Information inequalities from the book

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Outline

Searching for new entropy inequalities

How to do it?

The Zhang-Yeung method

1. split the random variables into two parts: $\vec{x}_1$ and $\vec{y}$,
2. make a copy $\vec{x}_2$ of $\vec{x}_1$ over $\vec{y}$ to get $\langle \vec{x}_1, \vec{x}_2, \vec{y} \rangle$,
3. iterate steps 1 and 2,
4. write up all Shannon inequalities for the final variable set,
5. search consequences for the original variables.

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Numerically intractable even after three iterations!

Searching for new entropy inequalities

Possible remedy

- Reduce the total number of auxiliary variables by
  - cutting the number of copied variables in $\vec{x}_2$
  - gluing together some variables in $\vec{x}_2$

Randall Dougherty et al computed all possibilities up to

3 iterations and

4 auxiliary variables

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Our approach

- Make several copies of $\vec{x}_1$ to get $\langle \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k, \vec{y} \rangle$
  - the high symmetry reduces the size of the computation,
  - equivalent to keeping the "over" variables $\vec{y}$ in $\log_2 k$ iterations.

The book

Definition

A *book* is a collection of variables $\vec{x}_1, \ldots, \vec{x}_k$, and $\vec{y}$ such that

- $\vec{x}_1\vec{y}$, $\vec{x}_2\vec{y}$, etc, $\vec{x}_k\vec{y}$ are identically distributed
- $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are totally independent over $\vec{y}$.

$\vec{y}$ is the *spine* of the book, and it has k *pages*, $\vec{x}_1, \ldots, \vec{x}_k$.

This book is not too interesting: all pages are the same.

Fact

Every almost entropic matroid has a k-page book extension.

Inequalities from the book

We have four random variables: a, b, c, and d.

Problem

Characterize all polymatroids on $abcd$ which have a k-book extension with spine ab.

That is

What are the (4-variable) information inequalities which can be extracted from a k-book extension?

Inequalities from the book

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What are the (4-variable) information inequalities which can be extracted from a k-book extension?

The case $k = 2$ was solved:

Theorem (F. Matus, 2007)

A polymatroid on abcd has a 2-page book extension at ab if and only if it satisfies six particular instances of the Zhang-Yeung inequality.

Outline

Natural coordinates

Every information inequality on *abcd* can be written as a linear combination of the following entropy expressions:

1	Ingleton: $[a,b,c,d] = -(a,b) + (a,b \mid c) + (a,b \mid d) + (c,d)$,
2, 3	$(a,b \mid c)$, $(a,b \mid d)$,
4–7	$(a,c \mid b)$, $(b,c \mid a)$, $(a,d \mid b)$, $(b,d \mid a)$,
8, 9	$(c,d \mid a)$, $(c,d \mid b)$,
10	(c,d),
11	$(a,b \mid cd)$,
12–15	$(a \mid bcd)$, $(b \mid acd)$, $(c \mid adb)$, $(d \mid abc)$,

This is an unimodular transformation of $\mathbb{R}^{15}$.

Natural coordinates

Theorem

An information inequality written in natural coordinates must have

- *non-negative natural coefficients with the exception of the Ingleton,*
- *zero coeffs for* $(a\,|\,bcd)$, $(b\,|\,acd)$, $(c\,|\,adb)$, $(d\,|\,abc)$
- *positive Ingleton coeff after a possible permutation of* a, b, c, d.

– The second statement is equivalent to T. Chan's result on balanced inequalities.
– If the Ingleton is not positive for all permutations, then the inequality is a consequence of the Shannon inequalities.

Corollary

Entropy inequalities have non-negative coefficients when written in natural coordinates.

