Approximation of smooth convex bodies by circumscribed polytopes with respect to the surface area

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Abstract

Let K be a convex body with C^2 boundary in the Euclidean d-space. Following the work of L. Fejes Tóth, R. Vitale, R. Schneider, P.M. Gruber, S. Glasauer and M. Ludwig, best approximation of K by polytopes of restricted number of vertices or facets is well-understood if the approximation is with respect to the volume or the mean width. In this paper we consider the circumscribed polytope $P_{(n)}$ of n facets with minimal surface area, and present an asymptotic formula in n for the difference of surface areas of $P_{(n)}$ and K.

Key words: polytopal approximation, extremal problems
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1 Introduction

For any notions related to convexity in this paper, consult P.M. Gruber [21], R. Schneider [27] or T. Bonnesen and W. Fenchel [2]. For any quadratic form q, we write $\operatorname{tr} q$ to denote the sum, and $\det q$ to denote the product of the eigenvalues of q, respectively. As usual we call a compact convex set in $\mathbb{E}^k$ with non-empty relative interior a convex body in $\mathbb{E}^k$. For a convex body K in $\mathbb{E}^d$, we write $V(K)$ to denote its volume, and $S(K)$ to denote its surface area. When integrating on the boundary ∂K, we always do it with respect

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to the $(d-1)$-dimensional Hausdorff measure (see K.J. Falconer [10] or C.A. Rogers [25] for definition and main properties). An $X \subset \partial K$ is called Jordan measurable if its relative boundary on ∂K is of $(d-1)$-measure zero. If K has C^2 boundary and $x \in \partial K$, then let Q_x denote the second fundamental form at x, let $H(x) = \operatorname{tr} Q_x$, and let $\kappa(x) = \det Q_x$ be the Gauß-Kronecker curvature. These notions naturally depend on K but we drop the reference to K. We say that K has C^k_+ boundary if ∂K is C^k and the Gauß-Kronecker is positive at each point.

Best approximation of a convex body K with C^2 boundary by circumscribed polytopes with respect to the volume or the mean width has very extensive literature since the 1970's (see the papers P.M. Gruber [16], [17], [18] and [20] for general surveys on related problems), and many of the major questions have been resolved. Here we only summarize the main results concerning the volume. P.M. Gruber [15] proved when ∂K is C^2_+, and K. Böröczky, Jr. [3] in the general case that if $P^{\rm vol}_{(n)}$ is a polytope containing K with n facets that has minimal volume, then as n tends to infinity, we have

$$V(P^{\rm vol}_{(n)}) - V(K) \sim \frac{\operatorname{div}_{d-1}}{2} \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, dx \right)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}} \tag{1}$$

where $\operatorname{div}_{d-1} > 0$ depends only on d. P.M. Gruber [14] determined that $\operatorname{div}_1 = \frac{1}{12}$ and $\operatorname{div}_2 = \frac{5\sqrt{3}}{18}$, and it follows from the work of P.L. Zador [28] that

$$\operatorname{div}_{d-1} = \tfrac{1}{2e\pi}\, d + O(\ln d) \tag{2}$$

where the implied constant in $O(\cdot)$ is an absolute constant.

In many cases important information available about the extremal bodies in polytopal approximation. If $\Xi_n \subset \partial K$ has cardinality n for $n \geq d+1$, and f is a non-negative measurable function on ∂K whose integral on ∂K is positive, then $\{\Xi_n\}$ is uniformly distributed with respect to f (compare S. Glasauer and R. Schneider [12]) provided that for any Jordan measurable $X \subset \partial K$, we have

$$\lim_{n\to\infty} \frac{\#(X \cap \Xi_n)}{n} = \frac{\int_X f(x)\, dx}{\int_{\partial K} f(x)\, dx}.$$

In addition $\{\Xi_n\}$ satisfies the Delone property (compare P.M. Gruber [19]) if there exists $\alpha, \beta > 0$ depending on K and the sequence $\{\Xi_n\}$ such that the distance between any two points of Ξ_n is at least $\alpha n^{\frac{-1}{d-1}}$, and for any $x \in \partial K$ there exists $y \in \Xi_n$ of distance at most $\beta n^{\frac{-1}{d-1}}$ from x.

Now let $\Xi^{\rm vol}_n$ be the family of points where the facets of $P^{\rm vol}_{(n)}$ touch $P^{\rm vol}_{(n)}$. S. Glasauer and P.M. Gruber [13] proved that if ∂K is C^2_+, then the $\{\Xi^{\rm vol}_n\}$

is uniformly distributed on ∂K with respect to $\kappa(x)^{\frac{1}{d+1}}$, and the result was extended to any convex body K with C^2 boundary by K. Böröczky, Jr. [3]. In addition, if ∂K is C^2_+ then P.M. Gruber [19] proved the Delone property for $\{\Xi_n^{\rm vol}\}$.

The analogous results are also known in the case of approximation by circumscribed polytopes with respect to the mean width (see P.M. Gruber [21]). However if closeness is measured in terms of the surface area then the asymptotic formula was only known if K is ball (in that case approximation with respect to the volume and with respect to the surface area are equivalent). The goal of this paper is to fill this gap. It is especially desirable because if best approximation of smooth convex bodies is replaced by random approximation then we have essentially the same amount of information for all these three quermassintegrals (see K. Böröczky, Jr. and M. Reitzner [6]).

We will assign certain number, which is denoted by ${\rm div}(q^*)$, to any positive definite quadratic form q in $d-1$ variables in Section 6. We prove (see (85))

$$\mathrm{div}_{d-1} \leq \mathrm{div}(q^*) \leq \tfrac{d}{d-1} \cdot \mathrm{div}_{d-1} \tag{3}$$

for any positive definite q where equality holds in the upper bound if the eigen values of q coincide. It follows by (2) that independently of the eigen values of q, we have

$$\mathrm{div}(q^*) = \tfrac{1}{2e\pi}\, d + O(\ln d) \tag{4}$$

where the implied constant in $O(\cdot)$ is an absolute constant. If q is a positive semi-definite quadratic form in $d-1$ variables that is not positive definite then we define ${\rm div}(q^*) = {\rm div}_{d-1}$. In particular while ${\rm div}(q^*)$ does depend on the eigen values of q, this dependence is rather "light". If $d = 2$, then the upper bound in (3) yields

$$\mathrm{div}(q^*) = 2\mathrm{div}_1 = 1/6. \tag{5}$$

Unfortunately if $d \geq 3$, then one cannot expect a "nice" closed formula for ${\rm div}(q^*)$ in terms of the eigen values of q. If $d = 3$ and $\tau_2 \geq \tau_1 > 0$ are the eigen values of q then K.J. Böröczky and B. Csikós [4] prove

$$\mathrm{div}(q^*) = \frac{2\tau_1 + \tau_2}{18(\tau_1 + \tau_2)} \cdot \frac{4\tau_1 + 8\tau_2 + (4\tau_1^2 - 2\tau_1\tau_2 + 7\tau_2^2)^{1/2}}{[2\tau_1 + 4\tau_2 + (4\tau_1^2 - 2\tau_1\tau_2 + 7\tau_2^2)^{1/2}]^{1/2}}. \tag{6}$$

This formula could be expressed in terms of the trace and the determinant of q, but the new formula would be even more complicated.

Theorem 1.1 *Given a convex body K in $\mathbb{E}^d$, $d \geq 2$, with C^2 boundary, if $P_{(n)}$ is a circumscribed polytope with n facets that has minimal surface area then as n tends to infinity,*

$$S(P_{(n)}) - S(K) \sim \frac{1}{2} \left(\int_{\partial K} \mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx \right)^{\frac{d+1}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}}.$$

We note that some cases of Theorem 1.1 have been already known. If K is a ball of radius r then $S(P_{(n)}) = dV(P_{(n)})/r$ and $S(K) = dV(K)/r$, and hence P.M. Gruber's (1) in [15] verifies Theorem 1.1. In addition if K is planar and its boundary is C^2_+, then Theorem 1.1 is due to D.E. McClure and R.A. Vitale [9].

If $K \subset P$ for the convex bodies K and P in $\mathbb{E}^d$ then their Hausdorff distance $\delta_H(P, K)$ is the maximal distance of the points of P from K (see Section 5). It is known (see K. Böröczky, Jr. [3]) that if K has C^2 boundary and P has at most n facets then $\delta_H(P, K) \geq \frac{\alpha}{n^{\frac{2}{d-1}}}$ where α is a positive constant depending on K.

Theorem 1.2 *For any convex body K in $\mathbb{E}^d$, $d \geq 2$, with C^2 boundary, if $P_{(n)}$ is a circumscribed polytope with n facets that has minimal surface area, and Ξ_n is a family of n points where the n facets of $P_{(n)}$ touch ∂K then*

(i) $\{\Xi_n\}$ is uniformly distributed with respect to $\mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}}$;

(ii) $\delta_H(P_{(n)}, K) \leq \beta n^{\frac{-2}{d-1}}$ where β depends on K;

(iii) if moreover ∂K is C^2_+ then $\{\Xi_n\}$ satisfies the Delone property.

The rough idea of the proof of Theorem 1.1 is the same like for the volume difference as it was initiated by P.M. Gruber [15], and developed also by M. Ludwig [23]; namely, one thinks patches on ∂K being patches on suitable paraboloids, and uses the fact the power diagrams in $\mathbb{E}^{d-1}$ correspond naturally to polytopal (piecewise linear) surfaces approximating paraboloids (see F. Aurenhammer [1]). In the cases of volume approximation, the problem is reduced (in a non-trivial way) to some properties of the second moment in $\mathbb{E}^{d-1}$ as follows. Let C be a convex body in $\mathbb{E}^{d-1}$. For finite $\Xi \subset C$, if $y \in \Xi$, then we define the Dirichlet-Voronoi cell Π_y of y to be the family of $x \in C$ with $\|x - y\| \leq \|x - z\|$ for all $z \in \Xi$, and assign the second moment integral

$$\sum_{y \in \Xi} \int_{\Pi_y} \|x - y\|^2 \, dx$$

to Ξ, where $\|\cdot\|$ denotes the Euclidean norm. Now for large n, the question is the asymptotics of the minimum of the integral above as Ξ runs through all subset of C of cardinality at most n. This problem was solved by L. Fejes Tóth [11] if C is planar, and by P.M. Gruber [15] for higher dimensions.

In the case of polytopal approximation with respect to the surface area, a similar problem arises in $\mathbb{E}^{d-1}$. Only in this case a positive definite quadratic form q in $d-1$ variables is given, and we integrate not $\|x-y\|^2$, but $q(x-y)$ above Π_y in the expression above. The fact that we define the Dirichlet-Voronoi cells with respect to the standard quadratic form, and integrate another quadratic form, causes much technical difficulty, especially when transforming the asymptotic result in $\mathbb{E}^{d-1}$ to polytopal approximation in $\mathbb{E}^d$.

Concerning the structure of the paper, we discuss the above version of the moment problem in Section 2. For the later study, we need to understand related properties of convex hypersurfaces, which is done in Sections 3 and 4. Using these properties, we establish Theorem 1.2 (ii) and (iii) in Section 5, and prove the core result about approximation of paraboloids in Section 6. Next we discuss polytopal approximation of convex hypersurfaces of positive curvature in Section 7. Finally Theorem 1.1 is proved in Section 8, and Theorem 1.2 (i) in Section 9 using the method of K. Böröczky, Jr. [3].

Let us summarize notation. We write o to denote the origin in $\mathbb{E}^d$, $\langle\cdot,\cdot\rangle$ to denote the scalar product. Moreover let B^d denote the unit ball centred at o. For non-colinear points u, v, w, the angle of the halflines vu and vw is $\angle(u, v, w)$. Given a set $X \subset \mathbb{E}^d$, the affine hull, the convex hull and the interior of X are denoted by aff$X$, conv$X$, and int$X$, respectively.

The $(d-1)$-dimensional Hausdorff measure of a measurable subset X of $\mathbb{E}^d$ is denoted by $|X|$. if X is a subset of the boundary of a closed convex set in $\mathbb{E}^d$ with non-empty interior then we write relint$X$ to denote its relative interior. In addition X is called Jordan measurable if it is bounded, and its relative boundary relbd$X$ is of $(d-1)$-measure zero. If in addition if X is the closure of relint$X$ then X is called a *convex hypersurface*.

Given two real functions f and g, we write $f = O(g)$ if $|f| \le c \cdot g$ for some constant c depending only on the dimension d, and $f = O_\Xi(g)$ if in addition c also depends on some object Ξ. Moreover $\lfloor t \rfloor$ and $\lceil t \rceil$ stand for the largest integer not larger, and the smallest integer not smaller, respectively, than $t \in \mathbb{R}$.

2 A version of the Moment Lemma

While this section can be read independently from the rest of the paper, we work in $\mathbb{E}^{d-1}$ because this is the setup how our main result Theorem 2.1 is applied in this paper. We write Υ_n to denote the family of all non-empty subsets of $\mathbb{E}^{d-1}$ of cardinality at most n. P.M. Gruber [15] proved the existence of $\mathrm{div}_{d-1} > 0$ depending only on d with the following property. If C is a Jordan measurable subset of $\mathbb{E}^{d-1}$, then as n tends infinity, we have

$$\min_{\Xi\in\Upsilon_n} \int_C \min_{y\in\Xi} \|x-y\|^2\,dx \sim \mathrm{div}_{d-1}\cdot|C|^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}}. \tag{7}$$

If C is planar then (7) follows from the celebrated Moment Lemma of L. Fejes Tóth [11]. Far reaching generalizations of (7) where $\|x-y\|^2$ is replaced by $f(\|x-y\|)$ for some increasing function f with a "growth condition" are proved in P.M. Gruber [20].

For $\Xi\in\Upsilon_n$ and $y\in\Xi$, the Dirichlet-Voronoi cell of y with respect to Ξ and C is

$$\Pi(y,\Xi,C)=\{x\in C\,:\,\|x-y\|\leq|x-z|\ \text{ for all }\ z\in\Xi\}.$$

In particular in (7), we have

$$\int_C \min_{y\in\Xi}\|x-y\|^2\,dx=\sum_{y\in\Xi}\int_{\Pi(y,\Xi,C)}\|x-y\|^2\,dx.$$

Now let q be a positive definite quadratic form in $d-1$ variables. For any finite subset Ξ of $\mathbb{E}^{d-1}$, we define

$$\Omega(C,\Xi,q)=\sum_{y\in\Xi}\int_{\Pi(y,\Xi,C)}q(x-y)\,dx.$$

It follows by compactness argument that there exists an extremal $\Xi_{q,C,n}\in\Upsilon_n$ such that $\Pi(y,\Xi_{q,C,n},C)\neq\emptyset$ for all $y\in\Xi_{q,C,n}$, and

$$\Omega(C,\Xi_{q,C,n},q)=\min_{\Xi\in\Upsilon_n}\Omega(C,\Xi,q)=\min_{\Xi\in\Upsilon_n}\sum_{y\in\Xi}\int_{\Pi(y,\Xi,C)}q(x-y)\,dx.$$

We note that it is not clear whether $\Xi_{q,C,n}$ has n elements. The reason is that if the eigenvalues of q are different then there exist C, Ξ and y such that

$$\Omega(C,\Xi\cup\{y\},q)>\Omega(C,\Xi,q).$$

Our main goal is to prove the following generalization of (7).

Theorem 2.1 *For any positive definite quadratic form q in $d-1$ variables there exists* $\operatorname{div}(q) > 0$ *with the following property. If C is a Jordan measurable subset of $\mathbb{E}^{d-1}$, then as n tends infinity, we have*

$$\Omega(C, \Xi_{q,M,n}, q) \sim \operatorname{div}(q) \cdot |C|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}.$$

Before proving Theorem 2.1, we verify two auxiliary statements used in the proof. The first estimate is a consequence of the fact that given $|M|$ for a Jordan measurable $M \subset \mathbb{E}^{d-1}$, $\int_M \|y\|^2\, dy$ is minimal if M is the $(d-1)$-ball centred at o.

Proposition 2.2 *If $M \subset \mathbb{E}^{d-1}$ is Jordan measurable then*

$$\int_M \|y\|^2\, dy \geq \frac{d-1}{(d+1) \cdot |B^{d-1}|^{\frac{2}{d-1}}} \cdot |M|^{\frac{d+1}{d-1}}.$$

The second estimate is the Hölder inequality for positive $a_1, \ldots, a_k$ and $n_1, \ldots, n_k$ in the form

$$a_1^{\frac{d+1}{d-1}} n_1^{\frac{-2}{d-1}} + \ldots + a_k^{\frac{d+1}{d-1}} n_k^{\frac{-2}{d-1}} \geq (a_1 + \ldots + a_k)^{\frac{d+1}{d-1}} (n_1 + \ldots + n_k)^{\frac{-2}{d-1}}, \qquad (8)$$

where equality holds if and only if $a_i/a_j = n_i/n_j$ for $i, j = 1, \ldots, k$.

