HOMOTOPY TYPES OF BOX COMPLEXES OF CHORDAL GRAPHS

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Abstract. We show that the box complex of a chordal graph is homotopy equivalent to a wedge of spheres. This complements the result that in general the box complex of a graph can be any free $\mathbb{Z}_2$-complex up to homotopy. The tools used to study the homotopy types of box complexes of chordal graphs might be of independent interest.

1. Introduction

Box complexes were introduced in [15], where Matoušek and Ziegler compared various topological lower bounds for the chromatic number. Lovász's original lower bound for the chromatic number [13] can be reformulated as $\chi(G) \geq \mathrm{ind}(\mathsf{B}(G)) \geq \omega(G)$. After that many interesting properties of box complexes were found. Kahle [9] studied the homology of box complexes of random graphs. Simonyi and Tardos [16] obtained a topological lower bound for the local chromatic number of a graph using box complexes. It was shown that box complexes are universal [4], for any simplicial complex with a free $\mathbb{Z}_2$-action there is a graph such that its box complex is (simple [6]) $\mathbb{Z}_2$-homotopy equivalent to the complex. This motivated Kamibeppu and Kawamura [10] to study a special graph class. They showed that the homology groups of box complexes of chordal graphs (every cycle of length at least 4 has a chord) are torsion free. Here we explain their discovery on the homotopy level. We show that box complexes of chordal graphs are homotopy equivalent to wedges of spheres. In Section 2 we review some definitions and collect the necessary tools for the proofs. In Section 3 we introduce a variation of the Fold Lemma; and show that the box complex of a graph is homotopy equivalent to the union of the box complexes of induced subgraphs. In Section 4 we prove the main theorem using the structure theorem of chordal graphs and the previous tools.

2. Preliminaries

In this section we recall some basic facts of graphs, simplicial complexes, and posets to fix notation. The interested reader is referred to [14, 2, 12] for more details.

Any graph G considered will be assumed to be finite, simple, connected, and undirected, i.e. G is given by a finite set $V(G)$ of *vertices* and a set of *edges* $E(G) \subseteq \binom{V(G)}{2}$. We will denote by $\omega(G)$ the size of the largest clique (complete subgraph) of G. In a graph G, the *common neighborhood* of $A \subseteq V(G)$, $\mathrm{N}_G(A)$ is the set of vertices which are adjacent to every vertex of A. If it is clear which G is meant, we just write $\mathrm{N}(A)$. For two disjoint sets of vertices $A, B \subseteq V(G)$ we define $G[A,B]$ as the (not necessarily induced) bipartite subgraph of G with $V(G[A,B]) = A \cup B$ and $E(G[A,B]) = \{\{a,b\} \in E(G) \colon a \in A, b \in B\}$.

A $\mathbb{Z}_2$-*space* is a pair (X,ν) where X is a topological space and $\nu\colon X \to X$, called the $\mathbb{Z}_2$-action, is a homeomorphism such that $\nu^2 = \nu \circ \nu = \mathrm{id}_X$. If (X_1,ν_1) and (X_2,ν_2) are $\mathbb{Z}_2$-spaces, a $\mathbb{Z}_2$-*map* between them is a continuous mapping $f\colon X_1 \to X_2$ such that

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$f \circ \nu_1 = \nu_2 \circ f$. The sphere S^n is understood as a $\mathbb{Z}_2$-space with respect to the antipodal homeomorphism $x \to -x$. A $\mathbb{Z}_2$-complex is *free* if the $\mathbb{Z}_2$-action ν has no fixed point. The *$\mathbb{Z}_2$-index* of a $\mathbb{Z}_2$-space (X, ν) is defined by $\mathrm{ind}(X) := \min\{n\colon \text{ there is a } \mathbb{Z}_2\text{-map } X \to S^n\}$.

A *simplicial complex* K is a finite hereditary set system.

The *neighborhood complex* (introduced in [13]) is $\mathcal{N}(G) = \{S \subseteq V(G)\colon \mathrm{N}(S) \neq \emptyset\}$.

