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EXPECTATION OF INTRINSIC VOLUMES OF RANDOM POLYTOPES

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Abstract

Let K be a convex body in $\mathbb{R}^d$, let $j \in \{1, \dots, d-1\}$, and let $K(n)$ be the convex hull of n points chosen randomly, independently and uniformly from K. If ∂K is C^2_+, then an asymptotic formula is known due to M. Reitzner (and due to I. Bárány if ∂K is C^3_+) for the difference of the jth intrinsic volume of K and the expectation of the jth intrinsic volume of $K(n)$. We extend this formula to the case when the only condition on K is that a ball rolls freely inside K.

1. Introduction

Throughout the paper, let K be a convex body (a compact convex set with non-empty interior) in Euclidean space $\mathbb{R}^d$, $d \geq 2$. Let B^d be the unit ball of $\mathbb{R}^d$ centered at the origin o, V the volume functional (d-dimensional Lebesgue measure), and $\kappa_d := V(B^d)$. The intrinsic volumes $V_j(K)$, $j = 0, \dots, d$, of a convex body K can be introduced as coefficients of the Steiner formula

$$V(K + \lambda B^d) = \sum_{j=0}^{d} \lambda^{d-j} \kappa_{d-j} V_j(K),$$

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where $K + \lambda B^d$ is the Minkowski sum of K and the ball λB^d of radius $\lambda \geq 0$; see P.M. Gruber [9] or R. Schneider [18]. In particular, V_d is the volume functional, $V_0(K) = 1$, V_1 is proportional to the mean width and V_{d-1} is a multiple of the surface area. Alternatively, intrinsic volumes can be interpreted as mean projection volumes. Specifically, for $j = 1, \dots, d-1$, it is well known that

$$V_j(K) = \frac{\binom{d}{j}\,\kappa_d}{\kappa_j\,\kappa_{d-j}} \int_{\mathcal{L}_j^d} V_j(K|L)\,\nu_j(dL), \tag{1}$$

where $\mathcal{L}_j^d$ is the Grassmannian of all j-dimensional linear subspaces of $\mathbb{R}^d$ equipped with the (unique) Haar probability measure ν_j and, for $L \in \mathcal{L}_j^d$, $K|L$ denotes the orthogonal projection of K onto L. Here, $V_j(K|L)$ is just the j-dimensional volume (Lebesgue measure) of $K|L$.

We call ∂K twice differentiable in the generalized sense at a boundary point $x \in \partial K$ if there exists a quadratic form Q on $\mathbb{R}^{d-1}$ with the following property: If K is positioned in such a way that $x = o$ and $\mathbb{R}^{d-1}$ is a support hyperplane of K, then in a neighbourhood of o, ∂K is the graph of a convex function f defined on a $(d-1)$-dimensional ball around o in $\mathbb{R}^{d-1}$ satisfying

$$f(z) = \tfrac{1}{2}\,Q(z) + o(\|z\|^2), \tag{2}$$

as $z \to o$, where $o(\cdot)$ denotes the usual Landau symbol. Thus f admits a convenient second order Taylor expansion at o. In this case, we write $\sigma_j(x)$ to denote the jth normalized elementary symmetric function of the eigenvalues of Q (the "generalized principal curvatures"). In particular, the generalized Gaussian curvature of K at x is $\sigma_{d-1}(x) = \det Q$. According to a classical result of Alexandrov (see P.M. Gruber [9] or R. Schneider [18]), the boundary ∂K is twice differentiable in the generalized sense at almost every boundary point with respect to the boundary measure of K. We say that ∂K is C_+^k, for some $k \geq 2$, if ∂K is a C^k manifold and its Gaussian curvature is positive everywhere.

In this paper, we study the expectation of intrinsic volumes of random polytopes given as the convex hull of random points chosen from a given convex body. We will consider random points in a given convex body K which follow the uniform probability distribution on K, hence their density with respect to Lebesgue measure on K is the function with the constant value $V(K)^{-1}$. Let $[x_1, \dots, x_n]$ denote the convex hull of $x_1, \dots, x_n \in \mathbb{R}^d$. For $n \geq 2$ and uniformly and independently distributed random points $x_1, \dots, x_n \in K$, the random polytope $K(n) = [x_1, \dots, x_n]$ is our basic model. An up-to-date account of classical and recent results on random polytopes is provided in the book by Schneider and Weil [19].

If K is a polytope, an asymptotic formula is known for $V(K) - \mathbb{E}V(K(n))$. This goes back to work by F. Affentranger and J.A. Wieacker [1] for simple polytopes, and by I. Bárány and Ch. Buchta [5] in the general case. For an arbitrary convex body, the asymptotic behaviour of $V(K) - \mathbb{E}V(K(n))$ is described by

$$\lim_{n\to\infty} \left(\frac{n}{V(K)}\right)^{\frac{2}{d+1}} [V(K) - \mathbb{E}V(K(n))] = c_d \int_{\partial K} \sigma_{d-1}(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx) \tag{3}$$

with a constant $c_d > 0$ depending only on d. Here and in the following, we write $\mathcal{H}^i$ for the i-dimensional Hausdorff measure in $\mathbb{R}^d$. The integral on the right-hand side of (3) is positive if and only if the generalized Gauss curvature of K is positive on a set of positive boundary measure. Relation (3) is due to J.A. Wieacker [24] if K is a ball, due to I. Bárány [3] if K has C^3_+ boundary, and due to C. Schütt [20] if K is arbitrary. The explicit value of the constant c_d is provided in [24].

Now we turn to the intrinsic volumes $V_j(K)$, $j = 1, \dots, d-1$. It is known that if the boundary ∂K of K is C^2_+, then we have

$$\lim_{n\to\infty} \left(\frac{n}{V(K)}\right)^{\frac{2}{d+1}} [V_j(K) - \mathbb{E}\, V_j(K(n))] = c_{d,j} \int_{\partial K} \sigma_{d-1}(x)^{\frac{1}{d+1}} \sigma_{d-j}(x)\, \mathcal{H}^{d-1}(dx), \tag{4}$$

with a constant $c_{d,j} > 0$ depending only on d and j. The formula is due to I. Bárány [3] if K has C^3_+ boundary, and due to M. Reitzner [16] if K has C^2_+ boundary. The goal of this paper is to extend (4) to a certain class of convex bodies for which the generalized Gauss curvature is allowed to be zero. We say that a ball rolls freely inside a convex body K in $\mathbb{R}^d$ if there exists some $r > 0$ such that any $x \in \partial K$ lies on the boundary of some ball B of radius r with $B \subset K$. The existence of a rolling ball is equivalent to saying that the exterior unit normal is a Lipschitz map on ∂K (see D. Hug [11]). In particular, already W. Blaschke observed that if ∂K is C^2, then K has a rolling ball (see [11] or K. Leichtweiß [13]).

Theorem 1.1. *Let $K \subset \mathbb{R}^d$ be a convex body in which a ball rolls freely, and let $j \in \{1, \dots, d-1\}$. Then* (4) *holds for K.*

Unlike in the case of the volume ($j = d$), we do need a condition similar to the existence of a rolling ball if $j < d/2$ in Theorem 1.1. If $j < d/2$ and P is a polytope, then

$$V_j(P) - \mathbb{E}V_j(P(n)) \gg n^{-\frac{1}{d-j+1}} \geq n^{-\frac{2}{d+3}}$$

according to I. Bárány [2]. There even exists a convex body K in $\mathbb{R}^d$ with $V(K) = 1$ such that ∂K is C^1 and, with the exception of one boundary point, is C^∞_+, and still

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} [V_j(K) - \mathbb{E}V_j(K(n))] = \infty.$$

For $j = 1$, such an example is described in K. J. Böröczky, F. Fodor, M. Reitzner, V. Vígh [7]. Actually, in [7] Theorem 1.1 is obtained for $j = 1$, in which case the proof is easier.

