Connected Graph Game

László Csirmaz*

Abstract

Two players claim alternatively edges of the complete graph on n vertices. The winner is the one who makes the spanned subgraph with the claimed edges connected. We determine, for each n, who wins the game.

1 Introduction

There is a huge literature for combinatorial games [2], the most basic one is, perhaps, [1]. In a typical family of these games two players occupy (or color) alternately edges of the complete graph on n vertices until the spanned subgraph does not have a certain property. The player who reaches the property is the winner of the game. In the *avoidance* game the player whose move makes the property true loses. In our case this property is the connectivity. The paper is organized as follows. Section 2 contains definitions and general theorems about positional games. In section 3 we use the results from the previous section to determine the outcome of the connected graph game.

2 Definitions

Combinatorial games are played by two players, I and II. Player I is the first, and they take moves alternately. A *play* of a game is a (possibly infinite) sequence of consecutive, legal moves. A play is *partial* if it can be continued, otherwise if is a complete, or *total play*. Each total play falls into exactly one of the following categories: it is a *win for I*, a *win for II*, or it is a *draw*. We say that this is the *outcome* of the play.

A *strategy S*, say for the first player, is a function which assigns a legal next move to all partial plays of even length (of odd length for the second player). A play *follows the strategy S*, if all even moves (odd moves if S is for player II) are the ones assigned by S to the partial play consisting of the previous moves. S is a *winning strategy*, if all total plays following S are wins for I (or wins for II if S is a strategy for II). S is a *draw*, if all such total plays are either a win or a draw for the corresponding player. A game is a *win for* I, if I has a winning strategy.

In the majority of combinatorial games all plays end after finitely many moves. It was *Ernst Zermelo* who proved in the 1920's that these finite games are *determined*, i.e. either I has a wining strategy, or II has a winning strategy, or both players have

*Central European University, Budapest

drawing strategies. "Solving a game" means to determine which is the case, and, if possible, also to give the strategy. Hales and Jewett in [3] studied generalized tic-tac-toe games on finite hypergraphs. In this very small subclass the second player cannot have a winning strategy. The proof is based on the so-called "strategy stealing argument," which was discovered well before. For example, in the 1940's J. F. Nash reinvented Hex, and proved that the second player cannot have a winning strategy and, consequently, it is a win for the first player. The argument gives no clue what this strategy would be, and for Hex as well as for several other games no explicit winning strategy is known[1].

In a *positional game* a set P of positions are given. For each position $p \in P$ there is a collection of possible successor positions; this forms a directed graph $G = \langle P, E\rangle$. We write $p \xrightarrow{E} q$ to denote that there is an edge from p to q in G. We have an *initial position* $s \in P$ and the disjoint sets of *winning and losing positions* W, and $L \subseteq P$, respectively.

Playing the positional game $\mathcal{G} = \langle P, E, s, W, L\rangle$ starts at position s, and a move consists of changing the present position into one of its possible successors. The game ends if the new position is either a winning or a losing position; and the player who took the last move wins in the first case, and loses in the second. To be consistent with this definition, we stipulate that if the starting position is in W (or in L), then I loses (wins) the game immediately. If the play stops because the reached position has no successor, or if the play continues indefinitely without either player winning or losing, then the play is a *draw*.

While positional games are special cases of general combinatorial games, arbitrary combinatorial game with no infinite plays can be turned easily into a positional game: simply choose partial plays as positions. Still it is worth to consider positional games separately as in the finitary case the number of positions can be significantly smaller than the number of partial plays.

The following statement tells us a way to solve a positional game.

Theorem 1 *Let $\mathcal{G} = \langle P, E, s, W, L\rangle$ be a positional game. Suppose there is a subset A of positions with the following properties:*

(i) *winning positions are in A, but losing positions are not;*

(ii) *legal moves lead out of A: if $p \in A$ and $p \xrightarrow{E} q$ then $q \notin A$;*

(iii) *from any position not in A we can move into A: if $p \notin A$ then for some $q \in A$ we have $p \xrightarrow{E} q$.*

If the initial position s is not in A then I *has a strategy which allows her either to win or to make a draw; however if $s \in A$ then* II *has such a strategy.*

Proof. If $s \notin A$ then I can always arrange that after her move the position will be in A. Thus she never will hit a losing positions, and, by (ii) above, II will never hit a winning position. Similar idea works if $s \in A$. $\square$

The condition in this theorem is not only sufficient but necessary: choose $A \subseteq P$ as the set of position starting from which I has *no* winning strategy. This set satisfies (i)–(iii) above.

