Geometry of the entropy region - III

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Outline

Private information

Definition

The **private info** of $a \in N$ is what a knows but nobody else does, that is, the difference between $\boldsymbol{H}(N)$ and $\boldsymbol{H}(N-a)$.

Claim

For an almost entropic point g, one can freely add and take away private info, and it still remains almost entropic.

Proof.

- $g + \lambda \boldsymbol{r}_a$ adds $\lambda \geq 0$ amount of private info to $a \in N$.
- Let $t = g(N) - g(N-a)$. Then $g{\downarrow}a = g{\downarrow}_t^a$ takes away all private info from a. □

Reminder:

$$g{\downarrow}_t^a(J) = \min\{g(aJ) - t, g(J)\} \quad \text{for all } J \subseteq N.$$

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Corollary (Reduction)

When investigating the entropy region, we may assume that no variable has private info.

Visualizing the entropy region of 3 random variables

For a view of the entropy region determined by $N = 3$ random variables choose another **coordinate system** determined by

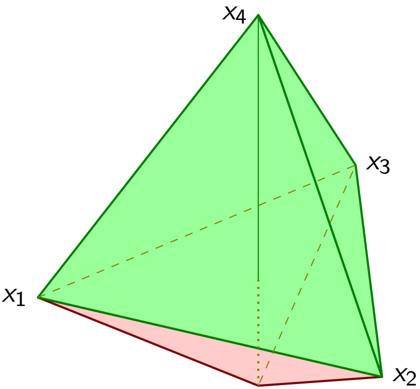
$$x_1 = (a, b|c),\ x_2 = (b, c|a),\ x_3 = (c, a|b),$$
$$x_4 = (a, b) - (a, b|c) = (b, c) - (b, c|a) = (a, c) - (a, c|b).$$
$$x_5 = (a|bc),\ x_6 = (b|ac),\ x_7 = (c|ab),$$

The last three coordinates are the **private info**, and can be discarded.

The rest determines a convex pointed cone in $\mathbb{R}^4$, which can be visualized by using (x_1, x_2, x_3, x_4) as **barycentric** coordinates: set weights (x_1, x_2, x_3, x_4) at vertices of a regular tetrahedron. $\Rightarrow$

Image of the 3-variable entropy region

Set weights (x_1, x_2, x_3, x_4) at vertices of a regular tetrahedron.



$x_4 = \boldsymbol{I}(a, b, c)$ can be negative (pink bottom part).

Outline

Shorthands for entropy expressions

We denote the four random variables by a, b, c, d. Letters $\boldsymbol{H}$ and $\boldsymbol{I}$ denoting entropy and mutual information are omitted:

- $(a, b) = \boldsymbol{I}(a, b)$ ⇐ mutual info
- $(a, b \,|\, c) = \boldsymbol{I}(a, b \,|\, c)$ ⇐ conditional mutual info
- $(a \,|\, bcd) = \boldsymbol{H}(a \,|\, bcd)$ ⇐ private info
- $[abcd\,] = -\boldsymbol{I}(a, b) + \boldsymbol{I}(a, b \,|\, c) + \boldsymbol{I}(a, b \,|\, d) + \boldsymbol{I}(c, d)$ ⇐ Ingleton expression

The **Ingleton** expression is symmetric in *ab* and *cd*:

$$[abcd\,] = [\overset{\curvearrowright}{ba}cd\,] = [ab\overset{\curvearrowright}{dc}\,] = [\overset{\curvearrowright}{ba}\overset{\curvearrowright}{dc}\,].$$

There are **six** non-equivalent Ingleton expressions:

$$[abcd\,] \qquad [acbd\,] \qquad [adbc\,] \qquad [bcad\,] \qquad [bdac\,] \qquad [cdab\,].$$

Why Ingleton is so important

Definition

$\boldsymbol{H}^{\square} \subset \bar{\Gamma}_4^*$ where **all six** Ingleton expressions are ≥ 0;

$\boldsymbol{H}^{\square}_{ab}$, $\boldsymbol{H}^{\square}_{ac}$, ... where the corresponding Ingleton is ≤ 0,
that Ingleton is *violated*.