The proof of Theorem 1.1 is based on arguments similar to those used in the proof of (3) in C. Schütt [20]. In Section 2, for a convex body K in $\mathbb{R}^d$, we introduce and discuss basic properties of a relative of the so-called convolution body that is adjusted to taking projections. In particular, we show that if n is large then $K(n)|L$ fills up most of $K|L$ with high probability, for all $L \in \mathcal{L}_j^d$. We start to use the rolling ball property for K in Section 3, and establish (see (12))

$$V_j(K) - \mathbb{E}V_j(K(n)) = \frac{\binom{d}{j}\,\kappa_d}{2\kappa_j\,\kappa_{d-j}} \int_{\mathcal{L}_j^d} \int_{\partial(K|L)} \varphi(n,K,L,z)\,\mathcal{H}^{j-1}(dz)\,\nu_j(dL) + o(n^{\frac{-2}{d+1}})$$

for a suitable quantity $\varphi(n,K,L,z)$ (as $n \to \infty$). Let us assume that $L \in \mathcal{L}_j^d$, $z \in \partial(K|L)$, ∂K is twice differentiable in the generalized sense at $x = x(z) \in \partial K$, and z is the orthogonal projection of x onto L. We prove in Section 4 that if $\sigma_{d-1}(x(z)) = 0$, then $\lim_{n\to\infty} n^{\frac{2}{d+1}}\varphi(n,K,L,z) = 0$, and if $\sigma_{d-1}(x(z)) > 0$, then the limit $\lim_{n\to\infty} n^{\frac{2}{d+1}}\sigma_{d-1}(x(z))^{-1}\varphi(n,K,L,z)$ is the same as it would be when K is a suitable ball. In particular, the known asymptotic formula (4) for balls leads to Theorem 1.1. This is shown in Section 5 and the argument is based on an integral geometric formula.

2. Macbeath regions and convolution bodies

In this section, we consider properties of convex compact sets where the property of having a rolling ball is not relevant. We work in a Euclidean space $\mathbb{R}^d$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

For a compact convex set C in $\mathbb{R}^d$, we write $\operatorname{relint} C$ for the relative interior (the interior with respect to the affine hull $\operatorname{aff} C$ of C), $\operatorname{int} C$ for the interior, and ∂C for the relative boundary of C. Moreover, for $x \in C$, the Macbeath region of C with respect to x is

$$M_C(x) := C \cap (2x - C),$$

which is symmetric through x and has the same affine hull as C if $x \in \operatorname{relint} C$.

This classical notion will now be extended. Let $L^\perp$ denote the orthogonal complement of a linear subspace L in $\mathbb{R}^d$, and let K be a convex body in $\mathbb{R}^d$. For $L \in \mathcal{L}_j^d$ and $z \in K|L$, we consider the Macbeath type region

$$M_K^L(z) := K \cap \left(L^\perp + M_{K|L}(z)\right).$$

Clearly, we have $M_K^L(z) = M_K(z)$ if $j = d$. We note that if $z = x|L$ and $x \in K$, then

$$M_K^L(z) = K \cap (2x - K + L^\perp). \tag{5}$$

The Macbeath region of a convex set and the above extensions are related as follows.

LEMMA 2.1. *Let K be a convex body in $\mathbb{R}^d$, and let $L \in \mathcal{L}_j^d$, $j \in \{1,\dots,d\}$. If $z \in \operatorname{relint}(K|L)$ and x is the centre of mass of $(z+L^\perp)\cap K$, then*

$$M_K(x) \subset M_K^L(z) \subset x + (2d-2j+1)(M_K(x)-x).$$

PROOF. The first relation immediately follows from (5). For the proof of the second inclusion, we may assume that $x = z = o$. Choose any point $y \in M_K^L(z) = M_K^L(o)$. Then there exists $w \in K$ such that $(y+w)/2 \in L^\perp \cap K$. Since $x = o$ is the centre of mass of $L^\perp \cap K$, we have

$$v := -\frac{y+w}{2(d-j)} \in L^\perp \cap K$$

(see [18, Lemma 2.3.3]), and hence

$$\frac{-y}{2d-2j+1} = \frac{1}{2d-2j+1}\,w + \frac{2d-2j}{2d-2j+1}\,v \in K.$$

In turn, we deduce $\frac{1}{2d-2j+1}y \in M_K(o)$, therefore $\frac{1}{2d-2j+1}\,M_K^L(o) \subset M_K(o)$. □

Let K be a convex body in $\mathbb{R}^d$. For $L \in \mathcal{L}_j^d$, $j \in \{1,\dots,d\}$, Fubini's theorem implies that

$$\begin{aligned}
&\int_{K|L} V(M_K^L(z))\,\mathcal{H}^j(dz)\\
&= \int_L \int_{L^\perp} \mathbf{1}\{y_1+y_2 \in K\} \int_L \mathbf{1}\{z \in \tfrac{1}{2}y_1 + \tfrac{1}{2}K|L\}\,\mathcal{H}^j(dz)\,\mathcal{H}^{d-j}(dy_2)\,\mathcal{H}^j(dy_1)\\
&= 2^{-j}V(K)V_j(K|L).
\end{aligned}$$

Therefore there exists a point $z \in K|L$ with $V(M_K^L(z)) \geq V(K)/2^j$ (see S. Stein [22] if $j = d$). For $t \in [0, 2^{-d}]$, we define the convolution body of K with respect to L as

$$K_t^L := \{z \in K|L : V(M_K^L(z)) \geq tV(K)\}.$$

If $L = \mathbb{R}^d$, K_t^L is the classical convolution body K_t. Let $z_1, z_2 \in K|L$ and $\lambda \in [0,1]$. Using (5) it is easy to check that $(1-\lambda)M_K^L(z_1) + \lambda M_K^L(z_2) \subset M_K^L((1-\lambda)z_1 + \lambda z_2)$. Hence, the Brunn–Minkowski inequality yields that K_t^L is convex. We also remark the scaling behaviour of Macbeath regions and convolution bodies. For any $\lambda > 0$ and $z \in K|L$, we have

$$M_{\lambda K}^L(\lambda z) = \lambda \cdot M_K^L(z) \qquad \text{and} \qquad (\lambda K)_t^L = \lambda \cdot K_t^L$$

as an immediate consequence of our definitions.

In addition to the convolution body, for $t \geq 0$ we use the floating body $K[t]$ of K, which is the set of all points $x \in K$ such that each closed halfspace H^+ with $x \in H^+$ satisfies $V(K \cap H^+) \geq tV(K)$ (the volume of each cap of K which contains x is at least $tV(K)$). It is well known that the floating body $K[t]$ and the convolution body K_t are closely related. For positive $t < (4d)^{-d}$, we have (see also I. Bárány [6], [2], [4])

$$K[(2d)^d t] \subset K_t \subset K[t/2]. \tag{6}$$

Here the second inclusion is trivial. The first inclusion follows from Lemma 2.1 by taking $j = 1$ and using the fact that if H is a hyperplane through $x \in \operatorname{int} K$ such that $V(K \cap H^+)$ is minimal, then x is the centre of mass of $K \cap H$.

I. Bárány and D. G. Larman [6] showed that $\mathbb{E}\, V(K(n))$ can be closely approximated in terms of the floating body $K[1/n]$. To describe the connection, we write $f \ll g$ or $f = O(g)$ for two functions $f, g\colon I \to \mathbb{R}$ with $I \subset \mathbb{R}$ if there exists a constant $c(K) > 0$ depending only on K such that $|f|(t) \leq c(K)g(t)$ for all $t \in I$, and we write $f \approx g$ if $f \ll g$ and $g \ll f$. Then, according to [6], for any convex body K we have

$$V(K) - \mathbb{E}V(K(n)) \approx V(K) - V(K[1/n]). \tag{7}$$

The convolution body is more useful in random approximation than the floating body because $C := M_K(x)$ is symmetric with respect to x. As a consequence, for $k \geq 2$, the probability $\mathbb{P}(x \notin C(k))$ can be explicitly expressed in terms of k according to J.G. Wendel [23]. In this paper, we do not use this formula directly, but rather its following consequence proved in I. Bárány and D.G. Larman [6].

Lemma 2.2. *Let K be a convex body in $\mathbb{R}^d$. If $x \in \operatorname{int} K$, $t \in [0, 2^{-d}]$ and $V(M_K(x)) = tV(K)$, then*

$$\mathbb{P}(x \notin K(n)) \leq 2\sum_{i=0}^{d-1} \binom{n}{i} \left(\frac{t}{2}\right)^i \left(1 - \frac{t}{2}\right)^{n-i}.$$

In turn we deduce the following.