Next we discuss two properties of $\Omega(M, \Xi_{q,M,n}, q)$ that will be also used in Section 6. The first is its homogeneity; namely, If $\lambda > 0$ and $n \geq 1$, then

$$\Omega(\lambda M, \Xi_{q,\lambda M,n}, q) = \lambda^{d+1} \Omega(M, \Xi_{q,M,n}, q) \qquad (9)$$

holds for any Jordan measurable M in $\mathbb{E}^{d-1}$. Now if M is a convex body in $\mathbb{E}^{d-1}$ with $o \in M$, then for any $\varepsilon \in (0,1)$, (9) yields

$$\Omega((1-\varepsilon)M, \Xi_{q,M,n}, q) \geq (1-\varepsilon)^{d+1} \Omega(M, \Xi_{q,M,n}, q).$$

In turn we deduce

$$\Omega(M \backslash (1-\varepsilon)M, \Xi_{q,M,n}, q) \leq \varepsilon \cdot 2d \cdot \Omega(M, \Xi_{q,M,n}, q). \qquad (10)$$

Proof of Theorem 2.1: There exists some $\omega \geq 1$ such that

$$\|z\|^2/\omega \leq q(z) \leq \omega \|z\|^2 \ \text{ for any } \ z \in \mathbb{E}^{d-1}.$$

Since q is fixed, we set $\Xi_{q,M,n} = \Xi_{M,n}$.

Let M be a Jordan measurable set that is the closure of its relative interior. It follows by Proposition 2.2 and the Hölder inequality (8), that if $\Xi \in \Upsilon_n$, then

$$\Omega(M, \Xi, q) \geq \sum_{y \in \Xi} \gamma_1 |\Pi(y, \Xi, M)|^{\frac{d+1}{d-1}} \geq \gamma_1 |M|^{\frac{d+1}{d-1}} n^{\frac{-2}{d-1}}, \qquad (11)$$

where γ_1 depends on ω and d. There exists $\tilde{n}$ depending on M such that if $n > \tilde{n}$, then in any face to face tiling of $\mathbb{E}^{d-1}$ by cubes of $(d-1)$-measure $2|M|/n$, the number of the tiles intersecting M is at most n. Taking $\widetilde{\Xi}$ to be the centres of these tiles shows that that there exists a $\tilde{\gamma}$ depending on ω and d such that if $n > \tilde{n}$, then

$$\Omega(M, \Xi_{M,n}, q) \leq \Omega(M, \widetilde{\Xi}, q) \leq \tilde{\gamma} n^{\frac{-2}{d-1}} |M|^{\frac{d+1}{d-1}}. \tag{12}$$

Next we show that

$$\lim_{n\to\infty} \max_{y\in\Xi_{q,M,n}} \operatorname{diam}[\{y\} \cup \Pi(y, \Xi_{M,n}, M)] = 0. \tag{13}$$

Since M is the closure of its interior, for any $\delta > 0$ there exists some $\eta > 0$ depending on δ and M such that if $x \in M$ then $|(x + \delta B^{d-1}) \cap M| > \eta$. Let us assume that $\operatorname{diam}[\{y_0\} \cup \Pi(y_0, \Xi_{M,n}, M)] \geq 4\delta$ for some $y_0 \in \Xi_{q,M,n}$. In particular there exists some $x_0 \in M$ such that $\|x_0 - y\| \geq 2\delta$ for all $y \in \Xi_{M,n}$. If $z \in (x + \delta B^{d-1}) \cap \Pi(y, \Xi_{M,n}, M)$ for some $y \in \Xi_{M,n}$ then $q(z - y) \geq \delta^2/\omega$, therefore

$$\Omega(M, \Xi_{M,n}, q) \geq \delta^2\eta/\omega. \tag{14}$$

In particular (12) yields (13).

For $W = [-\frac{1}{2}, \frac{1}{2}]^{d-1}$, we define

$$c_n = n^{\frac{2}{d-1}} \Omega(W, \Xi_{W,n}, q).$$

It follows by (11) and (12) that

$$\operatorname{div}(q) = \liminf_{n\to\infty} c_n$$

is a positive and finite real number.

Turning to C, we assume that C is the closure of its relative interior. It is suffient to prove that for any small $\varepsilon > 0$, there exist an N depending on ε, C, d and ω such that if $n > N$ then

$$\Omega(C, \Xi_{C,n}, q) = (1 + O_\omega(\varepsilon)) \cdot \operatorname{div}(q) \cdot |C|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}. \tag{15}$$

It follows by the definition $\operatorname{div}(q)$, and applying (13) to $M = W$, that there exists some N_ε depending on ε and d such that if $n \geq N_\varepsilon$, then

(i) $(1-\varepsilon)\operatorname{div}(q) \leq c_n \leq (1+\varepsilon)\operatorname{div}(q)$;

(ii) $\operatorname{diam}[\{y\} \cup \Pi(y, \Xi_{W,n}, W) < \varepsilon/2$ for $y \in \Xi_{W,n}$.

We start with the lower bound implied in (15). Choose pairwise disjoint homothetic copies $W_1,\ldots,W_k$ of W in relint$C$ whose total $(d-1)$-measure is at least $(1-\varepsilon)|C|$. It follows by (13) that there exists some N'_ε depending on ε, C and $W_1,\ldots,W_k$ such that for $n>N'_\varepsilon$ and $i=1,\ldots,k$, any Dirichlet-Voronoi cell $\Pi(y,\Xi_{C,n},C)$, $y\in\Xi_{C,n}$, intersects at most one W_i, and the the number n_i of Dirichlet-Voronoi cells intersecting W_i is at least N_ε for each i. In particular $n_1+\ldots+n_k\leq n$. Therefore the condition (i) and the Hölder inequality (8) yield in this case that

$$\begin{aligned}\Omega(C,\Xi_{C,n},q) &\geq \sum_{i=1}^{k}\Omega(W,\Xi_{C,n},q)\geq\sum_{i=1}^{k}(1-\varepsilon)\mathrm{div}(q)\cdot|W_i|^{\frac{d+1}{d-1}}\cdot n_i^{\frac{-2}{d-1}}\\ &\geq (1-\varepsilon)^2\mathrm{div}(q)\cdot|C|^{\frac{d+1}{d-1}}\cdot n^{\frac{-2}{d-1}}. \qquad (16)\end{aligned}$$

Next we choose $\lambda>0$, and homothets $w_i+\lambda W$, $i=1,\ldots,m$, with pairwise disjoint relative interiors such that the m homothetic copies of W cover C, and their total $(d-1)$-measure is at most $(1+\varepsilon)|C|$. We may assume that λ is small enough to ensure $mN_\varepsilon>1/\varepsilon$. If $n>mN_\varepsilon$ then let

$$\Xi_n=\cup_{i=1}^{m}(w_i+\lambda\Xi_{W,\lfloor n/m\rfloor}).$$

It follows by (ii) that if $x\in w_i+(1-\varepsilon)\lambda W$, $i=1,\ldots,m$, then any closest point of Ξ_n to x lies in $w_i+\lambda\Xi_{W,n}$. In particular

$$\begin{aligned}\sum_{i=1}^{m}\Omega(w_i+(1-\varepsilon)\lambda W,\Xi_n,q) &\leq mc_n\lambda^{d+1}\lfloor n/m\rfloor^{\frac{-2}{d-1}}\\ &\leq (1+\varepsilon)^3\mathrm{div}(q)|C|^{\frac{d+1}{d-1}}n^{\frac{-2}{d-1}}. \qquad (17)\end{aligned}$$

Let $W_\varepsilon=W\backslash(1-\varepsilon)W$, and hence (10) yields

$$\Omega(W_\varepsilon,\Xi_{W,\lfloor n/m\rfloor},q)\leq\varepsilon\cdot 2dc_{\lfloor n/m\rfloor}\lfloor n/m\rfloor^{\frac{-2}{d-1}}\leq\varepsilon\cdot 3d\,\mathrm{div}(q)(n/m)^{\frac{-2}{d-1}}.$$

For any $x\in w_i+\lambda W_\varepsilon$, $i=1,\ldots,m$, we consider a closest point y of Ξ_n to x, and closest point z of $w_i+\lambda\Xi_{W,n}$. It follows by the definition of ω that

$$q(x-y)\leq\omega\|x-y\|^2\leq\omega\|x-z\|^2\leq\omega^2q(x-z).$$

Therefore

$$\begin{aligned}\sum_{i=1}^{m}\Omega(w_i+\lambda W_\varepsilon,\Xi_n,q) &\leq \varepsilon\cdot m\omega^2 3d\,\mathrm{div}(q)\lambda^{d+1}(n/m)^{\frac{-2}{d-1}}\\ &\leq \varepsilon\cdot\omega^2 4d\,\mathrm{div}(q)|C|^{\frac{d+1}{d-1}}n^{\frac{-2}{d-1}}. \qquad (18)\end{aligned}$$

Finally combining (16), (17) and (18) yields (15), and in turn completes the proof of Theorem 2.1.

Q.E.D.

Next we discuss a weak stability version of Theorem 2.1 if C is convex.

Corollary 2.3 *Let $C \subset \mathbb{E}^{d-1}$ be convex body satisfying $\alpha r B^{d-1} \subset C \subset (r/\alpha)B^{d-1}$ for $\alpha, r > 0$, and let q be a quadratic form in $d-1$ variables satifying $\omega^{-1}\|z\| \leq q(z) \leq \omega\|z\|$ for $\omega \geq 1$. Then for any $\varepsilon \in (0,1)$, there exists n_0 depending only on the parameters d, ε, α and ω such that if $n > n_0$, then*

$$\Omega(C, \Xi_{q,C,n}, q) = (1 + O(\varepsilon))\mathrm{div}(q) \cdot |C|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}, \quad (19)$$

$$\Pi(y, \Xi_{q,C,n}, C) \subset y + \varepsilon C. \quad (20)$$

Proof: Let C_1, C_2 be convex bodies in $\mathbb{E}^{d-1}$, and let q_1, q_2 be positive definite quadratic forms in $d-1$ variables satisfying

$$(1+\varepsilon)^{-1}C_1 \subset C_2 \subset (1+\varepsilon)C_1 \ \text{ and } \ (1+\varepsilon)^{-1}q_1 \leq q_2 \leq (1+\varepsilon)q_1$$

for $\varepsilon \in (0,1)$. Writing $\Xi^i = \Xi_{q_i,C_i,n}$, we deduce by (9) that

$$\Omega(C_2, \Xi^2, q_2) \geq \Omega((1+\varepsilon)^{-1}C_1, \Xi^2, (1+\varepsilon)^{-1}q_1) \geq (1+\varepsilon)^{-(d+2)}\Omega(C_1, \Xi^1, q_1).$$

In particular for given n, $\Omega(C, \Xi_{q,C,n}, q)$ is a continuous function of q and a convex body C in $\mathbb{E}^{d-1}$.

To prove both statements in Corollary 2.3, we may assume that

$$\alpha B^{d-1} \subset C \subset (1/\alpha)B^{d-1} \quad (21)$$

by homogeneity. Under this assumption, the space of all possible C and q is compact, therefore (19) follows from Theorem 2.1. Since the η in (14) in the proof of Theorem 2.1 depends only on δ, d and α by (21), we deduce (20) from (14) and (19).

Q.E.D.

Remark 2.4 *It follows from Theorem 2.1 that if q_1 and q_2 are quadratic forms in $d-1$ variables satifying $(1+\varepsilon)^{-1}q_1 \leq q_2 \leq (1+\varepsilon)q_1$ for $\varepsilon \in (0,1)$, then $(1+\varepsilon)^{-1}\mathrm{div}(q_1) \leq \mathrm{div}(q_2) \leq (1+\varepsilon)\mathrm{div}(q_1)$.*

In general, we only have the following estimate for $\mathrm{div}(q)$ in terms of div_{d-1}.

Lemma 2.5 *If q is a positive definite quadratic form in $d-1$ variables whose minimal eigenvalue is τ, then*

$$\tau \cdot \mathrm{div}_{d-1} \leq \mathrm{div}(q) \leq \frac{\mathrm{tr}\, q}{d-1} \cdot \mathrm{div}_{d-1},$$

where equality holds in the upper bound if the eigen values of q coincide.

Proof: The lower bound readily holds. To prove the upper bound on $\mathrm{div}(q)$, we may assume that $q(y) = \sum_{j=1}^{d-1} \tau_j t_j^2$ for $y = (t_1, \ldots, t_{d-1})$. We write $p_j y$ to denote the j^{th} coordinate of $y \in \mathbb{E}^{d-1}$.

Let $W = [-\frac{1}{2}, \frac{1}{2}]^{d-1}$. According to (7), if $\varepsilon > 0$ and n is large then there exists $\Xi = \{y_1, \ldots, y_n\} \in \Upsilon_n$ such that

$$\int_W \min_{i=1}^{n} \|x - y_i\|^2 \, dx \leq (1 + \varepsilon) \cdot \mathrm{div}_{d-1} \cdot n^{\frac{-2}{d-1}}.$$

Writing $\Pi_i = \Pi(y_i, \Xi, W)$, we define

$$\alpha_j = \sum_{i=1}^{n} \int_{\Pi_i} [p_j(x - y_i)]^2 \, dx \quad \text{for } j = 1, \ldots, d-1.$$

Now there exists a permutation $\sigma : \{1, \ldots, d-1\} \to \{1, \ldots, d-1\}$ satisfying

$$\sum_{j=1}^{d-1} \tau_j \cdot \alpha_{\sigma(j)} \leq \frac{\mathrm{tr}\, q}{d-1} \cdot \sum_{j=1}^{d-1} \alpha_j.$$

Therefore writing Ψ to the linear transformation with $p_j \Psi y = p_{\sigma(j)} y$, we have $\Psi W = W$ and

$$\sum_{i=1}^{n} \int_{\Pi_i} q(x - y_i) \, dx \leq \frac{(1 + \varepsilon)\, \mathrm{tr}\, q}{d-1} \cdot \mathrm{div}_{d-1} \cdot n^{\frac{-2}{d-1}}.$$

In turn we conclude Lemma 2.5.

Q.E.D.

We note that $\mathrm{div}_1 = \frac{1}{12}$ and $\mathrm{div}_2 = \frac{5\sqrt{3}}{18}$ according to P.M. Gruber [14], and it follows from the work of P.L. Zador [28] that

$$\mathrm{div}_{d-1} = \tfrac{1}{2e\pi}\, d + O(\ln d) \tag{22}$$

where the implied constant in $O(\cdot)$ is an absolute constant.

It follows from Lemma 2.5 that if $d = 2$ and $q(x) = \kappa x^2$ then $\mathrm{div}(q) = \kappa/12$. In addition the value of $\mathrm{div}(q)$ has been determined in K.J. Böröczky, B. Csikós [4] if q has two variables and the eigen values are not too different: If q is a positive definite quadratic form in two variables with eigenvalues $\tau < \kappa \leq 2.4\tau$, then

$$\mathrm{div}(q) = \frac{\sqrt{\tau}[4\kappa + (4\kappa^2 - 6\tau\kappa + 3\tau^2)^{1/2}]}{18[2\kappa + (4\kappa^2 - 6\tau\kappa + 3\tau^2)^{1/2}]^{1/2}}. \tag{23}$$

3 Convex hypersurfaces

In this section we start the study of convex hypersurfaces. Let us discuss first some notions associated to a positive definite quadratic q form in $d-1$ variables. We can choose a orthonormal bases for $\mathbb{E}^{d-1}$ such that if $y = (y_1, \ldots, y_{d-1}) \in \mathbb{E}^{d-1}$ then $q(y) = \tau_1 y_1^2 + \ldots + \tau_{d-1} y_{d-1}^2$ where $\tau_1, \ldots, \tau_{d-1}$ are the associated eigen values. Using this notation, we assign the positive definite quadratic form q° to q defined by

$$q^\circ(y) = \tau_1^2 y_1^2 + \ldots + \tau_{d-1}^2 y_{d-1}^2.$$

We say that a convex hypersurface $X \subset \mathbb{E}^d$ is *proper* if $\mathrm{conv}X$ is a convex body in $\mathbb{E}^d$. In this case we write $u_X(x)$ to denote some exterior unit normal at $x \in \mathrm{relint}X$ that is unique for all $x \in \mathrm{relint}X$ but of a set of $(d-1)$-measure zero. When integrating over X, we always do it with respect to the $(d-1)$-dimensional Hausdorff measure. If the closest point x of $\mathrm{conv}X$ to some y lies in X then we write $\pi_X(y) = x$. We note that

$$\|\pi_X(y) - \pi_X(y')\| \leq \|y - y'\|,$$

hence if π_X is defined and injective on some convex hypersurface Y then $|\pi_X(Y)| \leq |Y|$, and $\pi_X(Y)$ is also a convex hypersurface. We will also meet the following setup: Given convex hypersurfaces X, Y such that $X = \pi_X(Y)$, let $Z \subset \mathrm{relint}X$ be a convex hypersurface. Then the subset Z' of Y satisfying $\pi_X(Z') = Z$ is a convex hypersurface, as well. We note if L is the boundary of a closed convex set in $\mathbb{E}^d$ with non-empty interior then we also write π_L to denote the closest point map into L.