Theorem (Matus – Studeny, 1995)

- $\bar{\Gamma}_4^* = \boldsymbol{H}^{\square} \cup \boldsymbol{H}^{\square}_{ab} \cup \boldsymbol{H}^{\square}_{ac} \cup \boldsymbol{H}^{\square}_{ad} \cup \boldsymbol{H}^{\square}_{bc} \cup \boldsymbol{H}^{\square}_{bd} \cup \boldsymbol{H}^{\square}_{cd}$.
- *Any two of the abopve parts have disjoint interior; common points are on the boundary of the core* $\boldsymbol{H}^{\square}$.
- $\boldsymbol{H}^{\square}$ *is a full dimensional closed polyhedral cone.*
- *Vertices and internal points of* $\boldsymbol{H}^{\square}$ *are linearly representable.*
- $\boldsymbol{H}^{\square}_{ab}$, ..., $\boldsymbol{H}^{\square}_{cd}$ *are isomorphic; isomorphisms are provided by permutations of* a, b, c, d. □

If . . .

If we know $\boldsymbol{H}_{ab}^{\Box}$,

then

we know everything*

If . . .

If we know $\boldsymbol{H}^{\square}_{ab}$,

then

we know everything*

*at least about Γ_4^*.

The case of five variables

Research problem

Give a similar decomposition of the 31-dimensional polymatroid cone Γ_5^* of five variables.

- Γ_5^* has a 120-fold symmetry;
- the enclosing Shannon polytope has 117978 vertices [1];
- the vertices fall into 1319 equivalence classes[1] (into 15 classes in case of four variables);
- the linearly representable core of Γ_5^* is known precisely[2].

[1] M. Studeny, R. R. Bouckaert, T. Kocka: *Extreme supermodular set functions over five variables*

[2] R. Dougherty, C. Freiling, K. Zeger: *Linear rank inequalities on five or more variables*

Bounding facets of $H^{\square}_{ab}$

$H^{\square}_{ab}$ is contained in the simplex determined by these facets:

1.	$[abcd]$,	$\Leftarrow$ the Ingleton facet
2, 3.	$(a,b\|c)$, $(a,b\|d)$,	$\Leftarrow$ Shannon facets
4, 5.	$(c,d\|a)$, $(c,d\|b)$,	$\Downarrow$
6–9.	$(a,c\|b)$, $(a,d\|b)$, $(b,c\|a)$, $(b,d\|a)$,	
10.	(c,d),	
11.	$(a,b\|cd)$,	
12–15.	$(a\|bcd)$, $(b\|acd)$, $(c\|abd)$, $(d\|abc)$.	

$H^{\square}_{ab}$ is on the ≤ 0 side of the Ingleton facet, and on the ≥ 0 side of the other 14 Shannon-facets.

$\Rightarrow$ The **base** of the simplex is in $H^{\square}$.

$\Rightarrow$ The base is entropic, the top is **not**.

Using "natural" coordinates

Definition

Use the facet equations as the coordinates of the entropy vector.

Example: entropies  of the ringing bells distribution:

a	b	c	d	Prob
0	0	0	0	1/4
1	0	0	1	1/4
1	0	1	0	1/4
1	1	1	1	1/4

Original entropy vector:

a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
.811	.811	1	1	1.5	1.5	1.5	1.5	1.5	2	2	2	2	2	2

The same in natural coordinates:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
-0.12	0	0	.19	.19	.19	.19	.19	.19	0	0	0	0	0	0

