Continuos submodular optimization

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Submodular functions

f is **submodular** over any lattice:

$$f(A) + f(B) \geq f(A \wedge B) + f(A \vee B).$$

In $\mathbb{R}^n$ this is the min and max, coordinatewise.

Diminishing return property (coordinatewise):

$$f(x + \varepsilon e_i) - f(x) \geq f(y + \varepsilon e_i) - f(y)$$

if $y = x + \lambda e_i$, $\lambda > 0$, and $\varepsilon > 0$.

(Investing the same amount of resouce, if you have more of that resource then the return is smaller.)

Entropy-like function

(a) f is defined on $\{x \in \mathbb{R}^n : x \geq 0\}$

(b) $f(0) = 0$ (pointed)

(c) non-decreasing: $0 \leq x \leq y \Rightarrow f(x) \leq f(y)$

(d) submodular: $f(x) + f(y) \geq f(x \wedge y) + f(x \vee y)$

(e) has the diminishing return property

Motivation: secret sharing of n groups.
Symmetric for any permutation fixing all groups.
$f(x_1, \ldots x_n)$ is the scaled entropy of the shares given to $x_i \cdot N$ people from the i-th group

Left and right partial derivatives

Left i-th partial derivative (if exists)

$$f_i^-(x) = \lim_{\varepsilon \to +0} \frac{f(x) - f(x - \varepsilon e_i)}{\varepsilon}$$

Right i-th partial derivative (if exists)

$$f_i^+(x) = \lim_{\varepsilon \to +0} \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}$$

Basic properties

1. f is continuous
2. concave along any positive direction: for $0 \leq x \leq y$

$$\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x) + (1-\lambda) y).$$

3. D.R. property holds for any $x \leq y$ (not only coordinatewise)
4. f has left and right partial derivatives everywhere inside
5. partial derivatives are ≥ 0 and decreasing along *positive* directions.

Proof of (2)

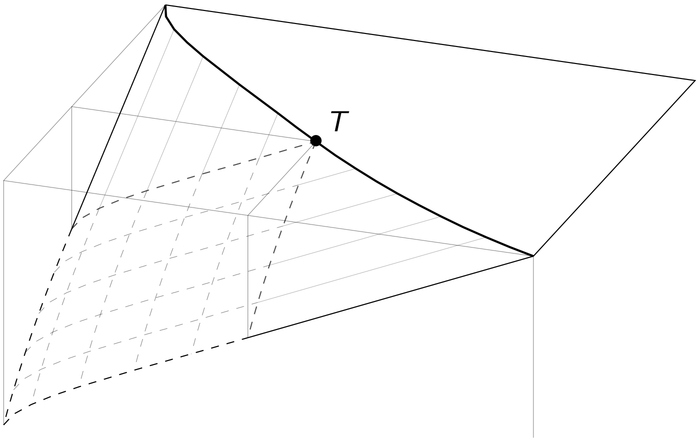
Concave along any coordinate by continuity and DR property.
By induction for points $(c, x, a) \leq (d, y, a)$:

$$\begin{aligned} \lambda f(c \phi d, x, a) + (1-\lambda) f(c \phi d, y, a) &\leq f(c \phi d, x \phi y, a), \\ \lambda^2 f(c, x, a) + \lambda(1-\lambda) f(d, x, a) &\leq \lambda f(c \phi d, x, a), \\ \lambda(1-\lambda) f(c, y, a) + (1-\lambda)^2 f(d, y, a) &\leq (1-\lambda) f(c \phi d, y, a), \\ \hline \lambda f(c, x, a) + (1-\lambda) f(d, y, a) &\leq f(c \phi d, x \phi y, a) \end{aligned}$$

Use submodularity $\lambda(1-\lambda)$ times:

$$f(c, x, a) + f(d, y, a) \leq f(c, y, a) + f(d, x, a)$$

A 2-dimensional example



The optimization problem

f is **feasible** for the $n-1$-dimensional surface S if in each point $x \in S$ the partial derivatives drop by at least 1:

$$f_i^-(x) - f_i^+(x) \geq 1 \quad (1 \leq i \leq n).$$

The **cost** of f is

$$\mathrm{Cost}(f) = \max\{\, f_1^+(0), f_2^+(0), \ldots, f_n^+(0)\,\},$$

and the optimization problem is:

$$\begin{cases} \text{minimize:} & \mathrm{Cost}(f) \\ \text{subject to:} & f \text{ is an } S\text{-feasible EL function.} \end{cases}$$