For sets A, B define $A \uplus B := \{(a, 1)\colon a \in A\} \cup \{(b, 2)\colon b \in B\}$. The *box complex* $\mathsf{B}(G)$ of a graph G (the one introduced by Matoušek and Ziegler [15]) is defined by

$$\mathsf{B}(G) := \left\{ A \uplus B \ : \begin{array}{l} A, B \subseteq V(G),\ A \cap B = \emptyset,\ G[A, B] \text{ is complete} \\ \text{bipartite},\ \mathrm{N}(A) \neq \emptyset \neq \mathrm{N}(B) \end{array} \right\}.$$

The vertices of the box complex are $V_1 := \{v \uplus \emptyset\colon v \in V(G)\}$ and $V_2 := \{\emptyset \uplus v\colon v \in V(G)\}$. The subcomplexes of $\mathsf{B}(G)$ induced by V_1 and V_2 are disjoint subcomplexes of $\mathsf{B}(G)$ that are both isomorphic to the neighborhood complex $\mathcal{N}(G)$. It is known that the neighborhood complex $\mathcal{N}(G)$ and the box complex $\mathsf{B}(G)$ are (simple [6]) homotopy equivalent [4].

A different *box$_0$ complex* $\mathsf{B}_0(G)$ [15] is defined by:

$$\mathsf{B}_0(G) \ = \{A \uplus B\colon A, B \subseteq V(G),\ A \cap B = \emptyset,\ G[A, B] \text{ is complete bipartite}\}.$$

Clearly $\mathsf{B}(G)$ is the subcomplex of $\mathsf{B}_0(G)$. It is known [4] that $\mathsf{B}_0(G)$ is homotopy equivalent to the suspension of $\mathsf{B}(G)$.

Examples. For the complete graph K_n its neighborhood complex $\mathcal{N}(K_n)$ is the boundary complex of the $(n-1)$-dimensional simplex. Its box$_0$ complex $\mathsf{B}_0(K_n)$ is the boundary complex of the n-dimensional cross polytope; while $\mathsf{B}(K_n)$ is the boundary complex of the n-dimensional cross polytope, with two opposite facets removed.

We recall that a *partially ordered set*, or *poset* for short, is a pair $(P, \preceq)$, where P is a set and $\preceq$ is a binary relation on P that is reflexive ($x \preceq x$), transitive ($x \preceq y$ and $y \preceq z$ imply that $x \preceq z$), and weakly antisymmetric ($x \preceq y$ and $y \preceq x$ imply $x = y$). When the order relation $\preceq$ is understood, we say only "a poset P." If $x \preceq y$ and $x \neq y$ then we write $x \prec y$. The *face poset* of a simplicial complex K is a set containing the simplices of K and the binary relation corresponds to containment.

We introduce the basics of Discrete Morse Theory, which was invented by Forman [7]. It provides a convenient language for describing homotopy equivalences.

Definition 1. *Let P be a poset with the order relation $\succeq$.*

- *We define a* partial matching *on P to be a set $\Sigma \subseteq P$, and an injective map $\mu : \Sigma \to P \setminus \Sigma$, such that $\mu(x) \succ x$, for all $x \in \Sigma$, and there is no c such that $\mu(x) \succ c \succ x$.*
- *The elements of $P \setminus (\Sigma \cup \mu(\Sigma))$ are called* critical.
- *Additionally, such a partial matching μ is called* acyclic *if there exists no sequence of distinct elements $x_1, \ldots, x_t \in \Sigma$, where $t \geq 2$, satisfying $\mu(x_1) \succ x_2$, $\mu(x_2) \succ x_3$, $\ldots$, $\mu(x_t) \succ x_1$.*

In many cases it is more convenient to define μ such that $\mu(x) \succ x$ or $\mu(x) \prec x$, and change the definition of the cycle accordingly. Since μ is injective we can assume that $\mu(\mu(x)) = x$. We will use the main theorem of Discrete Morse Theory.

Theorem 2 ([12, Theorem 11.13])**.** *Let K be a simplicial complex, and let M be an acyclic matching on the face poset of K. If the critical cells form a subcomplex K_c of K, then K_c and K are homotopy equivalent.*

3. Homotopy tools

The following lemma will be useful to understand the homotopy type of box complexes.