Outline

The case of the 2-page book

Theorem (F. Matus, 2007)

2-page extensions of abcd over ab are characterized by the following six instances of the Zhang-Yeung inequality:

$$
\begin{aligned}
&[abcd] + (a,b\,|\,c) + (a,c\,|\,b) + (b,c\,|\,a) \ge 0,\\
&[abdc] + (a,b\,|\,d) + (a,d\,|\,b) + (b,d\,|\,a) \ge 0,\\
&[bdac] + (b,d\,|\,a) + (a,b\,|\,d) + (a,d\,|\,b) \ge 0,\\
&[bcad] + (b,c\,|\,a) + (a,b\,|\,c) + (a,c\,|\,b) \ge 0,\\
&[adbc] + (a,d\,|\,b) + (a,b\,|\,d) + (b,d\,|\,a) \ge 0,\\
&[acbd] + (a,c\,|\,b) + (a,b\,|\,c) + (b,c\,|\,a) \ge 0.
\end{aligned}
$$

As the statement is symmetric for the $a \leftrightarrow b$ and $c \leftrightarrow d$ permutations, the condition must also be symmetric: the first two and the last four are the same up to symmetry.

The book conjecture

Conjecture

The k-page extensions are characterized by the following inequalities and their $a \leftrightarrow b$ and $c \leftrightarrow d$ symmetric versions:

$$[abcd] + \frac{1}{x_s}(a,b\,|\,c) + (1+\frac{y_s}{x_s})\big((a,c\,|\,b)+(b,c\,|\,a)\big) + \\ + \frac{z_s}{x_s}\big((a,d\,|\,b)+(b,d\,|\,a)\big) \ge 0,$$

$$[bdac] + \frac{1}{\ell}(b,d\,|\,a) + (1+\frac{\ell-1}{2})\big((a,b\,|\,d)+(a,d\,|\,b)\big) \ge 0,$$

where $\ell = 1,2,\ldots,k-1$, and either $\langle x_s, y_s, z_s\rangle$ or $\langle x_s, z_s, y_s\rangle$ is in $\bigcup\{G_\ell : \ell < k\}$, where G_ℓ is described next.

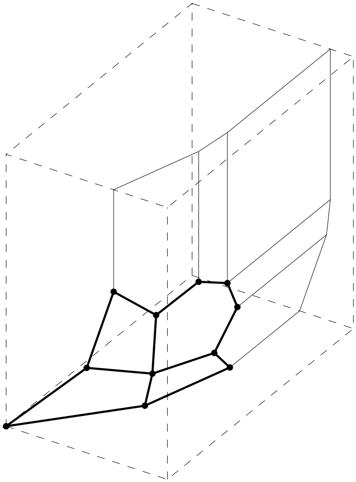
Look out for the the unexpected $y_s \leftrightarrow z_s$ symmetry.

The book conjecture – the coeffs

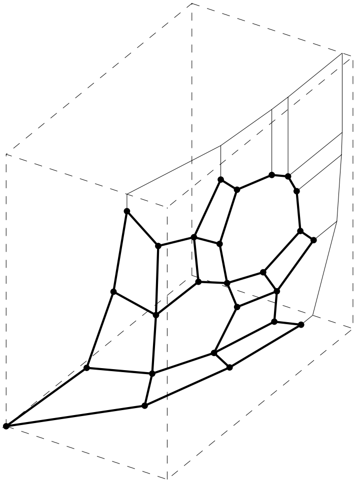
G_1	G_2	G_3	G_4	G_5
$\langle 1,0,0\rangle$	$\langle 2,1,0\rangle$	$\langle 3,3,0\rangle$	$\langle 4,6,0\rangle$	$\langle 5,10,0\rangle$
	$\langle 3,1,1\rangle$	$\langle 4,3,1\rangle$	$\langle 5,6,1\rangle$	$\langle 6,10,1\rangle$
		$\langle 6,5,3\rangle$	$\langle 7,8,3\rangle$	$\langle 8,12,3\rangle$
		$\langle 7,5,5\rangle$	$\langle 8,8,5\rangle$	$\langle 11,18,6\rangle$
			$\langle 10,14,6\rangle$	$\langle 12,18,8\rangle$
			$\langle 11,14,8\rangle$	$\langle 15,21,14\rangle$
			$\langle 14,17,14\rangle$	$\langle 15,30,10\rangle$
			$\langle 15,17,17\rangle$	$\langle 16,15,12\rangle$
				$\langle 16,21,17\rangle$
				$\langle 19,33,18\rangle$
				$\langle 20,33,21\rangle$
				$\dots$

The G_ℓ sets

The coefficients $\langle 1/x_s, y_s/x_s, z_s/x_s \rangle$



4-page book



5-page book

Outline

How to get the coefficients?

