Note on an inequality of Wegner

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January 2, 2007

Abstract

G. Wegner [12] gave a geometric characterization of all so–called Groemer packing of $n \geq 2$ unit discs in $\mathbb{E}^2$ that are densest packings of n unit discs with respect to the convex hull of the discs. In this paper we provide a number theoretic characterization of all n satisfying that such a "Wegner packing" of n unit discs exists, and show that the proportion of these n is $\frac{23}{24}$ among all natural numbers.

Acknowledgement: We are grateful to J.C. Lagarias for helpful discussions, and to an unknown referee whose remarks considerably improved the paper.

*Supported by OTKA grants T 042769, 043520 and 049301, and by the Marie Curie TOK project DiscConvGeo.

†Supported by OTKA grants T 025617, 038396 and 042750.

1 Introduction

For a convex compact set C in $\mathbb{E}^2$, we write $A(C)$ to denote the area and $P(C)$ to denote the perimeter of C where $P(C)$ is twice of its length if C is a segment. In addition B^2 denotes the unit disc centred at the origin. Given $n \geq 2$, we investigate the configurations that minimize the area of the convex hull of n non–overlapping discs. Equivalently we define the density of a finite packing of unit discs to be the ratio of the total area of the discs over the area of their convex hull, and we search for some packing of n unit discs of maximal density. Let us describe the known results.

L. Fejes Tóth proved in his paper [3] from 1949 that if the compact convex set D contains $n \geq 2$ non–overlapping unit discs then

$$A(D) > n \cdot 2\sqrt{3}. \tag{1}$$

The inequality (1) is optimal in the sense that no factor less than $2\sqrt{3}$ works for all n, but it is unfortunate from our point of view that we never have equality in (1). A decade later H. Groemer [7] and N. Oler [8] proved that if the convex compact set C contains the centres of n non–overlapping unit discs then

$$\tfrac{1}{2\sqrt{3}} A(C) + \tfrac{1}{4} P(C) + 1 \geq n, \tag{2}$$

where for any n there exists some C such that equality holds in (2) (see the discussion below). Since $C + B^2$ contains the n unit discs,

$$A(C + B^2) = A(C) + P(C) + \pi, \tag{3}$$

and $P(C) \geq 4$ for $n \geq 2$, the inequality (2) yields (1). We note that N. Oler [8] generalized (2) to Minkowski planes, and his inequality is known as the *Oler inequality*. Actually A. Thue proved certain weaker version of (2) already around 1900 (see [9], [10] and [11]). Therefore we call (2) the *Thue–Groemer inequality*. For other proofs of the Thue–Groemer inequality, see J.H. Folkman and R.L. Graham [6], or K. Böröczky, Jr. [1].

H. Groemer [7] also described the equality case in (2); namely, either C is a segment of length $2(n-1)$, or C is the convex hull of the n centres and can be triangulated using the n centres as vertices into regular triangles of edge length two (see Figure 1). In the latter case each side of C is parallel to some side of a fixed regular triangle from the tiling, hence C has at most six sides. If equality holds in (2) then the corresponding packing of n unit discs is called a *Groemer packing*.

The central question of this paper is whether any densest packing of n unit discs is some Groemer packing. In the following let $n \geq 2$, and let C_n always denote the convex hull of the centres in a Groemer packing of n unit discs. Since equality holds in (2) for C_n, (3) yields that the Groemer packings of highest densities are the ones that minimize $P(C_n)$. As C_n has at most six sides, we have $A(C_n) \leq \frac{\sqrt{3}}{24} P(C_n)^2$ according to the isoperimetric inequality for hexagons (see say A. Florian [5]). In addition $\frac{1}{2} P(C_n)$ is an integer, and equality holds in the Thue–Groemer inequality (2), therefore $P(C_n) \geq 2 \left\lceil \sqrt{12n-3} - 3 \right\rceil$.

