A triangle inequality for polymatroids

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1 Definitions

By a *polymatroid* we mean a lattice L with minimal element 0 and operations $\cup$ and $\cap$ endowed with a *rank* function $f : L \to \mathbb{R}$ satisfying the following conditions:

- $f(0) = 0$, and f is non-negative, i.e. $f(x) \geq 0$
- f is monotone, i.e. $x \leq y$ implies $f(x) \leq f(y)$
- f is submodular, i.e. for arbitrary $x, y \in L$,

$$f(x) + f(y) \geq f(x \cup y) + f(x \cap y).$$

If L is atomic, f takes only integer values, and $f(a) \leq 1$ for all atoms $a \in L$, then the pair $\langle L, f \rangle$ is a *matroid.*

By an observation of S. Fujishige [2], taking n random variables and the complete Boolean algebra on these variables as a lattice, and defining $f(x)$ as the (joint) entropy $H(x)$ of the subset x of the variables, we get a polymatroid. Following the information-theoretical conventions, the *conditional rank* is defined as $f(x|y) \stackrel{\text{def}}{=} f(x \cup y) - f(y)$. By the submodularity of the rank function, we have

$$0 \leq f(x|y) \leq f(x).$$

We can define the *distance* between two elements $x, y \in L$ as follows (cf. [1]):

$$d(x, y) \stackrel{\text{def}}{=} \frac{f(x|y) + f(y|x)}{f(x \cup y)} = \frac{2f(x \cup y) - f(x) - f(y)}{f(x \cup y)}.$$

By the submodularity it is immediate that $f(x, x) = 0$ and $0 \leq f(x, y) \leq 1$ for arbitrary x and $y \in L$. In some sense d measures, how independent x and y are. The interesting point is that d also satisfies the triangle inequality. This will be proved in the next section.

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2 The proof

We shall need the following trivial observation.

Lemma 2.1 *If u, v, $w \geq 0$ and $\frac{u+w}{v+w} \leq 1$ then $\frac{u}{v} \leq \frac{u+w}{v+w}$.* ∎

We prove a nice representation theorem for arbitrary polymatroid due to R. W. Yeung [3]. Essentially it expresses that the rank function be considered as a (signed) measure. From now on fix the elements A, B and C of the lattice L. Our aim is to show that $d(A,B) + d(B,C) \geq d(A,C)$.

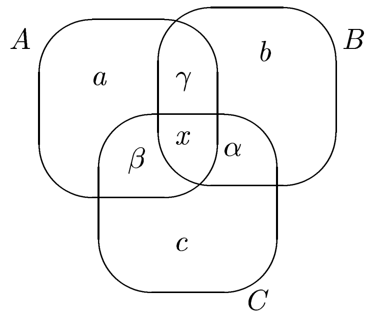

Figure 1: Representing the rank function

Lemma 2.2 *There are real numbers a, b, c, α, β, $\gamma \geq 0$ and x so that*

$$\begin{aligned} f(A) &= a + \beta + \gamma + x & \quad f(B \cup C) &= S - a \\ f(B) &= b + \alpha + \gamma + x & \quad f(A \cup C) &= S - b \\ f(C) &= a + \alpha + \beta + x & \quad f(A \cup B) &= S - c \end{aligned}$$

where $S = a + b + c + \alpha + \beta + \gamma + x = f(A \cup B \cup C)$. Moreover $\alpha + x$, $\beta + x$ and $\gamma + x$ are also non-negative (see Figure 1).

Proof. Since $a = f(A \cup B \cup C) - f(B \cup C)$, the monotonicity of f implies immediately that it is ≥ 0. Similarly,

$$\alpha = f(A \cup B) + f(A \cup C) - f(A \cup B \cup C) - f(A) \geq 0$$

by the submodularity. This proves the first half of the lemma. As for the other half, e.g.

$$\alpha + x = f(B) + f(C) - f(B \cup C) \geq 0,$$

as was claimed. ∎

Now we can turn to the proof of the triangle inequality.

Theorem 2.3 $d(A,B)+d(B,C)\geq d(A,C)$.

Proof. Using the notation from the lemma above, we get that

$$\begin{aligned} d(A,C) &= \frac{2S-2b-(a+\beta+\gamma+x+c+\alpha+\beta+x)}{S-b} \\ &= \frac{S-(b+\beta+x)}{S-b}. \end{aligned}$$

Using this we have to check

$$d(A,B)+d(B,C)=\frac{S-(c+\gamma+x)}{S-c}+\frac{S-(a+\alpha+x)}{S-a}\overset{?}{\geq}\frac{S-(b+\beta+x)}{S-b}.$$

The right hand side does not depend on b, and by Lemma 2.1 the left hand side doesn't increase if b is replaced by 0. Also, keeping $a+\alpha$ fixed and varying a and α, only the second term of the left hand side changes, and will be minimal if $a=0$. Thus we may also assume that $a=0$, and just in the same way that $c=0$. Therefore the inequality to be proved becomes

$$S-(\gamma+x)+S-(\alpha+x)\geq S-(\beta+x),$$

i.e. $2\beta\geq 0$ which holds by Lemma 2.2. ■

References

[1] I. Csiszár and J. Körner, *Information Theory. Coding Theorems for Discrete Memoryless Systems*, Academic Press, New York, 1981.

[2] S. Fujishige, Polymatroid dependence structure of a set of random variables, *Information and Control* 39(1978) pp. 55-72.

[3] R. W. Yeung, A new outlook on Shannon's information measures, *IEEE Trans. Inform. Theory* 37(1991) 466-474