If the convex hypersurface $Y \subset \mathbb{E}^d$ is the union of $F_1, \ldots, F_k$ such that each F_i is a Jordan measurable subset of some hyperplane and has positive $(d-1)$-measure, and $\mathrm{aff}F_1, \ldots, \mathrm{aff}F_k$ are pairwise different then we call Y a *convex polytopal hypersurface*, and $F_1, \ldots, F_k$ the facets of Y. If $\mathrm{aff}F_i$ for $i = 1, \ldots, k$ touches some proper convex hypersurface X then we say that Y is circumscribed around X.

For certain calculations it is useful to consider patches as graphs of functions. We think $\mathbb{E}^d$ as $\mathbb{E}^{d-1} \times \mathbb{R}$ where $x = (y, t)$ is the point of $\mathbb{E}^d$ corresponding to $y \in \mathbb{E}^{d-1}$ and $t \in \mathbb{R}$, and define $B^{d-1} = B^d \cap \mathbb{E}^{d-1}$. We write

$$\xi = (o, -1)$$

to denote the "downwards" unit normal to $\mathbb{E}^{d-1}$. If $C \subset \mathbb{E}^{d-1}$ has non-empty interior in $\mathbb{E}^{d-1}$, and $\theta : C \to \mathbb{R}$ is any function then the graph of θ is

$$\Gamma(\theta) = \{(y, \theta(y)) : y \in C\} \subset \mathbb{E}^d. \tag{24}$$

In particular the graph of any convex function defined on convex body in $\mathbb{E}^{d-1}$ is a convex hypersurface.

We say that a convex hypersurface X is a C^2 convex hypersurface if any point of X has an open neighbourhood on X that is congruent to the graph of some C^2 function. In order to define the principle curvatures at $x_0 \in \text{relint} X$, we may assume that $\mathbb{E}^{d-1}$ is the tangent hyperplane to X at $x_0 = (y_0, 0)$, and a neighbourhood $X_0 \subset X$ of x_0 is the graph of a C^2 function θ on an open convex $\Psi \subset \mathbb{E}^{d-1}$. Then the *principle curvatures* $\kappa_1(x_0), \ldots, \kappa_{d-1}(x_0)$ of X at x_0 are the eigenvalues of the symmetric matrix corresponding to the quadratic form representing the second derivative of θ at y_0. For $x \in X$, we define $\sigma_0(x) = 1$, and write $\sigma_j(x)$ to denote the j^{th} symmetric polynomial of the principal curvatures for $j = 1, \ldots, d-1$; namely,

$$\sigma_j(x) = \sum_{1 \le i_1 < \ldots < i_j \le d-1} \kappa_{i_1}(x) \cdot \ldots \cdot \kappa_{i_j}(x).$$

In particular, $H(x) = \sigma_1(x)$ and $\kappa(x) = \sigma_{d-1}(x)$. Naturally $\sigma_j(x)$ depends on X but what X is will be always clear from the context.

Let Y be a convex hypersurface such that π_X is defined on Y and is bijective. If $\pi_X(y) = x$ for $y \in \text{relint}\, Y$ then we write $y = x_Y$ and define $r_{X,Y}(x) = \|y - x\|$. Now the difference of the $(d-1)$-measures is (see K. Böröczky, Jr. and M. Reitzner [6])

$$\begin{aligned} |Y| - |X| \;\; = \;\; & \int_X \left(\frac{1}{\langle u_X(x), u_Y(x_Y) \rangle} - 1 \right) dx \\ & + \sum_{j=1}^{d-1} \int_X r_{X,Y}(x)^j \frac{\sigma_j(x)}{\langle u_X(x), u_Y(x_Y) \rangle}\, dx. \end{aligned} \tag{25}$$

We note that if Y is a compact convex $(d-1)$-dimensional set, $L = \text{aff} Y$ and $X \subset \partial K$ for some convex body K in $\mathbb{E}^d$ then sometimes abusing the notation we write $r_{X,Y}(x) = r_{\partial K, L}(x)$.

Let us present an application of (25). Let $K \subset P$ be convex bodies, and let K have C^2 boundary. We claim that there exist $\varrho_0, \xi^* > 0$ depending on K such that if $X \subset \partial K$ and $Y \subset \partial P$ are convex hypersurfaces satisfying $\pi_{\partial K} Y = X$ and $\delta_H(Y, X) \le \varrho$ for $\varrho \in (0, \varrho_0)$, then

$$|Y| - |X| \;\; \le \;\; \xi^* \varrho \cdot |X|. \tag{26}$$

We use that there exists a $\eta > 0$ depending on K with the following property (see say K. Leichtweiß [22]): If $x \in \partial K$ then there exists a ball that lies in K, is of radius η, and touches ∂K from inside at x. Since any tangent hyperplane to relint Y avoids int K, if $x \in \text{relint}\, X$ then

$$\langle u_X(x), u_Y(x_Y)\rangle \geq \tfrac{\eta}{\eta+\varrho}. \tag{27}$$

Therefore (25) yields (26).

4 Some basic properties of graphs of convex functions

The main goal of this section is to observe some basic properties of convex hypersurfaces that are needed in the paper. First we introduce the notions that we discuss until (45), and state the conditions (28) to (43) on these notions. Let q be a positive definite quadratic form in $d-1$ variables, and let $\omega > 1$ satisfy

$$\omega^{-1} \leq \cdot\|z\|^2 q(z) \leq \omega \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{d-1}. \tag{28}$$

In addition, let

$$\varepsilon \in (0, \tfrac{1}{20\omega^2}). \tag{29}$$

We investigate a non-negative C^2 function θ defined on the $(d-1)$-dimensional convex body C, where

$$C \subset \sqrt{\varepsilon}\, B^{d-1} \text{ with } o \in \text{relint} C. \tag{30}$$

We write l_y to denote the linear form and q_y to denote the quadratic form representing the second derivative of θ at $y \in C$. We assume that

$$\left.\begin{array}{c} \theta(o) = 0 \;\text{ and }\; l_o(z) = 0; \\ q(z) - \varepsilon \cdot \|z\|^2 \leq q_y(z) \leq q(z) + \varepsilon \cdot \|z\|^2 \end{array}\right\} \quad \text{for } z \in \mathbb{E}^{d-1}. \tag{31}$$

We define $X' = \Gamma(\theta)$. It follows by the Taylor formula for $y, z \in C$ that

$$\begin{aligned} \theta(z) - \theta(y) - l_y(z-y) &= \tfrac{1}{2}\, q_{y+t(z-y)}(z-y) \;\text{ for } t \in (0,1); && (32) \\ &= \tfrac{1}{2}\, q(z-y) + O(\varepsilon)\, \|y-z\|^2; && (33) \\ \|l_z - l_y\|^2 &= q^\circ(z-y) + O(\varepsilon\omega)\, \|z-y\|^2. && (34) \end{aligned}$$

If $y \in \text{relint} C$ and $x = (y, \theta(y))$ then

$$u_{X'}(x) = (1 + \|l_y\|^2)^{\frac{-1}{2}} \cdot (l_y, -1). \tag{35}$$

It follows by (30), (34) and (35) that if $x, x' \in \mathrm{relint} X'$ for $x = (y, \theta(y))$ and $x' = (y', \theta(y'))$ then

$$\langle u_{X'}(x), u_{X'}(x')\rangle = 1 - \tfrac{1}{2}\, q^{\circ}(y - y') + O(\varepsilon\omega^2)\, \|y - y'\|^2. \tag{36}$$

If $x' = o$ then we have a more precise formula. Since X' does not intersect the interior of the ball of radius $\frac{1}{2\omega}$ centred at $\frac{-1}{2\omega}\,\xi$, there is a point z on the boundary of this ball where the exterior unit normal is $u_{X'}(x)$ and $\|\pi_{\mathbb{E}^{d-1}} z\| \le 2\sqrt{\varepsilon}$. We conclude

$$\langle \xi, u_{X'}(x)\rangle \ge (1 + 8\omega^2\varepsilon)^{\frac{-1}{2}} \ge 1 - 4\omega^2\varepsilon \ \text{ for } x \in \mathrm{relint} X'. \tag{37}$$

It also follows that if $z \in C$ and $\pi_{X'} z$ is well defined then

$$\|z - \pi_{E^{d-1}} \pi_{X'} z\| \le \tilde{\aleph}\|z\|^3 \ \text{ for } \tilde{\aleph} > 1 \text{ depending on } \omega \text{ and } d. \tag{38}$$

Recalling that $\sigma_j(x)$ denotes the j^{th} symmetric polynomial of the principal curvatures $x \in \mathrm{relint} X'$ for $j = 1, \ldots, d-1$, we have

$$\begin{aligned} H(x) &= \mathrm{tr}\, q + O(\varepsilon); & (39)\\ \kappa(x) &= \det q + O(\varepsilon\omega^{d-2}); & (40)\\ \sigma_j(x) &= O(\omega^j). & (41) \end{aligned}$$

Next let Y be a convex hypersurface such that $\pi_{X'}$ is defined on Y and it is injective, and let $X = \pi_{X'}(Y)$. We will assume that if $x \in X$ then

$$\begin{aligned} r_{X,Y}(x) &\le \varrho \ \text{ where } \ \varrho \in (0, \varepsilon); & (42)\\ \|x - x'\| &> 2\sqrt{\varrho\omega} \ \text{ for } x \in X \text{ and } x' \in \mathrm{relbd}\, X'. & (43) \end{aligned}$$

Since all eigen values of q_y are at most 2ω for any $y \in C$, there is a ball of radius $\frac{1}{2\omega}$ touching X from inside at any $x \in X$ such that the ball intersects X only in x, which in turn yields

$$\langle u_X(x), u_Y(x_Y)\rangle^{-1} \le 1 + 2\varrho\omega. \tag{44}$$

As a rough estimate, (25) yields the analogue of (26); namely,

$$|Y| - |X| \;=\; O(\varrho\omega) \cdot |X|. \tag{45}$$

We also consider a polytopal hypersurface Z circumscribed around X' such that Z is the graph of a convex function h defined on a subset of C. Assume that a facet F of Z touches X' in $(y_0, \theta(y_0))$ for $y_0 \in C$, and let $x \in F$ of the form $x = (y, h(y))$, $y \in C$. Now $d(x, X')$ is at most the distance of x from $x' = (y, \theta(y))$, and at least the distance of x from the tangent plane to X' at x'. In particular the Taylor formula (32) and (37) yield

$$\tfrac{1-5\omega^2\varepsilon}{2} \cdot q(y-y_0) \le d(x, X') \le \tfrac{1+\omega\varepsilon}{2} \cdot q(y-y_0). \tag{46}$$

It also follows that if some facet of Z touches X' at $(z_0, \theta(z_0))$ for $z_0 \in C$ then

$$\tfrac{1}{2}\, q(y-z_0) \le q(y-y_0) \le 2q(y-z_0). \tag{47}$$

In the final part of the section, our main goal is to establish Lemma 4.2 that allows us to shift between patches on smooth convex hypersurfaces. First we verify a simple technical statement.

Proposition 4.1 *Let $z_1, z_2 \in \mathbb{E}^{d-1}$ such that $\|z_2 - z_1\| \le \tau$ for some $\tau > 0$, and let Y be the graph of a convex positive function on $z_1 + 2\tau B^{d-1}$ such that $\langle u_Y(y), \xi\rangle \ge \frac{\sqrt{3}}{2}$ for $y \in Y$. If $y_1, y_2 \in Y$ satisfy that $\langle \frac{z_i - y_i}{\|z_i - y_i\|}, \xi\rangle \ge \frac{\sqrt{3}}{2}$ for $i = 1, 2$ then*

$$\|y_1 - y_2\| \le 2 \cdot [\|z_1 - z_2\| + \|z_1 - y_1\| \cdot \angle(z_1 - y_1, o, z_2 - y_2)].$$

Proof: We define $y_1' \in Y$ by the property that the vectors $z_1 - y_1'$ and $z_2 - y_2$ are parallel, and prove

$$\|y_1 - y_1'\| \le 2\, \|z_1 - y_1\| \cdot \sin \angle(y_1, z_1, y_1'). \tag{48}$$

Let σ be the arc that is the intersection of the triangle $y_1 z_1 y_1'$ and Y, and let y be the point of σ farthest from the segment $y_1 y_1'$. Then the tangent line to σ at y is parallel to the line $y_1 y_1'$, hence $\langle u_Y(y), \xi\rangle \ge \frac{\sqrt{3}}{2}$ yields that the angle of $y_1' - y_1$ and ξ is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. Thus the angle of the triangle $z_1 y_1 y_1'$ at y_1' is between $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, therefore the law of sine implies (48).

Now an argument as above shows that $\|y_2 - y_1'\| \le 2\, \|z_2 - z_1\|$, which in turn yields Proposition 4.1.

Q.E.D.

Let us set up the notation used in Lemma 4.2. Let q be a positive definite quadratic form in $d-1$ variables with

$$\omega^{-1}\|z\| \le q(z) \le \omega\|z\| \quad \text{for } z \in \mathbb{E}^{d-1}.$$

For $\varepsilon \in (0, \frac{1}{16\omega^2})$ and $\varrho \in (0, \varepsilon^8)$, let the convex functions h_1, h_2, f_1, f_2 defined on $\frac{20\sqrt{\varrho}}{\varepsilon}\, B^{d-1}$ satisfy the following properties: We have $f_2(o) = 0$, $f_2'(o) = 0$, f_1 and f_2 are C^2. In addition if $y \in \frac{8\sqrt{\varrho}}{\varepsilon}\, B^{d-1}$ then

$$h_1(y) \le f_1(y) \le f_2(y) \le h_1(y) + \varrho \quad \text{and} \quad f_1(y) \ge 0, \tag{49}$$

and writing $q_{i,y}$ to denote the quadratic form representing the second derivative of f_i at y for $i = 1, 2$, we have

$$q(z) - \varepsilon^8 \cdot \|z\|^2 \leq q_{i,y}(z) \leq q(z) + \varepsilon^8 \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{d-1}.$$

For $i = 1, 2$, we define $Y_i = \Gamma(h_i)$ and $X_i = \Gamma(f_i)$ (see Figure 1). We assume that Y_i is a polytopal hypersurface circumscribed around X_i, and the affine hulls of the of the facets of Y_1 and Y_2 are in bijective correspondence in a way that the affine hulls of the corresponding facets are parallel. In particular

$$h_1(y) \leq h_2(y) \ \text{ for } y \in \tfrac{20\sqrt{\varrho}}{\varepsilon} B^{d-1}.$$

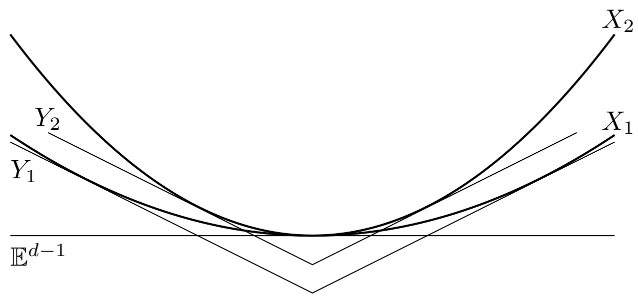

Figure 1:

Lemma 4.2 *Given $\omega > 1$ and d, there exists $\varepsilon_0 \in (0, \frac{1}{16\omega^2})$ depending on d and ω with the following properties. Using the notation as above, if $C \subset \mathbb{E}^{d-1}$ is a compact convex satisfying $\frac{\sqrt{\varrho}}{4\varepsilon} B^{d-1} \subset C \subset \frac{4\sqrt{\varrho}}{\varepsilon} B^{d-1}$, and $\widetilde{X}_i = \pi_{X_i}(C)$, moreover $\widetilde{Y}_i$ denote the subset of Y_i satisfying $\widetilde{X}_i = \pi_{X_i}(\widetilde{Y}_i)$ for $i = 1, 2$, then*

$$|\widetilde{X}_i| = [1 + O(\varepsilon)] \cdot |C| \quad \textit{for } i = 1, 2; \tag{50}$$

$$|\widetilde{Y}_1| - |\widetilde{X}_1| = |\widetilde{Y}_2| - |\widetilde{X}_2| + O(\varepsilon\varrho) \cdot |C|. \tag{51}$$

Proof: Readily if ε_0 is sufficiently small then

$$\pi_{\mathbb{E}^{d-1}}(\widetilde{X}_i), \ \pi_{\mathbb{E}^{d-1}}(\widetilde{Y}_i) \subset 2C, \quad i = 1, 2.$$

It follows by (37) that if ε_0 is sufficiently small then $\langle u_{X_i}(x), \xi \rangle \geq \frac{\sqrt{3}}{2}$ for any $x \in \text{relint} X_i$. In addition if $y = (z, h_i(z))$ for $z \in 2C$, $i = 1, 2$, and u is an exterior unit normal to Y_i at y then $d(y, X_i) \leq \varrho$ and (32) yield that there exists $x \in X_i \cap (y + 4\sqrt{\omega\varrho}\, B^d)$ with $u = u_{X_i}(x)$, hence $\langle u, \xi \rangle \geq \frac{\sqrt{3}}{2}$, as well.

