Tamás Mátrai, Department of Mathematics, Central European University, Budapest, Nádor utca 9., H-1051 Hungary (e-mail: matrait@renyi.hu)

Imre Z. Ruzsa, Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Pf. 127, H-1364 Hungary (e-mail: ruzsa@renyi.hu)

A Characterization of Essentially Ejective Sets

Abstract

We give three equivalent properties characterizing the essentially ejective sets of a compact commutative topological group.

1 Introduction

Let G be a compact commutative topological group with the normalized Haar measure μ. Our aim is to characterize those subsets H of G for which

"no measurable subset of G can be periodic by every element of H,"

that is, if for a measurable set $A \subset G$ we have that $\mu((A+h) \setminus A)$ is "small" for every $h \in H$, then $\mu(A)$ or $\mu(G \setminus A)$ is also "small". In other words, every measurable set $A \subset G$ can be "ejected out" of itself by some element of H. This property is described in the following definition.

Definition 1.1. Let H be an arbitrary subset of G. The function

$$\zeta_H : [0,1] \to [0,1],$$

$$\zeta_H(x) = \inf_{\mu(A)=x} \sup_{h \in H} \mu((A+h) \setminus A) \tag{1}$$

is the *measure of ejectivity* of the set H.

If $\zeta_H(x) > 0$ for some $x \in [0,1]$, then we say that H is *ejective*, while if $\zeta_H(x) > 0$ for every $x \in (0,1)$, then H is called *essentially ejective*. If $\zeta_H(x) = 0$ holds for every $x \in [0,1]$, then we say that H is *nonejective*.

Key Words: ejective set, weak Dirichlet set, difference property
Mathematical Reviews subject classification: 28A99, 43A25

Since the function $h \mapsto \mu((A+h) \setminus A)$ is continuous, a set H is (essentially) ejective if and only if its closure, cl H has the same property, so we can and will restrict our attention to closed sets.

According to our best knowledge, the notion of ejectivity has been introduced in [2]. Among other things, the authors discuss there some basic arithmetical properties of the function ζ_H, study the measure of ejectivity of the whole and give a criterion for the ejectivity/essential ejectivity of general sets. In this paper we focus on essential ejectivity. Later on, we will give more concrete references to [2] indicating the similarity of certain ideas and techniques.

It turns out that the notion of nonejectivity is closely related to the weak Dirichlet property. In the definition of this class of sets, for a probability measure ν on G, $\hat{\nu}$ stands for its Fourier-transform, that is

$$\hat{\nu}(\gamma) = \int_G \gamma \mathrm{d}\nu$$

for every character γ. The principle character is denoted by γ_0.

Definition 1.2. A Borel set $H \subset G$ is a *weak Dirichlet set* if for every probability measure ν supported by cl H,

$$\sup_{\gamma \neq \gamma_0} |\hat{\nu}(\gamma)| = 1.$$

We also define a function to measure how far is H from being a weak Dirichlet set.

Definition 1.3. Let $H \subset G$ be a Borel set and let $\mathcal{M}(H)$ denote the set of probability measures supported on cl H. Then let

$$\psi(H) = \sup_{\nu \in \mathcal{M}(H)} \inf_{\gamma \neq \gamma_0} \int_G |\gamma - 1|^2 \mathrm{d}\nu.$$

It is easy to show that H is weak Dirichlet if and only if $\psi(H) = 0$. For more information about weak Dirichlet sets and their relation to other thin classes of harmonic analysis see e.g. [5] or [6].

It is not surprising that no function can be "periodic" by every element of an essentially ejective set. We introduce a function to describe this phenomenon. We will use

$$\mathcal{F} = \left\{ f : G \to \mathbb{C} \,\middle|\, f \in L_2(G), ||f||_{L_2(G)} = 1, \int_G f \mathrm{d}\mu = 0 \right\},$$

where $\left(L_2(G), ||.||_{L_2(G)}\right)$ denotes the Hilbert-space of square integrable functions with the usual norm. For a function f and $h \in G$,

$$\Delta_h f(x) = f(x+h) - f(x)$$

denotes the difference function of f.

Definition 1.4. For an arbitrary subset H of G, let

$$\xi(H) = \inf_{f \in \mathcal{F}} \sup_{h \in H} ||\Delta_h f||^2_{L_2(G)}.$$

Our main result is the following equivalence.

Theorem 1.5. *Let $H \subset G$ be a compact set. Then the following properties are equivalent:*

1. *H essentially ejective;*
2. *H is not weak Dirichlet;*
3. *$\xi(H) > 0$.*

This result will be proved through the following theorems. The first gives a quantitative form to the implication 2. $\to$ 1.

Theorem 1.6. *Let $H \subset G$ be a Borel set. Then*

$$\zeta_H(x) \geq \frac{\psi(H)}{2}\left(x - x^2\right), \tag{2}$$

specially every not weak Dirichlet Borel set is essentially ejective.

