Typical faces of extremal polytopes with respect to a thin three-dimensional shell

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Dedicated to Károly Bezdek on his 50th birthday

Abstract

Given $r > 1$, we search for the convex body of minimal volume in $\mathbb{E}^3$ that contains a unit ball, and whose extreme points are of distance at least r from the centre of the unit ball. It is known that the extremal body is the regular octahedron and icosahedron for suitable values of r. In this paper we prove that if r is close to one then the typical faces of the extremal body are asymptotically regular triangles. In addition we prove the analogous statement for the extremal bodies with respect to the surface area and the mean width.

1 Notation and known results

Let us introduce the notation used throughout the paper. The implied constant in $O(\cdot)$ is always some absolute constant. For any notions related to convexity in this paper, consult R. Schneider [13] or P.M. Gruber [9]. We write o to denote the origin in $\mathbb{E}^n$, $\langle\cdot,\cdot\rangle$ to denote the scalar product, and $\|\cdot\|$ to denote the corresponding Euclidean norm. In addition for non–collinear points u, v, w, the angle of the half lines vu and vw is $\angle(u,v,w)$. Given a set $X \subset \mathbb{E}^n$, the affine hull and the interior of X are denoted by aff$X$ and int$X$, respectively. If X is compact convex then we write ∂X to denote the relative boundary of X with respect to aff$X$. Moreover let $[X_1,\ldots,X_k]$ stand for the convex hull of the objects $X_1,\ldots,X_k$.

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The unit ball centred at o in $\mathbb{E}^n$ is denoted by B^n, and the boundary of B^n is denoted by S^{n-1}. As usual we call a compact convex set C with non–empty interior a convex body, and write $V(C)$ to denote its volume, and $S(C)$ to denote its surface area. In addition a two-dimensional compact convex set is called a compact disc, and we write $A(\cdot)$ to denote the two–dimensional Hausdorff measure.

Given a compact convex set C in $\mathbb{E}^n$, its support function $h_C(u)$, $u \in \mathbb{E}^n$, is defined by

$$h_C(u) = \max_{x \in C} \langle x, u \rangle.$$

In particular for any $u \in S^{n-1}$, the width of C in the direction u is $h_C(u) + h_C(-u)$. Therefore the mean width of C is

$$M(C) = \frac{2}{S(B^n)} \int_{S^{n-1}} h_C(u)\, du. \tag{1}$$

In particular $M(B^n) = 2$, and if C is a convex disc then $M(C) = \frac{1}{\pi} S(C)$ according to the Cauchy formula.

The following statement is the starting point of our investigation:

Theorem 1.1 (Hajós lemma) *Among convex polygons, which contain B^2, and whose vertices are of distance at least r from o for some $r > 1$, the ones with minimal area are inscribed into rB^2 in a way that all but at most one side touch B^2. In addition the analogous statement holds for the minimal perimeter.*

Theorem 1.1 was proposed and proved by members of the seminar led by Gy. Hajós around 1960. It was inspired by an earlier result of L. Fejes Tóth (see for example [6]); namely, L. Fejes Tóth solved the case of minimal area for $r = \frac{2}{\sqrt{3}}$ when the optimal polygon is a regular hexagon. Actually his approach yields the similar characterization of any regular polygon.

Based on ideas of C.A. Rogers and J. Molnár, the following characterization of any regular polytopes in higher dimensions appeared in K. Böröczky and K. Máthéné Bognár [4]:

Theorem 1.2 (K. Böröczky) *Let M be a regular polytope in $\mathbb{E}^n$ whose circumcentre is o, and let r_i denote the distance of an i–face of M from o. If P is any polytope such that $o \in \mathrm{int} P$, and the distance of any i–face of P from o is at least r_i for $i = 0, \ldots, n$ then $V(P) \geq V(M)$. Moreover equality holds if and only if P is congruent to M, and its circumcentre is o.*

Recently Theorem 1.2 has been strengthened by K. Bezdek [1]:

Theorem 1.3 (K. Bezdek) *Let M be a regular polytope in $\mathbb{E}^n$ whose circumcentre is o, and let r_i denote the distance of an i–face of M from o. If P is any polytope such that $o \in \mathrm{int}P$, and the distance of any i–face of P from o is at least r_i for $i = 0, \dots, n$ then $S(P) \geq S(M)$. Moreover equality holds if and only if P is congruent to M, and its circumcentre is o.*

Theorem 1.3 implies Theorem 1.2 because we have $V(M) = \frac{r_{n-1}}{n} S(M)$ and $V(P) \geq \frac{r_{n-1}}{n} S(P)$.

One feels that possibly there are too many conditions on the polytope in Theorems 1.2 and 1.3 in higher dimensions. J. Molnár [10] asked the question whether for certain platonic solids in $\mathbb{E}^3$, the condition in Theorem 1.2 on the edges is superfluous. K. Böröczky, K. Böröczky, Jr. [2] verified the conjecture of J. Molnár in the case of the octahedron and the icosahedron.

Theorem 1.4 (K. Böröczky, K. Böröczky, Jr.) *Given $r = \sqrt{3}$ or $r = \sqrt{15 - 6\sqrt{5}}$, let M_r be the octahedron, or the icosahedron, respectively, circumscribed around B^3. If P is any polytope in $\mathbb{E}^3$ containing B^3, and each vertex of P is of distance at least r from o then $V(P) \geq V(M_r)$ and $S(P) \geq S(M_r)$. Moreover equality holds in either inequalities if and only if P is congruent to M_r, and its circumcentre is o.*

According to [2], cubes and dodecahedra are not optimal in their class, and no at least eight dimensional regular polytope is optimal in its class in the sense of Theorem 1.4. K. Böröczky, K. Böröczky, Jr. [2] conjecture that regular tetrahedra are optimal if $r = 3$. It is an open question whether the octahedron or the icosahedron could be characterized with respect to the mean width in the sense of Theorem 1.4.

In this paper our goal is to discuss optimal convex bodies with respect to thin shells, and to prove a weak generalization of the Hajós Lemma to dimension three. To define the corresponding class of convex bodies in $\mathbb{E}^3$, we recall that x is an extreme point of a convex compact set C if it does not lie in the relative interior of any segment contained in C. Actually the extreme points form the minimal subset of C whose convex hull is C. We write $\mathrm{ext}C$ to denote the family of extreme points of C.

Definition *Given $r > 1$, we write $\mathcal{F}_r$ to denote the family of convex bodies in $\mathbb{E}^3$, which contain B^3, and whose extreme points are of distance at least r from o. Moreover let $P_r \in \mathcal{F}_r$ have minimal volume, let $Q_r \in \mathcal{F}_r$ have minimal surface area, and let $Q_r \in \mathcal{F}_r$ have minimal mean width.*

The minima do exist according to the Blaschke Selection Theorem, and all extreme points of P_r, Q_r and W_r lie on rS^2 by the monotonicity of the volume, surface area and the mean width. Theorem 1.4 states that if $r = \sqrt{3}$ or $r = \sqrt{15-6\sqrt{5}}$ then P_r is an octahedron, or an icosahedron, respectively, circumscribed around B^3. Moreover the analogous statements hold for Q_r. It is not known whether P_r, Q_r and W_r are polytopes for all r. Actually if r is large then possibly all of P_r, Q_r and W_r are circular cylinders.

