Tamás Keleti, Department of Analysis, Eötvös Loránd University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary (e-mail: elek@cs.elte.hu)
Tamás Mátrai, Department of Mathematics, Central European University, Budapest, Nádor utca 9., H-1051 Hungary (e-mail: matrait@renyi.hu)

A nowhere convergent series of functions which is somewhere convergent after a typical change of signs *

Abstract

On any uncountable Polish space we construct a sequence of continuous functions (f_n) such that $\sum f_n$ is divergent everywhere, but for a typical sign sequence $(\varepsilon_n) \in \{-1,+1\}^{\mathbb{N}}$, the series $\sum \varepsilon_n f_n$ is convergent in at least one point. This answers a question of S. Konyagin in negative.

1 Introduction

Let X be a topological space, $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of continuous functions. One can ask for a condition on the order of magnitude of the sequence (f_n) which guarantees that for a "typical" choice of signs $\varepsilon_n \in \{-1,+1\}$, the signed series $\sum \varepsilon_n f_n$ diverges everywhere on X.

Such conditions are known for Fourier and Dirichlet series if "typical" means for almost every choice of signs in the product probability space $\Omega = \{-1,+1\}^{\mathbb{N}}$ (see [2], [1]). However, in this note we consider Ω as a product of discrete topological spaces and "typical" is understood in categorical sense.

In [1, Theorem 4.1] for $X = \mathbb{R}$ a condition on the divergence of the partial sums of $\sum f_n$ was given implying that $\sum \varepsilon_n f_n$ diverges everywhere for a dense G_δ set of sign sequences $(\varepsilon_n) \in \Omega$. Motivated by this result, S. Konyagin asked whether, in case of compact metric spaces X, the pure fact that $\sum f_n$ diverges everywhere could imply that $\sum \varepsilon_n f_n$ diverges everywhere for dense G_δ, hence residual set of sign sequences. We give a negative answer by the following example, which is the main result of this note.

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Theorem 1. *Consider $\mathcal{C} = \{-1, 0, 1\}^{\mathbb{N}}$ as the topological product of the discrete spaces (which is clearly homeomorphic to the Cantor set). There exists a sequence of continuous functions $f_n : \mathcal{C} \to [-1, 1]$ and a dense G_δ set $\Omega_0 \subset \Omega = \{-1, +1\}^{\mathbb{N}}$ such that the series $\sum f_n$ diverges everywhere on $\mathcal{C}$, but for every $(\varepsilon_n) \in \Omega_0$, the series $\sum \varepsilon_n f_n$ converges in at least one point of $\mathcal{C}$.*

Then we can easily get examples on any uncountable Polish space (so in particular on $\mathbb{R}$) as well:

Corollary 2. *On any uncountable Polish space (X, d) there exist a sequence of continuous functions $g_n : X \to \mathbb{R}$ such that $\sum g_n$ diverges everywhere on X but the sign sequences $(\varepsilon_n) \in \Omega = \{-1, 1\}^{\mathbb{N}}$ for which $\sum \varepsilon_n g_n$ diverges everywhere on X form a set of first category in Ω.*

Proof. It is well known (see e.g. in [3, Corollary 6.5]) that any uncountable Polish space contains a homeomorphic copy C of a Cantor set. Let $f_n : C \to [-1, 1]$ be the sequence of functions on C we get by Theorem 1, and for any $n \in \mathbb{N}$ let $\tilde{f}_n : X \to [-1, 1]$ be a continuous extension of f_n to X. Then the sequence of functions $g_n(x) = \tilde{f}_n(x) + n \cdot d(x, C)$ on X (where $d(x, C)$ denotes the distance of x from C) has all the required properties. □

Notation. In this note G_δ stands for the class of those sets that can be obtained as countable intersection of open sets; $\mathbb{N}$ and $\mathbb{R}^+$ stands for the set of nonnegative integers and nonnegative reals, respectively. On finite sets (e.g. $\{-1, 1\}$ or $\{-1, 0, 1\}$) the topology we consider is always the discrete topology. By Polish space we mean complete separable metric space.