The second theorem states 2. $\leftrightarrow$ 3. in a quantitative way.

Theorem 1.7. *Let $H \subset G$ be a compact set. Then*

$$\xi(H) = \psi(H),$$

that is

$$\inf_{f \in \mathcal{F}} \sup_{h \in H} ||\Delta_h f||^2_{L_2(G)} = \sup_{\nu \in \mathcal{M}(H)} \inf_{\gamma \neq \gamma_0} \int_G |\gamma - 1|^2 \, d\nu.$$

Moreover, for every $\delta > 0$ there is a real valued $f \in \mathcal{F}$ with

$$\sup_{h \in H} ||\Delta_h f||^2_{L_2(G)} \leq \psi(H) + \delta$$

such that the Fourier-transform of f, $\mathfrak{F}f$ has finite support and real coefficients.

Finally, we would like to remark that a suitable analogue of Theorem 1.5 holds in arbitrary compact groups, without assuming commutativity. This was pointed out to the authors by the referee; and together with other possible generalizations will be discussed elsewhere.

We will use $G^\star$ to denote the character group of G and $\mu^\star$ for the Haar measure on $G^\star$ normed so that the constant in Plancherel's formula is 1. The Fourier-transform on G mapping $L_2(G)$ to $L_2(G^\star)$ will be denoted by $\mathfrak{F}$.

For a function $f : G \to \mathbb{R}$ and $a, b \in \mathbb{R}$, let

$$[a \leq f \leq b] = \{x \in G : a \leq f(x) \leq b\}.$$

The sets $[f \leq u]$, etc. are defined analogously.

For a set $K \subset G$, χ_K denotes the characteristic function of K.

2 Proof of Theorem 1.6

The proof uses a refined version of an idea of de Bruijn [1] proving the weak difference property for the L_2 class on the circle group $\mathbb{R}/\mathbb{Z}$. A similar proof can be found in [2], Theorem 4.1, [4], Theorem 3.1 and in [3], Theorem 3.2.

Let $H \subset G$ be a Borel set. To prove the inequality (2) we have to show that for every Borel set $A \subset G$,

$$\sup_{h\in H} \mu\left(\left(A+h\right)\setminus A\right) \geq \frac{\psi\left(H\right)}{2}\left(\mu\left(A\right)-\mu\left(A\right)^2\right)$$

holds.

We have

$$\chi_A\left(t-h\right)-\chi_A\left(t\right) = \begin{cases} 1, \text{ if } t\in\left(A+h\right)\setminus A, \\ -1, \text{ if } t\in A\setminus\left(A+h\right), \\ 0, \text{ otherwise }. \end{cases}$$

Consequently,

$$\int_G \left|\chi_A\left(t-h\right)-\chi_A\left(t\right)\right|^2 \mathrm{d}\mu\left(t\right) = 2\mu\left(\left(A+h\right)\setminus A\right). \tag{3}$$

Since for the Fourier transform of the functions χ_A and $\chi_A\left(Id-h\right)$ we have

$$\left(\mathfrak{F}\chi_A\right)\left(\gamma\right) = \int_G \chi_A\left(t\right)\gamma\left(t\right)\mathrm{d}\mu\left(t\right)$$

and

$$\begin{aligned}\left(\mathfrak{F}\chi_A\left(Id-h\right)\right)\left(\gamma\right) &= \int_G \chi_A\left(t-h\right)\gamma\left(t\right)\mathrm{d}\mu\left(t\right) = \\ &= \int_G \chi_A\left(t\right)\gamma\left(t+h\right)\mathrm{d}\mu\left(t\right) = \\ &= \gamma\left(h\right)\int_G \chi_A\left(t\right)\gamma\left(t\right)\mathrm{d}\mu\left(t\right) = \gamma\left(h\right)\left(\mathfrak{F}\chi_A\right)\left(\gamma\right),\end{aligned} \tag{4}$$

by applying Plancherel's identity to (3) we get

$$2\mu\left(\left(A+h\right)\setminus A\right) = \int_{G^\star} \left|\left(\mathfrak{F}\chi_A\right)\left(\gamma\right)\right|^2 \left|\gamma\left(h\right)-1\right|^2 \mathrm{d}\mu^\star(\gamma). \tag{5}$$

By the definition of $\psi(H)$, for every $\eta > 0$ there is a $\nu_\eta \in \mathcal{M}\left(H\right)$ such that

$$\inf_{\gamma\neq\gamma_0} \int_G \left|\gamma(h)-1\right|^2 \mathrm{d}\nu_\eta(h) \geq \psi\left(H\right)-\eta.$$