Lemma 2.3. *Let K be a convex body in $\mathbb{R}^d$, and let $L \in \mathcal{L}_j^d$, $j \in \{1, \dots, d-1\}$. If $z \in \operatorname{relint}(K|L)$, $t \in [0, 2^{-d}]$ and $V(M_K^L(z)) = tV(K)$, then*

$$\mathbb{P}\left(z \notin K(n)|L\right) \leq 2\sum_{i=0}^{d-1} \binom{n}{i} \left(\frac{t}{2}\right)^i \left(1 - \frac{(2d)^{-d}t}{2}\right)^{n-i}.$$

PROOF. Let x be the centre of mass of $K \cap (z + L^{\perp})$. Since $x \in K(n)$ implies $z \in K(n)|L$,

$$\mathbb{P}\left(z \notin K(n)|L\right) \leq \mathbb{P}\left(x \notin K(n)\right).$$

Hence Lemmas 2.1 and 2.2 yield the required estimate.

□

Our next goal is to show that with high probability, $K(n)$ is very close to K.

LEMMA 2.4. *Let K be a convex body in $\mathbb{R}^d$, and let $t \in (0, 2^{-d})$. Then*

$$\mathbb{P}\left(K_t \not\subset K(n)\right) \leq 12^d t^{-1} e^{-tn6^{-d}} \frac{V(K \setminus K_t)}{V(K)}.$$

PROOF. For the proof we assume that $V(K) = 1$. The general assertion then follows from the scaling behaviour of the convolution bodies.

For any $x \in K$ and $\lambda > 0$, a blown-up version of the Macbeath region is defined by

$$M_K(x, \lambda) := x + \lambda(M_K(x) - x).$$

For any $y \in \partial K_t$, let H be a tangent hyperplane to K_t at y. Let $C(y)$ be the cap of K cut off by H which does not contain K_t. Fix a point $z = z(y) \in \partial K \cap C(y)$ such that $z - y + H$ is a tangent plane of ∂K. Then we define $w(y) = z + \frac{1}{6}(y - z)$. It follows that

$$V(M_K(w(y))) \geq V(M_K(y))/6^d = t/6^d \quad \text{and} \quad M_K(w(y), 5) \cap K \subset C(y).$$

Next let $y_1, \ldots, y_m \in \partial K_t$ be a maximal set of points such that the convex bodies $M_K(w(y_i), \frac{1}{2})$, $i = 1, \ldots, m$, are pairwise disjoint. In particular,

$$mt = \sum_{i=1}^{m} V(M_K(y_i)) \leq 6^d \sum_{i=1}^{m} V(M_K(w(y_i))) = 12^d \sum_{i=1}^{m} V(M_K(w(y_i), 1/2)),$$

and hence

$$m \leq 12^d t^{-1} V(K \setminus K_t).$$

Now let $x_1, \ldots, x_n \in K$ and assume that $\partial K_t \setminus [x_1, \ldots, x_n] \neq \emptyset$. Then we can find a point $y \in \partial K_t$ such that there is a support plane H of K_t with $y \in H$ and $[x_1, \ldots, x_n] \subset \operatorname{int} H^-$, where H^- denotes one of the two halfspaces bounded by H.

By maximality, $M_K(w(y), 1/2)$ intersects $M_K(w(y_i), 1/2)$, for some $i \in \{1, \ldots, m\}$, and hence $M_K(w(y_i)) \subset M_K(w(y), 5)$ by the basic observation of cap covering (see, e.g., (4.4) in I. Bárány and D.G. Larman [6] or [18, Lemma 2.3.4]). In particular, $M_K(w(y_i)) \subset C(y)$ is disjoint from $[x_1, \ldots, x_n]$. We deduce

$$\mathbb{P}(K_t \not\subset K(n)) \leq \sum_{i=1}^{m} \left(1 - V(M_K(w(y_i)))\right)^n \leq 12^d t^{-1} V(K \setminus K_t) \left(1 - \frac{t}{6^d}\right)^n.$$

The lemma follows by an application of the estimate $(1-x)^n \le e^{-nx}$, $x \in [0,1]$. □

If K is a convex body, then $V(K \setminus K_t) \le c(K) t^{\frac{2}{d+1}}$ by (6) and the corresponding estimate of C. Schütt and E. Werner [21] for floating bodies. Combining Lemmas 2.1 and 2.4 we deduce the following result.

LEMMA 2.5. *Let K be a convex body in $\mathbb{R}^d$. Let $L \in \mathcal{L}_j^d$, $j \in \{1, \dots, d\}$, and $t \in (0, 2^{-d})$. Then*

$$\mathbb{P}(K_t^L \not\subset K(n)|L) \ll e^{-tn(12d)^{-d}} \cdot t^{-\frac{d-1}{d+1}}.$$

PROOF. Assume that $K_{(2d)^{-d}t} \subset K(n)$, and let $z \in K_t^L$. Then $z \in K|L$ and $V(M_K^L(z)) \ge t$. Lemma 2.1 shows that there is some $x \in K$ with $z = x|L$ and $V(M_K(x)) \ge (2d)^{-d}t$, i.e. $x \in K_{(2d)^{-d}t}$. Therefore $x \in K(n)$ and thus $z \in K(n)|L$. Hence $K_t^L \not\subset K(n)|L$ implies that $K_{(2d)^{-d}t} \not\subset K(n)$. Now the proof can be completed by applying Lemma 2.4. □

3. Rolling ball property and random polytopes

Throughout this section, we consider a convex body K in $\mathbb{R}^d$ in which a ball of radius $r > 0$ rolls freely. Hence, K is smooth (i.e. K has a unique exterior unit normal vector at each boundary point) and the exterior unit normal $N_K(x)$ of K at $x \in \partial K$ is a Lipschitz map on ∂K. More generally, for a compact convex set $C \subset \mathbb{R}^d$, we write $N_C(x)$ to denote an exterior unit normal vector of C at x with respect to aff C. The orthogonal complement of $x \in \mathbb{R}^d \setminus \{o\}$ is denoted by $x^\perp$.

The following result is a version of Lemma 2 in C. Schütt [20]. The paper [20] refers to a similar statement in the unpublished notes by Schmuckenschläger [17] for a proof of Lemma 2 (a factor 1/2 is missing in [20, Lemma 2]). Therefore we provide a short proof of Lemma 3.1.

Let K be a smooth convex body in $\mathbb{R}^d$. We choose $\tau(K) \in (0, 2^{-d})$ so that the following property is satisfied: If $L \in \mathcal{L}_j^d$, $V(M_K^L(z)) \le \tau(K)$ for some $z \in \operatorname{relint}(K|L)$, and u, v are exterior unit normals to $M_K^L(z)$ at $x, y \in \partial M_K^L(z) \cap \operatorname{int} K$, then $\langle u, v\rangle > 0$. In particular, we thus ensure that K and $2z - K + L^\perp$ have a common interior point and intersect transversally (i.e. exterior unit normal vectors at common boundary points of K and $2z - K + L^\perp$ are linearly independent; cf. §2 in [12]). Hence the intersection $\partial K \cap \partial(2z - K + L^\perp)$ is a compact $(d-2)$-dimensional submanifold (cf. [12, Proposition 2.2]), and thus

$$\mathcal{H}^{d-1}(\partial K \cap \partial(2z - K + L^\perp)) = 0 \tag{8}$$

is satisfied. Note that for the following lemma smoothness (of class C^1) is a sufficient condition. However, the disintegration stated in the lemma may fail to be true in the given form, even for arbitrarily small t, if K is a simplex (for instance).