Lemma 3. *Let G be the union of two induced subgraphs G_1 and G_2. Then $\mathsf{B}(G)$ is homotopy equivalent to $\mathsf{B}(G_1) \cup \mathsf{B}(G_2)$.*

Proof. Clearly $\mathsf{B}(G_1) \cup \mathsf{B}(G_2) \subseteq \mathsf{B}(G)$. Let $S := \mathsf{B}(G) \setminus (\mathsf{B}(G_1) \cup \mathsf{B}(G_2))$ be the collection of simplices of $\mathsf{B}(G)$ which are not simplices of $\mathsf{B}(G_1) \cup \mathsf{B}(G_2)$. Let $L := G_1 \cap G_2$ be the intersection graph of G_1 and G_2. (L is the induced subgraph on $V(G_1) \cap V(G_2)$.) Let $A_1 \uplus A_2$ be a simplex of S. Because of the definition of S, $A_1 \cup A_2$ must contain a vertex $x \in V(G_1) \setminus V(L)$ and also a $y \in V(G_2) \setminus V(L)$. $x \in A_i$ and $y \in A_{3-i}$ ($i = 1$ or 2) cannot happen, since $xy \notin E(G)$. If $x, y \in A_i$ then A_{3-i} must be a subset of $V(L)$ since $G[A_1, A_2]$ is complete bipartite. So we obtained that if $A_1 \uplus A_2 \in S$ then either

(1) $A_1 \subseteq V(L)$, $A_2 \cap (V(G_1) \setminus V(L)) \neq \emptyset$ and $A_2 \cap (V(G_2) \setminus V(L)) \neq \emptyset$, or
(2) $A_2 \subseteq V(L)$, $A_1 \cap (V(G_1) \setminus V(L)) \neq \emptyset$ and $A_1 \cap (V(G_2) \setminus V(L)) \neq \emptyset$.

Assume that $V(L) = \{1, 2, \ldots, l\}$. Let $A \subset V(G)$ be such that $\mathrm{N}_G(A) \cap V(L) \neq \emptyset$. For such an A we define a vertex of $V(L)$ as $f(A) = \min\{\mathrm{N}_G(A) \cap V(L)\}$. We are now ready to define a matching on the simplices of $\mathsf{B}(G)$. If $A_1 \uplus A_2 \in S$ fulfills (1), then let

$$\mu(A_1 \uplus A_2) := \begin{cases} (A_1 \setminus f(A_2)) \uplus A_2 & \text{if } f(A_2) \in A_1, \\ (A_1 \cup f(A_2)) \uplus A_2 & \text{if } f(A_2) \notin A_1. \end{cases}$$

If $A_1 \uplus A_2 \in S$ fulfills (2), then symmetrically let

$$\mu(A_1 \uplus A_2) := \begin{cases} A_1 \uplus (A_2 \setminus f(A_1)) & \text{if } f(A_1) \in A_2, \\ A_1 \uplus (A_2 \cup f(A_1)) & \text{if } f(A_1) \notin A_2. \end{cases}$$

This matching is well defined. $f(A_i)$ is defined in the above cases, since $\mathrm{N}_G(A_i) \supseteq A_{3-i}$. We show that the obtained matching μ is acyclic. If we try to create a cycle, then we can assume that the first two simplices are $A_1 \uplus A_2$ and $\mu(A_1 \uplus A_2) = (A_1 \cup f(A_2)) \uplus A_2$. In this case $A_1 \subseteq V(L)$. When we go up by the matching we can only add a vertex of $V(L)$, so this cycle must be among simplices fulfilling (1). Now if we go down we have to delete something. If we delete a vertex from A_2 then since later we can not add by a matching anything to the second set we can not have a cycle. If we delete a vertex from A_1 then the resulting simplex of S is matched down by deleting $f(A_2)$.
The critical simplices of this matching are the simplices of $\mathsf{B}(G_1) \cup \mathsf{B}(G_2)$ so Theorem 2 completes the proof. □

The homology version for a special case was proved in [10]. In [10] the intersection of G_1 and G_2 was a complete graph K_l, and it was assumed that $\mathrm{N}_{G_1}(K_l) \neq \emptyset$ and $\mathrm{N}_{G_2}(K_l) \neq \emptyset$.

Clearly $\mathsf{B}(G_1 \cap G_2) \subseteq \mathsf{B}(G_1) \cap \mathsf{B}(G_2) \subseteq \mathsf{B}_0(G_1 \cap G_2)$. If $\mathrm{N}_{G_1}(G_1 \cap G_2) \neq \emptyset$ and $\mathrm{N}_{G_2}(G_1 \cap G_2) \neq \emptyset$, then $\mathsf{B}(G_1) \cap \mathsf{B}(G_2) = \mathsf{B}_0(G_1 \cap G_2)$.
We remark that the above lemma is not true if we just replace the box complex with the neighborhood complex, though they are homotopy equivalent.