For an $\eta > 0$, by integrating (5) with respect to this ν_η we get

$$\int_G 2\mu\left((A+h)\setminus A\right)\mathrm{d}\nu_\eta(h) =$$
$$= \int_G \int_{G^\star} |(\mathfrak{F}\chi_A)(\gamma)|^2 |\gamma(h)-1|^2 \mathrm{d}\mu^\star(\gamma)\,\mathrm{d}\nu_\eta(h) =$$
$$= \int_{G^\star} |(\mathfrak{F}\chi_A)(\gamma)|^2 \int_G |\gamma(h)-1|^2 \mathrm{d}\nu_\eta(h)\,\mathrm{d}\mu^\star(\gamma) \geq$$
$$\geq (\psi(H)-\eta)\int_{G^\star\setminus\gamma_0} |(\mathfrak{F}\chi_A)(\gamma)|^2 \mathrm{d}\mu^\star(\gamma).$$

By Plancherel's formula we also know that

$$\int_{G^\star} |(\mathfrak{F}\chi_A)(\gamma)|^2 \mathrm{d}\mu^\star(\gamma) = \mu(A),$$

while

$$(\mathfrak{F}\chi_A)(\gamma_0) = \mu(A).$$

Since ν_η is a probability measure, the ν_η mean of $2\mu((A+h)\setminus A)$ smaller than its supremum, thus

$$\sup_{h\in H} \mu((A+h)\setminus A) \geq \frac{\psi(H)-\eta}{2}(\mu(A)-\mu^\star(\gamma_0)\mu(A)^2),$$

so according to our normalization

$$\sup_{h\in H} \mu((A+h)\setminus A) \geq \frac{\psi(H)-\eta}{2}(\mu(A)-\mu(A)^2),$$

Letting $\eta \to 0$ we get the desired result.■

3 Proof of Theorem 1.7

First we show that $\psi(H) \leq \xi(H)$. Let $f \in \mathcal{F}$. By Plancherel's identity we have

$$\int_{G^\star} |\mathfrak{F}f(\gamma)|^2 \mathrm{d}\mu^\star(\gamma) = ||f||^2_{L_2(G)} = 1 \qquad (6)$$

and

$$\int_{G^\star} |\mathfrak{F}f(\gamma)|^2\, |\gamma(h)-1|^2 \mathrm{d}\mu^\star(\gamma) = ||\mathfrak{F}\Delta_h f||^2_{L_2(G^\star)} =$$
$$= ||\Delta_h f||^2_{L_2(G)} \leq \sup_{h\in H} ||\Delta_h f||^2_{L_2(G)},$$

thus for every $\nu \in \mathcal{M}(H)$ we have

$$\begin{aligned} &\int_{G^\star} |\mathfrak{F}f(\gamma)|^2 \int_G |\gamma(h)-1|^2 \,\mathrm{d}\nu(h)\mathrm{d}\mu^\star(\gamma) = \\ &= \int_G \int_{G^\star} |\mathfrak{F}f(\gamma)|^2 \,|\gamma(h)-1|^2 \,\mathrm{d}\mu^\star(\gamma)\mathrm{d}\nu(h) \leq \sup_{h\in H} ||\Delta_h f||^2_{L_2(G)}. \end{aligned} \tag{7}$$

Since $\mathfrak{F}f(\gamma_0)=0$ for our $f \in \mathcal{F}$, from (6) and (7) we get

$$\inf_{\gamma\neq\gamma_0} \int_G |\gamma(h)-1|^2 \mathrm{d}\nu(h) \leq \sup_{h\in H} ||\Delta_h f||^2_{L_2(G)},$$

which proves

$$\sup_{\nu\in\mathcal{M}(H)} \inf_{\gamma\neq\gamma_0} \int_G |\gamma(h)-1|^2 \mathrm{d}\nu(h) \leq \inf_{f\in\mathcal{F}} \sup_{h\in H} ||\Delta_h f||^2_{L_2(G)}.$$

For the other direction, let $\delta > 0$ be fixed. From the definition of $\psi(H)$ we have

$$\inf_{\gamma\neq\gamma_0} \int_G |\gamma(h)-1|^2 \mathrm{d}\nu(h) < \psi(H) + \delta \tag{8}$$

for every $\nu \in \mathcal{M}(H)$. For every $\gamma \in G^\star$, let

$$\mathcal{M}_\gamma = \left\{ \nu \in \mathcal{M}(H) : \int_G |\gamma(h)-1|^2 \mathrm{d}\nu(h) < \psi(H) + \delta \right\}.$$

Then (8) implies that

$$\bigcup_{\gamma\in G^\star\setminus\{\gamma_0\}} \mathcal{M}_\gamma = \mathcal{M}(H).$$