LEMMA 3.1. *Let K be a smooth convex body in $\mathbb{R}^d$. Let $L \in \mathcal{L}_j^d$, $j \in \{1,\dots,d\}$, and let $f\colon K|L \to \mathbb{R}$ be a nonnegative measurable function. Then, for $t \in (0,\tau(K))$, ∂K_t^L is smooth and*

$$\int_{(K|L)\setminus K_t^L} f(z)\,\mathcal{H}^j(dz) = \int_0^t \int_{\partial K_s^L} \frac{V(K)f(z)}{2V_{d-1}\left((\partial M_K^L(z)\cap \operatorname{int} K)|N_{K_s^L}(z)^\perp\right)}\,\mathcal{H}^{j-1}(dz)\,ds.$$

PROOF. For $z \in K|L$, we put $d(z) := V(M_K^L(z))/V(K)$, which defines a Lipschitz map. We determine the variation of $V(M_K^L(z))/V(K)$ if $V(M_K^L(z)) = s \in (0, 2^{-d})$. For this, we write $u(y)$ to denote the exterior unit normal to $2z - K + L^\perp$ at $y \in \partial(2z - K + L^\perp)$. Let $z \in K|L$ with $d(z) \le \tau(K)$. Then (5) implies that, as $h \in L$ tends to o,

$$\begin{aligned} V(M_K^L(z+h)) - &V(M_K^L(z)) \\ &= 2\int_{\partial(2z-K+L^\perp)\cap \operatorname{int} K} \langle u(y), h\rangle\,\mathcal{H}^{d-1}(dy) + \\ &\quad + 2\int_{\partial(2z-K+L^\perp)\cap \partial K} \min\{0, \langle u(y), h\rangle\}\,\mathcal{H}^{d-1}(dy) + o(\|h\|); \end{aligned}$$

cf. also [15, Lemma 2.1]. Now (8) implies that the map $z \mapsto d(z)$ is continuously differentiable at z and the differential is given by

$$Dd(z) = 2V(K)^{-1}\int_{\partial M_K^L(z)\cap \operatorname{int} K} u(y)\,\mathcal{H}^{d-1}(dy).$$

Since $N_{K_s^L}(z) = -\|Dd(z)\|^{-1}Dd(z)$, for the Jacobian $Jd(z)$ of $d(\cdot)$ at z we obtain

$$\begin{aligned} Jd(z) &= -2V(K)^{-1}\int_{\partial M_K^L(z)\cap \operatorname{int} K} \langle u(y), N_{K_s^L}(z)\rangle\,\mathcal{H}^{d-1}(dy) \\ &= \langle -N_{K_s^L}(z), Dd(z)\rangle > 0, \end{aligned}$$

and hence

$$Jd(z) = 2V(K)^{-1}V_{d-1}\left((\partial M_K^L(z)\cap \operatorname{int} K)|N_{K_s^L}(z)^\perp\right).$$

Thus, by the coarea formula [8]

$$\int_{(K|L)\setminus K_t^L} f(z)\,\mathcal{H}^j(dz) = \int_0^t \int_{d^{-1}(\{s\})} \frac{f(y)}{Jd(y)}\,\mathcal{H}^{j-1}(dy)\,ds.$$

Finally, $\partial K_t^L = d^{-1}(\{t\})$ and the implicit function theorem shows that ∂K_t^L is smooth.

□

Since a ball of radius $r > 0$ rolls freely inside the convex body K, the same is true for any projection of K onto a subspace L. Hence, for $L \in \mathcal{L}_j^d$, $j \in \{1, \dots, d\}$, $z \in \partial(K|L)$ and $t \in (0, r^d\kappa_d/(2V(K)))$, we define $z_t = z - sN_{K|L}(z)$, where $s \in (0, r)$ is chosen such that $V(M_K^L(z_t)) = tV(K)$. Clearly, z_t depends on K and L. We therefore also write $z_t^{K,L}$ or z_t^K to indicate this dependence. For instance, for $\lambda > 0$ we have $(\lambda z)_t^{\lambda K,L} = \lambda \cdot z_t^{K,L}$.

If $L \in \mathcal{L}_j^d$ and $z \in K|L$, then we define $x(z)$ to be the (unique) centre of the smallest ball containing $(z + L^\perp) \cap K$. Since $x(z) = \lim_{t\to 0} x(z_t)$ for $z \in \partial(K|L)$, $x(z)$ is a measurable function of $z \in \partial(K|L)$. Next we estimate the denominator in Lemma 3.1.

LEMMA 3.2. *Let K be a convex body in $\mathbb{R}^d$ in which a ball of radius $r > 0$ rolls freely. Let $j \in \{1, \dots, d\}$.*

(i) *For any $\varepsilon > 0$, there exists $t_\varepsilon \in (0, r^d\kappa_d/(2V(K)))$ such that, for $L \in \mathcal{L}_j^d$, $t \in (0, t_\varepsilon)$ and $z \in \partial(K|L)$,*

$$1 - \varepsilon < \frac{V_{d-1}((\partial M_K^L(z_t) \cap \operatorname{int} K)|N_{K_t^L}(z_t)^\perp)}{V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^\perp)} < 1 + \varepsilon.$$

(ii) *For $L \in \mathcal{L}_j^d$, $t \in (0, r^d\kappa_d(2V(K)))$ and $z \in \partial(K|L)$,*

$$V_{d-1}\left(M_K^L(z_t)|N_{K|L}(z)^\perp\right) \geq c \cdot t^{\frac{d-1}{d+1}},$$

where $c = \frac{1}{2} r^{\frac{d-1}{d+1}} (\kappa_{d-1}/d)^{\frac{2}{d+1}} V(K)^{\frac{d-1}{d+1}}$.

PROOF. For any $L \in \mathcal{L}_j^d$, $z \in \partial(K|L)$ and $t \in (0, r^d\kappa_d/(2V(K)))$, let $x_t(z) = x(z) + z_t - z$, and hence $x_t(z)|L = z_t$. The existence of the rolling ball at $x(z)$ yields that

$$x_t(z) + \left(N_{K|L}(z)^\perp \cap \sqrt{r \cdot \|z - z_t\|} B^d\right) \subset M_K^L(z_t), \tag{9}$$

since $\|z - z_t\| < r$, which is due to $t \in (0, r^d\kappa_d/(2V(K)))$, and since

$$\sqrt{r^2 - (r - \|z - z_t\|)^2} \geq \sqrt{r\|z - z_t\|}.$$

From (9) we now deduce that

$$tV(K) = V(M_K^L(z_t)) > \frac{1}{d}\|z - z_t\|\kappa_{d-1}\sqrt{r\|z - z_t\|}^{d-1},$$

hence

$$\|z - z_t\| < (d/\kappa_{d-1})^{\frac{2}{d+1}} r^{-\frac{d-1}{d+1}} (tV(K))^{\frac{2}{d+1}}. \tag{10}$$

Since $M_K^L(z_t)$ is contained in a strip $\Sigma(z, t)$ bounded by two hyperplanes orthogonal to $N_{K|L}(z)$ and having distance $2\|z - z_t\|$, Fubini's theorem yields

$$tV(K) = V(M_K^L(z_t)) \leq 2\|z - z_t\| V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^\perp), \tag{11}$$

and thus (ii) follows from (10) and (11).

Let $\tau(K)$ be defined as before Lemma 3.1. Since ∂K is smooth, there exists some $t_0 \in (0, \tau(K))$ depending on K such that if $L \in \mathcal{L}_j^d$, $z \in \partial(K|L)$ and $t \in (0, t_0)$, then

$$(\partial M_K^L(z_t) \cap \operatorname{int} K)|N_{K_t^L}(z_t)^\perp = M_K^L(z_t)|N_{K_t^L}(z_t)^\perp.$$

As t tends to zero, $N_{K_t^L}(z_t)$ tends uniformly to $N_{K|L}(z)$ for $L \in \mathcal{L}_j^d$ and $z \in \partial(K|L)$. Therefore combining (9), (10) and $M_K^L(z_t) \subset \Sigma(z,t)$, we obtain (i).

□

In what follows, K is a convex body in $\mathbb{R}^d$ in which a ball of radius $r > 0$ rolls freely. Moreover, we fix $j \in \{1, \ldots, d-1\}$.