Definition 4 (fold)**.** *$G - w$ is called a* fold *of a graph G if there exist $v \in V(G)$, $v \neq w$ such that $\mathrm{N}(v) \supseteq \mathrm{N}(w) \neq \emptyset$.*

It is known [3, 5, 11] that fold does not change the homotopy type of the box complex of graphs.

Lemma 5 (Fold Lemma). *Let G be a graph and $v, w \in V(G)$. If $\mathrm{N}(v) \supseteq \mathrm{N}(w)$ then $\mathsf{B}(G)$ is homotopy equivalent to $\mathsf{B}(G - w)$. (And $\mathcal{N}(G)$ is homotopy equivalent to $\mathcal{N}(G - w)$.)*

Let G be a graph, $v, w \in V(G)$. We define $G_{v=w}$ to be the graph obtained from G by deleting v and connecting w to the vertices of $\mathrm{N}_G(v)$. Clearly $G_{v=w}$ and $G_{w=v}$ are isomorphic graphs. In the situation of the Fold Lemma $G_{w=v} = G - w$. We will need the following variation of the Fold Lemma. (We remark that it is not true that $\mathcal{N}(G)$ and $\mathcal{N}(G_{w=v})$ are always homotopy equivalent.)

Theorem 6. *Let G be the union of two induced subgraphs G_1 and G_2. Let $L := G_1 \cap G_2$ be a complete graph and $w \in \mathrm{N}_{G_1}(L)$, $v \in \mathrm{N}_{G_2}(L)$ such that v is an isolated vertex in the induced subgraph of $\mathrm{N}_{G_2}(L)$. Then $\mathsf{B}(G)$ and $\mathsf{B}(G_{w=v})$ are homotopy equivalent.*

Proof. Since the box complex is homotopy equivalent to the neighborhood complex it is enough to show that $\mathcal{N}(G)$ and $\mathcal{N}(G_{w=v})$ are homotopy equivalent. Let $\Gamma(w) := \mathrm{N}_G(w) \setminus V(L)$ and $\Gamma(v) := \mathrm{N}_G(v) \setminus V(L)$. Let $\tilde{G}$ be the graph obtained from G by connecting v to the vertices of $\Gamma(w)$. $\tilde{G} - w = G_{w=v}$ is a fold of $\tilde{G}$. So it is enough to show that $\mathcal{N}(G)$ and

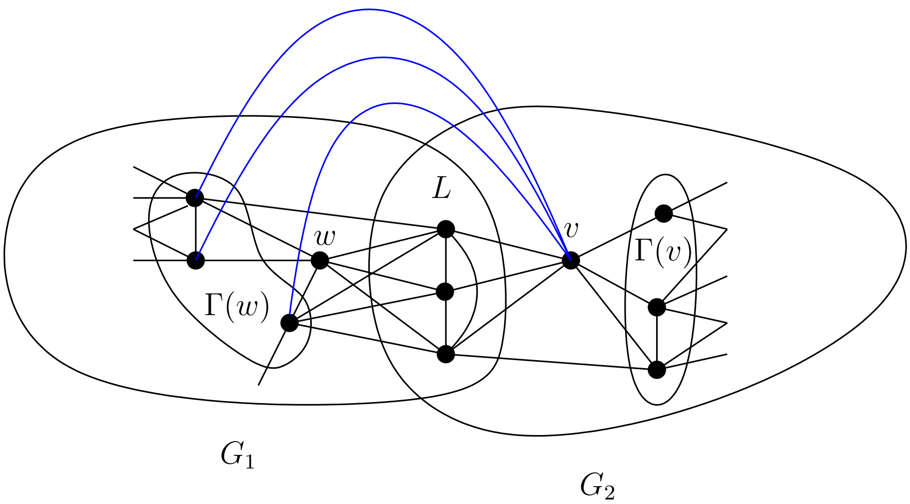

FIGURE 1. The graph $\tilde{G}$.