Transformations which preserve entropic points

1. $[a, b, c, d]$,

2, 3. $(a, b|c)$, $(a, b|d)$,

4, 5. $(c, d|a)$, $(c, d|b)$,

6–9. $(a, c|b)$, $(a, d|b)$, $(b, c|a)$, $(b, d|a)$,

10. (c, d),

11. $(a, b|cd)$,

12–15. $(a|bcd)$, $(b|acd)$, $(c|abd)$, $(d|abc)$

1. The private info can be discarded (replace them by zero).

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10.	(c, d),
11.	$(a, b\|cd)$,
12–15.	0, 0, 0, 0

1. The private info can be discarded (replace them by zero).

Transformations which preserve entropic points

1.	$[a, b, c, d]$,
2, 3.	$(a, b\vert c) + t$, $(a, b\vert d)$,
4, 5.	$(c, d\vert a)$, $(c, d\vert b)$,
6–9.	$(a, c\vert b)$, $(a, d\vert b)$, $(b, c\vert a)$, $(b, d\vert a)$,
10.	$(c, d) - t$,
11.	$(a, b\vert cd)$,
12–15.	0, 0, 0, 0

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2. Using $g{\uparrow}_t^c$, values from 10 can be moved to 2 (or 3).

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4, 5.	$(c, d\vert a) + u$, $(c, d\vert b)$,
6–9.	$(a, c\vert b)$, $(a, d\vert b)$, $(b, c\vert a)$, $(b, d\vert a)$,
10.	0,
11.	$(a, b\vert cd) - u$,
12–15.	0, 0, 0, 0

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2. Using $g{\uparrow}^c_t$, values from 10 can be moved to 2 (or 3).
3. Using $g{\downarrow}^a_t$, values from 11 can be moved to 4 (or 5).

Transformations which preserve entropic points

1.	$[a, b, c, d]$,
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6–9.	$(a, c\|b)$, $(a, d\|b)$, $(b, c\|a)$, $(b, d\|a)$,
10.	0,
11.	0,
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3. Using $g{\downarrow}_t^a$, values from 11 can be moved to 4 (or 5).

Transformations which preserve entropic points

1.	$[a, b, c, d]$,	$= -\alpha/4$
2, 3.	$(a, b\|c)^*$, $(a, b\|d)$,	$= \beta$
4, 5.	$(c, d\|a)*$, $(c, d\|b)$,	$= \gamma/2$
6–9.	$(a, c\|b)$, $(a, d\|b)$, $(b, c\|a)$, $(b, d\|a)$,	$= \delta$
10.	0,	
11.	0,	
12–15.	0, 0, 0, 0	

1. The private info can be discarded (replace them by zero).
2. Using $g{\uparrow}^c_t$, values from 10 can be moved to 2 (or 3).
3. Using $g{\downarrow}^a_t$, values from 11 can be moved to 4 (or 5).
4. As $\alpha + \beta + \gamma + \delta = \boldsymbol{H}(abcd)$, use them as **barycentric** coordinates to visualize the *central symmetrical part* of $\boldsymbol{H}^{\square}_{ab}$.