Fill the positive quadrant as follows:

3	1,	4,	10,	20,	35,
2	1,	3,	6,	10,	15,
1	1,	2,	3,	4,	5,
0	1,	1,	1,	1,	1,
	0	1	2	3	4

How to get the coefficients?

Fill the positive quadrant as follows:

3	1,0,	4,4,	10,20,	20,60,	35,140,
2	1,0,	3,3,	6,12,	10,30,	15,60,
1	1,0,	2,2,	3,6,	4,12,	5,20,
0	1,0,	1,1,	1,2,	1,3,	1,4,
	0	1	2	3	4

How to get the coefficients?

Fill the positive quadrant as follows:

3	1,0,3	4,4,12	10,20,30	20,60,60	35,140,105
2	1,0,2	3,3,6	6,12,12	10,30,20	15,60,30
1	1,0,1	2,2,2	3,6,3	4,12,4	5,20,5
0	1,0,0	1,1,0	1,2,0	1,3,0	1,4,0
	0	1	2	3	4

How to get the coefficients?

Fill the positive quadrant as follows:

3	1,0,3	4,4,12	10,20,30	20,60,60	35,140,105
2	1,0,2	3,3,6	6,12,12	10,30,20	15,60,30
1	1,0,1	2,2,2	3,6,3	4,12,4	5,20,5
0	1,0,0	1,1,0	1,2,0	1,3,0	1,4,0
	0	1	2	3	4

At $\langle i, j\rangle$ we have

$$\mathbf{v}_{i,j} = \binom{i+j}{j}\langle 1, i, j\rangle.$$

Good sequences

The sequence $s = \langle s_1, s_2, \ldots, s_k \rangle$ is *good* if

- $s_1 \ge s_2 \ge \cdots \ge s_{k-1} \ge s_k = 1$,
- $s_i - s_{i+1}$ is either 0 or 1.

Good sequences of different length are:

- 1;
- 11, 21;
- 111, 211, 221, 321;
- 1111, 2111, 2211, 2221, 3211, 3221, 3321, 4321;
- 11111, 21111, 22111, 22211, 22221, 32221, 33221, ...

To get the coefficients in G_k take all good sequences s of length k, mark the bottom s_i cells in column i, and add up the triplets.

And finally: the coefficients

sequence	sum	sequence	sum
1	$\langle 1,0,0\rangle$	211	$\langle 4,3,1\rangle$
11	$\langle 2,1,0\rangle$	2111	$\langle 5,6,1\rangle$
21	$\langle 3,1,1\rangle$	2211	$\langle 7,8,3\rangle$
111	$\langle 3,3,0\rangle$	3221	$\langle 11,14,8\rangle$
1111	$\langle 4,6,0\rangle$	4321	$\langle 15,17,17\rangle$
11111	$\langle 5,10,0\rangle$	43211	$\langle 16,21,17\rangle$

3	1,0,3	4,4,12	10,20,30	20,60,60	35,140,105
2	1,0,2	3,3,6	6,12,12	10,30,20	15,60,30
1	1,0,1	2,2,2	3,6,3	4,12,4	5,20,5
0	1,0,0	1,1,0	1,2,0	1,3,0	1,4,0
	0	1	2	3	4

Special cases

If s is the sequence of k ones, then $x_s = k$, $y_s = k(k-1)/2$, $z_s = 0$, and the first inequality is one from Matus' infinite lists:

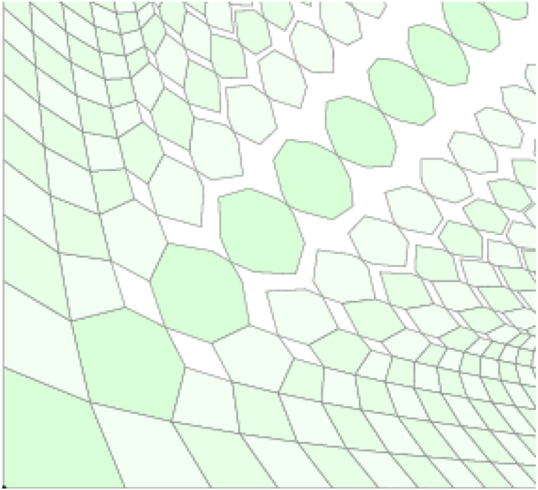
$$[abcd] + \frac{1}{k}(a,b\,|\,c) + \big(1 + \frac{k-1}{2}\big)\big((a,c\,|\,b + (b,c\,|\,a)\big) \ge 0$$

When s is the sequence $\langle k, k-1 \ldots, 2, 1\rangle$ then $x_s = 2^k - 1$, $y_s = z_s = (k-2)2^{k-1} + 1$, and the first inequality is Theorem 10 from Dougherty, Freiling and Zeger:

$$\begin{aligned}[abcd] &+ \frac{1}{2^k-1}(a,b\,|\,c) + \\ &+ \big(1 + \frac{(k-2)2^{k-1}+1}{2^k-1}\big)\big((a,c\,|\,b) + (b,c\,|\,a)\big) + \\ &+ \frac{(k-2)2^{k-1}+1}{2^k-1}\big((a,d\,|\,b) + (b,d\,|\,a)\big) \ge 0,\end{aligned}$$

Outline

2 pages



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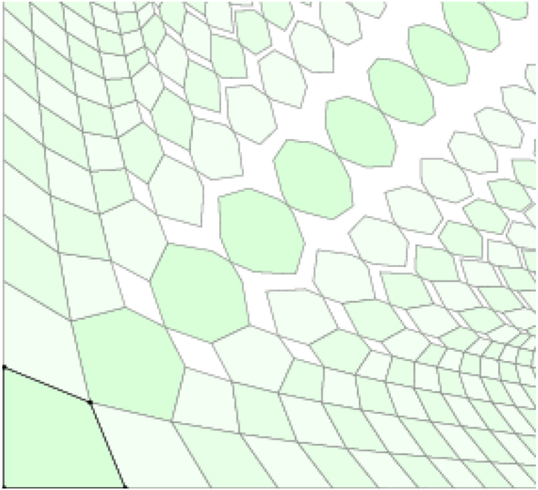

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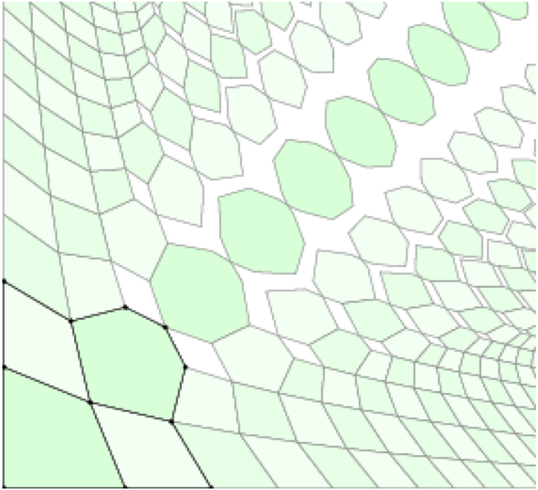

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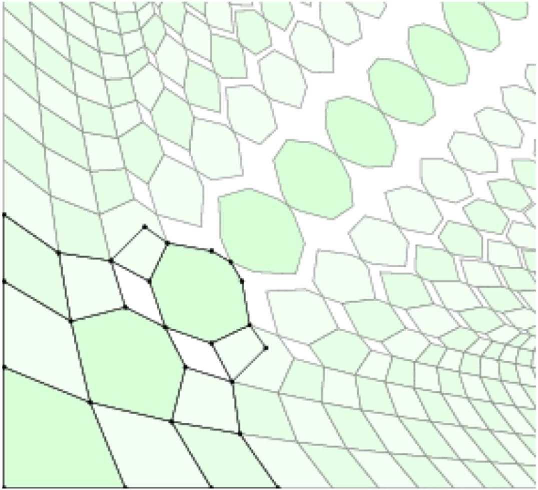