$\mathcal{N}(\tilde{G})$ are homotopy equivalent. Since G is a subgraph of $\tilde{G}$ we have that $\mathcal{N}(G) \subseteq \mathcal{N}(\tilde{G})$. Because of the construction of $\tilde{G}$ $\mathrm{N}_{\tilde{G}}(v) = \mathrm{N}_G(v) \cup \Gamma(w) = \Gamma(v) \cup V(L) \cup \Gamma(w)$. If $x \in \Gamma(w)$ then $\mathrm{N}_{\tilde{G}}(x) = \mathrm{N}_G(x) \cup v$. For every other vertex u of G we have $\mathrm{N}_{\tilde{G}}(u) = \mathrm{N}_G(u)$.
First we will look at the simplices of $S := \mathcal{N}(\tilde{G}) \setminus \mathcal{N}(G)$ coming from $\mathrm{N}_{\tilde{G}}(v)$. Such a simplex can be written in the form $A \cup B \cup C$, where $A \subseteq \Gamma(w)$, $B \subseteq V(L)$ and $C \subseteq \Gamma(v)$. If $A = \emptyset$ then the simplex $A \cup B \cup C$ would be a simplex of $\mathrm{N}_G(v)$, so could not be in S. If $C = \emptyset$ then symmetrically the simplex $A \cup B \cup C$ would be a simplex of $\mathrm{N}_G(w)$. So if we want to get a simplex in S, then $A \neq \emptyset$ and $C \neq \emptyset$. If $B = V(L)$ then such an $A \cup B \cup C$ will be in S. If $B \neq V(L)$ then for every $i \in V(L) \setminus B$ we must have that $\mathrm{N}_G(i) \not\supseteq A$ or $\mathrm{N}_G(i) \not\supseteq C$ for $A \cup B \cup C \in S$. This motivates the following. Assume that $V(L) = \{1, 2, \ldots, l\}$. For $C \subseteq \Gamma(v)$ let $f(C) := \min\{i \in V(L) \mid \mathrm{N}_G(i) \not\supseteq C\}$. This is well defined if $C \neq \emptyset$. If for every $i \in V(L)$ we get that $\mathrm{N}_G(i) \supseteq C$ then v would not be an isolated vertex in the induced subgraph of $\mathrm{N}_{G_2}(L)$. We are now ready to define a matching. Let $A \cup B \cup C \in S$.

$$\mu(A \cup B \cup C) := \begin{cases} A \cup (B \cup f(C)) \cup C & \text{if } f(C) \notin B, \\ A \cup (B \setminus f(C)) \cup C & \text{if } f(C) \in B. \end{cases}$$

This is well defined. If $A \cup B \cup C \in S$ then $A \cup (B \setminus f(C)) \cup C \in S$ because $\mathrm{N}_G(f(C)) \not\supseteq C$. This matching is acyclic. We can assume that $A \cup B \cup C$ and $\mu(A \cup B \cup C) = A \cup (B \cup f(C)) \cup C$ are the first two simplices of a cycle. Now if we delete a vertex from A or C we can not get that back by going up with the matching. So we have to delete a vertex $v \in B$, but then the next simplex of a cycle is $A \cup (B \cup f(C) \setminus v) \cup C$ which is matched down to $A \cup (B \setminus v) \cup C$. The critical simplices of this matching are the simplices of $X := \mathcal{N}(G) \cup_{x \in \Gamma(w)} (\mathrm{N}_G(x) \cup v)$ so Theorem 2 shows that $\mathcal{N}(\tilde{G})$ and X are homotopy equivalent.
Secondly we will look at the simplices of $T := X \setminus \mathcal{N}(G)$, they are simplices of $\mathrm{N}_{\tilde{G}}(x) = \mathrm{N}_G(x) \cup v$ for some $x \in \Gamma(w)$. Observe that each simplex of T contains the vertex v. This motivates to define the matching on the simplices of X as follows. Let $\sigma \in T$.

$$\mu(\sigma) := \begin{cases} \sigma \cup w & \text{if } w \notin \sigma, \\ \sigma \setminus w & \text{if } w \in \sigma. \end{cases}$$