Outline

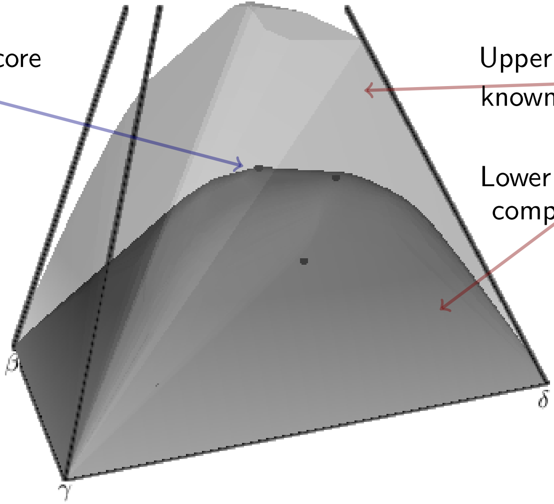
Upper and lower bounds

Best Ingleton score

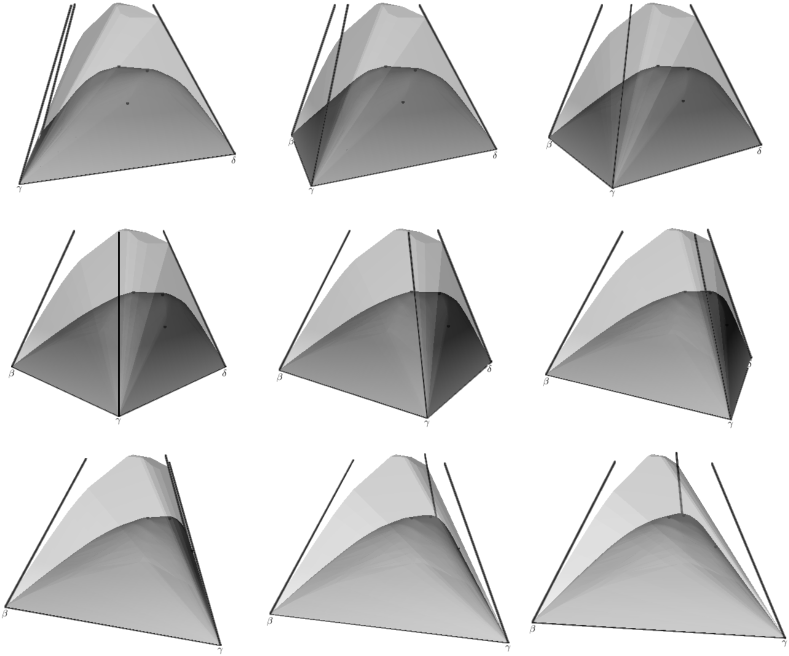
(so far)

Upper bound from known inequalities

Lower bound from computer search



https://www.youtube.com/watch?v=sam97F7oDnE


γ
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Outline

An entropy inequality

This is a **Shannon** inequality checked by xitip*:

$$[abcd] + (z, b \mid c) + (z, c \mid b) + (b, c \mid z) \geq -3(z, ad|bc).$$

As z and ad are independent in the black part, the Maximum Entropy Method (MAXE) says that in this case $(z, ad \mid bc) = 0$ can be assumed:

$$[abcd] + (z, b \mid c) + (z, c \mid b) + (b, c \mid z) \geq 0$$

is a five-variable entropy inequality.

Setting $z = a$ we get the Zhang-Yeung inequality.

* `http://xitip.epfl.ch`, or `https://github.com/lcsirmaz/minitip`

An entropy inequality

Theorem (Matus)

For each $k \geq 0$ this is a 5-variable entropy inequality:

$$k\,[abcd] \;+\; \frac{k(k-1)}{2}\big((a,b\,|\,c)+(a,c\,|\,b)\big) \;+ \\ +\; k\big((z,b\,|\,c)+(z,c\,|\,b)\big) \;+\; (b,c\,|\,z) \geq 0$$

For $k=0$ this is Shannon; for $k=1$ it is the previous inequality.

Proof.

By induction on k. By MAXE, $(z, ad\,|\,bc)=0$. Use the induction hypothesis for the variables az, bz, cz, d, az to get

$$k\,[az,bz,cz,d] + \frac{k(k-1)}{2}\big((az,bz\,|\,cz)+(az,cz\,|\,zb)\big) + \\ + k\big((az,bz\,|\,cz)+(az,cz\,|\,bz)\big) + (bz,cz\,|\,az) \geq 0$$

$\Rightarrow$

An entropy inequality

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$\Rightarrow$

An entropy inequality – proof

$\Rightarrow$

$$k\,[az,bz,cz,d\,] + \frac{k(k-1)}{2}\big((az,bz\,|\,cz)+(az,cz\,|\,zb)\big) + \\ + k\big((az,bz\,|\,cz)+(az,cz\,|\,bz)\big) + (bz,cz\,|\,az) \geq 0.$$