If r is close to 1 then it seems to be out of reach to determine P_r, Q_r or W_r. However we prove that in this case most part of the boundaries of P_r, Q_r and W_r are the union of triangles that are almost regular. In order to phrase this statement, in this paper we call F a face of a convex body C in $\mathbb{E}^3$ if F is the intersection of C and a supporting plane, moreover F is a convex disc. In addition we say that the convex discs M and N are ε–close for $\varepsilon > 0$ if there exist congruent copies M' and N' of M and N, respectively, satisfying

$$\tfrac{1}{1+\varepsilon} N' \subset M' \subset (1+\varepsilon) N'.$$

Theorem 1.5 *If $r > 1$ is close to one then all but at most $c(r-1)^{\frac{1}{9}}$ percent of the boundaries of all of P_r, Q_r and W_r are the union of faces that are $(r-1)^{\frac{1}{9}}$–close to the regular triangle of circumradius $\sqrt{r^2-1}$ where $c > 0$ is an absolute constant.*

The proof of Theorem 1.5 yields that if $r > 1$ tends to one then

$$V(P_r \backslash B^3) = \pi \cdot (r-1) + O\left((r-1)^{\frac{4}{3}}\right); \tag{2}$$

$$S(Q_r) - S(B^3) = 3\pi \cdot (r-1) + O\left((r-1)^{\frac{4}{3}}\right); \tag{3}$$

$$M(W_r^3) - M(B^3) = \tfrac{7}{6}(r-1) + O\left((r-1)^{\frac{4}{3}}\right). \tag{4}$$

What we still have in higher dimensions is the analogues of the asymptotic formulae (2), (3) and (4). Since the method of proof is quite different from the arguments below, these asymptotic results are proved in the accompanying paper K. Böröczky, K. Böröczky, Jr., C. Schütt and G. Wintsche [3].

Some of the ideas in this paper come from the theory of polytopal approximation (see P.M. Gruber [7], [8] and [9] for general surveys).

Let us review the structure of this paper. Below always $r > 1$ and is close to one. For a convex polytope L with $B^3 \subset P$ and $\text{ext}P \subset rS^2$, Section 2 presents some estimates for the differences of volume, surface area and mean width between L and B^3. These estimates use integrals based on the second moment over the faces of L. In turn certain extremal properties of the corresponding integral expressions over convex discs are determined in Sections 3 and 4. Next an element of $\mathcal{F}_r$ that is close to be optimal with respect to all of volume, surface area and mean width is constructed in Section 5. Finally Theorem 1.5, and the asymptotic formulae (2), (3) and (4) are proved in Section 6.

We close this section by introducing some notions and notations used in the arguments. We write $T(\rho)$ to denote a fixed regular triangle with circumradius $\rho > 0$. Next we write π_{S^2} to denote the radial projection from $\mathbb{E}^3 \backslash o$ onto S^2. Finally we define $\widetilde{\mathcal{F}}_r$ to be the family of all elements of $\mathcal{F}_r$ whose extreme points lie on rS^2, $r > 1$.

2 Some formulae for volume, surface area and mean width difference

In this section we provide formulae for the volume, surface area and mean width difference of a polytope in $\widetilde{\mathcal{F}}_r$ and B^3.

Let $r \in (1,2)$. We start with some observations concerning a convex disc $F \subset rB^3$ with $\mathrm{aff}F \cap \mathrm{int}B^3 = \emptyset$. First if $x, y \in F$ then

$$\|\pi_{S^2}(x) - \pi_{S^2}(y)\| \leq \|x - y\| \leq r^2 \cdot \|\pi_{S^2}(x) - \pi_{S^2}(y)\|. \tag{5}$$

In addition if $y \in S^2$ is normal to $\mathrm{aff}F$ and $(1+\nu)y \in \mathrm{aff}F$ for $\nu \geq 0$ then

$$A(F) = \int_{\pi_{S^2}(F)} \frac{(1+\nu)^2}{\langle x, y\rangle^3}\, dx; \tag{6}$$

$$V([o,F]) = \frac{1}{3}\int_{\pi_{S^2}(F)} \frac{(1+\nu)^3}{\langle x, y\rangle^3}\, dx. \tag{7}$$

Given a polytope $P \in \widetilde{\mathcal{F}}_r$, let $F_1, \ldots, F_k$ be the faces of P. For $i = 1, \ldots, k$, we write $x_i \in S^2$ to denote the unit exterior normal to F_i, and ν_i to denote the distance of $\mathrm{aff}F_i$ and B^3, moreover we define $z_i = (1+\nu_i)x_i \in \mathrm{aff}F_i$. Since if $x \in F_i$ then

$$\langle \pi_{S^2}(x), x_i\rangle = 1 - \tfrac{1}{2}\|x - z_i\|^2 + O((r-1)^2),$$

combining (5), (6) and (7) yields the formulae

$$S(P) - S(B^3) = \sum_{i=1}^{k} \int_{F_i} \left(\tfrac{3}{2}\|x - z_i\|^2 + 2\nu_i\right) dx + O((r-1)^2); \tag{8}$$

$$V(P) - V(B^3) = \sum_{i=1}^{k} \int_{F_i} \left(\tfrac{1}{2}\|x - z_i\|^2 + \nu_i\right) dx + O((r-1)^2). \tag{9}$$

Concerning the mean width, let $y_1, \ldots, y_l \in S^2$ be the points such that $ry_1, \ldots, ry_l$ are the vertices of P, hence

$$M(P) - M(B^3) = \frac{2}{S(B^3)} \int_{S^2} \left(\max_{j=1,\ldots,l} \left[\langle x, ry_j\rangle - 1\right]\right) dx. \tag{10}$$

It may happen that $x \in \pi_{S^2} F_i$ but there exists a y_j such that $\langle x, ry_j \rangle > \langle x, z \rangle$ for all $z \in F_i$. Therefore we cannot directly transfer the integral in (10) on S^2 onto ∂P. Instead we consider a related quantity over the individual faces of P.

Let F_i be a face of P, hence we deduce

$$A(F_i) = (1 + O(r-1)) \cdot A(\pi_{S^2} F_i) \tag{11}$$

by (5). If $y \in F_i$ and ry_j is a vertex of F_i then expanding $\langle \pi_{S^2} y, y_j \rangle$ in terms of $\|y_j - \pi_{S^2} y\|$ and using (5) leads to

$$\langle \pi_{S^2} y, ry_j \rangle - 1 = r - 1 + \tfrac{1}{2} \|y - ry_j\|^2 + O((r-1)^2).$$

Therefore simple argument yields

$$\begin{aligned} \int_{\pi_{S^2} F_i} \left(\max_{z \in \text{ext} F_i} [\langle x, z \rangle - 1] \right) dx &= A(\pi_{S^2} F_i)(r-1) - \frac{1}{2} \int_{F_i} \left(\min_{z \in \text{ext} F_i} \|y - z\|^2 \right) dy \\ &\quad + A(\pi_{S^2} F_i) O((r-1)^2). \end{aligned} \tag{12}$$

3 Estimates for convex discs related to volume and surface area

To prove the asymptotic formulae (2) and (3) about volume and surface area, an essential tool is the following estimate (compare (8) and (9)).

Lemma 3.1 *If $\rho > 0$ and C is a convex disc such that $\|x\| \geq \rho$ for any $x \in \text{ext} C$ then*

$$\int_C \|x\|^2 dx \geq \frac{\rho^2}{4} A(C),$$

with equality if and only if C is a regular triangle with circumradius ρ whose centroid is o.

What we need to prove Theorem 1.5 in the cases of volume and surface area is some stability version of Lemma 3.1.

Lemma 3.2 *There exist absolute constants $\varepsilon_0, c_1, c_2 > 0$ with the following properties: Let $\varepsilon \in (0, \varepsilon_0)$, $\rho > 0$ and let C be a convex disc with $\text{ext} C \subset \rho S^1$.*

(i) If C is ε-close to $T(\rho)$ then $\int_C \|x\|^2 dx \leq (\frac{1}{4} + c_1 \varepsilon) A(C) \cdot \rho^2$.