If $\sigma \in T$ then $\sigma \cup w \in T$, since $x \in \Gamma(w)$. We show that if $w \in \sigma \in T$ then $\sigma \setminus w \in T$. For contradiction assume that $\sigma \setminus w \notin T$. This means that there is a vertex $y \in V(G)$ such that $\sigma \setminus w \subseteq \mathrm{N}_G(y)$. Since $v \in \sigma$ we have that $y \in V(G_2)$. If $y \in V(L)$, then $w \in \mathrm{N}_G(y)$, so $\sigma \subseteq \mathrm{N}_G(y)$ would contradict to $\sigma \in T$. So $y \notin V(L)$. Since for each $\tau \in T$ we have that $\tau \cap (V(G_2) \setminus V(L)) = v$ we will look at $\sigma \cap V(L)$. Let $z \in V(L) \setminus (\sigma \cap V(L))$. If such a z exist then $\sigma \subseteq \mathrm{N}_G(z)$. This contradict to $\sigma \in T$. If such a z does not exist then $(V(L) \cup v) \subseteq \mathrm{N}_G(y)$. This contradict to the assumption that v is an isolated vertex in the induced subgraph of $\mathrm{N}_{G_2}(L)$. So the critical simplices of this matching are the simplices of $\mathcal{N}(G)$. This matching is acyclic since matched simplices differ always by w. By Theorem 2 we get that X and $\mathcal{N}(G)$ are homotopy equivalent which completes the proof. $\square$

4. Chordal graphs

A graph G is *chordal* if every cycle of length at least 4 has a chord. We will use the following characterization from [1].

Theorem 7 ([1, Theorem 9.20]). *Let G be a connected chordal graph, and let V_1 be a maximal clique of G. Then the maximal cliques of G can be arranged in a sequence $(V_1, V_2, \dots, V_k)$ such that $V_j \cap (\cup_{i=1}^{j-1} V_i)$ is a clique of G, $2 \le j \le k$.*

Such a sequence of maximal cliques is called the *simplicial decomposition* of a chordal graph. It is easy to see [1] that if a graph has a simplicial decomposition, then it is chordal.

Theorem 8. *The box complex of a chordal graph is homotopy equivalent to a wedge of spheres.*

Proof. We will prove it by induction on the number of vertices of a graph. For small graphs, e.g. for K_2 it is easy to check. Assume that we have a chordal graph G with n vertices. Let $V_1, V_2, \dots, V_k$ be its simplicial decomposition as in Theorem 7.
If $k = 1$ then $G = K_n$ and then its box complex is homotopy equivalent to S^{n-2}.
If $k > 1$ then we look at $X_k := V_k \cap (\cup_{i=1}^{k-1} V_i)$. If $V_k \setminus X_k$ contains only one vertex v, then since X_k is a proper subset of a maximal clique V_i (for some $i < k$) we get that $G - v$ is a fold of G. By Lemma 5 we get that the box complex of $G - v$ and G are homotopy equivalent. On the other hand $G - v$ is chordal, $V_1, V_2, \dots, V_{k-1}$ is its simplicial decomposition. By the inductive assumption we get that the box complex of G is homotopy equivalent to a wedge of spheres.
So we can assume that $V_k \setminus X_k$ contains at least two vertices. We define a set $S := \{i \colon 1 \le i \le k-1, X_k \subset V_i\}$. Because V_i is a maximal clique $|X_k| < |V_i|$ for each $i \in S$. Assume that there