Regard now $\mathcal{M}(H)$ as a subset of the dual space of continuous functions on H with the weak* topology. Since the functions $h \mapsto |\gamma(h)-1|^2$ are continuous,

$$\{\mathcal{M}_\gamma : \ \gamma \in G^\star \setminus \{\gamma_0\}\}$$

is an open cover of the compact set $\mathcal{M}(H)$, thus there is a finite set

$$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset G^\star \setminus \{\gamma_0\}$$

such that

$$\mathcal{M}(H) = \bigcup_{\gamma\in\Gamma} \mathcal{M}_\gamma,$$

that is

$$\inf_{\gamma\in\Gamma} \int_G |\gamma(h)-1|^2 \mathrm{d}\nu(h) < \psi(H) + \varepsilon, \ \forall \nu \in \mathcal{M}(H). \tag{9}$$

We assume that if Γ contains a character γ, then $\overline{\gamma} \in \Gamma$ as well.

Consider the Banach-space $(\mathbb{R}^n, ||.||_\infty)$ and its convex subset

$$K = \left\{ \left(\alpha \int_G |\gamma_1(t) - 1|^2 \mathrm{d}\nu(t), \alpha \int_G |\gamma_2(t) - 1|^2 \mathrm{d}\nu(t), \ldots \right.\right.$$
$$\left.\left. \ldots, \alpha \int_G |\gamma_n(t) - 1|^2 \mathrm{d}\nu(t) \right) : \nu \in \mathcal{M}(H), 0 \leq \alpha \leq 1 \right\}.$$

By (9), K is disjoint from the orthant $L = (\psi(H) + \varepsilon, \infty)^n$, so by the separation theorem of Hahn and Banach there is a linear functional

$$\phi \in (\mathbb{R}^n, ||.||_\infty)^\star = (\mathbb{R}^n, ||.||_1)$$

with

$$||\phi||_1 = 1 \tag{10}$$

such that

$$\phi(\underline{x}) \leq \phi(\underline{y}) \tag{11}$$

for every $\underline{x} \in K, \underline{y} \in L$. Let

$$\underline{y}_0 = (\psi(H) + \varepsilon, \psi(H) + \varepsilon, \ldots, \psi(H) + \varepsilon),$$

$$\underline{y}_i = (0, \ldots, 0, \underset{\substack{\smile \\ i}}{1}, 0, \ldots, 0), \ i = 1, \ldots, n.$$

We can identify ϕ with a sequence $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$. By applying (11) for $\underline{x} = \underline{0} = (0, \ldots, 0)$ and $\underline{y} = \underline{y}_0 + r\underline{y}_i$, $i = 1, \ldots, n$, $0 < r < \infty$ we get

$$0 = \phi(\underline{0}) \leq \phi(\underline{y}_0 + r\underline{y}_i) = \phi(\underline{y}_0) + rc_i, \ i = 1, 2, \ldots, n, \ 0 < r < \infty,$$

that is $0 \leq c_i, \ i = 1, \ldots n$.

Since $\|\phi\|_1 = 1$,

$$\sum_{i=1}^n c_i = \sum_{i=1}^n |c_i| = \|\phi\|_1 = 1, \tag{12}$$

while by applying (11) for $\underline{x} \in K$, $\underline{y} = \underline{y}_0$, from (11) and (12) we get

$$\int_G \sum_{i=1}^n c_i |\gamma_i(h) - 1|^2 \mathrm{d}\nu(h) < \psi(H) + \delta \tag{13}$$

for every $\nu \in \mathcal{M}(H)$.

If for a $\gamma \in \Gamma$ we have $\gamma = \gamma_i$, let $c(\gamma) = c_i$. With this convention let $f : G \to \mathbb{R}$ be defined as

$$f(h) = \sum_{\gamma \in \Gamma} \sqrt{\frac{c(\gamma) + c(\overline{\gamma})}{2}} \frac{\gamma + \overline{\gamma}}{2}.$$

This function f is obviously real valued, while from $\gamma_i \neq \gamma_0$ and (12) we get that

$$\int_G f(h) \mathrm{d}\mu(h) = (\mathfrak{F} f)(\gamma_0) = 0,$$

$$||f||^2_{L_2(G)} = ||\mathfrak{F} f||^2_{L_2(G^\star)} = \sum_{\gamma \in \Gamma} \frac{c(\gamma) + c(\overline{\gamma})}{2} = \sum_{i=1}^{n} c_i = 1,$$

while by applying (13) for the Dirac-measure $\nu = \delta_h$, $h \in H$ we have

$$||\Delta_h f||^2_{L_2(G)} = ||\mathfrak{F} \Delta_h f||^2_{L_2(G^\star)} =$$

$$= \sum_{\gamma \in \Gamma} |(\mathfrak{F} f)(\gamma)|^2 |\gamma(h) - 1|^2 = \sum_{\gamma \in \Gamma} \frac{c(\gamma) + c(\overline{\gamma})}{2} |\gamma(h) - 1|^2 =$$