These are Shannon inequalities, and by MAXE, $(z,ad\,|\,bc)=0$:

$$[abcd\,] + (b,c\,|\,z) + \\ + (z,b\,|\,c) + (z,c\,|\,b) \geq (bz,cz\,|\,az) - 3(z,ad\,|\,bc),$$

$$[abcd\,] + (z,b\,|\,c) + (z,c\,|\,b) \geq [az,bz,cz,d\,] - 3(z,ad\,|\,bc),$$

$$(a,b\,|\,c) \geq (az,bz\,|\,cz) - (z,ad\,|\,bc),$$

$$(a,c\,|\,b) \geq (az,cz\,|\,bz) - (z,ad\,|\,bc).$$

An entropy inequality – proof

$$\Rightarrow \quad k\,[az,bz,cz,d\,] + \frac{k(k-1)}{2}\big((az,bz\,|\,cz)+(az,cz\,|\,zb)\big) + \\ + k\big((az,bz\,|\,cz)+(az,cz\,|\,bz)\big)+(bz,cz\,|\,az)\geq 0.$$

These are Shannon inequalities, and by MAXE, $(z,ad\,|\,bc)=0$:

$$\Rightarrow 1 * [abcd\,] + (b,c\,|\,z) + \\ + (z,b\,|\,c)+(z,c\,|\,b)\geq (bz,cz\,|\,az)-3(z,ad\,|\,bc),$$

$$\Rightarrow k * [abcd\,] + (z,b\,|\,c)+(z,c\,|\,b)\geq [az,bz,cz,d\,]-3(z,ad\,|\,bc),$$

$$\Rightarrow k(k+1)/2 * \quad (a,b\,|\,c)\geq (az,bz\,|\,cz)-(z,ad\,|\,bc), \\ (a,c\,|\,b)\geq (az,cz\,|\,bz)-(z,ad\,|\,bc).$$

Sum them up; the LHS is the claim for $k+1$, the RHS is ≥ 0 by induction. □

A useful non-linear entropy inequality

Corollary

If $[abcd] \le 0$, then

$$(2(b,c\,|\,a) - 3[abcd])\big((a,b\,|\,c) + (a,c\,|\,b)\big) \ge [abcd]^2.$$

Proof.

$\mathcal{I} = [abcd]$, $\mathcal{B} = (b,c\,|\,a)$, $\mathcal{C} = (a,b\,|\,c) + (a,c\,|\,b)$. Setting $z = a$ in Matus' theorem we have

$$2k\mathcal{I} + 2\mathcal{B} + k(k+1)\mathcal{C} \ge 0.$$

By assumption, $\mathcal{I} \le 0$; choose $k \ge 0$ with $-1 - \mathcal{I}/\mathcal{C} < k \le -\mathcal{I}/\mathcal{C}$:

$$\Rightarrow \mathcal{C} * \qquad 2(-1 - \mathcal{I}/\mathcal{C})\mathcal{I} + 2\mathcal{B} + (-\mathcal{I}/\mathcal{C})(-\mathcal{I}/\mathcal{C} + 1)\mathcal{C} \ge 0,$$

$$2(-\mathcal{C} - \mathcal{I})\mathcal{I} + 2\mathcal{B}\mathcal{C} + \mathcal{I}(\mathcal{I} - \mathcal{C}) \ge 0,$$

$$-3\mathcal{I}\mathcal{C} + 2\mathcal{B}\mathcal{C} - \mathcal{I}^2 \ge 0.$$

□

A 2-dimensional view of $\bar{\Gamma}_4^*$

1. Start from this consequence of Matus' inequality:

$$\big(2(b,c\,|\,a)-3[abcd\,]\big)\big((a,b\,|\,c)+(a,c\,|\,b)\big)\geq [abcd\,]^2$$

2. Take the **cross-section** of $\bar{\Gamma}_4^*$ with the hyperplane

$$2(b,c\,|\,a)-3[abcd\,]=2.$$

Alternate view: norm the entropies according to this equation.