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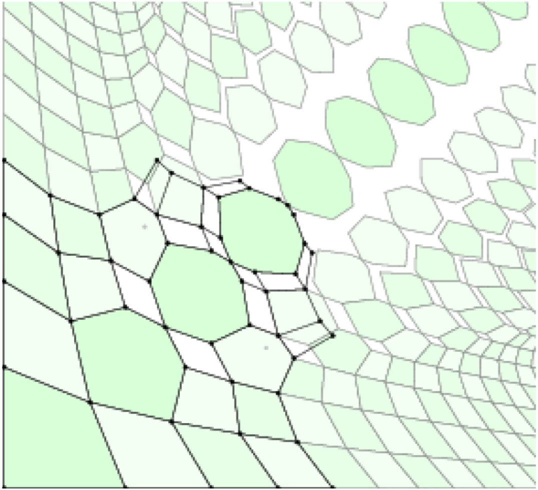

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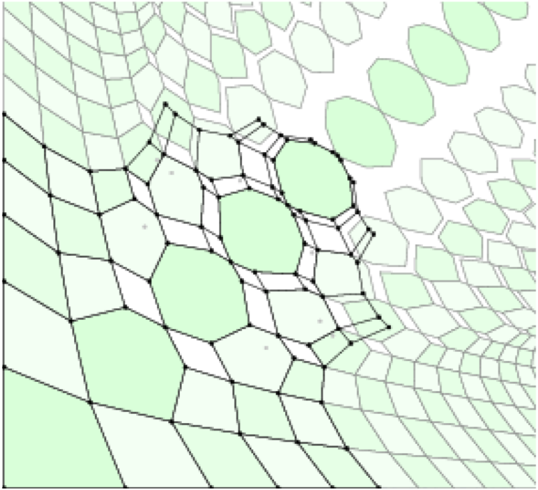

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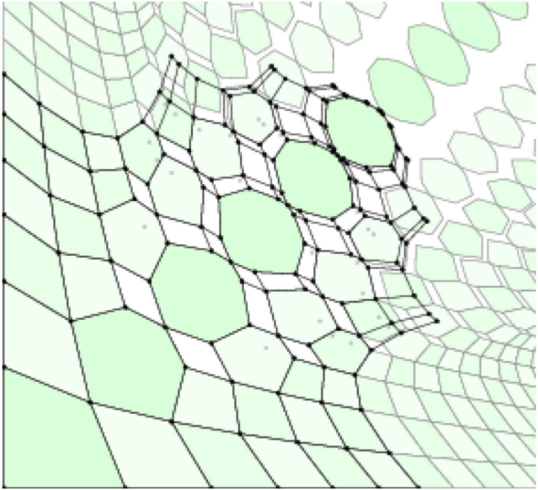

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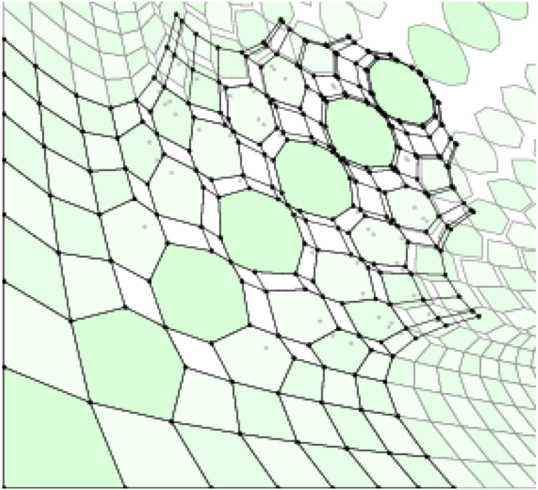

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