$$= \sum_{i=1}^{n} c_i |\gamma_i(h) - 1|^2 = \int_G \sum_{i=1}^{n} c_i |\gamma_i(t) - 1|^2 \mathrm{d}\delta_h(t) < \psi(H) + \varepsilon$$

for every $h \in H$. Thus $f \in \mathcal{F}$, it is real valued, $\mathfrak{F} f$ has finite support and real coefficients, and

$$\sup_{h \in H} ||\Delta_h f||^2_{L_2(G)} < \psi(H) + \varepsilon.$$

Letting $\delta \to 0$ we obtain

$$\inf_{f \in \mathcal{F}} \sup_{h \in H} ||\Delta_h f||^2_{L_2(G)} \leq \sup_{\nu \in \mathcal{M}(H)} \inf_{\gamma \neq \gamma_0} \int_G |\gamma(h) - 1|^2 \mathrm{d}\nu(h),$$

which completes the proof.■

4 Proof of Theorem 1.5

In order to prove the remaining implication 1. $\to$ 3. we have to show that no set H with $\xi(H) = 0$ can be essentially ejective. We do this by proving that if $\xi(H) = 0$ then there is a constant $C_0 > 0$ such that for every $\eta > 0$ one can find a Borel set $A_\eta \subset G$ with

$$C_0 \leq \mu(A_\eta), \mu(G \setminus A_\eta) \tag{14}$$

and

$$\sup_{h \in H} \mu((A_\eta + h) \setminus A_\eta) \leq \eta. \tag{15}$$

This implies that H is not essentially ejective, since then for an appropriate sequence $\eta_j \to 0$ and sequence (A_{η_j}) of Borel sets we have that (14), (15) holds and $\mu(A_{\eta_j})$ converges to some $x \in [C_0, 1-C_0]$. For every j sufficiently large, by removing or adding a set to A_{η_j} with measure $|x - \mu(A_{\eta_j})|$, we can obtain a sequence (B_j) of Borel sets such that

$$\mu(B_j) = x,$$

$$\lim_{j\to\infty} \sup_{h\in H} \mu((B_j + h) \setminus B_j) = 0,$$

which shows that H cannot be essentially ejective.

The sets A_η will be sublevel sets of appropriate functions. Before constructing them, we prove two technical lemmas.

Lemma 4.1. *Let $H \subset G$ be so that $\xi(H) = 0$, and for a $\delta > 0$ let f satisfy the conclusions of Theorem 1.7. That is f is real valued,*

$$f(t) = \sum_{j=1}^{n} c_j \gamma_j(t),\ \ c_j \in \mathbb{R}, j = 1, \ldots, n, \tag{16}$$

satisfying

$$\int_G f(t) d\mu(t) = 0, \tag{17}$$

$$||f||^2_{L_2(G)} = \sum_{j=1}^{n} c_j^2 = 1, \tag{18}$$

and

$$||\Delta_h f||^2_{L_2(G)} \le \delta \tag{19}$$

for every $h \in H$. Then there is a $\rho \in [-1/4, 1/4]$ for which

$$\sup_{h\in H} \mu([f \le \rho] + h) \setminus [f \le \rho]) \le 5\delta^{\frac{1}{3}}. \tag{20}$$

Proof. For every $u \in \mathbb{R}$ let $A_u = [f \le u]$. Since $v \le f(x+h) - f(x)$ whenever $x \in A_u \setminus (A_{u+v} - h)$, from (19) we get that

$$\begin{aligned} \mu(A_u + h \setminus A_{u+v}) &= \int_{A_u\setminus(A_{u+v}-h)} 1 \ \mathrm{d}\mu(x) \le \\ &\le \int_{A_u\setminus(A_{u+v}-h)} \frac{|f(x+h) - f(x)|^2}{v^2} \mathrm{d}\mu(x) \le \frac{||\Delta_h f||^2_{L_2(G)}}{v^2} \le \frac{\delta}{v^2} \end{aligned} \tag{21}$$

for every $u, v \in \mathbb{R}$ and $h \in H$.

Let $l \in \mathbb{N}$, its value will be chosen later. Since

$$\mu(A_{\frac{1}{4}} \setminus A_{-\frac{1}{4}}) \le \mu(G) = 1,$$

we have that

$$\mu(A_{-\frac{1}{4}+\frac{k+1}{2l}} \setminus A_{-\frac{1}{4}+\frac{k}{2l}}) \leq \frac{1}{l} \tag{22}$$

for some $k \in \{0, 1, \ldots l-1\}$. Let

$$\rho = -\frac{1}{4} + \frac{k}{2l}$$

with this k. Then for every $h \in H$, using

$$(A_\rho + h) \setminus A_\rho \subset \Big((A_\rho + h) \setminus A_{\rho+\frac{1}{2l}}\Big) \cup \Big(A_{\rho+\frac{1}{2l}} \setminus A_\rho\Big),$$