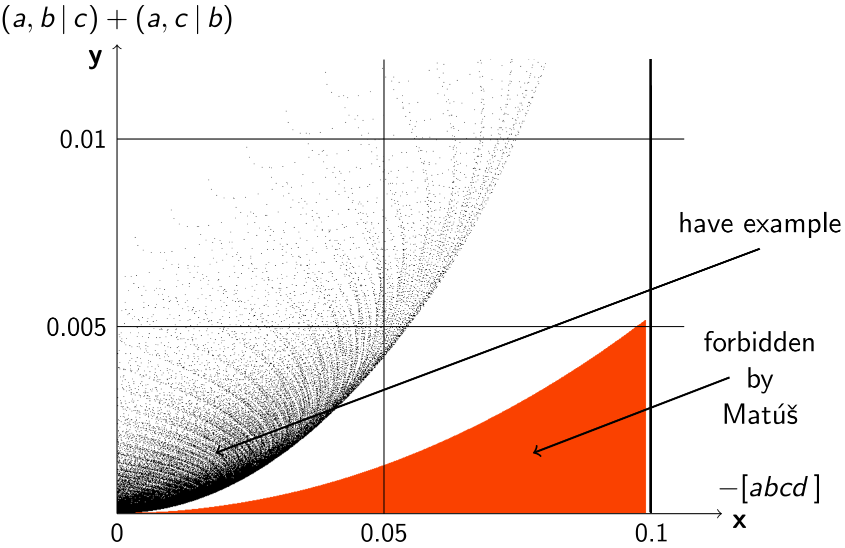
3. Consider the 2-dimensional plane spanned by the vectors

$$\mathbf{x}=-[abcd\,] \quad \text{and} \quad \mathbf{y}=(a,b\,|\,c)+(a,c\,|\,b).$$

4. **Project** the cross-section to this plane. Matus' inequality restricts where the projection can go: it must satisfy

$$2y\geq x^2, \quad \text{i.e.,} \quad y\geq x^2/2.$$

Picture


$(a,b\,|\,c)+(a,c\,|\,b)$
y
0.01
0.005
0
0.05
0.1
x
have example
forbidden
by
Matúš
$-[abcd]$

Where the examples are coming from?

Take the ringing bells  distribution with $s = 2 + 2\varepsilon$ and $\varepsilon \to 0$:

a	b	c	d	Prob
0	0	0	0	ε/s
1	0	0	1	$1/s$
1	0	1	0	ε/s
1	1	1	1	$1/s$

$$-[abcd] = \varepsilon/2 + O(\varepsilon^3),$$
$$(a, b \mid c) = 0,$$
$$(b, c \mid a) = 1 + O(\varepsilon \log \varepsilon),$$
$$(a, c \mid b) = \frac{1}{2 \ln 2}\varepsilon^2 + O(\varepsilon^3).$$

With these distributions $x \approx -[abcd] = \varepsilon/2 + O(\varepsilon^3)$, $y \approx (a, b \mid c) + (a, c \mid b) = \varepsilon^2/(2 \ln 2) + O(\varepsilon^3)$, which means

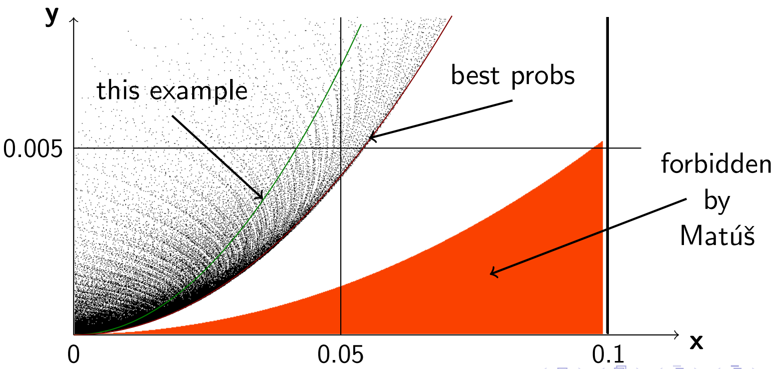
$$y = \frac{2}{\ln 2}x^2 + O(x^4) \approx 2.8854\, x^2.$$