In addition the conditions on h_1, h_2, f_1, f_2 and applying (32) to f_1, f_2 yield that

$$\begin{aligned} h_1(z) \geq h_2(z) &> 0 \;\text{ if } z \in (2C)\backslash(\tfrac{1}{2}C); & (52)\\ f_2(z) &\leq \tfrac{64\omega\varrho}{\varepsilon^2} \;\text{ if } z \in 2C; & (53)\\ f_2(z) - f_1(z) &\leq 64\varepsilon^6\varrho \;\text{ if } z \in 2C; & (54)\\ h_2(z) - h_1(z) &\leq 64\varepsilon^6\varrho \;\text{ if } z \in 2C. & (55)\end{aligned}$$

Therefore combining (45), (53) and $\frac{64\omega\varrho}{\varepsilon^2} \cdot \omega < \varepsilon$ leads to (50).

For $z \in C$, we define $\theta_{Y_i}(z) = Y_i \cap \text{conv}\{z, \pi_{X_i}(z)\}$ for $i = 1, 2$, $\theta'_{X_1}(z) = X_1 \cap \text{conv}\{z, \pi_{X_2}(z)\}$ and $\theta'_{Y_1}(z) = Y_1 \cap \text{conv}\{z, \pi_{X_2}(z)\}$, which points exist by (52). In particular $\widetilde{Y}_i = \theta_{Y_i}(C)$ for $i = 1, 2$, and the relative boundaries of $\widetilde{Y}_1$, $\theta'_{Y_1}(C)$, $\widetilde{X}_1$ and $\theta'_{X_1}(C)$ are the corresponding images of ∂C. We deduce by (45), (54) and (55) that if ε_0 is small enough then

$$\begin{aligned} |\theta'_{X_1}(C)| - |\widetilde{X}_2| &= O(\varepsilon\varrho) \cdot |C|; & (56)\\ |\theta'_{Y_1}(C)| - |\widetilde{Y}_2| &= O(\varepsilon\varrho) \cdot |C|. & (57)\end{aligned}$$

Now we prove

$$\begin{aligned} |\widetilde{X}_1| - |\theta'_{X_1}(C)| &= O(\varepsilon\varrho) \cdot |C|; & (58)\\ |\widetilde{Y}_1| - |\theta'_{Y_1}(C)| &= O(\varepsilon\varrho) \cdot |C|. & (59)\end{aligned}$$

Let $z \in \partial C$. It follows by (53) that $\|z - \theta_{X_1}(z)\| \leq \frac{64\omega\varrho}{\varepsilon^2}$, and the discussion above shows that $\langle \frac{z-\theta_{Y_1}(z)}{\|z-\theta_{Y_1}(z)\|}, \xi\rangle \geq \frac{\sqrt{3}}{2}$. In addition the analogous two inequalities hold for $\pi_{X_1}(z)$, θ'_{X_1} and θ'_{Y_1} in place of θ_{X_1}. Next let $x_i = \pi_{X_i}(z)$, $i = 1, 2$. Since $d(\theta'_{X_1}(z), X_2) \leq 64\varepsilon^6\varrho$ by (54), and there exists a ball of radius $\frac{1}{4\omega}$ touching X_2 from inside at x_2, we deduce that the angle α_2 of $u_{X_2}(x_2)$ and $u_{X_1}(\theta'_{X_1}(z))$ is $O(\varepsilon^3\omega\sqrt{\varrho})$. It follows that $\|\theta'_{X_1}(z) - x_1\| = O(\varepsilon^3\omega\sqrt{\varrho})\|\theta'_{X_1}(z) - z\| = O(\varepsilon\omega^2\varrho^{\frac{3}{2}})$, hence the angle α_1 of $u_{X_1}(x_1)$ and $u_{X_1}(\theta'_{X_1}(z))$ is $O(\varepsilon\omega^3\varrho^{\frac{3}{2}})$ according to (36). Therefore choosing ε_0 small enough, we have

$$\begin{aligned} \angle(z - \pi_{X_1}(z), o, z - \theta'_{X_1}(z)) &= \angle(z - \theta_{Y_1}(z), o, z - \theta'_{Y_1}(z))\\ &\leq \alpha_1 + \alpha_2 = O(\varepsilon^3\omega^3\sqrt{\varrho}) < \frac{\varepsilon^2\sqrt{\varrho}}{128\omega}. \quad (60)\end{aligned}$$

We provide the rest of argument only for (59), and (58) can be similarly proved. It follows by Proposition 4.1, (60) and $\|\theta_{Y_1}(z) - z\| \leq \frac{64\omega\varrho}{\varepsilon^2}$ that

$$\|\theta'_{Y_1}(z) - \theta_{Y_1}(z)\| \leq \varrho^{\frac{3}{2}}, \quad (61)$$

hence (59) is a consequence of

$$\left|Y_1 \cap \left(\theta_{Y_1}(\mathrm{relbd}C) + \varrho^{\frac{3}{2}}\, B^d\right)\right| = O(\varepsilon\varrho) \cdot |C|. \tag{62}$$

To prove (62), let $\tau = \frac{\sqrt{\varrho}}{4\varepsilon}$, and let $z_1, \ldots, z_k \in \partial C$ be a maximal family of points with the property that $\|z_i - z_j\| \geq 3\varrho^{\frac{3}{2}}$ for $i \neq j$. Since $z_i + \varrho^{\frac{3}{2}}\, B^{d-1}$ are pairwise disjoint for $i = 1, \ldots, k$, and each is contained in the difference of $(1 + \frac{\varrho^{\frac{3}{2}}}{\tau})\, C$ and $(1 - \frac{\varrho^{\frac{3}{2}}}{\tau})\, C$, we deduce that

$$k = O\left(\frac{\varrho^{\frac{3}{2}}}{\tau}\right) \cdot |C| \cdot (\varrho^{\frac{3}{2}})^{-(d-1)} = O(\varepsilon\varrho) \cdot |C| \cdot (\varrho^{\frac{3}{2}})^{-(d-1)}. \tag{63}$$

Now let $y \in Y_1$ satisfy that $\|y - \theta_{Y_1}(z)\| \leq \varrho^{\frac{3}{2}}$ for some $z \in \partial C$. There exists some z_i such that $\|z_i - z\| \leq 3\varrho^{\frac{3}{2}}$, hence $\|\pi_{X_1}(z_i) - \pi_{X_1}(z)\| \leq 3\varrho^{\frac{3}{2}}$. In particular (36) implies that the angle of $z_i - \theta_{Y_1}(z_i)$ and $z - \theta_{Y_1}(z)$, which is the angle of $u_{X_1}(\pi_{X_1}(z_i))$ and $u_{X_1}(\pi_{X_1}(z))$ is at most $4\omega\varrho^{\frac{3}{2}}$ (after choosing ε_0 small enough). Thus $\|z - \theta_{Y_1}(z)\| \leq \frac{1}{8\omega}$ and Proposition 4.1 yield that $\|\theta_{Y_1}(z_i) - \theta_{Y_1}(z)\| \leq 7\omega\varrho^{\frac{3}{2}}$, hence $\|\theta_{Y_1}(z_i) - y\| \leq 8\varrho^{\frac{3}{2}}$. We deduce by (63) that

$$\begin{aligned} \left|Y_1 \cap \left(\theta_{Y_1}(\partial C) + \varrho^{\frac{3}{2}}\, B^d\right)\right| &\leq \sum_{i=1}^{k} \left|Y_1 \cap \left(\theta_{Y_1}(z_i) + 8\varrho^{\frac{3}{2}}\, B^d\right)\right| \\ &\leq k \cdot S(8\varrho^{\frac{3}{2}}\, B^d) = O(\varepsilon\varrho) \cdot |C|. \end{aligned}$$

We conclude (62), and in turn (59) and (58).

Finally combining (56), (57), (58) and (59) yields (51), and in turn Lemma 4.2.

Q.E.D.

5 The Delone property of the extremal body

In this section we establish Theorem 1.2 (ii) and (iii). For $x \in \mathbb{E}^n$ and a compact $X \subset \mathbb{E}^n$, we write $d(x, X)$ to denote the minimal distance between x and the points of X. If X and Y are compact sets in $\mathbb{E}^n$ then their Hausdorff distance is

$$\delta_H(X, Y) = \max\left\{\max_{x\in X} d(x, Y), \max_{y\in Y} d(y, X)\right\}.$$

To verify our main result Lemma 5.2, we need Proposition 5.1. We write $\mathcal{H}^{d-2}(\cdot)$ to denote the $(d-2)$-dimensional Hausdorff measure.

Proposition 5.1 *If $M \subset N$ are convex bodies in $\mathbb{E}^d$ such that for some $x \in \partial M$, there exists a ball $B \subset M$ of radius η touching ∂M at x, and $x + \delta \cdot u_{\partial M}(x) \in N$ for $0 < \delta < \eta$ then*

$$S(N) - S(M) > c \cdot \eta^{\frac{d-3}{2}} \cdot \delta^{\frac{d+1}{2}}$$

where $c > 0$ depends only on d.

Proof: We may assume that $x = o$, $\mathbb{E}^{d-1}$ is the supporting hyperplane to M at x, and $u_{\partial M}(x) = \xi$. We define $C = \mathbb{E}^{d-1} \cap N$, $x_0 = x + \delta \cdot u_{\partial M}(x)$, and Y to be the convex hypersurface that is the union of the segments of the form $\text{conv}\{x_0, y\}$ for $y \in \partial C$. In addition we define $u_y \in \mathbb{E}^{d-1}$ to be an exterior unit normal to ∂C at $y \in \partial C$, and the radial function $\varrho(z) > 0$ by the property $\varrho(z) \cdot z \in \partial C$ for $z \in B^{d-1}$. The existence of B yields that $\varrho(z) > \frac{1}{2}\sqrt{\eta\delta}$ for all $z \in B^{d-1}$. Therefore

$$\begin{aligned} S(N) - S(M) &\geq |Y| - |C| = \frac{1}{d-1} \int_{\partial C} \sqrt{\langle u_y, y\rangle^2 + \delta^2} - \langle u_y, y\rangle \, dy \\ &> \frac{\delta^2}{4(d-1)} \int_{\partial C} \frac{1}{\langle u_y, y\rangle} \, dy = \frac{\delta^2}{4(d-1)} \int_{\partial B^{d-1}} \frac{\varrho(z)^{d-3}}{\langle u_{\varrho(z)\cdot z}, z\rangle^2} \, dz \\ &\geq \frac{\mathcal{H}^{d-2}(\partial B^{d-1})}{4(d-1)} \cdot \eta^{\frac{d-3}{2}} \cdot \delta^{\frac{d+1}{2}} \end{aligned}$$

where the integration always occurred with respect to $\mathcal{H}^{d-2}(\cdot)$. Q.E.D.

We prove Theorem 1.2 (ii) and (iii) as part of Lemma 5.2.

Lemma 5.2 *Let K be a convex body in $\mathbb{E}^d$ with C^2 boundary, let $P_{(n)}$ be a circumscribed polytope with $n \geq d+1$ facets that has minimal surface area, and let $X \subset \partial K$ be a convex hypersurface such that the Gauß-Kronecker curvature is positive at the points of X. Then $\delta_H(P_{(n)}, K) \leq \frac{\beta_0}{n^{\frac{2}{d-1}}}$, and if F is a facet of $P_{(n)}$ with $\pi_{\partial K}(F) \cap X \neq \emptyset$ and F touches K in x then* $\text{diam}F \leq \frac{\beta}{n^{\frac{1}{d-1}}}$, *and F contains a $(d-1)$-ball with centre x and radius $\frac{\alpha}{n^{\frac{1}{d-1}}}$ where α, β, β_0 are positive, β_0 depends on K, and α, β depend on X and K.*

Proof: Readily it is sufficient to consider the case when n is large. It is known (see say K. Leichtweiß [22]) that for suitable $\eta > 0$ depending on K, if $x \in \partial K$ then there exists a ball that lies in K, is of radius η, and touches ∂K from inside at x. In addition we write L_x to denote the supporting hyperplane to K at $x \in \partial K$, and L_x^+ to denote the half space containing K. During the proof of Lemma 5.2, $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$ denote positive constants that depend on X and K.

Our first task is to establish the order $S(P_{(n)}) - S(K)$ (see (64) and (69). According to K. Böröczky, Jr. [3] (or E.M. Bronšteĭn and L.D. Ivanov [8] in a more general framework), there exists a polytope $Q_{(n)}$ circumscribed around K with n facets satisfying $\delta_H(K, Q_{(n)}) < \frac{\beta_1}{n^{\frac{2}{d-1}}}$, therefore

$$S(P_{(n)}) - S(K) \leq S(Q_{(n)}) - S(K) < \frac{\beta_2}{n^{\frac{2}{d-1}}}. \tag{64}$$

Let $\delta = \delta_H(K, P_{(n)})$, let v be a vertex of $P_{(n)}$ with $d(v, K) = \delta$, and let $\tilde{x} = \pi_{\partial K}(v)$. We deduce by Proposition 5.1 that

$$S(P_{(n)}) - S(K) \geq \alpha_1 \cdot \delta^{\frac{d+1}{2}}. \tag{65}$$

Comparing with (64) shows that $\delta_H(K, P_{(n)})$ tends to zero; namely,

$$\delta = \delta_H(K, P_{(n)}) < \frac{\beta_3}{n^{\frac{4}{d^2-1}}}. \tag{66}$$

Now there exist convex hypersurfaces $X_1, X_2 \subset \partial K$ such that the principle curvatures at each point of X_2 are positive, and

$$X \subset \mathrm{relint} X_1 \subset X_1 \subset \mathrm{relint} X_2.$$

Next there exist $\gamma > 0$ and $\omega > 1$ depending on X and K with the following two properties: First if $y \in X_2$ then $H(y) = \sigma_1(y) > \omega^{-1}$. Secondly assuming that $\Phi_x = (x + \gamma B^d) \cap \partial K$ intersects X_1 for $x \in \partial K$, we have $\Phi_x \subset X_2$, and if $y \in \Phi_x$ then

$$\tan \angle(u_{\partial K}(x), o, u_{\partial K}(y)) \leq \omega \cdot \|y - x\|; \tag{67}$$
$$\omega^{-1} \cdot \|y - x\|^2 \leq r_{\partial K, L_x}(y) \leq \omega \cdot \|y - x\|^2. \tag{68}$$

We write $F_1, \dots, F_n$ to denote facets of $P_{(n)}$, and x_i to denote a point of ∂K where F_i touches K. We assume that $\pi_{\partial K}(F_i) \cap X \neq \emptyset$ if and only if $i \leq k$, and F_j intersects some F_i with $i \leq k$ if and only if $i \leq k_0$. According to (66) and (68), we may assume that n is large enough to ensure that if $i \leq k$ and F_j intersects F_i then

$$\pi_{\partial K}(F_i) \cup \pi_{\partial K}(F_j) \subset \Phi_{x_i}.$$

We write C_i to denote the orthogonal projection of $\pi_{\partial K}(F_i)$ into L_{x_i}, and deduce by (25) and Proposition 2.2 that if $i \leq k$ then

$$\begin{aligned} |F_i| - |\pi_{\partial K}(F_i)| &> \int_{\pi_{\partial K}(F_i)} r_{\partial K, L_{x_i}}(y) H(y)\, dy \\ &> \int_{\pi_{\partial K}(F_i)} \omega^{-2} \cdot \|y - x\|^2\, dy > \int_{C_i} \omega^{-2} \cdot \|z - x\|^2\, dz \\ &> \alpha_2 \cdot |C_i|^{\frac{d+1}{d-1}} > \alpha_3 \cdot |\pi_{\partial K}(F_i)|^{\frac{d+1}{d-1}}. \end{aligned}$$

Therefore the Hölder inequality (8) yields

$$S(P_{(n)}) - S(K) \geq \sum_{i=1}^{k} [|F_i| - |\pi_{\partial K}(F_i)|] \geq \frac{\alpha_3 \cdot |X|^{\frac{d+1}{d-1}}}{k^{\frac{2}{d-1}}}. \tag{69}$$

Comparing with (64) leads to $k > \alpha_4 n$.