Linear constraints

S is a hyperplane $c_1x_1 + c_2x_2 + \cdots + c_nx_n = M$. Search an optimal function among $k > 0$:

$$f(y) = k \cdot \min\left\{\sum c_i y_i, M\right\}.$$

Here $f_i^-(x) - f_i^+ = k \cdot c_i$ at points of S (f is linear), so $k \geq 1/\min\{c_i\}$. Also, $\mathrm{Cost}(f) = k \cdot \max\{c_i\}$, thus

$$\mathrm{OPT}(S) \leq \frac{\max\{c_i\}}{\min\{c_i\}}.$$

Theorems

Theorem (Lower bound)

For every s-surface S, inner point $x \in S$ and $1 \leq i, j \leq n$ the following inequality holds:

$$\mathrm{OPT}(S) \geq \frac{\nabla S_j(x)}{\nabla S_i(x)},$$

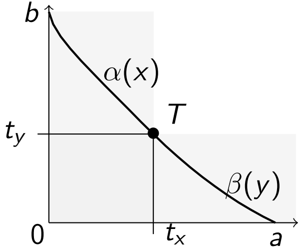
where $\nabla S(x)$ is the outward normal of S at $x \in S$.

Theorem (Existence)

Suppose S is smooth and $\mathrm{OPT}(S) < +\infty$. Then the optimal value is taken by some S-feasible function f, that is, $\mathrm{Cost}(f) = \mathrm{OPT}(S)$.

2-dimensional case

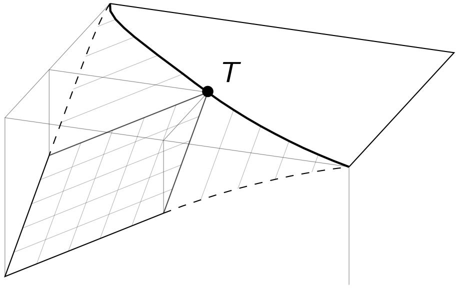
S is a strictly decreasing continuous curve.
$S = \{(x, \alpha(x)) : 0 \le x \le a\}$, and $S = \{(\beta(y), y) : 0 \le y \le b\}$.



S is either convex or concave $\Rightarrow$ $\nabla S_i(x)/\nabla S_j(x)$ is increasing or decreasing along the curve $\Rightarrow$ attains its maximal value at one of the endpoints.

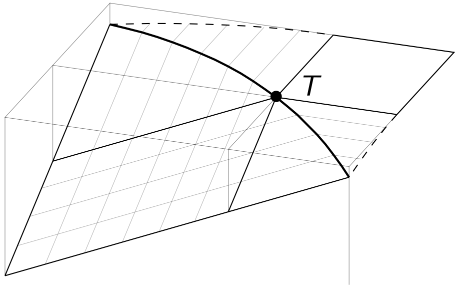
For strictly convex S the lower bound is tight

$$f(x,y) = \begin{cases} C + \min\{y - \alpha(x), 0\} & \text{if } x \geq t_x, \\ C + \min\{x - \beta(y), 0\} & \text{if } y \geq t_y, \\ x + y & \text{otherwise,} \end{cases}$$



For strictly concave S the lower bound in tight

$$f(x, y) = \begin{cases} y + \min\{x, \beta(y)\} & \text{if } x \geq t_x, \\ x + \min\{y, \alpha(x)\} & \text{if } y \geq t_y, \\ x + y & \text{otherwise.} \end{cases}$$



Questions

Problem (1)

For every smooth S with bounded normal there is a feasible function f.

Problem (1a)

Show that there a feasible function with finite cost.

Problem (2)

Find an S where the lower bound is not tight.

Problem (3)

Determine the cost of convex surfaces in dimensions > 2.