Observe that if $z \notin K(n)|L$, for some $z \in K_t^L$ and $t \in (0, 2^{-d})$, then $K_t^L \not\subset K(n)|L$. This remark will be applied with $t = n^{-1/2}$ and $n \geq 4^d$. Hence Lemma 2.5 yields

$$\int_{\mathcal{L}_j^d} \int_{K^L_{n^{-1/2}}} \mathbb{P}\left(z \notin K(n)|L\right)\, dz\, \nu_j(dL) \leq c_1(K) e^{-\sqrt{n}(12d)^{-d}} n^{\frac{1}{2}\frac{d-1}{d+1}} \ll o(n^{-k})$$

for all $k \in \mathbb{N}$. Thus, for all $k \in \mathbb{N}$, we get

$$\begin{aligned} V_j(K) - & \mathbb{E}V_j(K(n)) \\ &= \frac{\binom{d}{j}\kappa_d}{\kappa_j\,\kappa_{d-j}} \int_{\mathcal{L}_j^d} \int_{K|L} \mathbb{P}\left(z \notin K(n)|L\right)\, \mathcal{H}^j(dz)\, \nu_j(dL) \\ &= \frac{\binom{d}{j}\kappa_d}{\kappa_j\,\kappa_{d-j}} \int_{\mathcal{L}_j^d} \int_{(K|L)\setminus K^L_{n^{-1/2}}} \mathbb{P}\left(z \notin K(n)|L\right)\, \mathcal{H}^j(dz)\, \nu_j(dL) + o(n^{-k}), \end{aligned}$$

and therefore by Lemma 3.1

$$\begin{aligned} V_j&(K) - \mathbb{E}V_j(K(n)) \\ &= \frac{\binom{d}{j}\kappa_d}{2\kappa_j\,\kappa_{d-j}} \int_{\mathcal{L}_j^d} \int_0^{n^{-1/2}} \int_{\partial K_t^L} \frac{V(K)\mathbb{P}\left(z \notin K(n)|L\right)}{V_{d-1}((\partial M_K^L(z) \cap \operatorname{int} K)|N_{K_t^L}(z)^\perp)} \times \\ &\qquad \times \mathcal{H}^{j-1}(dz)\, dt\, \nu_j(dL) + o(n^{-k}). \end{aligned}$$

Since K has a rolling ball, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $L \in \mathcal{L}_j^d$, $t \in (0, \delta)$ and $z, w \in \partial(K|L)$ with $\|z - w\| < \delta$, then

$$(1-\varepsilon)\|z-w\| \leq \|z_t - w_t\| \leq (1+\varepsilon)\|z-w\|.$$

Therefore, applying the transformation $z \mapsto z_t$, Lemma 3.2 (i) and Fubini's theorem, we arrive at

$$\begin{aligned} V_j(K) - & \mathbb{E}V_j(K(n)) \\ &= \frac{\binom{d}{j}\kappa_d}{2\kappa_j\,\kappa_{d-j}} \int_{\mathcal{L}_j^d} \int_{\partial(K|L)} \int_0^{n^{-1/2}} \frac{V(K)\mathbb{P}\left(z_t \notin K(n)|L\right)}{V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^\perp)} \times \\ &\qquad \times dt\, \mathcal{H}^{j-1}(dz)\, \nu_j(dL) + o\left(n^{\frac{-2}{d+1}}\right). \end{aligned} \tag{12}$$

We have to estimate the order of magnitude of the integral above. For this, let $t \in (0, n^{-1/2})$ for large n, let $L \in \mathcal{L}_j^d$, $j \in \{1, \dots, d\}$, and $z \in \partial(K|L)$ (and thus $z_t \in \partial K_t^L$). Then, if $n \geq 2d$ and $\alpha \geq 0$, we deduce from Lemma 2.3 and Lemma 3.2

$$\begin{aligned}
\int_{\alpha/n}^{n^{-1/2}} & \frac{\mathbb{P}\left(z_t \notin K(n)|L\right)}{V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^\perp)}\, dt \\
&\leq c_1 \sum_{i=0}^{d-1} \binom{n}{i} \int_{\alpha/n}^{n^{-1/2}} t^{i-\frac{d-1}{d+1}} (1-(4d)^{-d}t)^{n-i}\, dt \\
&\leq c_1 \sum_{i=0}^{d-1} n^i \int_{\alpha/n}^{n^{-1/2}} t^{i-\frac{d-1}{d+1}} e^{-(4d)^{-d}nt/2}\, dt \\
&\leq c_2\, n^{-\frac{2}{d+1}} \sum_{i=0}^{d-1} \int_{\alpha/c}^{\infty} s^{i+\frac{2}{d+1}-1} e^{-s}\, ds,
\end{aligned} \tag{13}$$

where $c = 2(4d)^d$ and c_1, c_2 are constants depending only on K and d. Observe that the factor $\frac{1}{2}$ from $\frac{t}{2}$ is absorbed in the constants. The parameter α has been introduced in view of an application in Section 4. Putting $\alpha = 0$, we see that there is a constant $n(K)$ depending only on K such that if $n > n(K)$, then, for $L \in \mathcal{L}_j^d$ and $z \in \partial(K|L)$,

$$\int_0^{n^{-1/2}} \frac{\mathbb{P}\left(z_t \notin K(n)|L\right)}{V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^\perp)}\, dt \ll n^{\frac{-2}{d+1}}. \tag{14}$$

This estimate will allow us to apply the dominated convergence theorem in the final step of the proof in Section 5.

4. Comparing to balls

In this section, for given $L \in \mathcal{L}_j^d$ and $z \in \partial(K|L)$, we describe the asymptotic behaviour of the integral

$$\int_0^{n^{-1/2}} \frac{\mathbb{P}\left(z_t \notin K(n)|L\right)}{V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^\perp)}\, dt$$

(cf. (12)) as $n \to \infty$ by comparing it with the case where K is a suitable ball. If K is the ball $B(\rho)$ of radius $\rho > 0$ with $o \in \partial B(\rho)$ and exterior normal vector N at o, a comparison of the asymptotic formula (4) proved by I. Bárány [3] and M. Reitzner [16] and of relation (12) implies that

$$\lim_{n\to\infty} \int_0^{n^{-1/2}} \frac{n^{\frac{2}{d+1}} V(B(\rho)) \mathbb{P}\left(z_t^{B(\rho)} \notin B(\rho)(n)|L\right)}{V_{d-1}(M_{B(\rho)}^L(z_t^{B(\rho)})|N^\perp)}\, dt = \tilde{c}_{d,j} \cdot V(B(\rho))^{\frac{2}{d+1}} \rho^{-\frac{d-1}{d+1}}, \tag{15}$$

where

$$\tilde{c}_{d,j} = c_{d,j} \cdot \frac{2d\kappa_{d-j}}{j\binom{d}{j}}.$$

Equation (15) will be applied with $\rho = \sigma_{d-1}(x(z))^{-\frac{1}{d-1}}$ in the case where $\sigma_{d-1}(x(z)) > 0$.

Lemma 4.1. *Let K be a convex body in $\mathbb{R}^d$ in which a ball rolls freely. Let $L \in \mathcal{L}_j^d$, $j \in \{1, \ldots, d-1\}$, and let $z \in \partial(K|L)$ be such that ∂K is twice differentiable at $x(z)$ in the generalized sense. Then*

$$\lim_{n\to\infty} \int_0^{n^{-1/2}} \frac{n^{\frac{2}{d+1}} V(K)\mathbb{P}\left(z_t \notin K(n)|L\right)}{V_{d-1}\left(M_K^L(z_t)|N_{K|L}(z)^\perp\right)}\, dt = \tilde{c}_{d,j} \cdot V(K)^{\frac{2}{d+1}} \sigma_{d-1}(x(z))^{\frac{1}{d+1}}.$$

Proof. As before, we will use the more explicit notation $z_t^{K,L}$ or z_t^K whenever this is appropriate. We write $[X_1, \ldots, X_m]$ for the convex hull of sets $X_1, \ldots, X_m \subset \mathbb{R}^d$ (here we identify $x \in \mathbb{R}^d$ with $\{x\}$).

First, we consider the case where $\sigma_{d-1}(x(z)) = 0$. Then

$$V_{d-1}\left(M_K^L(z_t)|N_{K|L}(z)^\perp\right) \geq f(t) \cdot t^{\frac{d-1}{d+1}},$$

where $f\colon (0,\infty) \to (0,\infty)$ is a decreasing function with $\lim_{t\to 0} f(t) = \infty$. From Lemma 2.3 and proceeding as in the derivation of (13) we deduce

$$\begin{aligned}
\int_0^{n^{-1/2}} & \frac{\mathbb{P}\left(z_t \notin K(n)|L\right)}{V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^\perp)}\, dt \\
&\leq 2\sum_{i=0}^{d-1} \binom{n}{i} \int_0^{n^{-1/2}} t^i (1-(4d)^{-d}t)^{n-i} t^{-\frac{d-1}{d+1}} f(t)^{-1}\, dt \\
&\leq 2\sum_{i=0}^{d-1} \binom{n}{i} f(n^{-1/2})^{-1} \int_0^{n^{-1/2}} t^{i-\frac{d-1}{d+1}} (1-(4d)^{-d}t)^{n-i}\, dt \\
&\ll \frac{n^{\frac{-2}{d+1}}}{f(n^{-1/2})},
\end{aligned}$$

which yields the assertion in the present case.