We are ready to face directly the Delone property. It follows by (68) that

$$\|z - x_i\| < \sqrt{\omega\delta} \quad \text{if } i \leq k_0 \text{ and } z \in F_i, \tag{70}$$

which in turn yields that

$$\delta \geq \alpha_5 \cdot k^{\frac{-2}{d-1}} \geq \alpha_5 \cdot n^{\frac{-2}{d-1}}. \tag{71}$$

For any $i = 1, \ldots, k_0$, we write ν_i to denote the minimal distance of the $(d-2)$-faces of F_i from x_i, and define $\nu = \min_{i=1,\ldots,k} \nu_i$. Readily

$$\nu \leq \beta_4 \cdot k^{\frac{-1}{d-1}} \leq \beta_5 \cdot n^{\frac{-1}{d-1}}, \tag{72}$$

and let $m \leq k$ satisfy $\nu = \nu_m$. We observe that $\widetilde{P} = P_{(n)} \cap L^+_{\tilde{x}}$ has $n+1$ facets, and

$$S(P_{(n)}) - S(\widetilde{P}) > \alpha_6 \cdot \delta^{\frac{d+1}{2}} \tag{73}$$

according to Proposition 5.1. We define $\widetilde{F}_m = F_m \cap \widetilde{P}$, and

$$P' = L^+_{\tilde{x}} \cap \left(\cap_{\substack{i \neq m \\ i \leq n}} L^+_{x_i} \right),$$

which is a polytope circumscribed around K with n facets. We write Y to denote the part of $\partial \widetilde{P}$ cut off by L_{x_m}. If G is a facet of $\widetilde{P}$ intersecting $\widetilde{F}_m$ and touching K in y then $\|y - x_m\| < 2\sqrt{\omega\delta}$ (see (70)), hence (67) yields

$$\tan \angle(u_{\partial K}(x_m), o, u_{\partial K}(y)) \leq 2\omega^{\frac{3}{2}} \cdot \delta^{\frac{1}{2}}. \tag{74}$$

We define $w = x_m + 2\omega\delta \cdot u_{\partial K}(x_m)$, thus Y is contained in $C = \mathrm{conv}\{w, \widetilde{F}_m\}$. In addition there exists some x_j with $j \leq k_0$, $j \neq m$, such that $A = L_{x_m} \cap L_{x_j}$ contains a $(d-2)$-face of F_m, and the distance of x_m from A is ν. Since $\|x_j - x_m\| \leq (1+\omega^2)\nu$ follows by (68), (67) implies

$$\tan \angle(u_{\partial K}(x_m), o, u_{\partial K}(x_j)) \leq 2\omega^3 \cdot \nu. \tag{75}$$

We define $C' = C \cap L^+_{x_j}$ and $Y' = \partial C' \backslash \mathrm{relint}\widetilde{F}_m$, hence $Y \subset C'$ yields

$$S(P') - S(\widetilde{P}) = |Y| - |\widetilde{F}_m| \leq |Y'| - |\widetilde{F}_m|. \tag{76}$$

It follows by (74) that if $y \in \mathrm{relint}Y'$ and n is large then

$$\langle u_{\partial K}(x_m), u_{Y'}(y)\rangle^{-1} \leq 1 + 8\omega^3 \cdot \delta. \tag{77}$$

In addition if $y \in \mathrm{relint}(Y' \cap L_{x_j})$ then (75) yields

$$\langle u_{\partial K}(x_m), u_{Y'}(y)\rangle^{-1} \leq 1 + 8\omega^6 \cdot \nu^2. \tag{78}$$

Next we prove that if $\sqrt{\delta} > 2d\omega^3\nu$ then

$$\left(1 - \tfrac{2\omega^3\nu}{\sqrt{\delta}}\right) \cdot (\widetilde{F}_m - x_m) + x_m \subset \pi_{L_{x_m}}(Y' \cap L_{x_j}). \tag{79}$$

Let $x \in (1 - \frac{2\omega^3\nu\nu}{\sqrt{\delta}}) \cdot (\widetilde{F}_m - x_m) + x_m$, and let $y \in Y$ and $z \in Y' \cap L_{x_j}$ satisfy that $\pi_{L_{x_m}}(y) = \pi_{L_{x_m}}(z) = x$. Now the definition of Y shows that

$$\|z - x\| \geq \tfrac{2\omega^3\nu}{\sqrt{\delta}} \cdot 2\omega\delta = 4\omega^4 \cdot \nu\sqrt{\delta}. \tag{80}$$

Since the distance of x from A is at most $2\sqrt{\omega\delta}$, we deduce $\|y - x\| \leq 4\omega^{\frac{7}{2}}\nu\delta^{\frac{1}{2}}$ by (75), which inequality combined with (80) yield (79). In turn we conclude by (77), (78) and $|\widetilde{F}_m| \leq \beta_6\delta^{\frac{d-1}{2}}$ that if $\sqrt{\delta} > 2d\omega^3\nu$ then

$$|Y'| - |\widetilde{F}_m| \leq \beta_7\delta^{\frac{d-1}{2}} \cdot \left[\left(1 - \tfrac{2\omega^3\nu}{\sqrt{\delta}}\right)^{d-1} \cdot \nu^2 + \left(1 - \left(1 - \tfrac{2\omega^3\nu}{\sqrt{\delta}}\right)^{d-1}\right) \cdot \delta\right]. \tag{81}$$

Since $S(P_{(n)}) \leq S(P')$, combining (73), (76) and (81) leads to

$$\alpha_6 \cdot \delta^{\frac{d+1}{2}} < \beta_7\delta^{\frac{d-1}{2}} \cdot \left[\nu^2 + \tfrac{2d\omega^3\nu}{\sqrt{\delta}} \cdot \delta\right].$$

Therefore $\delta < \beta_8\nu^2$ in any case, hence the estimates (71) and (72) complete the proof of Lemma 5.2.

Q.E.D.

6 Approximating paraboloids

If K is a convex body with C^2 boundary, and the Gauß-Kronecker curvature is non-zero at $x \in \partial K$, then a small neighbourhood of x on ∂K can be very well approximated by a patch on a parabaloid. Using Lemma 4.2, polytopal approximation of this neighbourhood of x can be related to polytopal approximation of the patch on the paraboloid. Therefore in this section we consider the latter problem.

Let q be a positive definite quadratic form in $d-1$ variables, and let $\omega \geq 1$ satisfy

$$\omega^{-1} \cdot \|z\|^2 \leq q(z) \leq \omega \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{d-1}. \tag{82}$$

We choose a orthonormal bases for $\mathbb{E}^{d-1}$ such that if $y = (t_1, \ldots, t_{d-1}) \in \mathbb{E}^{d-1}$ then $q(y) = \tau_1 t_1^2 + \ldots + \tau_{d-1} t_{d-1}^2$, and hence $\omega^{-1} \leq \tau_i \leq \omega$ for each τ_i. We assign the positive definite quadratic form q^* to q defined by

$$q^*(y) = \|y\|^2 + \frac{1}{\operatorname{tr} q} \cdot q(y) = \sum_{i=1}^{d-1} \left(1 + \frac{\tau_i}{\tau_1 + \ldots + \tau_{d-1}}\right) \cdot t_i^2. \tag{83}$$

The role of q^* will be explained by Lemma 6.1. We frequently estimate $q^*(z)$ by

$$\|z\|^2 \leq q^*(z) \leq 2 \cdot \|z\|^2. \tag{84}$$

Since $\operatorname{tr} q^* = d$, Lemma 2.5 yields

$$\operatorname{div}_{d-1} \leq \operatorname{div}(q^*) \leq \tfrac{d}{d-1} \cdot \operatorname{div}_{d-1}, \tag{85}$$

where equality holds in the upper bound if the eigen values of q coincide.

In Lemma 6.1, we further consider the graph X' the graph of $\frac{1}{2}q$ above $\mathbb{E}^{d-1}$, and a compact convex set C in $\mathbb{E}^{d-1}$ with

$$rB^{d-1} \subset C \subset 16rB^{d-1} \tag{86}$$

for some $r > 0$. In addition let $X = \pi_{X'}(C)$.

Lemma 6.1 *For $\omega \geq 1$ and $d \geq 2$, there exist $\varepsilon_0, \aleph > 0$ with the following properties. If $\varepsilon \in (0, \varepsilon_0)$, and q, C, X' and X as above, and $C \subset \sqrt{\varepsilon}\, B^{d-1}$, and $n > n_0$ where n_0 depends on ω, d and ε, then we have (i) and (ii) below.*

(i) If Y is a polytopal convex hypersurface circumscribed around X' satisfying $X = \pi_{X'}(Y)$, and Y has at most n facets then

$$|Y| - |X| \geq (1 - \aleph\varepsilon) \frac{\operatorname{div}(q^*)}{2} \cdot (\operatorname{tr} q) \cdot (\det q)^{\frac{1}{d-1}} |C|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}.$$

(ii) There exists a polytopal convex hypersurface Y circumscribed around X' with at most n facets such that $X = \pi_{X'}(Y)$, and

$$\begin{aligned} |Y| - |X| &\leq (1+\aleph\varepsilon)\frac{\operatorname{div}(q^*)}{2} \cdot (\operatorname{tr} q) \cdot (\det q)^{\frac{1}{d-1}} |C|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}, && (87) \\ \delta_H(Y, X) &\leq \aleph n^{\frac{-2}{d-1}} |C|^{\frac{2}{d-1}}. && (88) \end{aligned}$$

Proof: We write l_y to denote the linear function representing the derivative of $\frac{1}{2}q$ at $y \in \mathbb{E}^{d-1}$. The bases of the argument is the Taylor formula (32); namely, for $z, y \in \mathbb{E}^{d-1}$,

$$\tfrac{1}{2}\, q(z) = \tfrac{1}{2}\, q(y) + l_y(z-y) + \tfrac{1}{2}\, q(z-y). \tag{89}$$

First we prove an estimate for $|Y| - |X|$ if Y is a polytopal convex hypersurface circumscribed around X' with $X = \pi_{X'}(Y)$ and at most n facets, and Y satisfies the following:

($*$) If a facet F of Y touches X' at y, then $\pi_{\mathbb{E}^{d-1}}(F) \subset \pi_{\mathbb{E}^{d-1}}(y) + \varepsilon C$.

If ε_0 is chosen small enough then we deduce

$$\max_{y \in \mathrm{relbdY}} \|y - \pi_{X'}(y)\| < \min_{z \in \mathrm{relbdC}} \|z - \pi_{X'}(z)\|. \tag{90}$$

We write $F_1, \ldots, F_k$ to denote the facets of Y. We define x_i to be the point where affF_i touches X', $y_i = \pi_{\mathbb{E}^{d-1}}(x_i)$ and $\Pi_i = \pi_{\mathbb{E}^{d-1}}(F_i)$ for $i = 1, \ldots, k$. In particular, if $i = 1, \ldots, k$ then (89) yields

$$\Pi_i = \{z \in \pi_{\mathbb{E}^{d-1}}(Y) :\ q(z - y_i) \leq q(z - y_j) \quad \text{for } j = 1, \ldots, k\}. \tag{91}$$

We define

$$f(z) = \operatorname{tr} q \cdot q(z) + q^\circ(z),$$

and claim that

$$|Y| - |X| = \frac{1 + O_\omega(\varepsilon)}{2} \sum_{i=1}^{k} \int_{\Pi_i} f(y - y_i)\, dy. \tag{92}$$

To prove (92), we observe that $C \subset \sqrt{\varepsilon}\, B^{d-1}$ and (90) yield

$$\pi_{\mathbb{E}^{d-1}}(X), \pi_{\mathbb{E}^{d-1}}(Y) \subset \sqrt{\varepsilon}\, B^{d-1}. \tag{93}$$

We have $\|l_y\| \leq \omega \cdot \|y\|$, and hence if $\|y\| < 2\sqrt{\varepsilon}$, then

$$\langle \xi, u_{X'}(x) \rangle \geq 1 - \tfrac{1}{2}\, \omega^2 \|y\|^2 \geq 1 - 2\,\omega^2 \varepsilon \ \text{ for } x = (y, q(y)). \tag{94}$$

Since $q(z) \le \omega\|z\|^2$, a ball of radius $\frac{1}{\omega}$ rolls above X'. In other words, for any $x \in X'$ there exists a ball of radius $\frac{1}{\omega}$ that touches X' at x and lies in convX'.

For $z \in F_i$, let $x = \pi_X(z)$ and $y = \pi_{\mathbb{E}^{d-1}}(z)$. In particular $\|z-x\| = O_\omega(\varepsilon)$ by the condition (*) and the existence of the rolling ball. It follows from the estimates on $\|l_y\|$ and the existence of the rolling ball that

$$r_{X,Y}(x) = \tfrac{1+O_\omega(\varepsilon)}{2}\, q(y-y_i) = O_\omega(\varepsilon).$$

In addition (36) yields

$$\langle u_X(x), u_Y(z)\rangle^{-1} = \langle u_X(x), u_{X'}(x_i)\rangle^{-1} = 1 + \tfrac{1+O_\omega(\varepsilon)}{2}\, q^\circ(y-y_i) = 1 + O_\omega(\varepsilon).$$

Since the Jacobian of the map $\pi_X : F_i \to X$ is $1 + O_\omega(\varepsilon)$, we deduce

$$|Y| - |X| = \frac{1+O_\omega(\varepsilon)}{2} \sum_{i=1}^{k} \int_{F_i} f(\pi_{\mathbb{E}^{d-1}}(z) - y_i)\, dz,$$

by (25) and (39), and hence we conclude (92).

Before continuing to prove (i) and (ii), we observe that if $z \in B^{d-1}$ and $z' = \pi_{\mathbb{E}^{d-1}}\pi_{X'}(z)$, then

$$\|z - z'\| = O_\omega(\|z\|^2). \tag{95}$$

It follows by (95) that if ε_0 is small enough and $(*)$ holds, then (90) yields

$$(1-\varepsilon)C \subset \pi_{\mathbb{E}^{d-1}}(X), \pi_{\mathbb{E}^{d-1}}(Y) \subset (1+\varepsilon)C. \tag{96}$$

Let us prove (i) first. We may assume that

$$|Y| - |X| \le \tfrac{\mathrm{div}(q^*)}{2} \cdot (\mathrm{tr}\, q) \cdot (\det q)^{\frac{1}{d-1}} |C|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}. \tag{97}$$

To verify that (*) holds for large n, we assume only $C \subset B^{d-1}$ instead of $C \subset \sqrt{\varepsilon}B^{d-1}$ for the time being. It follows by the condition (86) that there exists some $\alpha > 0$ depending only on d such that if $y \in C$ and $\varrho \in (0, r)$, then

$$|(x + \varrho B^{d-1}) \cap C| > \alpha_1 \varrho^{d-1}.$$

Thus the existence of the rolling ball of radius $\frac{1}{\omega}$ for X' yields the existence of $t_0 > 0$ and α_2 depending on d and ω with the following property. If $x_0 \in X$ satisfies $r_{X,Y}(x_0) \ge t \in (0, t_0)$ then there exists some $A \subset X$ with $|A| \ge \alpha_2 t^{\frac{d-1}{2}}$ such that $r_{X,Y}(x) \ge t/2$ for $x \in A$. In particular (25) yields that $|Y| - |X| \ge \alpha_3 t^{\frac{d+1}{2}}$ for some $\alpha_3 > 0$ depending on d, ω, and hence we deduce by (97) that

$$\delta_H(Y, X) = O_\omega(n^{\frac{-4}{d^2-1}})|C|^{\frac{2}{d-1}}. \tag{98}$$

Therefore the condition (*) holds for $n > n_1$ where n_1 depends on d, ω and ε.

Let $\Pi_i' = \Pi_i \cap (1-\varepsilon)C$. Some of the Π_i's might be the empty, therefore we renumber $F_1, \dots, F_k$ in a way such that $\Pi_i' \neq \emptyset$ if and only $i \leq k'$ for some $k' \leq k$. In particular, if $i = 1, \dots, k'$ then

$$\Pi_i' = \{z \in (1-\varepsilon)C : \; q(z-y_i) \leq q(z-y_j) \quad \text{for } j = 1, \dots, k'\}. \tag{99}$$

It follows from (92) that

$$|Y| - |X| \geq \frac{1 - O_\omega(\varepsilon)}{2} \sum_{i=1}^{k'} \int_{\Pi_i'} f(y - y_i)\, dy.$$

Next let Ψ_q be the linear transformation of $\mathbb{E}^{d-1}$ defined by

$$\Psi_q z = (\sqrt{\tau_1} t_1, \dots, \sqrt{\tau_{d-1}} t_{d-1}) \text{ for } z = (t_1, \dots, t_{d-1}), \tag{100}$$

and hence

$$\operatorname{tr} q \cdot q^*(\Psi_q z) = \operatorname{tr} q \cdot q(z) + q^\circ(z) = f(z) \text{ and } \|\Psi_q z\|^2 = q(z). \tag{101}$$

It follows that

$$\omega^{-\frac{1}{2}} r B^{d-1} \subset \Psi_q C \subset 16 \omega^{\frac{1}{2}} r B^{d-1}. \tag{102}$$

Writing $\Xi = \Psi_q\{y_1, \dots, y_{k'}\}$, we have (compare (99)),

$$|Y| - |X| \geq \frac{1 - O_\omega(\varepsilon)}{2} \cdot (\det q)^{\frac{-1}{2}} \cdot \operatorname{tr} q \cdot \Omega(\Psi_q((1-\varepsilon)C), \Xi, q^*).$$

Here Ξ has at most n elements. According to Corollary 2.3, if $n > n_2$ where $n_2 > n_1$ depends on d, ω and ε, then

$$\begin{aligned} |Y| - |X| \;&\geq\; \frac{1 - O_\omega(\varepsilon)}{2} \cdot (\det q)^{\frac{-1}{2}} \cdot \operatorname{tr} q \cdot \operatorname{div}(q^*) |\Psi_q((1-\varepsilon)C)|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}} \\ &\geq\; (1 - O_\omega(\varepsilon)) \frac{\operatorname{div}(q^*)}{2} \cdot (\operatorname{tr} q) \cdot (\det q)^{\frac{1}{d-1}} |C|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}, \end{aligned}$$

completing the proof of (i).