is an $s \in S$ such that $|X_k| + 2 \leq |V_s|$. Let G_1 be the subgraph of G induced by the vertices of $V_1 \cup V_2 \cup \cdots \cup V_{k-1}$, and G_2 be the clique V_k. $X_k = G_1 \cap G_2$. By Lemma 3 we get that $\mathsf{B}(G)$ is homotopy equivalent to $\mathsf{B}(G_1) \cup \mathsf{B}(G_2)$. Now $\mathsf{B}(G_1) \cap \mathsf{B}(G_2) = \mathsf{B}_0(X_k)$ is homotopy equivalent to a sphere $S^{|X_k|-1}$. Since the cliques V_s and V_k have at least 2 more vertices then X_k we get that $\mathsf{B}_0(X_k) \subset \mathsf{B}(V_s) \subseteq B(G_1)$ and $\mathsf{B}_0(X_k) \subset \mathsf{B}(V_k) = B(G_2)$. Since $\mathsf{B}(V_s)$ and $\mathsf{B}(V_k)$ are higher dimensional spheres than $\mathsf{B}_0(X_k)$ we get that $\mathsf{B}_0(X_k)$ is null-homotopic in both $\mathsf{B}(G_1)$ and $\mathsf{B}(G_2)$. This gives that $\mathsf{B}(G_1) \cup \mathsf{B}(G_2)$ is homotopy equivalent to the wedge $\mathsf{B}(G_1) \vee \mathsf{B}(G_2) \vee S^{|X_k|}$ (see example 0.14 in [8]). Now $\mathsf{B}(G_2) = \mathsf{B}(V_k)$ is homotopy equivalent to $S^{|V_k|-2}$. G_1 is chordal, $V_1, V_2, \ldots, V_{k-1}$ is its simplicial decomposition. By the inductive assumption we get that the box complex of G_1 is a wedge of spheres. This completes this case.
The last case is when for every $s \in S$ we have $|X_k| + 1 = |V_s|$. Let $w \in V_k \setminus X_k$ be a vertex. We fix $s \in S$ and let $v \in V_s \setminus X_k$. We can choose G_1 and G_2 as before and by Theorem 6 we get that $\mathsf{B}(G)$ is homotopy equivalent to $\mathsf{B}(G_{w=v})$. It is easy to see that $\mathsf{B}(G_{v=w})$ is chordal, $V_1, V_2, \ldots, V_{k-1}$ is its simplicial decomposition, where we have to replace V_s by a clique of size $|V_k|$. The induction completes the last case of the proof. □

Remark. *There are several things to note after the above proof:*

Since there is a free $\mathbb{Z}_2$-action on the box complex we get that the number of spheres in the wedge must be odd. This can be seen from the inductive proof as well.

We can choose V_1 to be the largest clique in G. The inductive proof gives that $S^{\omega(G)-2}$ will be the largest dimensional sphere in the wedge. From this we get that for a chordal graph G the ith dimensional homology vanishes, $\mathrm{H}_i(\mathsf{B}(G)) = 0$ if $i \geq \omega(G) - 1$. This was observed in [10].

It is easy to see that the neighborhood complex is connected if the graph contains a clique with at least 3 vertices. So if we have S^0 in the wedge then the maximal cliques are the edges of the graph. Since G is chordal, it must be a tree. A tree always folds to K_2 so in this case the neighborhood complex is homotopy equivalent to S^0.
On the other hand look at the wedge $\vee_{i=1}^{2r+1} S^{a_i}$, $a_i \geq 1$, $a_i \geq a_{i+1}$. There is a chordal graph such that its box complex is this wedge of spheres. The simplicial decomposition of such a graph $V_1, V_2, \ldots, V_{r+1}$ can be obtained as follows. V_i is a clique with $a_{2i-1} + 2$ vertices. We choose those cliques so that $|V_1 \cap V_i| = a_{2i-2}$ if $i > 1$. The vertices of $V_i \setminus V_1$ are disjoint for different i. From the above proof one can easily see that the box complex of this chordal graph is the wedge $\vee_{i=1}^{2r+1} S^{a_i}$.

Finally, the dimensions of the spheres in the wedge can be obtained from the simplicial decomposition $V_1, V_2, \ldots, V_k$ of the chordal graph G. Let a_j denote the number of vertices of V_j. We define $X_j := V_j \cap (\cup_{i=1}^{j-1} V_i)$ for $j = 2, 3, \ldots, k$. Let b_j denote the number of vertices of X_j. Now the spheres of the wedge are amongst the spheres $S^{a_1-2}, S^{a_2-2}, S^{a_3-2}, \ldots, S^{a_k-2}$ and $S^{b_2}, S^{b_3}, \ldots, S^{b_k}$. From the above inductive proof it can be seen recursively which spheres are in the wedge.

Remark 9. *After Theorem 8 one might suspect that something like that might be true for a bigger graph class, for* perfect *graphs as well. (A graph G is called perfect if for each induced subgraph H we have that $\chi(H) = \omega(H)$.) Unfortunately that is not true. Let K be the disjoint union of two homeomorphic projective planes. The free $\mathbb{Z}_2$-action interchanges the two copies. In* [4, 6] *a graph G is constructed such that $\mathsf{B}(G)$ is homotopy equivalent to K. From that construction it is easy to see that G is bipartite, which means that G is* *perfect as well. But* $\mathsf{B}(G)$ *is not even torsion free. If one glues a triangle to* G *such that the intersection is a vertex then one can get a non-bipartite example, too.*

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