(ii) If C is not ε-close to $T(\rho)$ then $\int_C \|x\|^2 dx \geq (\frac{1}{4} + c_2 \varepsilon^2) A(C) \cdot \rho^2$.

We only prove Lemma 3.2 because it directly yields Lemma 3.1. In turn we prepare the proof of Lemma 3.2 by a series of small observations, the last of which is Proposition 3.5.

For an acute angle α, we define

$$\gamma(\alpha) = \frac{4\sin 2\alpha + \sin 4\alpha}{12\sin 2\alpha} = \frac{1}{3} + \frac{\cos 2\alpha}{6}.$$

This definition is motivated by the following observation. If $M = [v,u,w]$ is a triangle such that the angle at v is α and the angle at u is $\frac{\pi}{2}$ then

$$A(M) = \frac{\|w-v\|^2 \sin 2\alpha}{4} \tag{13}$$

$$\int_M \|x-v\|^2 dx = \frac{\|w-v\|^4 (4\sin 2\alpha + \sin 4\alpha)}{48} \tag{14}$$

$$= \gamma(\alpha) \cdot \|w-v\|^2 A(K). \tag{15}$$

We readily have

Proposition 3.3 *The function* $\gamma(\alpha)$ *is decreasing on* $(0, \frac{\pi}{2})$*, and satisfies* $\gamma(\alpha) + \gamma(\frac{\pi}{2} - \alpha) = \frac{2}{3}$.

Next we average the second moment over two triangles with right angle.

Proposition 3.4 *Let M and K be triangles with right angle that intersect in their common longest sides whose length is* ρ*, and let v be an endpoint of this common side. Writing* α *and* β *to denote the angle of M and K, respectively, at v, we have*

$$\frac{\int_{M\cup K} \|x-v\|^2 dx}{A(M \cup K)} \begin{cases} \geq \frac{\rho^2[2+\cos(\alpha+\beta)]}{6} & \text{if } \alpha+\beta > \frac{\pi}{2}; \\ \leq \frac{\rho^2[2+\cos(\alpha+\beta)]}{6} & \text{if } \alpha+\beta < \frac{\pi}{2} \end{cases}$$

with equality in both inequalities if $\alpha = \beta$.

Proof: It follows by (13) and (14), moreover by applying the addition formulae for sine and cosine that

$$\frac{\int_{M\cup K} \|x-v\|^2 dx}{A(M \cup K)} = \frac{\rho^2 (4\sin 2\alpha + 4\sin 2\beta + \sin 4\alpha + \sin 4\beta)}{12(\sin 2\alpha + \sin 2\beta)}$$

$$= \frac{\rho^2}{6} \cdot \left[2 + \cos(\alpha+\beta) \cdot \frac{2\cos^2(\alpha-\beta) - 1}{\cos(\alpha-\beta)} \right].$$

Since $\frac{2\cos^2 t - 1}{\cos t} = 1 - \frac{(1+2\cos t)(1-\cos t)}{\cos t}$ holds if $|t| < \frac{\pi}{2}$, we conclude Proposition 3.4.

Q.E.D.

The last preparatory statement gives an estimate for the second moment over any triangle:

Proposition 3.5 *If $M=[v,u,w]$ is a triangle such that the angle at v is φ then*

$$\int_M \|x-v\|^2 dx \geq \gamma(\tfrac{\varphi}{2})\cdot\|u-v\|\cdot\|w-v\|\cdot A(M).$$

Proof: We may assume that $\|u-v\|\cdot\|w-v\|=1$ and $v=o$. Writing $\|u\|=\lambda$, we define $\tilde{u}=\frac{1}{\lambda}u$ and $\tilde{w}=\lambda w$, hence $\|\tilde{u}\|=\|\tilde{w}\|=1$. Since $A(\widetilde{M})=A(M)$ and $\int_{\widetilde{M}}\|x\|^2dx=\gamma(\frac{\varphi}{2})A(\widetilde{M})$ for $\widetilde{M}=[o,\tilde{u},\tilde{w}]$, Proposition 3.5 is equivalent to

$$\int_M \|x\|^2dx \geq \int_{\widetilde{M}}\|x\|^2dx. \tag{16}$$

We consider the linear transforms A and B defined by

$$\begin{aligned} A\tilde{u}=u=\lambda\tilde{u} \quad &\text{and} \quad A\tilde{w}=w=\tfrac{1}{\lambda}\tilde{w};\\ B\tilde{u}=\tilde{w} \quad &\text{and} \quad B\tilde{w}=\tilde{u}. \end{aligned}$$

We observe that $M=A\widetilde{M}$ and $\det A=1$; moreover B is a reflection with $B\widetilde{M}=\widetilde{M}$. We define the norm $\|\cdot\|_*$ by $\|x\|_*=\|Ax\|$ for $x\in\mathbb{R}^2$. Writing $x=t\tilde{u}+s\tilde{w}$, $t,s\in\mathbb{R}$, we have

$$\begin{aligned} \|x\|_*^2+\|Bx\|_*^2 &= \left(\lambda^2+\tfrac{1}{\lambda^2}\right)(t^2+s^2)+4ts\langle\tilde{u},\tilde{w}\rangle\\ &\geq 2(t^2+s^2)+4ts\langle\tilde{u},\tilde{w}\rangle=\|x\|^2+\|Bx\|^2. \end{aligned}$$

Using the substitution $y=Ax$, we deduce

$$\begin{aligned} \int_M\|y\|^2dy &= \int_{\widetilde{M}}\|x\|_*^2dx=\frac{1}{2}\int_{\widetilde{M}}\|x\|_*^2+\|Bx\|_*^2dx\\ &\geq \frac{1}{2}\int_{\widetilde{M}}\|x\|^2+\|Bx\|^2dx=\int_M\|x\|^2dx. \end{aligned}$$

We conclude (16), and in turn Proposition 3.5. Q.E.D.

Proof of Lemma 3.2: Since $\int_{T(\rho)}\|x\|^2dx=\frac{\rho^2}{4}A(T(\rho))$, simple argument yields (i).

Therefore we turn to the proof of (ii). We may assume $\rho=1$. During the argument $\omega_1,\omega_2,\ldots$ denote positive absolute constants.

Case 1. C is triangle whose angles are all acute

In this case the main idea is to deform C into an isosceles triangle. Let $C=[a_1,a_2,a_3]$, and let α_i denote the angle at a_i, $i=1,2,3$, in a way such that $\alpha_1\leq\alpha_2\leq\alpha_3$. In addition we define $M_1=[o,a_2,a_3]$, $M_2=[o,a_1,a_3]$ and $M_3=[o,a_1,a_2]$. Since the angle of M_i at o is $2\alpha_i$, we have $\int_{M_i}\|x\|^2dx=\gamma(\alpha_i)A(M_i)$ for

$i=1,2,3$. Next let $a_1' \in \partial B^2$ be the point such that the triangle $C'=[a_1',a_2,a_3]$ contains o and satisfies $\|a_1'-a_2\| = \|a_1'-a_3\|$. In addition we define $M_2' = [o,a_1',a_3]$ and $M_3' = [o,a_1',a_2]$. Now the angle of the congruent triangles M_2' and M_3' at o is $\alpha_2+\alpha_3$, hence $\int_{M_i'} \|x\|^2 dx = \gamma(\frac{\alpha_2+\alpha_3}{2})A(M_i')$, $i=2,3$. As we have $\alpha_2+\alpha_3 \geq 2\alpha_1$, Proposition 3.3 yields

$$\gamma' = \frac{\int_{M_2'\cup M_3'} \|x\|^2 dx}{A(M_2'\cup M_3')} \leq \gamma(\alpha_1).$$