2 The example

In this section we prove Theorem 1.
For each fixed $a = (a_j) \in \mathcal{C} = \{-1, 0, 1\}^{\mathbb{N}}$ we define the sequence $(f_n(a))$ together with a sequence $(m_k(a))$ by induction:
Let $m_0(a) = 0$. Suppose that $k \in \mathbb{N}$ and the numbers $m_0(a) < \ldots < m_k(a)$ and $f_0(a), \ldots, f_{m_k(a)-1}(a)$ are already defined. Then let

$$f_{m_k(a)}(a) = f_{m_k(a)+1}(a) = \ldots = f_{m_k(a)+2^k-1}(a) = \frac{1}{2^k}, \tag{1}$$

$$m_{k+1}(a) = \min\{j \geq m_k(a) + 2^k : a_j = 0\}, \tag{2}$$

$$f_n(a) = \frac{a_n}{2^k} \qquad \text{for } m_k(a) + 2^k \leq n < m_{k+1}(a). \tag{3}$$

(If $\{j \geq m_k(a) + 2^k : a_j = 0\}$ is empty then $m_{k+1}(a) = \infty$ and after defining $f_n(a) = \frac{a_n}{2^k}$ for every $n \geq m_k(a) + 2^k$ the procedure terminates.)

Claim 1. *Every function f_n ($n \in \mathbb{N}$) is continuous on $\mathcal{C}$.*

Proof. This is clear since $f_n(a)$ depends only on $a_1, \ldots, a_n$. □

Claim 2. *The series $\sum f_n(a)$ diverges for every $a \in \mathcal{C}$*

Proof. If $m_{k+1}(a) = \infty$ for some $k \in \mathbb{N}$ then $|f_n(a)| = 2^{-k}$ for every $n \geq m_k + 2^k$, so $f_n(a)$ does not even converge to zero. Otherwise - by (1) - infinitely many blocks of sum 1 appears in $\sum f_n(a)$, so it cannot be convergent. □

Put

$$\Omega_0 = \bigcap_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{j=m}^{m+k} \{(\varepsilon_n) \in \{-1,+1\}^{\mathbb{N}} : \varepsilon_j = (-1)^j\}. \tag{4}$$

Claim 3. *The set Ω_0 is dense G_δ in the product space $\{-1,+1\}^{\mathbb{N}}$.*

Proof. This is clear since $\{(\varepsilon_n) \in \{-1,+1\}^{\mathbb{N}} : \varepsilon_j = (-1)^j\}$ is open for any j and $\bigcup_{m \in \mathbb{N}} \bigcap_{j=m}^{m+k} \{(\varepsilon_n) \in \{-1,+1\}^{\mathbb{N}} : \varepsilon_j = (-1)^j\}$ is dense for any k. □

Claim 4. *For every $(\varepsilon_n) \in \Omega_0$ there exist an $a \in \mathcal{C}$ such that $\sum \varepsilon_n f_n(a)$ converges.*

Proof. For a fixed $(\varepsilon_n) \in \Omega_0$ let

$$J = \{j \in \mathbb{N} \ : \varepsilon_j = (-1)^j\}. \tag{5}$$

Since $(\varepsilon_n) \in \Omega_0$, the set J contains arbitrarily long finite sequences of consecutive integers. Thus there exists a sequence $0 = m_0 < m_1 < \ldots$ such that

$$m_{k+1} \geq m_k + 2^k \quad \text{and} \tag{6}$$

$$m_k, m_k + 1, \ldots, m_k + 2^k - 1 \in J \quad (\forall k \in \mathbb{N}). \tag{7}$$

Let

$$a_j = \begin{cases} 0 & \text{if } j = m_k \text{ for some } k \in \mathbb{N} \\ (-1)^j / \varepsilon_j & \text{otherwise} \end{cases} \tag{8}$$

We have $m_k(a) = m_k$ ($k \in \mathbb{N}$) since $m_0 = 0$ and the sequence (m_k) satisfies (2).
For every $k \in \mathbb{N}$ and $m_k \leq j < m_k + 2^k$ we have (by (1)) that $f_j(a) = 1/2^k$ and (by (7) and (5)) that $\varepsilon_j = (-1)^j$, thus $\varepsilon_j f_j(a) = (-1)^j / 2^k$.
For every $k \in \mathbb{N}$ and $m_k + 2^k \leq j < m_{k+1}$ we have (by (3)) that $f_j(a) = a_j / 2^k$ and (by (8)) that $a_j = (-1)^j / \varepsilon_j$, thus again $\varepsilon_j f_j(a) = (-1)^j / 2^k$.
Therefore $\sum \varepsilon_n f_n(a)$ is a Leibniz series, so it is convergent. □

The four Claims above (together with the clear fact that, by definition, every f_n maps into $[-1, 1]$) complete the proof of Theorem 1.

References

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