Improving the constant

Fine tuning the probabilities in the  bells distribution, the constant 2.8854 in

$$y = \frac{2}{\ln 2} x^2 + O(x^3) \approx 2.8854\, x^2$$

can be lowered to around 1.688.



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Research Problem

Find a sequence of distributions which improve this constant. You need to look beyond the ringing bells distribution.

Outline

Outline of the attack

Mimic the idea of the proof that $\bar{\Gamma}_4^*$ is not polyhedral:

1. **Find** a good cross-section of $\bar{\Gamma}_4^*$.
2. **Project** the cross-section to a well-chosen two-dimensional plane.
3. **Find** an entropy inequality which excludes the pointset $\mathcal{X}$ of the plane.
4. **Find** distributions in the cross-section whose projection to the plane give $\mathcal{D}$.
5. **Prove** that $\mathcal{X}$ and $\mathcal{D}$ cannot be separated by a semi-algebraic curve.

Outline of the attack

For each point there are good candidates, but more work is needed.

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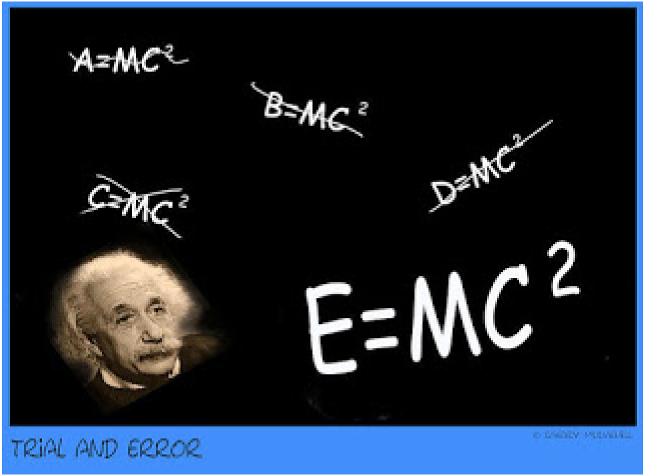
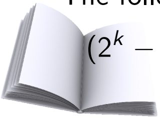


#3: A useful entropy inequality

The following **book inequality** was discovered by Dougherty *et al.*



$$(2^k-1)\,[abcd\,] + (a,b\,|\,c) + k2^{k-1}\,\big((a,c\,|\,b)+(b,c\,|\,a)\big)$$
$$+\,(k2^{k-1}-2^k+1)\,\big((a,d\,|\,b)+(b,d\,|\,a)\big) \ge 0.$$

Using $\mathcal{I}=-[abcd\,]$, $\mathcal{B}=(a,b\,|\,c)$, $\mathcal{C}=(a,c\,|\,b)+\cdots+(b,d\,|\,a)$,

$$-(2^k-1)\,\mathcal{I}+\mathcal{B}+k2^{k-1}\mathcal{C}\ge 0.$$

Take the **cross-section** defined by $\mathcal{I}+\mathcal{B}=1$; then $1+k2^{k-1}\mathcal{C}\ge 2^k\mathcal{I}$. Assuming $\mathcal{I}$ is positive, choose $2\le 2^k\mathcal{I}$. Then

$$\underset{1+\log_2(2/\mathcal{I})}{k\ge}\quad \underset{(1/\mathcal{I})}{2^{k-1}\ge}\quad \mathcal{C}\ge 1+k2^{k-1}\mathcal{C}\ge 2^k\mathcal{I}\ge 2,$$