Now we assume that $\sigma_{d-1}(x(z)) > 0$. For the subsequent investigation, we can assume that $x(z) = 0 = z$, and we put $\sigma := \sigma_{d-1}(x(z))$. Further, we put $N := N_{K|L}(o)$, $e_d := -N$ and we identify $\mathbb{R}^{d-1}$ with $N^\perp$. Let $e_1, \ldots, e_{d-1}$ be an orthonormal basis of eigenvectors of the quadratic form Q associated with K at o with corresponding eigenvalues $k_1, \ldots, k_{d-1}$ (generalized principal curvatures). Then, in particular, $e_1, \ldots, e_d$ is an orthonormal basis of $\mathbb{R}^d$. We denote by B_σ the ball with radius $\sigma^{-\frac{1}{d-1}}$ and center at $\sigma^{-\frac{1}{d-1}} e_d$. Let Ψ be the osculating paraboloid of B_σ at o, which is the graph of $\frac{1}{2}\sigma^{\frac{1}{d-1}}\|x\|^2$, $x \in \mathbb{R}^{d-1}$. The linear transformation φ which is defined by $\varphi(e_d) := e_d$ and $\varphi(e_i) := (k_i\sigma^{-\frac{1}{d-1}})^{\frac{1}{2}} e_i$, $i \in \{1, \ldots, d-1\}$, is volume preserving. Then $\widetilde{K} := \varphi(K)$ has the same volume as K and Ψ is the osculating paraboloid of $\widetilde{K}$ at o. We define $\widetilde{L} := \varphi(L^\perp)^\perp$. Then

$$\mathbb{P}(z_t^{K,L} \notin K(n)|L) = \mathbb{P}(z_t^{\widetilde{K},\widetilde{L}} \notin \widetilde{K}(n)|\widetilde{L})$$

and

$$\varphi(M_K^L(z_t^{K,L})) = M_{\widetilde{K}}^{\widetilde{L}}(z_t^{\widetilde{K},\widetilde{L}}).$$

From the special form of the map φ we therefore conclude that

$$\begin{aligned}\lim_{n\to\infty} \int_0^{n^{-1/2}} & \frac{n^{\frac{2}{d+1}} V(K)\mathbb{P}(z_t^{K,L} \notin K(n)|L)}{V_{d-1}(M_K^L(z_t^{K,L})|N^\perp)}\, dt \\ &= \lim_{n\to\infty} \int_0^{n^{-1/2}} \frac{n^{\frac{2}{d+1}} V(\widetilde{K})\mathbb{P}(z_t^{\widetilde{K},\widetilde{L}} \notin \widetilde{K}(n)|\widetilde{L})}{V_{d-1}(M_{\widetilde{K}}^{\widetilde{L}}(z_t^{\widetilde{K},\widetilde{L}})|N^\perp)}\, dt.\end{aligned} \tag{16}$$

In the following, we write again K, L for $\widetilde{K}, \widetilde{L}$.

Next we introduce for $t > 0$ the linear map A_t of $\mathbb{R}^d$ which is defined by $A_t(e_d) = t^{-\frac{2}{d+1}} e_d$ and $A_t(x) = t^{-\frac{1}{d+1}} x$ for $x \in \mathbb{R}^{d-1}$. Let $o^* := -sN$ for some $s > 0$ satisfy

$$V(M^*) = 1 \qquad \text{for} \qquad M^* := [\Psi] \cap (2o^* - [\Psi] + L^\perp).$$

Since $\det(A_t) = t^{-1}$ and $V(M_K^L(z_{t/V(K)}^{K,L})) = t$, we have $V(A_t M_K^L(z_{t/V(K)}^{K,L})) = 1$. Moreover, $A_t M_K^L(z_{t/V(K)}^{K,L})$ converges in the Hausdorff metric, as $t \to 0$, and therefore also in the symmetric difference metric, to M^*. Using the special form of the linear map A_t, we obtain

$$\begin{aligned}\frac{V(M^*)^{\frac{d-1}{d+1}}}{V_{d-1}(M^*|N^\perp)} &= \lim_{t\to 0} \frac{V(A_t M_K^L(z_{t/V(K)}^K))^{\frac{d-1}{d+1}}}{V_{d-1}(A_t M_K^L(z_{t/V(K)}^K)|N^\perp)} \\ &= \lim_{t\to 0} \frac{t^{-\frac{d-1}{d+1}} V(M_K^L(z_{t/V(K)}^K))^{\frac{d-1}{d+1}}}{t^{-\frac{d-1}{d+1}} V_{d-1}(M_K^L(z_{t/V(K)}^K)|N^\perp)} = \lim_{t\to 0} \frac{V(M_K^L(z_t^K))^{\frac{d-1}{d+1}}}{V_{d-1}(M_K^L(z_t^K)|N^\perp)}.\end{aligned}$$

Since B_σ has the same osculating paraboloid as K at o, we deduce

$$\lim_{t\to 0} \frac{(V(K)t)^{\frac{d-1}{d+1}}}{V_{d-1}(M_K^L(z_t^K)|N^\perp)} = \lim_{t\to 0} \frac{(V(B_\sigma)t)^{\frac{d-1}{d+1}}}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)},$$

and therefore

$$\lim_{t\to 0} \frac{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}{V_{d-1}(M_K^L(z_t^K)|N^\perp)} = \left(\frac{V(B_\sigma)}{V(K)}\right)^{\frac{d-1}{d+1}}.$$

Now we can continue with

$$\lim_{n\to\infty}\int_0^{n^{-1/2}} \frac{n^{\frac{2}{d+1}}V(K)\mathbb{P}(z_t^K\notin K(n)|L)}{V_{d-1}(M_K^L(z_t^K)|N^\perp)}\,dt$$
$$=\left(\frac{V(K)}{V(B_\sigma)}\right)^{\frac{2}{d+1}}\lim_{n\to\infty}\int_0^{n^{-1/2}} \frac{n^{\frac{2}{d+1}}V(B_\sigma)\mathbb{P}(z_t^K\notin K(n)|L)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt.$$

In the remaining part of the proof we will show that on the right-hand side $\mathbb{P}(z_t^K\notin K(n)|L)$ can be replaced by $\mathbb{P}(z_t^{B_\sigma}\notin B_\sigma(n)|L)$. Once this is accomplished, we can conclude the proof by applying (15) with $\rho=\sigma^{-\frac{1}{d-1}}$.

In the following, we can assume that $V(K)=1$, since $\mathbb{P}((\lambda z)_t^{\lambda K}\notin(\lambda K)(n)|L)$ is independent of $\lambda>0$. Constants involved in $O(\cdot)$ or in $\ll$ depend on K and z. Let $\varepsilon\in(0,1)$. We choose $\alpha=\alpha(\varepsilon,d)>1$ sufficiently large such that the sum in (13) is smaller than ε. Moreover, we always assume that n is large enough for our purposes. For instance, we require n to satisfy $\alpha(\varepsilon,d)n^{-1}<n^{-1/2}<r^d\kappa_d/2$ (recall that $V(K)=1$), further conditions will be mentioned in the course of the proof.

First, we show that for estimating the integral

$$\int_0^{n^{-1/2}} \frac{n^{\frac{2}{d+1}}\mathbb{P}\left(z_t^K\notin K(n)|L\right)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt$$

up to an error of order $O(\varepsilon)$, we can restrict the integration to $t\in[\varepsilon^{\frac{d+1}{2}}/n,\alpha(\varepsilon,d)/n]$ (see (17) and (18)).