Moreover we deduce by Proposition 3.4 that

$$\gamma_0 = \frac{\int_{M_2\cup M_3} \|x\|^2 dx}{A(M_2\cup M_3)} \geq \gamma'.$$

Since $A(M_2'\cup M_3') \geq A(M_2\cup M_3)$, it follows that

$$\begin{aligned}\frac{\int_{C'} \|x\|^2 dx}{A(C')} &= \frac{\gamma(\alpha_1)A(M_1)+\gamma' A(M_2'\cup M_3')}{A(M_1)+A(M_2'\cup M_3')} \leq \frac{\gamma(\alpha_1)A(M_1)+\gamma' A(M_2\cup M_3)}{A(M_1)+A(M_2\cup M_3)} \\ &\leq \frac{\gamma(\alpha_1)A(M_1)+\gamma_0 A(M_2\cup M_3)}{A(M_1)+A(M_2\cup M_3)} = \frac{\int_C \|x\|^2 dx}{A(C)}.\end{aligned}$$

With the help of (13) and (14), we obtain

$$\frac{\int_C \|x\|^2 dx}{A(C)} \geq \frac{\int_{C'} \|x\|^2 dx}{A(C')} = \frac{\int_{M_1} \|x\|^2 dx + 2\int_{M_2'} \|x\|^2 dx}{A(M_1)+2A(M_2')} = \frac{1}{4}+\frac{1}{3}\left(\cos\alpha_1 - \frac{1}{2}\right)^2.$$

Since C is not ε-close to $T(1)$, we have $\alpha_1 \leq \frac{\pi}{3} - \omega_1\varepsilon$, which in turn yields $\cos\alpha_1 - \frac{1}{2} \geq \omega_2\varepsilon$. In particular we deduce (ii) in this case.

Case 2. C is triangle that has an angle at least $\frac{\pi}{2}$

In this case we prove

$$\frac{\int_C \|x\|^2 dx}{A(C)} \geq \frac{47}{168} = \frac{1}{4}+\frac{5}{168}, \tag{17}$$

which estimate readily yields (ii). Let $C=[a_1,a_2,a_3]$ where the angle a_1 is at least $\frac{\pi}{2}$. We define $p=\frac{a_2+a_3}{2}$, hence $\langle p, x-p\rangle \geq 0$ for $x\in C$. Writing $\varphi = \angle(a_1,p,a_2)$, Proposition 3.5 applied to the triangles $[a_1,p,a_2]$ and $[a_1,p,a_3]$ implies

$$\begin{aligned}\int_C \|x\|^2 dx &= \int_C \left[p^2 + 2\langle p,x-p\rangle + (x-p)^2\right] dx \\ &\geq \left(\|p\|^2 + \|a_1-p\|\cdot\|a_2-p\|\cdot\frac{\gamma(\frac{\varphi}{2})+\gamma(\frac{\pi}{2}-\frac{\varphi}{2})}{2}\right)\cdot A(C)\end{aligned}$$

If $\|p\| \geq \frac{2}{3}$ then $\|p\|^2 \geq \frac{4}{9}$ directly yields (17). Therefore let $\|p\| \leq \frac{2}{3}$, hence $\|a_2 - p\| = \sqrt{1-\|p\|^2} \geq 1 - \frac{\|p\|}{2}$. Since $\|a_1 - p\| \geq 1 - \|p\|$ and $\gamma(\frac{\varphi}{2}) + \gamma(\frac{\pi}{2} - \frac{\varphi}{2}) = \frac{2}{3}$ (see Proposition 3.3), we have

$$\frac{\int_C \|x\|^2 dx}{A(C)} \geq \|p\|^2 + \frac{1}{3}\left(1 - \frac{\|p\|}{2}\right)(1 - \|p\|) = \frac{47}{168} + \frac{7}{6}\left(\|p\| - \frac{3}{14}\right)^2 \geq \frac{47}{168}.$$

In particular we have proved (17).

Case 3. C is a convex disc and is not a triangle

We may assume by approximation that the convex disc C is a polygon, and has at least four sides. We triangulate C into the triangles $T_1, \ldots, T_k$, $k \geq 2$, such that any vertex of some T_i is a vertex of C. If no T_i is $\frac{\varepsilon}{2}$-close to $T(1)$ then we conclude (ii) by Cases 1 and 2 above. Thus we assume that T_1 is $\frac{\varepsilon}{2}$-close to $T(1)$, hence $o \in \mathrm{relint} T_1$. It follows that all T_i with $i \geq 2$ has some obtuse angle. We have $\sum_{i=2}^{k} A(T_i) \geq \omega_3 \varepsilon A(C)$ because C is not ε-close to $T(1)$. Therefore (17) completes the proof of Lemma 3.2 (ii).

Q.E.D.

We have the following consequence of Lemma 3.2.

Corollary 3.6 *There exist absolute constants* $\varepsilon^*, c_1^*, c_2^* > 0$ *and* $r^* > 1$ *with the following properties: Let* $\varepsilon \in (0, \varepsilon^*)$ *and* $r \in (1, r^*)$*, moreover let* C *be a face of some element of* $\widetilde{F}_r$*. In addition let* z *be the centre of the circular disc* $\mathrm{aff} C \cap rB^3$*, let* $\nu = \|z\| - 1$*, and let* $\lambda \in [1,2]$*.*

(i) If C *is* ε*-close to* $T(\sqrt{r^2-1})$ *then*

$$\int_C (\|x - z\|^2 + \lambda\nu)dx \leq (\tfrac{1}{2} + c_1^*\varepsilon)A(\pi_{S^2}C)\cdot(r-1) + O((r-1)^2)A(\pi_{S^2}C).$$

(ii) If C *is not* ε*-close to* $T(\sqrt{r^2-1})$ *then*

$$\int_C (\|x - z\|^2 + \lambda\nu)dx \geq (\tfrac{1}{2} + c_2^*\varepsilon^2)A(\pi_{S^2}C)\cdot(r-1).$$

Remark *In any case* $\int_C (\|x - z\|^2 + \lambda\nu)dx \geq \frac{1}{2}A(\pi_{S^2}C)\cdot(r-1)$.

Proof: During the argument, $\omega_1, \omega_2, \omega_3$ denote positive constants. We define $\rho = \sqrt{r^2 - (1+\nu)^2}$, hence $\mathrm{ext} C$ lies in a circle of radius ρ. We note that

$$(1 - \tfrac{2\nu}{r-1})\sqrt{r^2-1} \leq \rho \leq (1 - \tfrac{\nu}{3(r-1)})\sqrt{r^2-1}. \qquad (18)$$

According to (5), we also have the useful estimates

$$2(r-1)A(\pi_{S^2}C) \leq (r^2-1)A(C) \leq 2(r-1)A(\pi_{S^2}C) + O((r-1)^2)A(\pi_{S^2}C). \quad (19)$$

If C is ε-close to $T(\sqrt{r^2-1})$ then (18) yields $\nu \leq \omega_1\varepsilon(r-1)$, hence (i) follows by (19) and Lemma 3.2 (i).