L. Fejes Tóth conjectured in [4] that if $n = 6\binom{k}{2} + 1$ for some $k \geq 2$ then in the densest packing of n unit discs, the convex hull of the centres is the regular hexagon of side length $2(k-1)$. G. Wegner [12] proved this conjecture in a more general form. To state his result, we call a Groemer packing of n unit discs a *Wegner packing* if the convex hull C_n of the centres satisfies $P(C_n) = 2 \left\lceil \sqrt{12n-3} - 3 \right\rceil$. In particular any Wegner packing of n unit discs is a densest Groemer packing. A typical example is when $n = 6\binom{k}{2} + 1$ for some $k \geq 2$ and C_n is the regular hexagon of side length $2(k-1)$. On the one hand there exist two non–congruent Wegner packings of 18 unit discs, on the other hand there may not exist any Wegner packing for a given n where the smallest such n is 121 (see G. Wegner [12]). Now G. Wegner [12] proved L. Fejes Tóth's conjecture in the following form:

Theorem 1.1 (Wegner's Inequality) *If D is the convex hull of n non–overlapping unit discs then*

$$A(D) \geq 2\sqrt{3} \cdot (n-1) + (2-\sqrt{3}) \cdot \left\lceil \sqrt{12n-3} - 3 \right\rceil + \pi.$$

Equality holds if and only if the packing is a Wegner packing.

The lower bound of Theorem 1.1 is a very good estimate even if strict inequality holds, as we prove

Theorem 1.2 *For any $n \geq 2$, there exists a Groemer packing of n unit discs whose convex hull D satisfies*

$$A(D) \leq 2\sqrt{3} \cdot (n-1) + (2-\sqrt{3}) \cdot \left\lceil \sqrt{12n-3} - 3 \right\rceil + \pi + 2 - \sqrt{3}.$$

Our main results are to characterize all n such that a Wegner packing of n unit discs exists, and to show that the proportion of these n is $\frac{23}{24}$ among all natural numbers:

Theorem 1.3 *A Wegner packing of $n \geq 2$ unit discs exists if and only if for any positive $k,m \in \mathbb{Z}$, we have $\lceil \sqrt{12n-3} \rceil^2 + 3 - 12n \neq (3k-1) \cdot 9^m$.*

Theorem 1.4 *Given $N \geq 2$, let $f(N)$ be the number of $2 \leq n \leq N$ such that there exists a Wegner packing of n unit discs. Then*

$$\lim_{N\to\infty} \frac{f(N)}{N} = \frac{23}{24}.$$

The results above support the following conjecture:

Conjecture 1.5 *Any densest packing of n unit discs is some Groemer packing. In other words if D is the convex hull of n unit discs in a densest packing then equality holds either in Theorem 1.1 or in Theorem 1.2.*

2 Proofs

Proof of Theorem 1.3: First we present certain formulae that hold for any Groemer packing of n unit discs provided the convex hull C of the centres is a polygon. There exists a regular triangle T circumscribed around C such that at least two sides of T contain some side of C (see Figure 1). We write v_1, v_2, v_3 to denote the vertices of T. If v_i, $i = 1,2,3$, is not a vertex of C then it is the vertex of a regular triangle $T_i \subset T$ such that $C \cap T_i$ is a common side. If v_i is a vertex of C then we define $T_i = \{v_i\}$, which we consider a degenerate regular triangle of side length zero. We define the non–negative $a,b,c \in \mathbb{Z}$ by the property that the side lengths of T_1, T_2 and T_3 are $2a$, $2b$ and $2c$, respectively, and define $d \in \mathbb{Z}$ by the property that the side length of T is $2(a+b+c+d)$. We note that if C is a hexagon then three of its sides with lengths $2(a+d)$, $2(b+d)$, $2(c+d)$ are contained in the boundary of T, and the other three sides are of lengths $2a$, $2b$ and $2c$. In any case we have