To prove (ii), we take a reverse path. It follows by (102) that there exist $\vartheta > 0$ depending on d and ω, moreover $n_3 > n_2$ depending on d, ω and ε with the following property. For $n > n_3$, one finds a set Ξ_0 of cardinality at most $\vartheta \varepsilon n$ such that

$$\Xi_0 \subset \Psi_q((1+\varepsilon)C) \backslash \Psi_q((1-3\varepsilon)C) \subset \Xi_0 + |\Psi_q C|^{\frac{1}{d-1}} n^{\frac{-1}{d-1}} B^{d-1}.$$

According to Corollary 2.3 and (10), if $n > n_4$ where $n_4 > n_3$ depends on d, ω and ε, there exists Ξ' of cardinality at most $(1-\vartheta\varepsilon)n$ such that

$$\Omega(\Psi_q((1+\varepsilon)C), \Xi', q^*) \leq (1+O_\omega(\varepsilon))\mathrm{div}(q^*)|\Psi_q((1+\varepsilon)C)|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}, \quad (103)$$

$$\Omega(\Psi_q((1+\varepsilon)C)\backslash\Psi_q((1-19\omega\varepsilon)C), \Xi', q^*) \leq O_\omega(\varepsilon)\cdot\Omega(\Psi_q((1+\varepsilon)C), \Xi', q^*), \quad (104)$$

$$\Pi(y, \Xi', \Psi_q C) \subset y + \varepsilon\Psi_q C \ \text{ for } y \in \Xi'. \quad (105)$$

It follows from (103) and applying Corollary 2.3 to $(1-3\varepsilon)C$ that $\Xi' \cap (1-3\varepsilon)C$ has at least $(1-O_\omega(\varepsilon))n$ points, and hence

$$\Xi'\backslash(1-3\varepsilon)C \ \text{ has } O_\omega(\varepsilon n) \text{ points.} \quad (106)$$

Let $\widetilde{\Xi} = \Xi' \cup \Xi_0$, and hence the cardinality of $\widetilde{\Xi}$ is at most n. For any $x \in \Psi_q((1+\varepsilon)C)$, let y be a closest point of $\widetilde{\Xi}$ to x, and let z be a closest point of Ξ' to x. First we assume $x \in (1-19\omega\varepsilon)C$. It follows by (102) and (105) that

$$\|z-x\| \leq \|y-x\| \leq \varepsilon \cdot 16r\omega^{\frac{1}{2}},$$

and hence again (102) yields

$$z \in x + \varepsilon \cdot 16\omega\Psi_q C \subset (1-\varepsilon 3\omega)\Psi_q C.$$

Since $\Xi_0 \cap (1-\varepsilon 3\omega)\Psi_q C = \emptyset$, we conclude that if $x \in (1-6\omega\varepsilon)C$, then $q^*(x-y) = q^*(x-z)$. On the other hand, if $x \notin (1-19\omega\varepsilon)C$, then

$$q^*(x-y) \leq 2\|x-y\|^2 \leq 2\|x-z\|^2 \leq 2q^*(x-z).$$

Therefore (103) and (104) imply

$$\begin{aligned}\Omega(\Psi_q((1+\varepsilon)C), \widetilde{\Xi}, q^*) &\leq (1+O_\omega(\varepsilon))\Omega(\Psi_q((1+\varepsilon)C), \Xi', q^*)\\ &\leq (1+O_\omega(\varepsilon))\mathrm{div}(q^*)|\Psi_q((1+\varepsilon)C)|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}.\end{aligned} \quad (107)$$

Let $\Xi^* = \Psi_q^{-1}\widetilde{\Xi}$, and let Y' be the polytopal convex hypersurface whose facets touch X' in the points whose projection into $\mathbb{E}^{d-1}$ is Ξ^*. Finally, let $Y^* \subset Y'$ satisfy $\pi_{X'}(Y^*) = X$. It follows by (105) that Y^* satisfies (*). Therefore (92) and (107) yields

$$|Y^*| - |X| \leq \frac{1+O_\omega(\varepsilon)}{2}\mathrm{div}(q^*)(\mathrm{tr}\, q)\cdot(\det q)^{\frac{1}{d-1}}|C|^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}. \quad (108)$$

In particular Y^* satisfies (87); namely, the half of (ii).

Next we aim at (88). We show that at least points near the relative boundary of Y^* satisfy (88). We define $\Xi_0^* = \Psi_q^{-1}\widetilde{\Xi}\backslash(1-3\varepsilon)C$, and write m to denote the cardinality of Ξ_0^*. It follows from (106) and the definition of Ξ_0 that $m = O_\omega(\varepsilon n)$. The choice of Ξ_0 also yields that for any $x \in (1+\varepsilon)C\backslash(1-3\varepsilon)C$, there exists $z \in \Xi_0^*$ satisfying

$$q(x-z) \leq O_\omega\left(|C|^{\frac{2}{d-1}} n^{\frac{-2}{d-1}}\right). \tag{109}$$

It follows from (109) that we only need to modify Y^* in the "inner" part to get (88).

We write Υ_{n-m}^C to denote the family of all $\Xi \subset (1-3\varepsilon)C$ of cardinality at most $n-m$. For $\Xi \in \Upsilon_{n-m}^C$, let Y'_Ξ be the polytopal convex hypersurface whose facets touch X' in the points whose projection into $\mathbb{E}^{d-1}$ is $\Xi \cup \Xi_0^*$, and let $Y_\Xi \subset Y'_\Xi$ satisfy $\pi_{X'} Y_\Xi = X$. It follows from (109) that if H is a hyperplane that touches X in a point x with $\pi_{\mathbb{E}^{d-1}}(x) \in (1-3\varepsilon)C$ then

$$\pi_{\mathbb{E}^{d-1}}(H \cap Y_\Xi) \subset (1-2\varepsilon)C. \tag{110}$$

Let $Y_{(n)} = Y_{\Xi(n)}$ for $\Xi(n) \in \Upsilon_{n-m}^C$ such that

$$|Y_{(n)}| - |X| = \min\{|Y_\Xi| - |X| : \Xi \in \Upsilon_{n-m}^C\}.$$

Since $Y^* = Y_\Xi$ for some $\Xi \in \Upsilon_{n-m}^C$, $Y_{(n)}$ satisfies (87). It follows from (i) and (87) that $Y_{(n)}$ has at least $(1-O_\omega(\varepsilon))n$ facets, thus $\Xi(n) \geq (1-O_\omega(\varepsilon))n$. It particular the minimal distance between points of $\Xi(n)$ is $O_\omega(|C|^{\frac{1}{d-1}} n^{\frac{-1}{d-1}})$. It follows from (110) that we may apply the the argument in Lemma 5.2 to $Y_{(n)}$, and the extremality of $Y_{(n)}$ yields that for any $x \in (1-3\varepsilon)C$ there exists a $y \in \Xi(n) \cup \Xi_0^*$ with $q(x-y) = O_\omega(|C|^{\frac{2}{d-1}} n^{\frac{-2}{d-1}})$. Combining this estimate with (109) completes the proof of (88).

Q.E.D.

7 The proof of Theorem 1.1 if ∂K is C_+^2

Let K be a convex body in $\mathbb{E}^d$ with C^2 boundary. In this section we prove Theorem 7.1 about polytopal approximation of a compact Jordan measurable subset X such that $\kappa(x) > 0$ for $x \in X$. In particular if ∂K is C_+^2, and hence $X = \partial K$ can be assumed, then Theorem 7.1 proves Theorem 1.1. In addition, Theorem 7.1 forms the core of approximating ∂K also in the case when the Gauß-Kronecker curvature is allowed to be zero.

Theorem 7.1 *Given a convex body K in $\mathbb{E}^d$ with C^2 boundary, and a convex hypersurface $X \subset \partial K$ such that the Gauß-Kronecker curvature is positive at any $x \in X$, there exist $\varepsilon_0, \tilde{\beta} > 0$ depending on K and X with the following properties: Let $\varepsilon \in (0, \varepsilon_0)$.*

(i) If n is large and $P_{(n)}$ is a circumscribed polytope with n facets that has minimal surface area, and the $Y \subset \partial P_{(n)}$ with $\pi_{\partial K} Y = X$ has k facets then

$$|Y| - |X| \geq \frac{1 - O_{K,X}(\varepsilon)}{2} \left(\int_X \mathrm{div}(Q_x)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx \right)^{\frac{d+1}{d-1}} k^{\frac{-2}{d-1}}.$$

(ii) If k is large then there exists a polytopal convex hypersurface Y circumscribed around ∂K with at most k facets such that $\pi_{\partial K} Y = X$, $\delta_H(Y, X) \leq \tilde{\beta} \cdot k^{\frac{-2}{d-1}}$ and

$$|Y| - |X| \leq \frac{1 + O_{K,X}(\varepsilon)}{2} \left(\int_X \mathrm{div}(Q_x)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx \right)^{\frac{d+1}{d-1}} k^{\frac{-2}{d-1}}.$$

Remark: How large n should be depends on ε, X, K.

7.1 The common parameters for the proofs of (i) and (ii)

There exists $\omega > 3$ such that the principal curvatures at each $x \in X$ lie between $\frac{3}{\omega}$ and $\frac{\omega}{3}$. We choose a convex hypersurface $X' \subset \partial K$ such that $X \subset \mathrm{relint} X'$, and the principal curvatures at each $x \in X'$ lie between $\frac{2}{\omega}$ and $\frac{\omega}{2}$. In particular $X' = \partial K$ if $X = \partial K$. There exist $\varrho_0 > 0$ and $\aleph_0 > 1$ depending on X and K such that if H is the tangent hyperplane at some $x \in X'$, and $\|y - \pi_{\partial K} y\| \leq \varrho$ for a $y \in H$ and $\varrho \in (0, \varrho_0)$, then

$$\|y - x\| \leq \aleph_0 \sqrt{\varrho} B^d. \tag{111}$$

Given d and the ω of the previous paragraph, we choose the corresponding $\varepsilon_0 > 0$ in a way such that it is small enough for Lemmas 4.2 and 6.1, moreover

$$\varepsilon_0 < (2^{20} \tilde{\aleph} \cdot \aleph_0 \cdot \omega^2)^{-1}$$

where $\tilde{\aleph}$ comes from (38), and $\aleph_0$ comes from (111). We choose $\gamma > 1$ depending on ω and d such that

$$\gamma^{-1} < \mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} < \gamma$$

for $x \in X'$. Let $\varepsilon \in (0, \varepsilon_0)$, and let n_0 depending on ω, d and ε come from Lemma 6.1. In addition let $\nu \in (0, \varepsilon]$ be maximal with the properties

$$\nu^{-(d-1)} \geq 8^{d-1}\aleph_0^{d-1} \cdot n_0, \tag{112}$$
$$\nu^{-(d-1)} \geq 2^d |B^{d-1}|^{-1} \cdot n_0. \tag{113}$$

These inequalities will ensure that when we apply Lemma 6.1, the number of facets of the corresponding polytopal convex hypersurface is at least n_0.

We choose convex hypersurfaces $Z_1, Z_2 \subset \partial K$ such that $Z_1 \subset \text{relint}X$, $X \subset \text{relint}Z_2$ and $Z_2 \subset \text{relint}X'$, and

$$\frac{\int_{Z_2} \text{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx}{\int_{Z_1} \text{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx} < 1 + \varepsilon. \tag{114}$$

We note that if $X = \partial K$ then we simply choose $Z_1 = Z_2 = \partial K$.

It follows *via* an compactness argument that there exists $\delta \in (0, \sqrt{\varepsilon})$ depending on K with the following properties: For $x \in Z_2$, let H be the tangent hyperplane to K at x. After identifying x with o and H with $\mathbb{E}^{d-1}$ in a way such that K lies above $\mathbb{E}^{d-1}$, there exists a convex C^2 function f on $\delta\, B^{d-1}$ whose graph is part of X', and writing q_y to denote the quadratic form representing the second derivative of f at $y \in \text{relint}\, \delta\, B^{d-1}$, we have

$$Q_x(z) - \tfrac{1}{2}\nu^8 \cdot \|z\|^2 \leq q_y(z) \leq Q_x(z) + \tfrac{1}{2}\nu^8 \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{d-1}. \tag{115}$$

It follows from Remark 2.4 and (35) for $\text{div}(Q_x^*)$, and (39) and (40) for $H(x)$ and $\kappa(x)$ that if $y \in \text{relint}\, \delta\, B^{d-1}$ and $x' = (y, f(y))$ then

$$\text{div}(Q_{x'}^*) = (1 + O_{K,X}(\varepsilon^8)) \cdot \text{div}(Q_x^*); \tag{116}$$
$$H(x') = H(x) + O_{K,X}(\varepsilon^8); \tag{117}$$
$$\kappa(x') = \kappa(x) + O_{K,X}(\varepsilon^8). \tag{118}$$

In addition for the map $\pi_{\partial K} : \text{relint}\, \delta\, B^{d-1} \to \partial K$, we deduce from (26) that

$$\text{the Jacobian is } 1 + O_K(\|y\|^2) \text{ at each } y \in \text{relint}\, \delta\, B^{d-1}. \tag{119}$$

When we say that n or k is large enough then we mean a threshold that depends on ε, X and K.

7.2 The proof of Theorem 7.1 (i)

According to Lemma 5.2, $\delta_H(P_{(n)}, K) \leq \beta_0 n^{\frac{-2}{d-1}}$ where β_0 depends on K. For large n, we define

$$\varrho = 2^{20}\beta_0 n^{\frac{-2}{d-1}}. \tag{120}$$

Readily $\varrho < \nu^8$, $\varrho < \varrho_0$ and $\frac{20\sqrt{\varrho}}{\nu} < \delta$ if n is large.

We choose a maximal family $s_1, \ldots, s_{m'} \in \partial K$ with the property that $\|s_i - s_j\| \geq 2\frac{\sqrt{\varrho}}{\nu}$ for $i \neq j$, and we write $C_1^*, \ldots, C_{m'}^*$ to denote the facets of the circumscribed polytope whose facets touch K at $s_1, \ldots, s_{m'}$. Let $X_i^* = \pi_{\partial K} C_i^*$, $i = 1, \ldots, m'$, and let us reindex $s_1, \ldots, s_{m'}$ in a way such that

$$X_i^* \cap Z_1 \neq \emptyset \text{ if and only if } i \leq m \text{ for some } m \leq m'.$$

We write B_i to denote the unit $(d-1)$-ball that is centred at s_i, and contained in the tangent hyperplane to K at s_i. If n is large and $i = 1, \ldots, m$, then $X_i^* \subset X$ and

$$s_i + \tfrac{\sqrt{\varrho}}{2\nu} B_i \subset C_i^* \subset s_i + 3\tfrac{\sqrt{\varrho}}{\nu} B_i.$$

Since $\delta_H(P_{(n)}, K) \leq \varrho$, (111) yields that if $\pi_{\partial K} F$ intersects X for a facet F of $P_{(n)}$, then $\mathrm{diam} F \leq 2\aleph_0 \sqrt{\varrho}$. We define

$$C_i = s_i + (1 - 8\aleph_0 \nu)(C_i^* - s_i),\ i = 1, \ldots, m.$$

In particular if n is large and $i = 1, \ldots, m$, then

$$s_i + \tfrac{\sqrt{\varrho}}{4\nu} B_i \subset C_i \subset s_i + 3\tfrac{\sqrt{\varrho}}{\nu} B_i,$$

and if $\pi_{\partial K} F$ intersects $\pi_{\partial K} C_i$ for a facet F of $P_{(n)}$, then it is disjoint from $\pi_{\partial K} C_j$ for $j \neq i$. For $i = 1, \ldots, m$, let $\pi_{\partial K} C_i = X_i$, and let $Y_i \subset \partial P_{(n)}$ satisfy that $\pi_{\partial K} Y_i = X_i$. Therefore writing k_i to denote the number facets of Y_i, we have $k_1 + \ldots + k_m \leq k$. Since the projections of the facets of Y_i into C_i cover $s_i + \frac{\sqrt{\varrho}}{8\nu} B_i$, (111) and (112) yield that $k_i > n_0$. According to Lemma 5.2, any facet F of $P_{(n)}$ with $\pi_{\partial K} F \subset X$ contains a $(d-1)$-ball of radius $\alpha n^{\frac{-1}{d-1}}$ where α depends on K and X. Therefore

$$n_0 < k_i \leq O_{K,X}(n|C_i|). \tag{121}$$

Let $\pi_{\partial K} C_i^* = X_i^*$ for $i = 1, \ldots, m$.