In fact, since B_σ is a ball, we can apply Lemma 3.2 for $t\in(0,1/2)$. Hence, for $n\geq 2$,

$$\int_0^{\varepsilon^{\frac{d+1}{2}}/n} \frac{\mathbb{P}\left(z_t^K\notin K(n)|L\right)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt\leq\int_0^{\varepsilon^{\frac{d+1}{2}}/n}\frac{1}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt$$
$$\ll\int_0^{\varepsilon^{\frac{d+1}{2}}/n} t^{-\frac{d-1}{d+1}}\,dt=\varepsilon\cdot n^{\frac{-2}{d+1}}. \tag{17}$$

In the range $t>\alpha/n$, we use Lemma 2.3 to estimate the numerator and we treat the denominator as before. Thus, for $n\geq 2$, as in the derivation of (13) we get

$$\int_{\alpha/n}^{n^{-1/2}} \frac{\mathbb{P}\left(z_t^K\notin K(n)|L\right)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt\leq 2\sum_{i=0}^{d-1}\binom{n}{i}\int_{\alpha/n}^{n} t^i(1-(4d)^{-d}t)^{n-i}t^{-\frac{d-1}{d+1}}\,dt$$
$$\ll\varepsilon\cdot n^{\frac{-2}{d+1}} \tag{18}$$

by the choice of α. Thus we have shown that, as $n\to\infty$,

$$\int_0^{n^{-1/2}} \frac{n^{\frac{2}{d+1}}\mathbb{P}(z_t^K\notin K(n)|L)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt=\int_{\varepsilon^{\frac{d+1}{2}}/n}^{\alpha/n}\frac{n^{\frac{2}{d+1}}\mathbb{P}(z_t^K\notin K(n)|L)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt+O(\varepsilon).$$

It remains to show that up to a term of order $O(\varepsilon)$

$$I(n) := \int_{\varepsilon^{\frac{d+1}{2}}/n}^{\alpha/n} \frac{n^{\frac{2}{d+1}}\mathbb{P}\left(z_t^K \notin K(n)|L\right)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt$$

remains asymptotically (as $n \to \infty$) unchanged if in the numerator K is replaced by B_σ.

For this, up to an error of order $O(\varepsilon)$ we approximate the remaining integral expression by an analogous expression involving $[\Psi]$ instead of K. Here we assume that $n > n_0$, where n_0 depends on K, z and ε. For that purpose, we first define $\beta = \beta(\varepsilon, d) > (4(d-1))^{d+1}$ and then an integer $k = k(\varepsilon, d) > 1$ so as to satisfy

$$2^{d-1}e^{-(2d)^{-2d-2}\beta^{\frac{1}{d+1}}\varepsilon^{\frac{d+1}{2}}} < \varepsilon/\alpha^{\frac{2}{d+1}} \qquad \text{and} \qquad (\alpha\beta)^k/k! < \varepsilon/\alpha^{\frac{2}{d+1}}.$$

The reason for prescribing $\varepsilon/\alpha^{\frac{2}{d+1}}$ instead of just ε on the right-hand sides is that the derivations of (27) and (28) involve a factor $\alpha^{\frac{2}{d+1}}$ coming from an estimate of the form

$$\int_{\varepsilon^{\frac{d+1}{2}}/n}^{\alpha/n} \frac{n^{\frac{2}{d+1}}}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\,dt \ll \int_{\varepsilon^{\frac{d+1}{2}}/n}^{\alpha/n} n^{\frac{2}{d+1}}t^{-\frac{d-1}{d+1}}\,dt \ll \alpha^{\frac{2}{d+1}}.$$

For $t \in (0, r^d\kappa_d/(2\beta))$, we define $C(\varepsilon, t) = H^+ \cap K$ where H^+ is the halfspace with exterior unit normal $-N$ and

$$V(C(\varepsilon, t)) = \beta \cdot t.$$

As before, we write $x_1, \dots, x_n$ to denote n random points chosen uniformly and independently from K. For $i \in \mathbb{N}$, $C(\varepsilon, t)(i)$ denotes the convex hull of i random points chosen uniformly and independently from $C(\varepsilon, t)$.

Next we show that $C(\varepsilon, t)$ contains at most $k(\varepsilon, d)$ points out of $x_1, \dots, x_n$ (see (19)) with probability at least $1 - \alpha^{\frac{-2}{d+1}}\varepsilon$, and it remains to be checked whether z_t is contained in the projection to L of the convex hull of these at most $k(\varepsilon, d)$ points (see (21)). Here the cardinality of a finite set F is denoted by $\#F$. In fact, if $t \leq \alpha(\varepsilon, d)/n$, then

$$\begin{aligned}\mathbb{P}(\#(\{x_1, \dots, x_n\} \cap C(\varepsilon, t)) \geq k) &\leq \binom{n}{k} V(C(\varepsilon, t))^k \\ &\leq \binom{n}{k}\left(\frac{\alpha\beta}{n}\right)^k < \frac{\varepsilon}{\alpha^{\frac{2}{d+1}}}.\end{aligned} \tag{19}$$

Let $C^* = [\Psi] \cap H^*$, where H^* is a halfspace whose exterior unit normal is $-N$ and such that

$$V(C^*) = \beta.$$

We write Δ for the symmetric difference of two sets. Since Ψ is the osculating paraboloid of K at o, $A_tC(\varepsilon,t) \to C^*$ in the Hausdorff metric as $t \to 0$, and hence

$$\lim_{t\to 0} V(C^* \,\Delta\, A_tC(\varepsilon,t)) = 0. \tag{20}$$

Subsequently, we prove that if $t \in [\varepsilon^{\frac{d+1}{2}}/n, \alpha/n]$, then

$$0 \le \mathbb{P}\left(z_t \notin [\{x_1,\ldots,x_n\} \cap C(\varepsilon,t)]|L\right) - \mathbb{P}\left(z_t \notin K(n)|L\right) < \varepsilon/\alpha^{\frac{2}{d+1}}. \tag{21}$$

To verify (21) for all t in the given range, we construct sets $\Xi_1,\ldots,\Xi_{2^{d-1}} \subset K$, depending on t, such that

$$V(\Xi_i) \ge (2d)^{-2d-2}\beta^{\frac{1}{d+1}}t, \qquad i=1,\ldots,2^{d-1}, \tag{22}$$

and such that if $z_t \in K(n)|L$ but $z_t \notin [\{x_1,\ldots,x_n\} \cap C(\varepsilon,t)]|L$, then $\Xi_i \cap \{x_1,\ldots,x_n\} = \emptyset$ for some $i \in \{1,\ldots,2^{d-1}\}$. Once this has been accomplished, we will conclude (21), since

$$\sum_{i=1}^{2^{d-1}} (1-V(\Xi_i))^n \le 2^{d-1}e^{-(2d)^{-2d-2}\beta^{\frac{1}{d+1}}\varepsilon^{\frac{d+1}{2}}} < \varepsilon/\alpha^{\frac{2}{d+1}}$$

by the choice of β.

We put $B^{d-1} := \mathbb{R}^{d-1} \cap B^d$. The coordinate hyperplanes in $\mathbb{R}^{d-1}$ divide $\mathbb{R}^{d-1}$ into 2^{d-1} congruent cones, which we denote by $\Theta_1,\ldots,\Theta_{2^{d-1}}$. Each Θ_i contains a unit vector w_i whose first $d-1$ coordinates have absolute value $(d-1)^{\frac{-1}{2}}$. For any $y \in \mathbb{R}^d$, let $\widetilde{H}^+(y)$ be the halfspace whose exterior unit normal is $-N$ and whose bounding hyperplane $\widetilde{H}(y)$ contains y. In particular, if n is sufficiently large (and therefore $t > 0$ is small enough), then

$$(2d)^{-d-1}t < V(\widetilde{H}^+(z_t) \cap K) < 0.9t. \tag{23}$$

In fact, since Ψ is the osculating paraboloid of K at o, $A_t(z_t) \to o^*$ and

$$V(\widetilde{H}^+(o^*) \cap [\Psi]) = \frac{1}{2}V([\Psi] \cap (2o^* - [\Psi])) = \frac{1}{2}V(M_{[\Psi]}(o^*)),$$

we get

$$\frac{V(\widetilde{H}^+(z_t)\cap K)}{V(M_K(z_t))} = \lim_{t\to 0}\frac{V(A_t(\widetilde{H}^+(z_t)\cap K))}{V(A_t(M_K(z_t)))} = \frac{1}{2}.$$

Using Lemma 2.1 we conclude (for n sufficiently large) that

$$V(\widetilde{H}^+(z_t)\cap K) \le 0.9V(M_K(z_t)) \le 0.9V(M_K^L(z_t)) = 0.9t$$

and

$$V(\widetilde{H}^+(z_t)\cap K)\geq\frac{(2d-1)^d}{(2d)^{d+1}}V(M_K(z_t))\geq\frac{1}{(2d)^{d+1}}V(M_K^L(z_t))=(2d)^{-d-1}t,$$

which yields the asserted estimates.