[1] I am indebted to the referee for pointing out information about the history of combinatorial games.

Now let $\mathcal{G}_1 = \langle P_1, E_1, s_1, W_1, L_1 \rangle$ and $\mathcal{G}_2 = \langle P_2, E_2, s_2, W_2, L_2 \rangle$ be two positional games. The function $\varphi : P_1 \to P_2$ is a *homomorphism*, if the following holds:

(i) φ keeps the initial, winning, and losing positions: $\varphi(s_1) = s_2$, $\varphi(W_1) \subseteq W_2$, $\varphi(L_1) \subseteq L_2$;

(ii) if $p_1 \stackrel{E_1}{\to} q_1$ is a legal move in $\mathcal{G}_1$, then $\varphi(p_1) \stackrel{E_2}{\to} \varphi(q_1)$ is a legal move in $\mathcal{G}_2$;

(iii) if $\varphi(p_1) = p_2$ and $p_2 \stackrel{E_2}{\to} q_2$ is a legal move in $\mathcal{G}_2$, then for some $q_1 \in P_1$, $\varphi(q_1) = q_2$ and $p_1 \stackrel{E_1}{\to} q_1$ is a legal move in $\mathcal{G}_1$.

Theorem 2 *If $\mathcal{G}_2$ is the homomorphic image of $\mathcal{G}_1$ then $\mathcal{G}_1$ and $\mathcal{G}_2$ has the same outcome.*

Proof. The player with a strategy should mimic the play in the other game; by (ii) and (iii) it can be done; and by (i) the outcome of both plays will be the same. □

We say that $\varphi : P_1 \to P_2$ is a *weak homomorphism* from $\mathcal{G}_1$ to $\mathcal{G}_2$, if instead of (ii) above we have

(ii′) if $p_1 \stackrel{E_1}{\to} q_1$ is a legal move in $\mathcal{G}_1$, then *either* $\varphi(p_1) \stackrel{E_2}{\to} \varphi(q_1)$ is a legal move in $\mathcal{G}_2$, *or* for some $r_1 \in P_1$, $\varphi(r_1) = \varphi(p_1)$ and $q_1 \stackrel{E_1}{\to} r_1$ is a legal move in $\mathcal{G}_1$.

In other words, either we could mimic the move in $\mathcal{G}_2$, or its effect can be undone. With some restrictions Theorem 2 remains true for weak homomorphisms.

Theorem 3 *Suppose that there is no infinite play in $\mathcal{G}_1$, moreover positions in W_1 and L_1 have no successors. If there is a weak homomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$ then $\mathcal{G}_1$ and $\mathcal{G}_2$ has the same outcome.*

Proof. The extra assumptions ensure that the second possibility in (ii′) cannot be applied when reaching a winning or losing position in $\mathcal{G}_1$ – thus the original and the mimicked plays reach winning and losing positions simultaneously –, nor can it be applied infinitely many times which might force a draw in the original play. □

3 The Game

With the arsenal set up in the previous section, we are ready to attack the game in the title. The game starts with $n \geq 2$ points. The players alternately connect two so far unconnected points. The winner (the loser in the avoidance game) is the one who makes the produced graph on these points connected.

This game obviously is a positional game, the positions being all the graphs on n vertices. An important measure of the game is the number of the connected components. At the beginning it is n, and the game ends when this number goes down to 1. We shall introduce another positional game, which is easier to analyze. After establishing a weak isomorphism from the graph game into this one we shall be able to decide who is the winner.

The positions on the second game are quadruples $\langle a,b,c,d\rangle$ of non-negative integers with $a+b+c+d>0$. We view this position as a heap of letters A, B, C and D where the heap contains exactly a copies of A's, b copies of B, etc. The game ends when only one letter remains in the heap. In a single move the following possibilities are: you can trade a single A for a B; a single C for a D; or you can trade two letters for a third one according to Table 1. In the language of quadruples it translates to the