Proposition 7.2 *If n is large and $i = 1, \ldots, m$ then*

$$|Y_i^*| - |X_i^*| \geq \frac{1 - O_{K,X}(\varepsilon)}{2} \left(\int_{X_i^*} \mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}}\, dx \right)^{\frac{d+1}{d-1}} \cdot k_i^{\frac{-2}{d-1}}.$$

Proof: We may assume that $s_i = o$, $\mathrm{aff}\, C_i = \mathbb{E}^{d-1}$ and K lies above $\mathbb{E}^{d-1}$. According to (115), there exists a convex C^2 function f_1 on $\sqrt{\nu}\, B^{d-1}$ whose graph is part of ∂K, and writing q_y to denote the quadratic form representing the second derivative of f_1 at y, we have that if $y \in \frac{20\sqrt{\varrho}}{\nu}\, B^{d-1}$ and n is large then

$$Q_{s_i}(z) - \tfrac{1}{2}\,\nu^8 \cdot \|z\|^2 \leq q_y(z) \leq Q_{s_i}(z) + \tfrac{1}{2}\,\nu^8 \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{d-1}.$$

We define q by

$$q(z) = Q_{s_i}(z) + \tfrac{1}{2}\,\nu^8 \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{d-1},$$

which satisfies that if $y \in \frac{20\sqrt{\varrho}}{\nu}\, B^{d-1}$ then

$$q(z) - \nu^8 \cdot \|z\|^2 \leq q_y(z) \leq q(z) \quad \text{for } z \in \mathbb{E}^{d-1}. \tag{122}$$

In addition all eigen values of q lie between $\frac{1}{\omega}$ and ω.

We write $X_i' \subset \partial K$ to denote the convex hypersurface that is the graph of a convex function above $\frac{20\sqrt{\varrho}}{\nu}\, B^{d-1}$, hence $X_i \subset X_i'$. In addition let $\widetilde{X}_i'$ be the graph of $f_2 = \frac{1}{2}\, q$ above $\frac{20\sqrt{\varrho}}{\nu}\, B^{d-1}$, and let $\widetilde{X}_i = \pi_{\widetilde{X}_i'} C_i$. We observe that $\widetilde{X}_i'$ lies "above" X_i' according to (122), and if $x \in X_i'$ with $\|\pi_{\mathbb{E}^{d-1}} x\| \leq \frac{10\sqrt{\varrho}}{\nu}$ then (122), $\varrho \in (0, \nu^8)$, and the Taylor formula (32) yield

$$d(x, \widetilde{X}_i') \leq 50\nu^6 \varrho. \tag{123}$$

It follows from (38), $\varrho < \nu^8$ and the conditions on ε_0 that if $y = \pi_{X_i'} x$ and $z = \pi_{\widetilde{X}_i'} x$ for $x \in \partial C_i$ then

$$\sqrt{q(x - \pi_{\mathbb{E}^{d-1}} y)} + \sqrt{q(x - \pi_{\mathbb{E}^{d-1}} z)} < \nu\sqrt{\varrho}. \tag{124}$$

During the argument, we frequently apply that $\sqrt{q(\cdot)}$ is a norm.

In order to apply Lemma 4.2, we need to extend Y_i to a suitable polytopal convex hypersurface, whose facets touch X_i', and that is the graph a convex function h_1 on $\frac{20\sqrt{\varrho}}{\nu}\, B_i$. Let Z be the family of points $z \in \frac{10\sqrt{\varrho}}{\nu}\, B_i$ such that $q(z - y) \geq \varrho/64$ for all $y \in \pi_{E^{d-1}} Y_i$. In addition let Ξ be the family of projections into $\mathbb{E}^{d-1}$ of the points where the facets of Y_i touch X_i', and let Ξ' be a maximal family of points in Z such that $q(z - y) \geq \nu^2 \varrho$ for different $z, y \in \Xi'$. Let Y_i' be the convex polytopal surface circumscribed around X_i' that is the graph of the convex piecewise linear function h_1 on $\frac{20\sqrt{\varrho}}{\nu}\, B_i$ such that projections of the points of tangency into $\mathbb{E}^{d-1}$ is $\Xi \cup \Xi'$. For $y \in Y_i$, Lemma 5.2 and the definition of ϱ yield that $d(y, X_i) \leq 2^{-20}\varrho$, and hence there exists $x \in \Xi$ with

$$q(x - \pi_{E^{d-1}} y) < 2^{-18} \varrho \tag{125}$$

according to (46). Since $q(x-\pi_{E^{d-1}}y) > 2^{-7}\varrho$ for any $x \in \Xi'$ by (124), we deduce that $Y_i \subset Y'_i$ by (47). In addition for any $z \in \frac{8\sqrt{\varrho}}{\nu}\,B_i$, there exists an x_0 with $q(z-x_0) \leq \varrho/64$ such that either $x_0 \in Z$ or $x_0 \in \pi_{E^{d-1}}Y_i$. Therefore there exists an $x \in \Xi \cup \Xi'$ with $q(z-x) \leq \varrho/8$, and hence (49) is satisfied.

For any $y \in \frac{10\sqrt{\varrho}}{\nu}\,B_i$, let $\tilde{y} \in \frac{20\sqrt{\varrho}}{\nu}\,B_i$ be the point such that the exterior unit normals to X'_i at $x = (y, f_1(y))$ and to $\widetilde{X}'_i$ at $(\tilde{y}, f_2(\tilde{y}))$ coincide. Inasmuch as (123) yields that $(\tilde{y}, f_2(\tilde{y}))$ is contained in the cap of K bounded by the hyperplane parallel to the tangent at x and of distance $50\nu^6\varrho$ from x, it follows from the Taylor formula (32) and (46) that

$$q(y-\tilde{y}) \leq 200\nu^6\varrho. \tag{126}$$

Next let $\widetilde{\Xi} = \{\tilde{y} : y \in \Xi\}$, and let $\widetilde{\Xi}' = \{\tilde{y} : y \in \Xi'\}$. If $\widetilde{Y}'_i$ is the convex polytopal hypersurface circumscribed around $\widetilde{X}'_i$ whose facets are in bijective correspondence with the facets of Y_i in a way such that the corresponding facets are parallel, and $\widetilde{Y}'_i$ is the graph of the convex function h_2 over $\frac{20\sqrt{\varrho}}{\nu}\,B_i$, then the projections of the points of tangency into $\mathbb{E}^{d-1}$ is $\widetilde{\Xi} \cup \widetilde{\Xi}'$. Let $\widetilde{Y}_i \subset \widetilde{Y}'_i$ satisfy $\pi_{\widetilde{X}'_i}\widetilde{Y}_i = \widetilde{X}_i$. If $y \in \widetilde{Y}_i$ then combining (124), (125) and (126) shows that there exists $x \in \widetilde{\Xi}$ with $q(x-\pi_{E^{d-1}}y) < 2^{-17}\varrho$, while $q(z-\pi_{E^{d-1}}y) > 2^{-8}\varrho$ for any $z \in \widetilde{\Xi}'$. It follows by (47) that the projections of the points where the facets $\widetilde{Y}_i$ touch $\widetilde{X}'_i$ into $\mathbb{E}^{d-1}$ all land in $\widetilde{\Xi}$, therefore $\widetilde{Y}_i$ has at most k_i facets.

Next we apply Lemma 4.2, where C_i, X_i, Y_i, $\widetilde{X}_i$, $\widetilde{Y}_i$ play the role of C, $\widetilde{X}_1$, $\widetilde{Y}_1$, $\widetilde{X}_2$, $\widetilde{Y}_2$. We deduce using (121) that

$$\begin{aligned} |X_i| - |Y_i| &\geq |\widetilde{X}_i| - |\widetilde{Y}_i| - O_{K,X}(\nu \cdot n^{\frac{-2}{d-1}}) \cdot |C_i| \\ &\geq |\widetilde{X}_i| - |\widetilde{Y}_i| - O_{K,X}(\nu \cdot k_i^{\frac{-2}{d-1}}) \cdot |C_i|^{\frac{d+1}{d-1}}. \end{aligned}$$

Since $k_i > n_0$, we may apply Lemma 6.1 (i) to $|\widetilde{Y}_i| - |\widetilde{X}_i|$, and conclude

$$\begin{aligned} |Y_i| - |X_i| &\geq (1 - O_{K,X}(\varepsilon)) \cdot \tfrac{\mathrm{div}(q^*)}{2}\,(\mathrm{tr}q)(\det q)^{\frac{1}{d-1}}|C_i|^{\frac{d+1}{d-1}} \cdot k_i^{\frac{-2}{d-1}} \\ &\geq (1 - O_{K,X}(\varepsilon)) \cdot \frac{\mathrm{div}(Q^*_{s_i})}{2}\,H(s_i)\kappa(s_i)^{\frac{1}{d-1}}|C^*_i|^{\frac{d+1}{d-1}} \cdot k_i^{\frac{-2}{d-1}}. \end{aligned}$$

Since $|Y^*_i| - |X^*_i| \geq |Y_i| - |X_i|$, the estimates (116), (117), (118) and (119) complete the proof of Proposition 7.2. Q.E.D.

We have $X^*_i \subset X$ for $i = 1, \ldots, m$, and the union of $X^*_1, \ldots, X^*_m$ covers Z_1. It follows from Proposition 7.2, the Hölder inequality (8) and $k_1 + \ldots + k_m \leq k$ that

$$|Y| - |X| \geq \sum_{i=1}^{m} |Y^*_i| - |X^*_i|$$

$$\geq \frac{1-O_{K,X}(\varepsilon)}{2}\sum_{i=1}^{m}\left(\int_{X_i^*}\mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}}H(x)^{\frac{d-1}{d+1}}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot k_i^{\frac{-2}{d-1}}$$
$$\geq \frac{1-O_{K,X}(\varepsilon)}{2}\left(\int_{Z_1}\mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}}H(x)^{\frac{d-1}{d+1}}\kappa(x)^{\frac{1}{d+1}}\,dx\right)^{\frac{d+1}{d-1}}\cdot k_i^{\frac{-2}{d-1}}.$$

Therefore (114) completes the proof of Theorem 7.1 (i).

7.3 The proof of Theorem 7.1 (ii)

For large k, we define

$$\varrho = 2^{24}\aleph\gamma^{\frac{4}{d-1}}|X|^{\frac{2}{d-1}}k^{\frac{-2}{d-1}}. \tag{127}$$

where the $\aleph > 1$ depending on ω and d comes from Lemma 6.1. Readily $\varrho < \nu^8$, $\varrho < \varrho_0$ and $\frac{20\sqrt{\varrho}}{\nu} < \delta$ if k is large.

We choose a maximal family $s_1,\dots,s_{m'}\in\partial K$ with the property that $\|s_i - s_j\| \geq 2\frac{\sqrt{\varrho}}{\nu}$ for $i\neq j$, and we write $C_1,\dots,C_{m'}$ to denote the facets of the circumscribed polytope whose facets touch K at $s_1,\dots,s_{m'}$. Let $X_i = \pi_{\partial K}C_i$, $i=1,\dots,m'$, and let us reindex $s_1,\dots,s_{m'}$ in a way such that

$$X_i\cap X\neq\emptyset \text{ if and only if } i\leq m \text{ for some } m\leq m'.$$

We write B_i to denote the unit $(d-1)$-ball that is centred at s_i, and contained in the tangent hyperplane to K at s_i. If k is large and $i=1,\dots,m$, then $X_i\subset Z_2$ and

$$s_i + \frac{\sqrt{\varrho}}{2\nu}B_i \subset C_i \subset s_i + 3\frac{\sqrt{\varrho}}{\nu}B_i. \tag{128}$$

For $i=1,\dots,m$, let

$$k_i = \left\lfloor \frac{\int_{X_i}\mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}}H(x)^{\frac{d-1}{d+1}}\kappa(x)^{\frac{1}{d+1}}\,dx}{\int_{X}\mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}}H(x)^{\frac{d-1}{d+1}}\kappa(x)^{\frac{1}{d+1}}\,dx}\cdot k\right\rfloor$$

It follows from (127) that

$$\varrho = 2^{24}\aleph\left(\frac{\gamma^2|X|}{k}\right)^{\frac{2}{d-1}} \geq 2^{24}\aleph\left(\frac{|X_i|}{2k_i}\right)^{\frac{2}{d-1}} > 2^{20}\aleph\left(\frac{|C_i|}{k_i}\right)^{\frac{2}{d-1}}. \tag{129}$$

We also deduce using (129), (128), and (113) in this order that

$$k_i > \varrho^{\frac{-(d-1)}{2}}|C_i| \geq \nu^{-(d-1)}2^{-d}|B^{d-1}| \geq n_0.$$

In summary,

$$n_0 < k_i \leq O_{K,X}(k|C_i|). \tag{130}$$

Proposition 7.3 *If k is large and $i = 1, \ldots, m$ then then there exists a convex polytopal hypersurface Y_i circumscribed around ∂K such that $\pi_{\partial K} Y_i = X_i$, Y_i has at most k_i facets, $\delta_H(Y_i, X_i) \leq \varrho$, and*

$$|Y_i| - |X_i| \leq \frac{1 + O_{K,X}(\varepsilon)}{2} \left(\int_{X_i} \operatorname{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx \right)^{\frac{d+1}{d-1}} \cdot k_i^{\frac{-2}{d-1}}.$$

Proof: We may assume that $s_i = o$, $\operatorname{aff} C_i = \mathbb{E}^{d-1}$ and K lies above $\mathbb{E}^{d-1}$. According to (115), there exists a convex C^2 function f on $\sqrt{\nu}\, B^{d-1}$ whose graph is part of ∂K, and writing q_y to denote the quadratic form representing the second derivative of f at y, we have that if $y \in \frac{20\sqrt{\varrho}}{\nu}\, B^{d-1}$ and k is large then

$$Q_{s_i}(z) - \tfrac{1}{2}\, \nu^8 \cdot \|z\|^2 \leq q_y(z) \leq Q_{s_i}(z) + \tfrac{1}{2}\, \nu^8 \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{d-1}.$$

We define q by

$$q(z) = Q_{s_i}(z) - \tfrac{1}{2}\, \nu^8 \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{d-1},$$

which satisfies that if $y \in \frac{20\sqrt{\varrho}}{\nu}\, B^{d-1}$ then

$$q(z) \cdot \|z\|^2 \leq q_y(z) \leq q(z) + \nu^8 \quad \text{for } z \in \mathbb{E}^{d-1}. \tag{131}$$

In addition all eigen values of q lie between $\frac{1}{\omega}$ and ω.

We write $X_i' \subset \partial K$ to denote the convex hypersurface that is the graph of a convex function above $\frac{20\sqrt{\varrho}}{\nu}\, B^{d-1}$, hence $X_i \subset X_i'$. In addition let $\widetilde{X}_i'$ be the graph of $f_2 = \frac{1}{2}\, q$ above $\frac{20\sqrt{\varrho}}{\nu}\, B^{d-1}$, and let $\widetilde{X}_i = \pi_{\widetilde{X}_i'} C_i$. We observe that X_i' lies "above" $\widetilde{X}_i'$ according to (131), and if $x \in \widetilde{X}_i'$ with $\|\pi_{\mathbb{E}^{d-1}} x\| \leq \frac{10\sqrt{\varrho}}{\nu}$ then (131), $\varrho \in (0, \nu^8)$, and the Taylor formula (32) yield

$$d(x, X_i') \leq 50 \nu^6 \varrho. \tag{132}$$

Since $k_i > n_0$, and (129) yields $\varrho > 2^{20} \aleph |C_i|^{\frac{2}{d-1}} k_i^{\frac{-2}{d-1}}$, Lemma 6.1 (ii) yields the existence of a convex polytopal surface $\widetilde{Y}_i$ circumscribed around $\widetilde{X}_i'$ such that $\widetilde{X}_i = \pi_{\widetilde{X}_i'} \widetilde{Y}_i$, $\widetilde{Y}_i$ has at most k_i facets, $\delta_H(\widetilde{Y}_i, \widetilde{X}_i) \leq 2^{-20} \varrho$, and

$$|\widetilde{Y}_i| - |\widetilde{X}_i| \leq (1 + O_{K,X}(\varepsilon)) \cdot \tfrac{\operatorname{div}(q^*)}{2}\, (\operatorname{tr} q)(\det q)^{\frac{1}{d-1}} |C_i|^{\frac{d+1}{d-1}} \cdot k_i^{\frac{-2}{d-1}}.$$

It follows from (116), (117), (118) and (119) that

$$|\widetilde{Y}_i| - |\widetilde{X}_i| \leq \frac{1 + O_{K,X}(\varepsilon)}{2} \left(\int_{X_i} \operatorname{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx \right)^{\frac{d+1}{d-1}} \cdot k_i^{\frac{-2}{d-1}}.$$

Finally, the same way how based on Lemma 4.2, the $\widetilde{Y}_i$ was constructed knowing Y_i in the proof of Proposition 7.2, one can construct the Y_i for Proposition 7.3 knowing the $\widetilde{Y}_i$ above.

Q.E.D.

