X_{m-s+1}$ with compact closure in $V_{s-1} \cap X_{m-s+1}$. We apply Lemma 16 for $X' = \operatorname{cl} N_s$ and $U = U_s = V_{s-1} \setminus \operatorname{cl} X_{m-s+2}$, $\varepsilon = \eta/2^s$. This can be done, since from conclusion (3) of Lemma 14 with $i = m-s+1$, $j = m-s+2$ we know that

$$\operatorname{cl} N_s \subset V_{s-1} \cap X_{m-s+1} \subset V_{s-1} \setminus \operatorname{cl} X_{m-s+2} = U_s,$$

in particular $U_s \cap \operatorname{cl} N_s$ is nonempty.

Let V_s denote the resulting open set, k_s, l_s be the resulting integers,

$$G_1, G_2, \ldots, G_{l_s}$$

denote the resulting sequence of open sets. By applying Corollary 17 with $B = G_i^t$ for every $1 \leqslant t \leqslant s-1$, $1 \leqslant i \leqslant q_t$ separately, V_s and G_j, $1 \leqslant j \leqslant k_s$ can be chosen such that for every $1 \leqslant t \leqslant s-1$, $1 \leqslant i \leqslant q_t$ and $1 \leqslant j \leqslant k_s$ either $G_j \subset G_i^t$ or $G_j \cap G_i^t = \emptyset$. Let $G_1^s, \ldots, G_{q_s}^s$ be the subsequence of $G_1, \ldots, G_{k_s}$ consisting of the sets disjoint of $G^1 \cup \cdots \cup G^{s-1}$, and let

$$G^s = \bigcup_{i=1}^{q_s} G_i^s.$$

We have that (5), (6) and (7) hold.

Let $x_s \in V_s \cap \operatorname{cl} N_s$. We prove that

$$\sum_{y \in G^s} \left|\mu_y(x_s)\right| \geqslant \frac{1}{L(H)}. \tag{12}$$

Consider a function $f \in C_p(X)$ satisfying $\|f\|_\infty = f(x_s) = 1$ and $\operatorname{supp} f \subset V_s$. From conclusion (6) of Lemma 16, we have

$$\begin{aligned} 1 = f(x_s) &= \sum_{y \in I^n} \mu_y(x_s)(Hf)(y) = \sum_{y \in G^1 \cup \cdots \cup G^s} \mu_y(x_s)(Hf)(y) \\ &\leqslant \left| \sum_{y \in G^1 \cup \cdots \cup G^{s-1}} \mu_y(x_s)(Hf)(y) \right| + \left| \sum_{y \in G^s} \mu_y(x_s)(Hf)(y) \right|. \end{aligned} \tag{13}$$

From $V_s \subset V_{s-1} \subset \cdots \subset V_1$ and conclusion (7) of Lemma 16 we know that if $y \in G^j$ for some $1 \leqslant j \leqslant s-1$, then

$$\operatorname{supp} \nu(y) \cap V_s \subset \operatorname{supp} \nu(y) \cap V_j \subset \operatorname{cl} N_j \subset \operatorname{cl} X_{m-j+1}.$$

Since by Lemma 14(2) we have $\operatorname{cl} X_{m-j+1} \subset \operatorname{cl} X_{m-s+2}$, and

$$V_s \subset U_s = V_{s-1} \setminus \operatorname{cl} X_{m-s+2},$$

we get $\operatorname{supp} \nu(y) \cap V_s = \emptyset$ whenever $y \in G^1 \cup \cdots \cup G^{s-1}$. Thus $(Hf)(y) = 0$ for every $y \in G^1 \cup \cdots \cup G^{s-1}$. This, $\|Hf\|_\infty \leqslant L(H)$ and (13) imply

$$\sum_{y \in G^s} \left|\mu_y(x_s)\right| \geqslant \frac{1}{L(H)}.$$

Just as in the case of V_1, conclusion (8) of Lemma 16 and $q_s \leqslant k_s$ imply that for every $x \in V_s$,

$$\sum_{y \in G_i^s} \left|\mu_y(x)\right| \geqslant \mu_{[\operatorname{supp} \mu(x_s)] \cap G_i^s}(x_s) - \frac{\eta}{2^s q_s}, \quad 1 \leqslant i \leqslant q_s. \tag{14}$$

Thus from (12) we get that

$$\sum_{i=1}^{q_s} \sum_{y \in G_i^s} \left|\mu_y(x)\right| \geqslant \frac{1}{L(H)} - \frac{\eta}{2^s}.$$

Since $V_s \subset V_j$ whenever $1 \leqslant j \leqslant s-1$, we have that (8) holds for $1 \leqslant t \leqslant s$ whenever $x \in V_s$. This proves the recursive step and completes the proof. □

We immediately get the first part of Theorem 6 as a corollary.

Corollary 19. *Suppose that the topological space X is l-equivalent to I^n. Then $X^{[[m,n]]} = \emptyset$ for some $m \in \mathbb{N}$.*

Proof. If

$$H : C_p(X) \to C_p\bigl(I^n\bigr)$$

is a linear homeomorphism, then by Theorem 18, any m with $m > L(H)^2$ fulfills the requirements. □

5. Conclusions

Combining the results stated in the introduction and proved here we obtain the following corollaries. As usual, ω denotes the first infinite ordinal.

Corollary 20. *Let X be a topological space. If X is l-equivalent to I^n, then*

(1) *X is compact, metrizable and n-dimensional*;
(2) *every nonempty closed subset of X contains a relatively open nonempty subset which can be embedded into I^n*;
(3) *$X^{[[m,n]]} = \emptyset$ for some $m < \omega$.*

Proof. The statement immediately follows from Theorem 1 and the first part of Theorem 6. □

The following corollary contains the second part of Theorem 6.

Corollary 21. *A topological space X is l-equivalent to I^1 if and only if*

(1) *X is one-dimensional, compact, metrizable*;
(2) *$X^{[[m,1]]} = \emptyset$ for some $m < \omega$.*

Proof. Since the class of compact metrizable spaces and dimension are preserved under l-equivalence, from Theorem 18 we get that the conditions are necessary, while Lemma 15 and Theorem 2 prove sufficiency. □

This corollary immediately implies that for compact metric spaces, the property of having empty mth one-dimensional strong embedding derivative for some finite m is preserved by l-equivalence.

Corollary 22. *Let X and Y be l-equivalent compact metric spaces . Then $X^{[[m,1]]} = \emptyset$ for some $m \in \mathbb{N}$ if and only if $Y^{[[m',1]]} = \emptyset$ for some $m' \in \mathbb{N}$.*

Theorem 6 also answers Problem 4.6. of [6].

Corollary 23. *For an ordinal α, the space $I^n \times [1, \alpha]$ is l-equivalent to I^n if and only if $\alpha < \omega^\omega$.*

As mentioned above, for $n \geqslant 2$ there are compact metric spaces which have empty mth n-dimensional strong embedding derivative for some finite m such that they do not vanish after taking a finite number of n-dimensional embedding derivatives. For these spaces our theorems do not work. As a typical representative of the problems this generates, we ask the following question. M_k^n denotes the k-dimensional Menger sponge in I^n (for the definition see [5, p. 121]).

Problem 24. Let $n \geqslant 2$ and consider the space

$$X = M_{n-1}^n \times \left[1, \omega^\omega\right] \cup I^n \times \left\{\omega^\omega + 1\right\}.$$

Is X l-equivalent to I^n?

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