	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	B	A
D	D	C	A	B

Table 1: Trading letters

$\langle -1, 1, 0, 0\rangle$	$A \to B$
$\langle 0, 0, -1, 1\rangle$	$C \to D$
$\langle -1, 0, 0, 0\rangle$	$AX \to X$
$\langle 1, -2, 0, 0\rangle$	$BB \to A$
$\langle 0, -1, -1, 1\rangle$	$BC \to D$
$\langle 0, -1, 1, -1\rangle$	$BD \to C$
$\langle 0, 1, -2, 0\rangle$	$CC \to B$
$\langle 0, 1, 0, -2\rangle$	$DD \to B$
$\langle 1, 0, -1, -1\rangle$	$CD \to A$

Table 2: Add-on vectors

following: to any quadruple $\langle a,b,c,d\rangle$ we can add one of the vectors in Table 2 provided no entry becomes negative, and the sum $a+b+c+d$ remains positive. From the table it is also clear that there is no infinite play since in each step either the sum $a+b+c+d$ decreases, or the sum remains the same and then the quadruple decreases lexicographically.

There are exactly four positions with no successors: those where one of the coordinates is 1 and the other three ones are zero. In the game the set W of winning positions consists of this four quadruples, and the set L of losing positions is empty. Table 3

	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
$b=4$	01	23	01	23	01	$\cdots$
$b=3$	01	23	01	23	01	$\cdots$
$b=2$	1	23	01	23	01	$\cdots$
$b=1$	12	3	01	23	01	$\cdots$
$b=0$	12	03	1	23	01	$\cdots$
$a=$	0	1	2	3	4	$\cdots$

$+$	$-$
$0,0,0,3$	$0,0,0,2$
$0,0,1,0$	$0,0,1,2$
$0,0,2,1$	$0,0,2,0$
$0,1,0,0$	$0,0,3,0$
	$0,1,0,1$
	$0,1,1,2$
	$1,0,0,3$
	$1,0,1,0$
	$1,0,2,1$

Table 3: Winning positions

shows the set of *winning positions*, i.e. a subset A of all positions satisfying the conditions of Theorem 1. The table should be read as follows. The entries of the left hand side matrix contain one or two values from the set $\{0,1,2,3\}$. The entries are indexed by a and b. Columns continue upwards with the topmost value; other columns (i.e. columns with $a\geq 5$) consist of entirely 23 for a odd, and 01 for a even. The quadruple

$\langle a,b,c,d\rangle$ is in the set A if $d-c$ is congruent mod 4 to one of the values given in the entry indexed by a and b. There are several exceptions, however, these are enlisted next to the matrix: those which should be added to this set, and those which should be taken away.

Among the conditions of Theorem 1 (i) is immediate, the other two has been checked by a computer program for all quadruples where none of a, b, c, and d exceed 7. For other quadruples a move cannot result in an exceptional state (as a coordinate decreases by at most 2), thus it is enough to concentrate on the left hand table only. By a painful case by case checking we see that starting from a quadruple in A all moves lead out of A; finally Table 4 indicates that from a non-exceptional quadruple not in A which move gets into A. The entries in each cell indicate the four congruence classes

	$\cdots$		$\cdots$		$\cdots$		$\cdots$		$\cdots$		
$b=4$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$\cdots$
	$BB\to A$	$BB\to A$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	
$b=3$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$\cdots$
	$CC\to B$ $DD\to B$	$BB\to A$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	
$b=2$	$BC\to D$ $BD\to C$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$\cdots$
	$CC\to B$ $DD\to B$	$BC\to D$ $BD\to C$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	
$b=1$	$BC\to D$ $BD\to C$	$\otimes$	$CC\to B$ $DD\to B$	$AB\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$\cdots$
	$\otimes$	$BC\to D$ $BD\to C$	$AB\to B$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	
$b=0$	$C\to D$ $DD\to B$	$\otimes$	$\otimes$	$A\to B$	$AA\to A$	$\otimes$	$A\to B$	$A\to B$	$\otimes$	$\otimes$	$\cdots$
	$\otimes$	$C\to D$ $DD\to B$	$A\to B$	$\otimes$	$CC\to B$ $DD\to B$	$A\to B$	$\otimes$	$\otimes$	$A\to B$	$A\to B$	
	$a=0$		$a=1$		$a=2$		$a=3$		$a=4$		

Table 4: Winning moves

modulo 4 for $d-c$. $\otimes$ is a winning position, otherwise it is indicated which letter (or pair of letters) out of A, B, C, and D should be replaced. When more than one possibility is enlisted, at least one of them is applicable. This claim is true when either the number of C's or the number of D's exceed 1, and as we are checking non-exceptional cases only, i.e. when at least one coordinate exceeds 6, this will always hold.