Thus we assume that C is not ε-close to $T(\sqrt{r^2-1})$. We will use the constant c_2 occurring in Lemma 3.2 (ii). If $\rho \geq (1-\frac{c_2}{4}\varepsilon^2)\sqrt{r^2-1}$ then C is not $\frac{\varepsilon}{2}$-close to $T(\rho)$ for small enough ε^*. We deduce by Lemma 3.2 (ii) that

$$\begin{aligned}\int_C (\|x-z\|^2 + \lambda\nu)dx &\geq (\tfrac{1}{4} + c_2\tfrac{\varepsilon^2}{4})(1-\tfrac{c_2}{4}\varepsilon^2)^2(r^2-1)A(C)\\ &\geq (\tfrac{1}{4} + \tfrac{c_2}{9}\varepsilon^2)(r^2-1)A(C) \geq (\tfrac{1}{2} + \tfrac{c_2}{5}\varepsilon^2)(r-1)A(\pi_{S^2}C).\end{aligned}$$

Finally if $\rho \leq (1-\frac{c_2}{4}\varepsilon^2)\sqrt{r^2-1}$ then $\nu \geq \omega_2\varepsilon^2(r-1)$ according to (18). Thus Lemma 3.1 and $\rho^2 \geq r^2-1-3\nu$ yield

$$\begin{aligned}\int_C (\|x-z\|^2 + \lambda\nu)dx &\geq (\tfrac{1}{4}\rho^2 + \lambda\nu)A(C) \geq (\tfrac{1}{4}(r^2-1) + \tfrac{1}{4}\nu)\cdot A(C)\\ &\geq (\tfrac{1}{2} + \omega_3\varepsilon^2)(r-1)A(\pi_{S^2}C),\end{aligned}$$

completing the proof of Corollary 3.6. Q.E.D.

4 Estimates for convex discs related to the mean width

For any $y \in \mathbb{R}^2$ and compact $X \subset \mathbb{R}^2$, we write $d(y,X)$ to denote the minimal distance of y from the points of X. In particular if X is fixed then $d(y,X)$ is continuous in y. The formulae (10) and (12) motivate the following definition. If C is a convex disc then let

$$I(C) = \int_C d(x, \mathrm{ext}C)^2\, dx,$$

which expression is is well-defined because ext$C$ is compact. We recall that $T(\rho)$ is a fixed regular triangle with circumradius ρ. The proof of the asymptotic formula (4) and Theorem 1.5 in the case of mean width is based on

Lemma 4.1 *There exist absolute constants* $\varepsilon_0, c_1, c_2 > 0$ *with the following properties: Let* $\varepsilon \in (0,\varepsilon_0)$*,* $\rho > 0$ *and let* C *be a convex disc with* ext$C \subset \rho S^1$*. Then*

(i) $I(C) \leq \frac{5}{12}A(C)\cdot\rho^2$ *with equality if* $C = T(\rho)$*;*

(ii) if C is not ε–close to $T(\rho)$ then $I(C) \le (\frac{5}{12} - c_1\varepsilon^2)A(C)\cdot\rho^2$;

(iii) if C is ε–close to $T(\rho)$ then $I(C) \ge (\frac{5}{12} - c_2\varepsilon)A(C)\cdot\rho^2$.

The proof of Lemma 4.1 borrows many ideas and observations from the proof of Lemma 3.2. In particular we start by verifying two auxiliary statements. We note that $I(C)$ is not a continuous function of convex discs. Still we have

Proposition 4.2 *Given a convex disc M, $I(C)$ is a continuous function of convex discs C with* $\mathrm{ext}C \subset \partial M$.

Proposition 4.3 *If $K = [v_1, v_2, v_3]$ be a triangle that has a right angle at v_3, and an angle $\alpha \le \frac{\pi}{4}$ at v_1. If the perpendicular bisector of the side $[v_1, v_2]$ intersect the side $[v_1, v_3]$ in w then $\|w - v_1\| = \|w - v_2\| = \frac{\|v_1 - v_2\|}{2\cos\alpha}$, and*

$$\int_K \min_{i=1,2} \|x - v_i\|^2\, dx = \|w - v_1\|^2 \left[\frac{5}{12} - \frac{1}{3}\left(\frac{1}{2} - \cos 2\alpha\right)^2\right]\cdot A(K).$$

Proof: Writing m to denote the midpoint of $[v_1, v_2]$, and K_1, K_2 and M to denote $[w, m, v_1]$, $[w, m, v_2]$ and $[w, v_2, v_3]$, respectively, direct calculations based on (14) yield

$$\begin{aligned}\frac{\int_K \min_{i=1,2}\|x - v_i\|^2\, dx}{A(K)} &= \frac{\int_{K_1}\|x - v_1\|^2 dx + \int_{K_2}\|x - v_2\|^2 dx + \int_M \|x - v_2\|^2 dx}{A(K)} \\ &= \frac{\|w - v_1\|^2[1 + \cos 2\alpha - \cos^2 2\alpha]}{3}.\end{aligned}$$

Therefore we are done by the identity $1 + t - t^2 = \frac{5}{4} - (\frac{1}{2} - t)^2$. Q.E.D.

Proof of Lemma 4.1: During the argument $\omega_1, \omega_2, \ldots$ denote positive absolute constants. We may assume $\rho = 1$.

First we assume that C is ε-close to $T(1)$. In particular we may also assume that for any extremal point of C, there exists a vertex of $T(1)$ of distance at most $\omega_1\varepsilon$, and reverse. Since $I(T(1)) = \frac{5}{12}A(T(1))$, we conclude (iii).

Since (i) readily follows from (ii) and (iii), we verify (ii) in the rest of the proof.

Case 1. C is triangle whose angles are all acute

In this case the main idea is to deform C into an isosceles triangle. Let $C = [a_1, a_2, a_3]$, and let α_i denote the angle at a_i, $i = 1, 2, 3$, in a way such that $\alpha_1 \le \alpha_2 \le \alpha_3$. To shorten formulae, we write $\beta_i = \frac{\pi}{2} - \alpha_i$, $i = 1, 2, 3$ where $\beta_1 \ge \beta_2 \ge \beta_3$. Moreover let $m_1 = \frac{a_2 + a_3}{2}$, $m_2 = \frac{a_1 + a_3}{2}$ and $m_3 = \frac{a_1 + a_2}{2}$, and let $K_1 = [o, a_2, m_1]$,

$K_2=[o,a_1,m_2]$ and $K_3=[o,a_1,m_3]$. These triangles satisfy that the angles of K_1, K_2, K_3 at a_2, a_1 and a_1 are β_1, β_2 and β_3, respectively, and

$$\frac{I(C)}{A(C)}=\frac{\int_{K_1}\|x-a_2\|^2dx+\int_{K_2\cup K_3}\|x-a_1\|^2dx}{A(K_1)+A(K_2\cup K_3)}.$$

Next let $a_1'\in\partial B^2$ be the point such that the triangle $C'=[a_1',a_2,a_3]$ contains o and satisfies $\|a_1'-a_2\|=\|a_1'-a_3\|$. In addition let $m_2'=\frac{a_1+a_3'}{2}$ and $m_3'=\frac{a_1+a_2'}{2}$, moreover let $K_2'=[o,a_1',m_2']$ and $K_3=[o,a_1',m_3']$. Now the angle of the congruent triangles K_2' and K_3' at v_1' is $\frac{\alpha_1}{2}\leq\beta_1$, hence Proposition 3.3 yields

$$\gamma(\tfrac{\alpha_1}{2})=\frac{\int_{K_2'\cup K_3'}(x-v_1')^2dx}{A(K_2'\cup K_3')}\geq\gamma(\beta_1).$$

Moreover we deduce by Proposition 3.4 that

$$\gamma_0=\frac{\int_{K_2\cup K_3}(x-v_1)^2dx}{A(K_2\cup K_3)}\leq\gamma(\tfrac{\alpha_1}{2}).$$

Since $A(K_2'\cup K_3')\geq A(K_2\cup K_3)$, it follows that

$$\begin{aligned}\frac{I(C')}{A(C')}&=\frac{\gamma(\beta_1)A(K_1)+\gamma(\frac{\alpha_1}{2})A(K_2'\cup K_3')}{A(K_1)+A(K_2'\cup K_3')}\geq\frac{\gamma(\beta_1)A(M_1)+\gamma(\frac{\alpha_1}{2})A(K_2\cup K_3)}{A(K_1)+A(K_2\cup K_3)}\\&\geq\frac{\gamma(\beta_1)A(K_1)+\gamma_0A(K_2\cup K_3)}{A(K_1)+A(K_2\cup K_3)}=\frac{I(C)}{A(C)}.\end{aligned}$$

Now Proposition 4.3 implies

$$\frac{I(C)}{A(C)}\leq\frac{I(C')}{A(C')}=\frac{5}{12}-\frac{1}{3}\left(\cos\alpha_1-\frac{1}{2}\right)^2.$$

Since C is not ε-close to $T(1)$, we have $\alpha_1\leq\frac{\pi}{3}-\omega_2\varepsilon$, which in turn yields $\cos\alpha_1-\frac{1}{2}\geq\omega_3\varepsilon$. Therefore we conclude (ii) in this case.