It follows from Proposition 7.3 and the definition of k_i that if

$$\Delta = \left(\int_X \operatorname{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}}\, dx\right)^{\frac{2}{d-1}} \cdot k^{\frac{-2}{d-1}},$$

and $i = 1, \ldots, m$, then

$$|Y_i| - |X_i| \le \frac{1 + O_{K,X}(\varepsilon)}{2} \int_{X_i} \operatorname{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}}\, dx \cdot \Delta. \quad (133)$$

Let Y be the polytopal hypersurface circumscribed around ∂K such that the set of affine hulls of its facets is the union of the affine hulls of the facets of $Y_1, \ldots, Y_m$, and $\pi_{\partial K} Y = X$. It follows that Y has at most k facets, and

$$\delta_H(Y, X) \le \varrho = O_{K,X}(k^{\frac{-2}{d-1}}). \quad (134)$$

For $i = 1, \ldots, m$, let $Y_i^\diamond \subset Y$ such that $\pi_{\partial K} Y_i^\diamond = X_i$. We claim that

$$|Y_i^\diamond| - |X_i| \le \frac{1 + O_{K,X}(\varepsilon)}{2} \int_{X_i} \operatorname{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}}\, dx \cdot \Delta. \quad (135)$$

To show (135), we define

$$C_i^* = s_i + (1 - 8\aleph_0 \nu)(C_i - s_i),$$

$X_i^* = \pi_{\partial K} C_i^*$, and $Y_i^* \subset Y_i^\diamond$ such that $\pi_{\partial K} Y_i^* = X_i^*$. If F is a facet of Y, then $\operatorname{diam} F \le 2\aleph_0 \sqrt{\varrho}$ by (111) and (134). If in addition $\operatorname{aff} F$ is the affine hull of some facet of a Y_j with $j \ne i$, then (128) yields that

$$\pi_{\partial K} C_i^* \cap \pi_{\partial K} F = \emptyset.$$

Therefore $Y_i^* \subset Y_i$, and we deduce by (133) that

$$|Y_i^*| - |X_i^*| \le \frac{1 + O_{K,X}(\varepsilon)}{2} \int_{X_i} \operatorname{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}}\, dx \cdot \Delta.$$

On the other hand (26) and (134) imply

$$\begin{aligned} |Y_i^\diamond \backslash Y_i^*| - |X_i \backslash X_i^*| &= O_{K,X}(\varrho) \cdot |X_i \backslash X_i^*| = O_{K,X}(\varrho) \cdot |C_i \backslash C_i^*| \\ &= O_{K,X}(\varepsilon\varrho) \cdot |C_i| = O_{K,X}(\varepsilon\Delta) \cdot |X_i|. \end{aligned}$$

In turn we conclude (135).

Adding (135) for $i = 1, \ldots, m$ leads to

$$\begin{aligned} |Y| - |X| &\leq \frac{1 + O_{K,X}(\varepsilon)}{2} \int_{X_i} \mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx \cdot \Delta \\ &= \frac{1 + O_{K,X}(\varepsilon)}{2} \left(\int_X \mathrm{div}(Q_x^*)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}} \, dx \right)^{\frac{d+1}{d-1}} \cdot k^{\frac{-2}{d-1}}. \end{aligned}$$

With this, the proof of Theorem 7.1 is complete. Q.E.D.

8 The proof of Theorem 1.1

In this section, K is a convex body with C^2 boundary. For $x \in \partial K$, we define

$$\varphi(x) = \mathrm{div}(Q_x)^{\frac{d-1}{d+1}} H(x)^{\frac{d-1}{d+1}} \kappa(x)^{\frac{1}{d+1}}. \tag{136}$$

The proof of Theorem 1.1 is along the line set up in K. Böröczky, Jr. [3]. It is equivalent to proving that for certain $\varepsilon_0 > 0$ depending on K, if $\varepsilon \in (0, \varepsilon_0)$, then there exists n_0 depending on K and ε with the following properties: If $n > n_0$ and P is a polytope circumscribed around K with at most n facets, then

$$S(P) - S(K) \geq \frac{1 - \varepsilon}{2} \left(\int_{\partial K} \varphi(x) \, dx \right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}, \tag{137}$$

and in addition there exists a polytope $P'_{(n)}$ circumscribed around K with at most n facets, such that

$$S(P'_{(n)}) - S(K) \leq \frac{1 + \varepsilon}{2} \left(\int_{\partial K} \varphi(x) \, dx \right)^{\frac{d+1}{d-1}} n^{\frac{-2}{d-1}}. \tag{138}$$

To prove the lower bound (137), let $X \subset \partial K$ be a convex hypersurface such that the Gauß-Kronecker curvature is positive at any $x \in X$, and

$$\int_X \varphi(x) \, dx > \left(1 - \frac{\varepsilon}{2}\right) \int_{\partial K} \varphi(x) \, dx.$$

Let P be a polytope circumscribed around K with at most n facets, and let $Y \subset \partial P$ satisfy $\pi_{\partial K} Y = X$. In particular Y has at most n facets. According to Theorem 7.1 (i), there exists n_0 depending on K and ε such that if $n > n_0$ then

$$|Y| - |X| > \frac{1-\frac{\varepsilon}{2}}{2}\left(\int_{\partial K}\varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}$$

Since $S(P) - S(K) \geq |Y| - |X|$, we conclude (137).

Turning to (138), if the Gauß-Kronecker curvature is positive at any $x \in \partial K$ then (138) is a direct consequence of Theorem 7.1 (ii), hence we assume that there exists some point of ∂K such that at least one principal curvature is zero. For $\mu > 0$, let $\Sigma(\mu)$ denote the set of points on ∂K such that the minimal principal curvature is less than μ. We observe that $\Sigma(\mu)$ is Jordan measurable, and hence a convex hypersurface for all but countably many μ. It follows by Lemma 1 in K. Böröczky, Jr. [3] that there exist $\mu > 0$ and m_0 depending on ε and K with the following properties: $\Sigma(\mu)$ is a convex hypersurface, and if $m > m_0$, then there exists a polytopal convex surface Y_m circumscribed around ∂K with at most m facets such that $\pi_{\partial K} Y_m = \Sigma(\mu)$ and

$$\delta_H(Y_m, \Sigma(\mu)) < \varepsilon_0 \cdot \varepsilon^{\frac{d+1}{d-1}} \cdot m^{\frac{-2}{d-1}}. \tag{139}$$

It follows form (26) that choosing ε_0 depending on K small enough then

$$|Y_m| - |\Sigma(\mu)| < \frac{18^{\frac{-2}{d-1}}}{12}\left(\int_{\partial K}\varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \varepsilon^{\frac{d+1}{d-1}} \cdot m^{\frac{-2}{d-1}}. \tag{140}$$

We define X to be the closure of $\partial K \backslash \Sigma(\mu)$. According to Theorem 7.1 (ii), choosing ε_0 depending K small enough, we have the following properties. For $\varepsilon \in (0, \varepsilon_0)$, there exist $\gamma, n_1 > 1$ depending on K and ε such that if $n > n_1$ then $\lfloor \frac{\varepsilon}{18} n \rfloor > m_0$, and there exists a polytopal convex hypersurface $\widetilde{Y}_{(n)}$ circumscribed around ∂K with at most $\lceil (1 - \frac{\varepsilon}{18})n \rceil$ facets such that $\pi_{\partial K}\widetilde{Y}_{(n)} = X$, and

$$\begin{aligned}
\delta_H(\widetilde{Y}_{(n)}, X) &< \gamma \cdot n^{\frac{-2}{d-1}} & (141)\\
|\widetilde{Y}_{(n)}| - |X| &< \frac{1+\frac{\varepsilon}{9}}{2}\left(\int_{\partial K}\varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot \left(1 - \frac{\varepsilon}{18}\right)^{\frac{-2}{d-1}} n^{\frac{-2}{d-1}} \\
&< \frac{1+\frac{\varepsilon}{3}}{2}\left(\int_{\partial K}\varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}. & (142)
\end{aligned}$$

Next let $Y'_{(n)} = Y_m$ for $m = \lfloor \frac{\varepsilon}{18} n \rfloor$. In particular (139) and (140) yield that choosing ε_0 depending K small enough, we have

$$\delta_H(Y'_{(n)}, \Sigma(\mu)) < \gamma \cdot n^{\frac{-2}{d-1}} \tag{143}$$

$$|Y'_{(n)}| - |\Sigma(\mu)| < \frac{\varepsilon}{6} \cdot \left(\int_{\partial K}\varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}. \tag{144}$$

We consider all supporting halfspaces to K determined by the affine hulls of the facets of either $Y'_{(n)}$ or $\widetilde{Y}_{(n)}$, and define $P'_{(n)}$ to be the intersection of all these halfspaces. Then $P'_{(n)}$ is a polytope with at most n facets, and (141) and (143) yield

$$\delta_H(P'_{(n)},K) < \gamma \cdot n^{\frac{-2}{d-1}}. \tag{145}$$

Let $Z \subset \partial K$ be a convex hypersurface whose relative interior contains $\partial X = \partial\Sigma(\mu)$, the Gauß-Kronecker is positive at each $x \in Z$, and

$$|Z| < \frac{\varepsilon}{6\xi^*\gamma}\left(\int_{\partial K}\varphi(x)\,dx\right)^{\frac{d+1}{d-1}},$$

where ξ^* comes from (26). Let $Y^\circ_{(n)} \subset P'_{(n)}$ such that $\pi_{\partial K}Y^\circ_{(n)} = Z$. We deduce by (26) that

$$|Y^\circ_{(n)}| - |Z| < \frac{\varepsilon}{6}\left(\int_{\partial K}\varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}}. \tag{146}$$

It follows from (111) that there exists $n_0 > n_1$ (depending on K and ε) such that if $n > n_0$ then

$$\partial P'_{(n)} \backslash Y^\circ_{(n)} \subset Y'_{(n)} \cup \widetilde{Y}_{(n)}.$$

Therefore combining (142), (144) and (146) implies (138). In turn we conclude Theorem 1.1.

Q.E.D.

9 The proof of Theorem 1.2 (i)

Let K be convex a body in $\mathbb{E}^d$ with C^2 boundary. In order to prove Theorem 1.2 (i), we use again the desity function φ from (136). For each facet of $P_{(n)}$, we choose a point where the facet touches K, and we write $\Xi_n \subset \partial K$ to denote the set of these points. In particular Ξ_n has n elements.

For a Jordan measurable $X \subset \partial K$, we should prove that

$$\lim_{n\to\infty}\frac{\#(X \cap \Xi_n)}{n} = \frac{\int_X \varphi(x)\,dx}{\int_{\partial K}\varphi(x)\,dx}. \tag{147}$$

We wite $m(n) = \#(X \cap \Xi_n)$, and distinguish three cases.

Case 1 $\int_X \varphi(x)\,dx = \int_{\partial K}\varphi(x)\,dx$

In this case it is equivalent to prove that for any $\varepsilon \in (0,1)$, if $n > n_0$ where n_o depends on K, X and ε, then

$$m(n) > (1-\varepsilon)\, n. \tag{148}$$

Choose a convex hypersurface $Z \subset \mathrm{relint} X$ such that the Gauß-Kronecker curvature is positive on Z, and

$$\int_Z \varphi(x)\, dx > \left(1 - \frac{2\varepsilon}{3(d+1)}\right) \int_{\partial K} \varphi(x)\, dx.$$

Let $Y_{(n)} \subset \partial P_{(n)}$ satisfy that $\pi_{\partial K} Y_{(n)} = Z$. Since $\delta_H(P_{(n)}, K)$ tends to zero, there exists n_1 such that if $n > n_1$ then all facets of $Y_{(n)}$ touch at some point of X, and hence $Y_{(n)}$ has at most $m(n)$ facets. It follows by Theorem 1.1, $|Y_{(n)}| - |Z| < S(P_{(n)}) - S(K)$, and Theorem 7.1 (i) that if $n > n_0$ for suitable $n_0 > n_1$ then

$$|Y_{(n)}| - |Z| \;<\; \frac{(1+\frac{\varepsilon}{3})^{\frac{2}{d-1}}}{2} \cdot \left(\int_{\partial K} \varphi(x)\, dx\right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}} \tag{149}$$

$$|Y_{(n)}| - |Z| \;>\; \frac{(1-\frac{\varepsilon}{3})^{\frac{2}{d-1}}}{2} \cdot \left(\int_{Z} \varphi(x)\, dx\right)^{\frac{d+1}{d-1}} \cdot m(n)^{\frac{-2}{d-1}}. \tag{150}$$

In particular we conclude (148) as

$$\frac{m(n)}{n} > \frac{1-\frac{\varepsilon}{3}}{1+\frac{\varepsilon}{3}} \left(1 - \frac{2\varepsilon}{3(d+1)}\right)^{\frac{d+1}{2}} > 1-\varepsilon.$$

Case 2 $\int_X \varphi(x)\, dx = 0$
Since $\lim_{n\to\infty} \frac{\#((\partial K \setminus X)\cap \Xi_n)}{n} = 1$ by Case 1, we conclude $\lim_{n\to\infty} \frac{\#(X\cap \Xi_n)}{n} = 0$.

Case 3 $0 < \int_X \varphi(x)\, dx < \int_{\partial K} \varphi(x)\, dx$

In this case it is equivalent to prove that for any $\varepsilon \in (0, \frac{1}{2})$, if $n > n_0$ where n_o depends on K, X and ε, then

$$(1-\varepsilon) \cdot \frac{\int_X \varphi(x)\, dx}{\int_{\partial K \setminus X} \varphi(x)\, dx} < \frac{m(n)}{n - m(n)} < (1-\varepsilon)^{-1} \cdot \frac{\int_X \varphi(x)\, dx}{\int_{\partial K \setminus X} \varphi(x)\, dx}. \tag{151}$$

Let $\omega \geq 2$ be maximal with the property that

$$2/\omega \leq \frac{\int_X \varphi(x)\, dx}{\int_{\partial K \setminus X} \varphi(x)\, dx} \leq \omega/2$$

It follows from the equality case of the Hölder inequality (8), that there exists $\nu \in (0, \varepsilon)$ such that if $a_1, a_2, n_1, n_2 > 0$ satisfies $\omega^{-1} \le a_1/a_2 \le \omega$ and

$$a_1^{\frac{d+1}{d-1}} n_1^{\frac{-2}{d-1}} + a_2^{\frac{d+1}{d-1}} n_2^{\frac{-2}{d-1}} \le (1+\nu)(a_1+a_2)^{\frac{d+1}{d-1}}(n_1+n_2)^{\frac{-2}{d-1}}, \qquad (152)$$

then

$$\left(1-\frac{\varepsilon}{2}\right) \cdot \frac{a_1}{a_2} \le \frac{n_1}{n_2} \le \left(1-\frac{\varepsilon}{2}\right)^{-1} \cdot \frac{a_i}{a_j}.$$

We choose a convex hypersurfaces $Z_1 \subset \mathrm{relint} X$ and $Z_2 \subset \mathrm{relint}(\partial K \backslash X)$ such that the Gauß-Kronecker curvature is positive on Z_1 and Z_2, and

$$\frac{\int_{Z_1} \varphi(x)\,dx}{\int_X \varphi(x)\,dx} > 1 - \frac{\nu}{27} \quad \text{and} \quad \frac{\int_{Z_2} \varphi(x)\,dx}{\int_{\partial K \backslash X} \varphi(x)\,dx} > 1 - \frac{\nu}{27}. \qquad (153)$$

For $i = 1, 2$, let $Y^i_{(n)} \subset \partial P_{(n)}$ satisfy that $\pi_{\partial K} Y^i_{(n)} = Z_i$. Since $\delta_H(P_{(n)}, K)$ tends to zero, there exists n_1 such that if $n > n_1$ then all facets of $Y^1_{(n)}$ and $Y^2_{(n)}$ touch at some point of X or at $\partial K \backslash X$, respectively, and hence $Y^1_{(n)}$ has at most $m(n)$ facets, and $Y^2_{(n)}$ has at most $n - m(n)$ facets. It follows by Theorem 1.1, $\frac{d+1}{d-1} \le 3$ and Theorem 7.1 (i) that if $n > n_0$ for suitable $n_0 > n_1$ then

$$\begin{aligned}
\sum_{i=1,2} (|Y^i_{(n)}| - |Z_i|) &< \frac{1+\frac{\nu}{9}}{2} \cdot \left(\int_{\partial K} \varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}} \\
&< \frac{1+\frac{\nu}{3}}{2} \cdot \left(\int_{Z_1 \cup Z_2} \varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}} \\
|Y^1_{(n)}| - |Z_1| &> \frac{1-\frac{\nu}{3}}{2} \cdot \left(\int_{Z_1} \varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot m(n)^{\frac{-2}{d-1}} \\
|Y^2_{(n)}| - |Z_2| &> \frac{1-\frac{\nu}{3}}{2} \cdot \left(\int_{Z_2} \varphi(x)\,dx\right)^{\frac{d+1}{d-1}} \cdot (n - m(n))^{\frac{-2}{d-1}}.
\end{aligned}$$

In particular the choice of ν (see (152)) yields that

$$\left(1-\frac{\varepsilon}{2}\right) \cdot \frac{\int_{Z_1} \varphi(x)\,dx}{\int_{Z_2} \varphi(x)\,dx} \le \frac{m(n)}{n-m(n)} \le \left(1-\frac{\varepsilon}{2}\right)^{-1} \cdot \frac{\int_{Z_1} \varphi(x)\,dx}{\int_{Z_2} \varphi(x)\,dx}.$$

Therefore we conclude (151) by (153).

Q.E.D.

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