This gives the **forbidden region** for $\langle \mathcal{I}/(\mathcal{I}+\mathcal{B}),\mathcal{C}/(\mathcal{I}+\mathcal{B})\rangle$:

$$\mathcal{X}=\left\{\langle x,y\rangle \;:\; y\ge \frac{x}{1-\log_2 x} > -0.5\,\frac{x}{\log_2 x}\right\}.$$

$\Rightarrow$

#4 and #5: sample distributions

$\Rightarrow$
$$\mathcal{X} = \left\{ \langle x, y \rangle \,:\, y \geq \frac{x}{1 - \log_2 x} > -0.5\, \frac{x}{\log_2 x} \right\}.$$

Looking at the  ringing bells distribution,

$$\mathcal{B} = (a, b \,|\, c) = 0,$$

whatever probabilities are chosen.

#4 and #5: sample distributions

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$$\mathcal{X} = \left\{ \langle x, y \rangle \ : \ y \geq \frac{x}{1 - \log_2 x} > -0.5 \, \frac{x}{\log_2 x} \right\}.$$

Using $\mathcal{I} = -[abcd]$, $\mathcal{B} = (b, c \,|\, a)$, $\mathcal{C} = (a, b \,|\, c) + (a, c \,|\, b) + (a, b \,|\, d) + (a, d \,|\, b)$ and the cross-section $\mathcal{I} + \mathcal{B} = 1$, **we can do better . . .** (see next page)

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$\Rightarrow$
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Using $\mathcal{I} = -[abcd\,]$, $\mathcal{B} = (b, c \,|\, a)$, $\mathcal{C} = (a, b \,|\, c) + (a, c \,|\, b) + (a, b \,|\, d) + (a, d \,|\, b)$ and the cross-section $\mathcal{I} + \mathcal{B} = 1$,
we can do better . . . (see next page)

No corresponding entropy inequality is known. (But probably exists.)

Tweaking the bells distribution

$\mathcal{X} \Rightarrow$

$$\mathcal{X} = \left\{ \langle x, y \rangle \; : \; y \geq \frac{x}{1-\log_2 x} > -0.5 \, \frac{x}{\log_2 x} \right\}.$$

Using the probabilities below with $s = (1+\varepsilon)^2$,

a	b	c	d	Prob
0	0	0	0	ε^2/s
1	0	0	1	ε/s
1	0	1	0	ε/s
1	1	1	1	$1/s$

$$\mathcal{I} = -(1+o(1))\,\varepsilon^2 \log_2 \varepsilon,$$
$$(a,b\,|\,c) = (a,b\,|\,d) = 0,$$
$$\mathcal{B} = -(1+o(1))\,\varepsilon \log_2 \varepsilon,$$
$$\mathcal{C} = (2+o(1))\,\varepsilon^2,$$

which gives the example dataset for $\mathcal{I}/(\mathcal{I}+\mathcal{B})$ and $\mathcal{C}/(\mathcal{I}+\mathcal{B})$:

$\mathcal{D} \Rightarrow$

$$\mathcal{D} = \left\{ \langle x, y \rangle : y = -(2+o(x)) \frac{x}{\log_2 x} \right\}.$$

Observe: $\mathcal{X}$ and $\mathcal{D}$ are **inseparable** by algebraic curves.

Conclusion

To prove that $\bar{\Gamma}_4^*$ is **not** semi-algebraic,

Research problem #3

Prove this variant of the book inequality à la Matúš:

$$(2^k-1)\,[abcd\,] + (b,c\,|\,a) + k2^{k-1}\big((a,b\,|\,c)+(a,c\,|\,b)\big) \\ + (k2^{k-1}-2^k+1)\big((a,b\,|\,d)+(a,d\,|\,b)\big) \ge 0.$$

Or,

Research problem #4

Show that the quoted book inequality is **essentially sharp** by giving examples where

$$y \le \text{const}\,\frac{x}{\log_2(1/x)}$$

for small values of x.

Thank you for your attention