Case 2. C is triangle that has an angle at least $\frac{\pi}{2}$

In this case we prove

$$I(C)\leq(\tfrac{5}{12}-\omega_4)A(C),\tag{20}$$

which estimate readily yields (ii). Let $C=[a_1,a_2,a_3]$ where the angle a_3 is at least $\frac{\pi}{2}$, and the smallest angle α' of C lies at a_1. Let p be the point on the side $[a_1,a_2]$ such that a_3-p is perpendicular to a_1-a_2. The segment $[a_3,p]$ cuts C into the triangles $K' = [a_3, p, a_1]$ and $K'' = [a_3, p, a_2]$ with right angles, and the smallest angle of K' is α'. We write α'' to denote the smallest angle of K'', which angle might be either at a_2 or at a_3. In addition we define $\rho' = \frac{\|v_1 - v_3\|}{2\cos\alpha'}$ and $\rho'' = \frac{\|v_2 - v_3\|}{2\cos\alpha''}$, moreover $\lambda' = \frac{A(K')}{A(K')+A(K'')}$ and $\lambda'' = \frac{A(K'')}{A(K')+A(K'')}$, which satisfy $\lambda' + \lambda'' = 1$ and $\rho', \rho'' \leq 1$. We deduce by Proposition 4.3 that

$$\begin{aligned}
\frac{I(C)}{A(C)} &\leq \frac{\int_{K'} \min_{i=1,3} \|x - a_i\|^2\,dx + \int_{K''} \min_{i=2,3} \|x - v_i\|^2\,dx}{A(K') + A(K'')} \\
&= \lambda' \rho'^2 [\tfrac{5}{12} - \tfrac{1}{3}(\tfrac{1}{2} - \cos 2\alpha')^2] + \lambda'' \rho''^2 [\tfrac{5}{12} - \tfrac{1}{3}(\tfrac{1}{2} - \cos 2\alpha")^2] \\
&\leq \frac{5}{12} - \frac{\lambda'}{3}\left(\frac{1}{2} - \cos 2\alpha'\right)^2 - \frac{5\lambda''}{12}(1 - \rho''^2).
\end{aligned}$$

Writing $\beta = \angle(a_3, a_2, a_1)$, we have $\frac{\lambda'}{\lambda''} = \frac{A(K')}{A(K'')} = \frac{\tan\beta}{\tan\alpha'} \geq 1$, thus $\lambda' \geq \frac{1}{2}$ follows by $\lambda' + \lambda'' = 1$. If $|\alpha' - \frac{\pi}{6}| \geq \frac{\pi}{24}$ then $(\frac{1}{2} - \cos 2\alpha')^2 \geq \omega_5$, hence we deduce (20) in this case. Therefore let $|\alpha' - \frac{\pi}{6}| \leq \frac{\pi}{24}$. On the one hand we have $\beta \leq \frac{3\pi}{8}$, hence $\lambda'' \geq \omega_6$. On the other hand $\alpha'' \leq \frac{\pi}{4}$ yields

$$\rho'' = \frac{\|a_2 - a_3\|}{2\cos\alpha''} = \frac{\sin\alpha'}{\cos\alpha''} \leq \frac{\sin\frac{5\pi}{24}}{\cos\frac{\pi}{4}} = 1 - \omega_7.$$

In turn we conclude (20).

Case 3. C is a convex disc and is not a triangle

According to Proposition 4.2, we may assume that the convex disc C is a polygon, and has at least four sides. We triangulate C into the triangles $T_1, \ldots, T_k$, $k \geq 2$, such that any vertex of some T_i is a vertex of C. If no T_i is $\frac{\varepsilon}{2}$-close to $T(1)$ then we conclude (ii) by Cases 1 and 2 above. Thus we assume that T_1 is $\frac{\varepsilon}{2}$-close to $T(1)$, hence $o \in \text{relint}T_1$. It follows that all T_i with $i \geq 2$ has some obtuse angle. In addition $\sum_{i=2}^{k} A(T_i) \geq \omega_8 \varepsilon A(C)$ because C is not ε-close to $T(1)$. Therefore (20) completes the proof of Lemma 4.1 (ii). Q.E.D.

Lemma 4.1 leads to

Corollary 4.4 *There exist absolute constants $\varepsilon^*, c_1^*, c_2^* > 0$ and $r^* > 1$ with the following properties: Let $\varepsilon \in (0, \varepsilon^*)$ and $r \in (1, r^*)$, moreover let C be a face of some element of $\widetilde{F}_r$.*

(i) If C is ε-close to $T(\sqrt{r^2 - 1})$ then

$$I(C) \geq (\tfrac{5}{6} - c_1^* \varepsilon) A(\pi_{S^2} C) \cdot (r - 1).$$

(ii) If C is not ε-close to $T(\sqrt{r^2-1})$ then

$$I(C) \leq (\tfrac{5}{6} - c_2^* \varepsilon^2) A(\pi_{S^2} C) \cdot (r-1) + O((r-1)^2) A(\pi_{S^2} C).$$

Remark *In any case* $I(C) \leq \frac{5}{6} A(\pi_{S^2} C) \cdot (r-1) + O((r-1)^2) A(\pi_{S^2} C)$.
Proof: During the argument, $\omega_1, \omega_2, \ldots$ denote positive constants. We write ρ to denote the radius of $rS^2 \cap \mathrm{aff} C$. According to (5), we have

$$2(r-1)A(\pi_{S^2}C) \leq (r^2-1)A(C) \leq 2(r-1)A(\pi_{S^2}C) + O((r-1)^2)A(\pi_{S^2}C). \quad (21)$$

If C is ε-close to $T(\sqrt{r^2-1})$ then $\rho \geq (1-\omega_1\varepsilon)\sqrt{r^2-1}$ and C is $\omega_2\varepsilon$-close to $T(\rho)$. Therefore using the constant c_2 of Lemma 4.1 (iii), we have

$$I(C) \geq (\tfrac{5}{12} - c_2\omega_2\varepsilon)(1-\omega_1\varepsilon)^2(r^2-1)A(C) \geq (\tfrac{5}{6} - \omega_3\varepsilon)A(\pi_{S^2}C) \cdot (r-1).$$

Thus we assume that C is not ε-close to $T(\sqrt{r^2-1})$. If $\rho \leq (1-\varepsilon^2)\sqrt{r^2-1}$ then Lemma 4.1 (i) and (21) yield

$$I(C) \leq \tfrac{5}{12}(1-\varepsilon)^2(r^2-1)A(C) \leq (\tfrac{5}{6} - \omega_4\varepsilon^2)A(\pi_{S^2}C) \cdot (r-1) + O((r-1)^2)A(\pi_{S^2}C).$$

Finally if $\rho \geq (1-\varepsilon^2)\sqrt{r^2-1}$ then C is not $\frac{\varepsilon}{2}$-close to $T(\rho)$ for small enough ε^*. We deduce using the constant c_1 of Lemma 4.1 (ii) that

$$I(C) \leq (\tfrac{5}{12} - c_1\tfrac{\varepsilon^2}{4})(r^2-1)A(C) \leq (\tfrac{5}{6} - \omega_5\varepsilon^2)A(\pi_{S^2}C) \cdot (r-1) + O((r-1)^2)A(\pi_{S^2}C),$$

completing the proof of Corollary 4.4. Q.E.D.