from (21) with $u = \rho$, $v = \frac{1}{2l}$ and from (22) we have

$$\mu((A_\rho + h) \setminus A_\rho) \leq \delta 4l^2 + \frac{1}{l}. \tag{23}$$

For $l = \delta^{-\frac{1}{3}}$, this shows (20) and proves the lemma.■

Lemma 4.2. *Let $f \in L_2(G)$ be real valued of the form (16) satisfying (17) and (18). For every $\underline{\varepsilon} = (\varepsilon_j) \in \{-1, +1\}^n$, we define*

$$f_{\underline{\varepsilon}}(t) = \sum_{j=1}^{n} \varepsilon_j c_j \gamma_j(t).$$

Let $\mathcal{E}$ denote the set of those signings where $f_{\underline{\varepsilon}}$ is real valued, that is where a character and its conjugate get the same sign.

1. *For a couple $a, b \in \mathbb{R}$ with $0 < b < a$, let*

$$\begin{aligned}\Theta(x) = -(x-a+b)(x-a-b)(x+a)^2 = \\ = -x^4 + (b^2 + 2a^2)x^2 + 2ab^2x + a^2b^2 - a^4.\end{aligned}$$

   Then for every $\underline{\varepsilon} \in \mathcal{E}$,

$$\begin{aligned}b^2 + 2a^2 + a^2b^2 - a^4 - \int_G f_{\underline{\varepsilon}}^4(t)\,d\mu(t) \leq \\ \leq \left[\max_{x\in[a-b,a+b]} \Theta(x)\right] \min\big\{\mu([a-b \leq f_{\underline{\varepsilon}} \leq a+b]), \\ \mu([-a-b \leq f_{\underline{\varepsilon}} \leq -a+b])\big\}.\end{aligned}$$

2. *For an appropriate $\underline{\varepsilon}_0 \in \mathcal{E}$,*

$$\int_G f_{\underline{\varepsilon}}^4(t)\,d\mu(t) \leq 12. \tag{24}$$

Proof. 1. We estimate the integral

$$\int_G \Theta(f_{\underline{\varepsilon}}(t))\mathrm{d}\mu(t)$$

for every signing $\underline{\varepsilon} \in \mathcal{E}$.

Since $\Theta(x) > 0$ if and only if $x \in (a-b, a+b)$,

$$\int_G \Theta(f_{\underline{\varepsilon}}(t))\mathrm{d}\mu(t) \le \left[\max_{x\in[a-b,a+b]} \Theta(x)\right] \mu([a-b \le f_{\underline{\varepsilon}} \le a+b]).$$

On the other hand,

$$\int_G \Theta(f_{\underline{\varepsilon}}(t))\mathrm{d}\mu(t) = -\int_G f_{\underline{\varepsilon}}^4(t)\mathrm{d}\mu(t) + \\ + (b^2+2a^2)\int_G f_{\underline{\varepsilon}}^2(t)\mathrm{d}\mu(t) + 2ab^2\int_G f_{\underline{\varepsilon}}(t)\mathrm{d}\mu(t) + a^2b^2 - a^4.$$

Since by (17) and (18)

$$\int_G f_{\underline{\varepsilon}}(t)\mathrm{d}\mu(t) = 0, \quad \int_G f_{\underline{\varepsilon}}^2(t)\mathrm{d}\mu(t) = 1,$$

this means

$$b^2 + 2a^2 + a^2b^2 - a^4 - \int_G f_{\underline{\varepsilon}}^4(t)\mathrm{d}\mu(t) \le \\ \le \left[\max_{x\in[a-b,a+b]} \Theta(x)\right] \mu([a-b \le f_{\underline{\varepsilon}} \le a+b]).$$

Repeating the same calculation for $\widetilde{\Theta}(x) = \Theta(-x)$, we get

$$b^2 + 2a^2 + a^2b^2 - a^4 - \int_G f_{\underline{\varepsilon}}^4(t)\mathrm{d}\mu(t) \le \\ \le \left[\max_{x\in[-a-b,-a+b]} \widetilde{\Theta}(x)\right] \mu([-a-b \le f_{\underline{\varepsilon}} \le -a+b]) = \\ = \left[\max_{x\in[a-b,a+b]} \Theta(x)\right] \mu\left([-a-b \le f_{\underline{\varepsilon}} \le -a+b]\right)$$

which proves the statement.