$$p = \tfrac{1}{2}P(C) = 2a+2b+2c+3d, \tag{4}$$

$$n = \tfrac{1}{2\sqrt{3}}A(C) + \tfrac{1}{2}p + 1, \tag{5}$$

$$A(C) = \sqrt{3} \cdot ((a+b+c+d)^2 - a^2 - b^2 - c^2). \tag{6}$$

Thus the isoperimetric deficit $p^2 - 2\sqrt{3}A(C)$ can be expressed as

$$(p+3)^2 + 3 - 12n = (2a-b-c)^2 + 3(b-c)^2 + 3d^2. \tag{7}$$

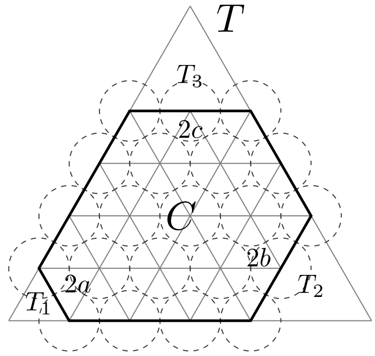

Figure 1:

Let us consider a Wegner packing of n unit discs. We may assume that $n \geq 3$, hence the convex hull C of the centres is a polygon. Using the notion as above, we have $p+3 = \lceil\sqrt{12n-3}\rceil$. As $x^2+3y^2+3z^2$, $x,y,z \in \mathbb{Z}$ is never of the form $(3k-1)\cdot 9^m$ for positive $k,m \in \mathbb{Z}$, (7) yields the necessity condition in Theorem 1.3.

To prove the sufficiency of the condition in Theorem 1.3, we assume that $p = \lceil\sqrt{12n-3}-3\rceil$ satisfies that $\Omega = (p+3)^2+3-12n$ is not of the form $(3k-1)\cdot 9^m$ for positive $k,m \in \mathbb{Z}$. It is known that Wegner packings do exist for $n \leq 30$ (see G. Wegner [12]), hence let $n \geq 31$.

Now 3Ω is not of the form $(9k-3)\cdot 9^m$, therefore

$$3\Omega = 3x^2+\tilde{y}^2+\tilde{z}^2 \tag{8}$$

for some $x,\tilde{y},\tilde{z} \in \mathbb{Z}$ (see L.E. Dickson [2], p.97). Checking remainders modulo 3, we deduce that $\tilde{y} = 3y$ and $\tilde{z} = 3z$ for some $y,z \in \mathbb{Z}$, and hence

$$(p+3)^2+3-12n = x^2+3y^2+3z^2. \tag{9}$$

Changing x to $-x$ if necessary, we may assume that $p \equiv x \pmod 3$. Since a square is 0 or 1 $\pmod 4$, and the roles of y and z are symmetric, we may assume that $p \equiv z \pmod 2$, and hence $x \equiv y \pmod 2$. We define

$$a = \tfrac{p+2x-3z}{6},\ b = \tfrac{p+3y-x-3z}{6},\ c = \tfrac{p-3y-x-3z}{6},\ d = z. \tag{10}$$

The divisibility properties imply that a,b,c,d are integers, and readily

$$x = 2a-b-c,\ y = b-c,\ z = d.$$

We want to make a hexagon (namely $\frac{1}{2}C$) with sides a, b, c, $a+d$, $b+d$ and $c+d$; to this end we need that they are all positive. We have $p < \sqrt{12n-3}-2$, hence (9) yields that

$$x^2+3y^2+3z^2 < (p+3)^2-(p+2)^2 = 2p+5. \tag{11}$$

Here $p \geq 17$ follows by $n \geq 31$, and hence the Cauchy–Schwarz inequality and (11) imply

$$\begin{aligned} 2|x|+3|z| &= 2\cdot|x|+\sqrt{3}\cdot(\sqrt{3}\,|z|) \leq \sqrt{7(2p+5)} < p; \\ |x|+3|y|+3|z| &= 1\cdot|x|+\sqrt{3}\cdot(\sqrt{3}\,|y|)+\sqrt{3}\cdot(\sqrt{3}\,|z|) \leq \sqrt{7(2p+5)} < p. \end{aligned}$$

Therefore a, b, c, $a+d$, $b+d$ and $c+d$ are all positive.