5 Constructing a close to be optimal element of $\widetilde{\mathcal{F}}_r$

In this section we prove the upper bounds related to the asymptotic formulae (2), (3) and (4).

Lemma 5.1 *If $1 < r < r_0$ then there exists a $K_r \in \widetilde{\mathcal{F}}_r$ satisfying*

$$V(K_r) - V(B^3) \leq \pi \cdot (r-1) + O\left((r-1)^{\frac{4}{3}}\right); \quad (22)$$

$$S(K_r) - S(B^3) \leq 3\pi \cdot (r-1) + O\left((r-1)^{\frac{4}{3}}\right); \quad (23)$$

$$M(K_r) - M(B^3) \leq \tfrac{7}{6}(r-1) + O\left((r-1)^{\frac{4}{3}}\right) \quad (24)$$

where $r_0 > 1$ is an absolute constant.

Proof: We observe that if H is a tangent plane to B^3 then the radius of $H \cap rB^3$ is $\sqrt{r^2-1}$. We choose the maximal number of points $x_1, \dots, x_k \in S^2$ in a way that the distance between any two is at least $\sqrt[6]{r-1}$. Let H_i be the tangent plane to B^3 at x_i for $i = 1, \dots, k$. Then $H_1, \dots, H_k$ bound the auxiliary polytope P, and let F_i be the face of P contained in H_i for $i = 1, \dots, k$. Next we consider an edge to edge tiling of H_i by congruent copies of $\frac{1}{r} T(\sqrt{r^2-1})$, and write Σ_i to denote the family of the tiles that intersect F_i. We define $\Sigma = \cup_{i=1,\dots,k} \Sigma_i$. If $T \in \Sigma$ then let T' be the triangle whose vertices are the radial projections into rS^2 of the vertices of T, let $z(T')$ be the centre of the circular disc $\mathrm{aff} T' \cap rB^3$, and let $\nu(T') = \|z(T')\| - 1$. We define K_r to be polytope whose vertices are the family of vertices of all T' as T runs through the elements of Σ.

Since π_{S^2} decreases distance, if T is a triangle of some Σ_i then T' is of circumradius at most $\sqrt{r^2-1}$. Thus $B^3 \subset K_r$, hence $K_r \in \widetilde{\mathcal{F}}_r$.

Next we observe that F_i contains a circular disc of centre x_i and of radius $\frac{1}{2}\sqrt[6]{r-1}$, and if r_0 is close to one then any element of Σ_i is contained in a circular disc of centre x_i and of radius $2\sqrt[6]{r-1}$. Let $T \in \Sigma_i$. It follows that T' is $\omega\sqrt[3]{r-1}$–close to $T(\sqrt{r^2-1})$ for some positive absolute constant ω.

We call a face G of K_r a *proper face* if any face of K_r intersecting G is of the form T' for some $T \in \Sigma$. In particular any proper face G of K_r is $\omega\sqrt[3]{r-1}$–close to $T(\sqrt{r^2-1})$, if $x \in \pi_{S^2} G$ then

$$\max_{v \in \mathrm{ext} K_r} \langle x, v \rangle = \max_{v \in \mathrm{ext} G} \langle x, v \rangle. \tag{25}$$

We want to show that the typical faces of K_r are proper, thus let

$$F_i' = x_i + (1 - 32\sqrt[3]{r-1})(F_i - x_i), \; i = 1, \dots, k.$$

Writing $X = \cup_{i=1}^k \pi_{S^2} F_i$, we have

$$A(S^2 \backslash X) \le O\left((r-1)^{\frac{1}{3}}\right). \tag{26}$$

In addition if $T \in \Sigma$ then T' is of diameter at most $2\sqrt{r^2-1}$, hence any face G of K_r with $\pi_{S^2} G \cap X \neq \emptyset$ is proper.

Next we present some crude estimate for any face G of K_r. Let $z(G)$ denote the centre of the circular disc $\mathrm{aff} G \cap rB^3$, and let $\nu(G) = \|z(G)\| - 1$. We deduce using (5) the inequalities

$$\int_G \|x - z(G)\|^2 dx \;\le\; (r^2-1)A(G) \le 3(r-1) \cdot A(\pi_{S^2} G) \tag{27}$$

$$\nu(G)A(G) \;\le\; 2(r-1)A(\pi_{S^2} G). \tag{28}$$

In addition if $x \in \pi_{S^2} G$ then

$$\max_{v \in \mathrm{ext} K_r} \langle x, v \rangle \le r. \tag{29}$$

We have everything to prove the estimates of Lemma 5.1. In the rest of the proof, G always denotes some face of K_r. We start with the case of the mean width, where (10) yields

$$\frac{A(S^2)}{2}[M(K_r)-M(B^3)] = \left[\sum_{\pi_{S^2}G\cap X\neq\emptyset}\int_{\pi_{S^2}G}\left(\max_{v\in\mathrm{ext}K_r}[\langle x,v\rangle-1]\right)dx\right]+ \quad (30)$$

$$\left[\sum_{\pi_{S^2}G\cap X=\emptyset}\int_{\pi_{S^2}G}\left(\max_{v\in\mathrm{ext}K_r}[\langle x,v\rangle-1]\right)dx\right]. \quad (31)$$

We may apply (12) to the sum in (30) according to (25). Applying (26) and (29) to the sum in (31), and later Corollary 4.4 (i) to proper faces, we deduce

$$\begin{aligned}\frac{A(S^2)}{2}[M(K_r)-M(B^3)] &\leq A(S^2)(r-1)-\frac{1}{2}\left[\sum_{\pi_{S^2}G\cap X\neq\emptyset}I(G)\right]+O\left((r-1)^{\frac{4}{3}}\right)\\ &\leq A(S^2)(r-1)-\tfrac{5}{12}A(S^2)(r-1)+O\left((r-1)^{\frac{4}{3}}\right),\end{aligned}$$

which in turn yields (24).

For the case of the volume, we use first (9), and afterwards Corollary 3.6 (i) to proper faces and the crude estimates above. We have

$$\begin{aligned}V(K_r)-V(B^3) &= \left[\sum_{\pi_{S^2}G\cap X\neq\emptyset}\int_G\left(\tfrac{1}{2}\|x-z(G)\|^2+\nu(G)\right)dx\right]+\\ &\quad\left[\sum_{\pi_{S^2}G\cap X=\emptyset}\int_G\left(\tfrac{1}{2}\|x-z(G)\|^2+\nu(G)\right)dx\right]+O((r-1)^2)\\ &\leq \tfrac{1}{4}A(S^2)(r-1)+O\left((r-1)^{\frac{4}{3}}\right)=\pi(r-1)+O\left((r-1)^{\frac{4}{3}}\right),\end{aligned}$$

which in turn yields (22). Finally (23) follows from (22) as $S(K_r)\leq 3V(K_r)$. Q.E.D.