2. In the following, for two characters γ and γ' we write $\gamma \asymp \gamma'$ if and only if $\gamma = \gamma'$ or $\gamma = \overline{\gamma'}$. We prove this part of the lemma by averaging on the signings in $\mathcal{E}$. We have

$$\int_G f_{\underline{\varepsilon}}^4(t)\mathrm{d}\mu(t) = \\ = \sum \varepsilon_{j_1}\varepsilon_{j_2}\varepsilon_{j_3}\varepsilon_{j_4}c_{j_1}c_{j_2}c_{j_3}c_{j_4}\int_G \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\mathrm{d}\mu(t) = \\ = \sum_{\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}=\gamma_0} \varepsilon_{j_1}\varepsilon_{j_2}\varepsilon_{j_3}\varepsilon_{j_4}c_{j_1}c_{j_2}c_{j_3}c_{j_4}. \tag{25}$$

If in a term of the right-hand side of (25) there is an index j_k such that $\gamma_{j_k} \not\asymp \gamma_{j_l}$ for $k \neq l$, then by symmetry the averaging on the signings in $\mathcal{E}$ cancels this term. If this is not the case, that is the indices can be matched on a way that the characters corresponding to the indices in a pair are $\asymp$-equivalent, then by the constraint that conjugate characters must get the same sign the term appears with multiplicity equal to $|\mathcal{E}|$, the cardinality of $\mathcal{E}$. Let J be the set of the index quartets (j_1, j_2, j_3, j_4) of the non-canceling terms satisfying $\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} = \gamma_0$, so we have

$$\frac{1}{|\mathcal{E}|}\sum_{\underline{\varepsilon}\in\mathcal{E}}\int_G f_{\underline{\varepsilon}}^4(t)\mathrm{d}\mu(t) = \sum_{(j_1,j_2,j_3,j_4)\in J} c_{j_1}c_{j_2}c_{j_3}c_{j_4}. \tag{26}$$

We show that

$$\sum_{(j_1,j_2,j_3,j_4)\in J} c_{j_1}c_{j_2}c_{j_3}c_{j_4} \leq 12\left(\sum_{j=1}^{n} c_j^2\right)^2. \tag{27}$$

This will complete the proof since combining (26) with (27), from (18) we get

$$\frac{1}{|\mathcal{E}|}\sum_{\underline{\varepsilon}\in\mathcal{E}}\int_G f_{\underline{\varepsilon}}^4(t)\mathrm{d}\mu(t) \leq 12\left(\sum_{j=1}^{n} c_j^2\right)^2 = 12,$$

so for at least one $\underline{\varepsilon}_0 \in \mathcal{E}$ we have

$$\int_G f_{\underline{\varepsilon}_0}^4(t)\mathrm{d}\mu(t) \leq 12.$$

To see (27), note that since f is real valued, $c_i = c_j$ if $\gamma_i \asymp \gamma_j$. To simplify our counting we change on both sides of (27) the indices corresponding to conjugate characters to the *smaller* index, so we can write

$$\sum_{(j_1,j_2,j_3,j_4)\in J} c_{j_1}c_{j_2}c_{j_3}c_{j_4} = \sum_{1\leq i<j\leq n} a_{i,j}c_i^2c_j^2 + \sum_{1\leq i\leq n} a_i c_i^4$$

and

$$\left(\sum_{j=1}^{n} c_j^2\right)^2 = \sum_{1\leq i<j\leq n} b_{i,j}c_i^2c_j^2 + \sum_{1\leq i\leq n} b_i c_i^4 \tag{28}$$

with appropriate coefficients a_i, b_i and $a_{i,j}, b_{i,j}$. That is, to prove (27) it is enough to show that $a_i \leq 12b_i$ and $a_{i,j} \leq 12b_{i,j}$ for every $1 \leq i, j \leq n$.

The computation of a_i, b_i, $a_{i,j}$ and $b_{i,j}$ is simple but lengthy, so we write out the details only for the b_i's. For the other coefficients, the calculations being similar, we present only the results.

By the choice of the indexing, $b_i = 0$ if and only if γ_i has a conjugate character of smaller index, that is $\overline{\gamma_i} = \gamma_j$ for some $j < i$. If γ_i is selfadjoint, c_i^4 appears with multiplicity one in (28), while if $\overline{\gamma_i} = \gamma_j$ for some $j > i$, then using $c_i = c_j$ we get that $c_i^4 = c_j^4 = c_i^2 c_j^2 = c_j^2 c_i^2$, so c_i^4 appears with multiplicity four in (28). To summarize the $b_i \neq 0$ case,

$$b_i = \begin{cases} 1, & \text{if } \gamma_i = \overline{\gamma_i}; \\ 4, & \text{if } \gamma_i \neq \overline{\gamma_i}. \end{cases}$$

Similarly, $b_{i,j} = 0$ if and only if γ_i or γ_j has a conjugate character of smaller index, else

$$b_{i,j} = \begin{cases} 2, & \text{if } \gamma_i = \overline{\gamma_i},\ \gamma_j = \overline{\gamma_j}; \\ 4, & \text{if } \gamma_i \neq \overline{\gamma_i},\ \gamma_j = \overline{\gamma_j}; \\ 4, & \text{if } \gamma_i = \overline{\gamma_i},\ \gamma_j \neq \overline{\gamma_j}; \\ 8, & \text{if } \gamma_i \neq \overline{\gamma_i},\ \gamma_j \neq \overline{\gamma_j}. \end{cases}$$

To estimate the a_i's and the $a_{i,j}$'s, consider the following partition of J:

$$J_1 = \{(j_1, j_2, j_3, j_4) \in J : \gamma_{j_1} \asymp \gamma_{j_2} \asymp \gamma_{j_3} \asymp \gamma_{j_4}\}$$

and $J_2 = J \setminus J_1$.