Starting from the regular triangle of side length $2(a+b+c+d)$, we can construct a Groemer packing of certain number of discs such that the convex hull C of the centres has side lengths $2a$, $2(c+d)$, $2b$, $2(a+d)$, $2c$ and $2(b+d)$ in this order. We deduce (4) by (10), hence (9) yields (7). Therefore we have constructed a Wegner packing of n unit discs, completing the proof of Theorem 1.3. □

Proof of Theorem 1.4: Let $g(N) = N - f(N)$. It is sufficient to show that for any $\varepsilon > 0$ and for N large,

$$g(N) = \tfrac{1+O(\varepsilon)}{24}\cdot N \tag{12}$$

where the implied constant in $O(\cdot)$ is some absolute constant. Given $n \geq 2$, we set $s = \lceil\sqrt{12n-3}\rceil$ and define t by the formula

$$12n-3 = s^2-t.$$

The condition $s = \lceil\sqrt{12n-3}\rceil$ is equivalent to $s^2-t > (s-1)^2$, and hence to $s > \frac{t+1}{2}$. On the other hand $n \leq N$ is equivalent to $s \leq \sqrt{12N-3+t}$. Therefore $g(N)$ is the number of "good" pairs $s,t \in \mathbb{N}$ such that $t = l\cdot 9^m$ for some positive $l,m \in \mathbb{Z}$ with $l \equiv -1 \pmod 3$, $n = \frac{s^2-t+3}{12}$ is an integer, and

$$\tfrac{t+1}{2} < s \leq \sqrt{12N-3+t}. \tag{13}$$

If N is large then $t < (1+\varepsilon)2\sqrt{12N}$ and $m < (1+\varepsilon)\log_9 2\sqrt{12N}$ follows by (13). Now n is an integer if and only if either $l \equiv -4 \pmod{12}$ and $s \equiv \pm 3 \pmod{12}$, or $l \equiv -1 \pmod{12}$ and $s \equiv 0,6 \pmod{12}$. We observe that if t is fixed and $t < (1-\varepsilon)2\sqrt{12N}$ then a "good" pair s of t occurs uniformly and with density $\frac{1}{6}$. Therefore after fixing a large N and an m satisfying $1 \leq m \leq (1-\varepsilon)\log_9 2\sqrt{12N}$, the number of the corresponding "good" pairs $s,t \in \mathbb{N}$ is

$$(1+O(\varepsilon)) \sum_{\substack{2 \leq l \leq \frac{(1-\varepsilon)2\sqrt{12N}}{9^m} \\ l \equiv -1,-4 \pmod{12}}} \frac{1}{6} \cdot \left(\sqrt{12N} - \frac{l \cdot 9^m}{2}\right) = (1+O(\varepsilon)) \cdot \frac{N}{3 \cdot 9^m}.$$

We conclude that if N is large then

$$g(N) = (1+O(\varepsilon)) \cdot \sum_{m \geq 1} \frac{N}{3 \cdot 9^m} = (1+O(\varepsilon)) \cdot \frac{N}{24},$$

completing the proof of Theorem 1.4. □

Proof of Theorem 1.2: If there exists no Wegner packing for some n then $p = \lceil \sqrt{12n-3} - 3 \rceil$ is divisible by 3 according to Theorem 1.3. Thus $(p+1+3)^2 + 3 - 12n$ is not equal $(3k-1) \cdot 9^m$ for any positive $k,m \in \mathbb{Z}$. Applying the proof of the sufficiency of the condition in Theorem 1.3 to $q = p+1$ in the place of p yields the existence of a Groemer packing of n unit discs with $p+1$ boundary discs. In turn we conclude Theorem 1.2. □

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