6 Proof of Theorem 1.5 and the asymptotic formulae

Since the optimal convex bodies in $\widetilde{\mathcal{F}}_r$ may not be polytopes, we will need Lemma 6.1. We note that the inradius of $T(\rho)$ is $\frac{1}{2}\rho$.

Lemma 6.1 *There exist absolute constants r_0 and c_0 with the following property. For $L \in \widetilde{\mathcal{F}}_r$, $r \in (1,2)$, let $F_1, \dots, F_k$ be the faces of L whose inradius is at least $\frac{1}{4}\sqrt{r^2-1}$. For each F_i, let z_i be the centre of the circular disc* $\mathrm{aff} F_i \cap rB^3$, *and let $\nu_i = \|z_i\| - 1$. Writing $X = S^2 \backslash (\cup_{i=1}^k \pi_{S^2} F_i)$, we have*

$$\begin{aligned}
V(L) - V(B^3) &\geq (\tfrac{1}{4}+c_0)A(X)(r-1) + O((r-1)^2) + \sum_{i=1}^k \int_{F_i} (\tfrac{1}{2}\|x-z_i\|^2 + \nu_i)dx; \\
S(L) - S(B^3) &\geq (\tfrac{3}{4}+c_0)A(X)(r-1) + O((r-1)^2) + \sum_{i=1}^k \int_{F_i} (\tfrac{3}{2}\|x-z_i\|^2 + 2\nu_i)dx; \\
M(L) - M(B^3) &\geq (\tfrac{7}{6}+c_0)\tfrac{A(X)}{A(S^2)}(r-1) + O((r-1)^2) + \\
&\qquad \tfrac{2}{A(S^2)}\sum_{i=1}^k [A(\pi_{S^2}F_i)(r-1) - \tfrac{1}{2}I(F_i)].
\end{aligned}$$

Remark *If the inradius of all faces of L are less than $\frac{1}{4}\sqrt{r^2-1}$ then $X = S^2$, and no sums occur in the estimates above. Moreover if $\partial L = \cup_{i=1}^k F_i$ then we set $A(X) = 0$ above.*

Proof: Let ω be a positive absolute constant such that if F is a convex disc that is ω-close to some $T(\rho)$ then the inradius of F is at least $\frac{1}{4}\rho$. We may assume that ω is less than the ε^* of Corollaries 3.6 and 4.4.

Next we define a suitable sequence $L^{(m)}$, $m \geq 1$, of polytopes in $\widetilde{\mathcal{F}}_r$ that tends to L. First for each F_i, we choose a sequence $F_i^{(m)}$ of polygons that tend to F_i in a way such that $\mathrm{ext} F_i^{(m)} \subset (rS^2 \cap \mathrm{aff} F_i)$. Next let H_i^+ be the open half space bounded by $\mathrm{aff} F_i$ and containing L, $i = 1, \dots, k$, moreover let $Y = \mathrm{ext} L \cap H_1^+ \cap \dots \cap H_k^+$. For each $m \geq 1$, we choose a finite $Y^{(m)} \subset Y$ in a way such that for any $x \in Y$ there exists $y \in Y^{(m)}$ satisfying $\|x-y\| \leq \frac{1}{m}$. We define

$$L^{(m)} = [Y^{(m)}, F_1^{(m)}, \dots, F_k^{(m)}],$$

which readily tends to L. In addition $X^{(m)} = S^2 \backslash (\cup_{i=1}^k \pi_{S^2} F_i^{(m)})$ satisfies $\lim_{m\to\infty} A(X^{(m)}) = A(X)$.

We write $\mathcal{L}^{(m)}$ to denote the family of faces of $L^{(m)}$. For any $G \in \mathcal{L}^{(m)}$, we write $z(G)$ to denote the centre of $rB^3 \cap \mathrm{aff}$, and define $\nu(G) = z(G) - 1$. It is easy to see that there exists m_0 such that if $m \geq m_0$ and $G \in \mathcal{L}^{(m)}$ is ω-close to $T(\sqrt{r^2-1})$ then G is one of $F_1^{(m)}, \dots, F_k^{(m)}$. In particular let $G \in \mathcal{L}^{(m)}$ be different from $F_1^{(m)}, \dots, F_k^{(m)}$. It follows by Corollary 4.4 (ii) that

$$I(G) \leq (\tfrac{5}{6} - \omega_0)A(\pi_{S^2}G) \cdot (r-1) + O((r-1)^2)A(\pi_{S^2}G), \tag{32}$$

and if $\lambda \in [1,2]$ then Corollary 3.6 (ii) yields

$$\int_G (\|x - z(G)\|^2 + \lambda\nu)dx \geq (\tfrac{1}{2} + \omega_0)A(\pi_{S^2}C) \cdot (r-1) \tag{33}$$

where ω_0 is a positive absolute constant.

To prove the formulae of Lemma 6.1, we start with the case of volume and surface area. It follows by (8), (9) and (33) that the corresponding formulae of Lemma 6.1 hold for $L^{(m)}$ and $F_i^{(m)}$ in place of L and F_i, $i = 1, \ldots, k$, $m \geq m_0$, therefore we deduce Lemma 6.1 by approximation in these cases.

For the mean width, we deduce by (10) and (12) that

$$M(L^{(m)}) - M(B^3) \geq O((r-1)^2) + \tfrac{2}{A(S^2)} \sum_{G \in \mathcal{L}^{(m)}} [A(\pi_{S^2}G)(r-1) - \tfrac{1}{2}I(G)].$$

Therefore Lemma 6.1 follows by (32) and approximation. Q.E.D.

Proof of the asymptotic formulae: We observe that P_r, Q_r and W_r are all elements of $\widetilde{\mathcal{F}}_r$. Combining Lemma 6.1 with Corollary 3.6 in the cases of volume and surface area, and with Corollary 4.4 in the case of mean width leads to

$$\begin{aligned} V(P_r) - V(B^3) &\geq \pi \cdot (r-1) + O((r-1)^2); \\ S(Q_r) - S(B^3) &\geq 3\pi \cdot (r-1) + O((r-1)^2); \\ M(W_r) - M(B^3) &\geq \tfrac{7}{6} \cdot (r-1) + O((r-1)^2). \end{aligned}$$

These lower bounds together with the upper bounds of Lemma 5.1 yield (2), (3) and (4). Q.E.D.

Proof of Theorem 1.5: Let $\varepsilon = \sqrt[9]{r-1}$. We start with the case of mean width. Let $F_1, \ldots, F_l$ be the faces of W_r that are ε–close to $T(\sqrt{r^2-1})$, and let $Z_r = \partial W_r \backslash (\cup_{i=1}^{l} F_i)$. If $Z_r = \emptyset$ then we are done, therefore we assume that $Z_r \neq \emptyset$.

Since $W_r \in \widetilde{\mathcal{F}}_r$, Lemma 6.1 and Corollary 4.4 yield the lower bound

$$M(W_r) - M(B^3) \geq \tfrac{7}{6} \cdot (r-1) + \omega A(\pi_{S^2}Z_r) \cdot \varepsilon^2 \cdot (r-1) + O((r-1)^2)$$

for some absolute constant $\omega > 0$. On the other hand we have the upper bound

$$M(W_r) - M(B^3) \leq \tfrac{7}{6} \cdot (r-1) + O\left((r-1)^{\frac{4}{3}}\right)$$

according to Lemma 5.1. Comparing these two bounds leads to $A(Z_r) \leq 2A(\pi_{S^2}Z_r) = O(\sqrt[9]{r-1})$, as it is required by Theorem 1.5.

We omit the argument in the case of the volume and surface area because it runs as above. The only alterations are the obvious change of constants, and to replace Corollary 4.4 by Corollary 3.6 in the argument. Q.E.D.

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