Take now a_i for some $1 \leq i \leq n$. These terms come from J_1. Again, $a_i = 0$ if γ_i has a conjugate character of smaller index, else

$$a_i \leq \begin{cases} 1, & \text{if } \gamma_i = \overline{\gamma_i}; \\ 16, & \text{if } \gamma_i \neq \overline{\gamma_i}. \end{cases}$$

Note that we can give only an upper bound given the restriction $(j_1, j_2, j_3, j_4) \in J$.

Similarly, $a_{i,j} = 0$ if γ_i or γ_j has conjugate character of smaller index, else

$$a_{i,j} \leq \begin{cases} 12, & \text{if } \gamma_i = \overline{\gamma_i},\ \gamma_j = \overline{\gamma_j}; \\ 24, & \text{if } \gamma_i \neq \overline{\gamma_i},\ \gamma_j = \overline{\gamma_j}; \\ 24, & \text{if } \gamma_i = \overline{\gamma_i},\ \gamma_j \neq \overline{\gamma_j}; \\ 96, & \text{if } \gamma_i \neq \overline{\gamma_i},\ \gamma_j \neq \overline{\gamma_j}. \end{cases}$$

So the inequalities $a_i \leq 12 b_i$ and $a_{i,j} \leq 12 b_{i,j}$ for every $1 \leq i, j \leq n$ hold. This finishes the proof.■

Now we turn to prove the existence of the set satisfying (14) and (15), so let $\eta > 0$ be fixed. By Theorem 1.7 there is a function $f \in L_2(G)$ satisfying (16), (17), (18) and (19) for $\delta = (\frac{\eta}{5})^3$. For every $\underline{\varepsilon} \in \mathcal{E}$, $f_{\underline{\varepsilon}}$ also satisfies these conditions. By Lemma 4.2 2., there is an $\underline{\varepsilon}_0 \in \mathcal{E}$ such that

$$\int_G f_{\underline{\varepsilon}_0}^4(t) \mathrm{d}\mu(t) \leq 12.$$

This, by Lemma 4.2, 1. for $a = 4$ and $b = 3.75$ implies that

$$3 \leq b^2 + 2a^2 + a^2b^2 - a^4 - \int_G f^4_{\underline{\varepsilon}_0}(t)\mathrm{d}\mu(t) \leq$$
$$\leq \left[\max_{x\in[a-b,a+b]} \Theta(x)\right] \min\left\{\mu\left(\left[\frac{1}{4} \leq f_{\underline{\varepsilon}_0} \leq 7 + \frac{1}{4}\right]\right),\right.$$
$$\left.\mu\left(\left[-7 - \frac{1}{4} \leq f_{\underline{\varepsilon}_0} \leq -\frac{1}{4}\right]\right)\right\},$$

that is with

$$C_0 = \frac{3}{\max\limits_{x\in[a-b,a+b]} \Theta(x)}$$

we have

$$C_0 \leq \mu\left(\left[f_{\underline{\varepsilon}_0} \leq -\frac{1}{4}\right]\right),\ \mu\left(G \setminus \left[f_{\underline{\varepsilon}_0} \leq \frac{1}{4}\right]\right). \tag{29}$$

We can apply Lemma 4.1 for $f_{\underline{\varepsilon}_0}$ to obtain a $\rho \in [-\frac{1}{4}, \frac{1}{4}]$ for which

$$\sup_{h\in H} \mu\left(\left([f_{\underline{\varepsilon}_0} \leq \rho] + h\right) \setminus [f_{\underline{\varepsilon}_0} \leq \rho]\right) \leq 5\delta^{\frac{1}{3}} = \eta.$$

For $A_\eta = [f_{\underline{\varepsilon}_0} \leq \rho]$, this shows (15), while (14) follows from (29), since for $\rho \in [-\frac{1}{4}, \frac{1}{4}]$ we have

$$\left[f_{\underline{\varepsilon}_0} \leq -\frac{1}{4}\right] \subset A_\eta = [f_{\underline{\varepsilon}_0} \leq \rho] \subset \left[f_{\underline{\varepsilon}_0} \leq \frac{1}{4}\right].$$

So the proof is complete.■

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