Consequently, by Theorem 1, if the initial position is not in A, then player I has a strategy playing which she cannot loose. As all plays end after finitely many moves, and there is no draw, this means that this strategy is, in fact, a winning strategy. Similarly, if the initial position is in A, then player II has a winning strategy in this game.

Theorem 4 *Playing the Connected Graph Game on $n \geq 2$ points, player I has a winning strategy if $n = 2$ or $n \geq 4$ and n is congruent to 0 or 3 modulo 4. In the other cases player II has a winning strategy.*

Proof. We establish a weak isomorphism from the connected graph game $\mathcal{G}_1$ to the game $\mathcal{G}_2$ discussed above, then we apply Theorem 3. The positions in $\mathcal{G}_1$ are the set of spanned subgraphs on n vertices. For such a graph G, we consider its connected components. A component is *of type* $\langle i, j\rangle$, where i and j are either "even" or "odd," if the parity of the number of its vertices is i, and the parity of the missing edges is j. The homomorphic image $\varphi(G)$ of the position G will be $\langle a,b,c,d\rangle$, where a is the number of connected components of G of type $\langle \text{even}, \text{odd}\rangle$, b is the number of connected components of type $\langle \text{even}, \text{even}\rangle$, c is the number of components of type $\langle \text{odd}, \text{odd}\rangle$, and finally, d is the number of components of type $\langle \text{odd}, \text{even}\rangle$.

To see that this is a weak homomorphism indeed, first note that a play in $\mathcal{G}_1$ ends when after a move only one component remains; this translates to $a+b+c+d = 1$. Given a position G of $\mathcal{G}_1$, a move means to draw a new edge. It may connect two connected components – this corresponds to trading two letters for one in game $\mathcal{G}_2$. Indeed, for example connecting two components of type $\langle \text{odd}, \text{odd}\rangle$ (type C), the number of vertices in the new component will be even, and the missing edges will be those which were missing in the original components, plus the new edges between the two components, minus the edge just drawn. This is odd + odd + even, i.e. even. The resulting component is of type $\langle \text{even}, \text{even}\rangle$.

If a component has an odd number of missing edges, then drawing an edge in that component corresponds to either an $A \to B$, or to a $C \to B$ move in $\mathcal{G}_2$. If the component, however, has an even number of missing edges, then this move can be "undone" by drawing another edge into the same component.

The initial position of $\mathcal{G}_1$ is the empty graph on n vertices, i.e. n components of type $\langle \text{odd}, \text{even}\rangle$, which corresponds to the position $\langle 0,0,0,n\rangle$ in game $\mathcal{G}_2$. By Table 3 among these quadruples those listed in the Theorem are winnings for player I, the others are winnings for player II. Now theorem 3 gives the result. □

Theorem 5 *In the avoidance case the Connected Graph Game is a win for player I if $n \geq 3$ and n is congruent to 0 or 3 modulo 4. Otherwise it is a win for player II.*

Proof. Similarly to the previous case, the same mapping gives a weak homomorphism from $\mathcal{G}_1^a$ to $\mathcal{G}_2^a$, however in this case there are no winning positions in $\mathcal{G}_2^a$, and the set L of losing positions are all quadruples $\langle a,b,c,d\rangle$ with $a+b+c+d = 1$. Table 5 shows the winning positions for this game. From here the claim of the theorem follows exactly the same way as in the previous case. □

References

[1] E. R. Berlakamp, J.H. Conway, R.K. Guy: *Winning Ways*, Academic Press, New York (1982)

	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
$b=4$	01	23	01	23	01	$\cdots$
$b=3$	01	23	01	23	01	$\cdots$
$b=2$	01	23	01	23	01	$\cdots$
$b=1$	01	23	01	23	01	$\cdots$
$b=0$	01	23	01	23	01	$\cdots$
$a=$	0	1	2	3	4	$\cdots$

$+$	$-$
$0,0,0,2$	$0,0,0,1$
$0,0,2,0$	$0,0,1,1$
	$0,1,0,0$
	$0,1,1,1$
	$1,0,0,2$

Table 5: Winning positions for the avoidance game

[2] J. Beck: On positional Games, *Journal of Combinatorial Theory, series A*, **30** (1981) 117–133

[3] A.W.Hales, R.I. Jewett: Regularity and positional games, *Transactions of the American Mathematical Society* **106** (1963) 222–229