Let $\omega>0$ be chosen such that $\omega z_t\in\partial C(\varepsilon,t)$. Then ω satisfies (for n sufficiently large)

$$\omega>\beta^{\frac{2}{d+1}}\qquad\text{and}\qquad\widetilde{H}(\omega z_t)\cap K\subset\omega z_t+2\sqrt{\varrho\,\omega\|z_t\|}B^{d-1}.\tag{24}$$

The inclusion follows, since Ψ is the osculating paraboloid of K and since $\frac{1}{2\varrho}\|x\|^2=\omega\|z_t\|$ for some $x\in\mathbb{R}^{d-1}$ implies that

$$\|x\|=\sqrt{2\varrho\omega\|z_t\|}\leq 2\sqrt{\varrho\omega\|z_t\|}.$$

To justify the inequality, we deduce from $V(\widetilde{H}^+(z_t)\cap K)<0.9t$ that

$$V(\widetilde{H}^+(\beta^{\frac{2}{d+1}}z_t)\cap A_{\beta^{-1}}K)=V(A_{\beta^{-1}}(\widetilde{H}^+(z_t)\cap K))\leq\beta\cdot 0.9t.\tag{25}$$

Since $A_{\beta^{-1}}K$ and K both have Ψ as their osculating paraboloid, we also have

$$\frac{V(\widetilde{H}^+(\beta^{\frac{2}{d+1}}z_t)\cap K)}{V(\widetilde{H}^+(\beta^{\frac{2}{d+1}}z_t)\cap A_{\beta^{-1}}K)}<\frac{10}{9}.\tag{26}$$

Hence, from (25) and (26) it follows that

$$V(\widetilde{H}^+(\beta^{\frac{2}{d+1}}z_t)\cap K)<\beta t,$$

which yields that $\omega>\beta^{\frac{2}{d+1}}$ (all this for n sufficiently large).

Next we define

$$p:=\left(1+\frac{\omega-1}{\sqrt{\omega}}\right)\cdot z_t.$$

We put $\lambda:=\sqrt{\varrho\sqrt{\omega}\|z_t\|}$. Then $p+\lambda w_i\in K$, for $i=1,\dots,2^{d-1}$, since Ψ is the osculating paraboloid of K and

$$\frac{\lambda^2}{\|p\|}=\varrho\frac{\omega}{\sqrt{\omega}+\omega-1}<\varrho.$$

We now define

$$\Xi_i:=[p+\lambda w_i,K\cap(z_t+\Theta_i)],\qquad i=1,\dots,2^{d-1}.$$

Then (22) is satisfied, since

$$V(\Xi_i) = \frac{\omega - 1}{d\sqrt{\omega}} \cdot \|z_t\| \cdot \mathcal{H}^{d-1}(K \cap (z_t + \Theta_i)) > \frac{\sqrt{\omega}}{2d\,2^d} \cdot \|z_t\| \cdot \mathcal{H}^{d-1}(K \cap \widetilde{H}(z_t))$$
$$> \frac{\beta^{\frac{1}{d+1}}}{(2d)^{d+1}} V(\widetilde{H}^+(z_t) \cap K) > \frac{\beta^{\frac{1}{d+1}}}{(2d)^{2d+2}} \cdot t.$$

For the first estimate we use that $(\omega - 1)/\sqrt{\omega} \geq \sqrt{\omega}/2$ which follows from $\omega > \beta^{\frac{2}{d+1}} \geq 2$. Moreover, we also use that the osculating paraboloid of K at o is symmetric with respect to rotations around its axis and therefore

$$\mathcal{H}^{d-1}(K \cap (z_t + \Theta_i)) \geq 2^{-d}\mathcal{H}^{d-1}(K \cap \widetilde{H}(z_t))$$

if t is small enough (n is sufficiently large). The second estimate is based on (24) and a simple geometric estimate. The final estimate follows from (23).

We still have to prove that if $z_t \in K(n)|L$ but $z_t \notin [\{x_1, \ldots, x_n\} \cap C(\varepsilon, t)]|L$, then $\Xi_i \cap \{x_1, \ldots, x_n\} = \emptyset$ for some $i \in \{1, \ldots, 2^{d-1}\}$. Clearly, $z_t \in [a, b]$ for some $a \in [\{x_1, \ldots, x_n\} \cap C(\varepsilon, t)]|L$ and $b \in (K \setminus \operatorname{int} C(\varepsilon, t))|L$. Therefore there exists a hyperplane H containing $z_t + L^\perp$ determining the closed halfspaces H^+ and H^- such that $[\{x_1, \ldots, x_n\} \cap C(\varepsilon, t)] \subset \operatorname{int} H^+$ and $b \in H^-$. The halfspace H^- intersects $p + 2\sqrt{\varrho\|z_t\|}B^{d-1}$ in some point q. In fact, in $H^- \cap (p + \mathbb{R}^{d-1})$ there is a point having distance at most

$$u = \frac{\|p\| - \|z_t\|}{\omega\|z_t\| - \|z_t\|} \cdot d$$

from the line $\{s \cdot z_t : s \in \mathbb{R}\}$, where $d \leq 2\sqrt{\varrho\omega\|z_t\|}$ by (24). This shows that $u \leq 2\sqrt{\varrho\|z_t\|}$.

There exists some $i \in \{1, \ldots, 2^{d-1}\}$ such that $z_t + \Theta_i \subset H^-$, and hence also $q + \Theta_i \subset H^-$. This in turn yields $p + \lambda w_i \in H^-$, since $2\sqrt{\varrho\|z_t\|} < \frac{\lambda}{\sqrt{d-1}}$. The latter condition is equivalent to $\omega > 4^2(d-1)^2$, which follows from $\omega \geq \beta^{\frac{2}{d+1}} > (4(d-1))^2$. Therefore $\Xi_i \cap \{x_1, \ldots, x_n\} = \emptyset$, concluding the proof of (21).

Combining (19) and (21), we get

$$I(n) = \int_{\varepsilon^{\frac{d+1}{2}}/n}^{\alpha/n} \frac{n^{\frac{2}{d+1}} \sum_{i=0}^{k} \binom{n}{i} (\beta t)^i (1 - \beta t)^{n-i} \mathbb{P}(z_t \notin C(\varepsilon, t)(i)|L)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\, dt + O(\varepsilon). \quad (27)$$

Finally, it follows from (20) and $\lim_{t\to 0} A_t z_t = o^*$ that

$$I(n) = \int_{\varepsilon^{\frac{d+1}{2}}/n}^{\alpha/n} \frac{n^{\frac{2}{d+1}} \sum_{i=0}^{k} \binom{n}{i} (\beta t)^i (1 - \beta t)^{n-i} \mathbb{P}(o^* \notin C^*(i)|L)}{V_{d-1}(M_{B_\sigma}^L(z_t^{B_\sigma})|N^\perp)}\, dt + O(\varepsilon). \quad (28)$$

This last formula holds not only for K at $x(z)$, but for B_σ at o, as well. Thus we conclude Lemma 4.1.

□

5. The proof of Theorem 1.1

We start with a lemma which will help us to transfer an integral over an average of projections of a convex body to a boundary integral. For an introduction to curvature measures $C_j(K,\cdot)$, $j \in \{0,\dots,d-1\}$, of convex bodies K, we refer to [18].

LEMMA 5.1. *Let $K \subset \mathbb{R}^d$ be a convex body in which a ball rolls freely, let $f\colon \partial K \to [0,\infty)$ be nonnegative and measurable, and let $j \in \{1,\dots,d-1\}$. Then*

$$\frac{j\kappa_j}{d\kappa_d}\int_{\partial K} f(x)\sigma_{d-j}(x)\,\mathcal{H}^{d-1}(dx) = \int_{\mathcal{L}_j^d}\int_{\partial(K|L)} f(x(z))\,\mathcal{H}^{j-1}(dz)\,\nu_j(dL).$$

PROOF. Let $\beta \subset \partial K$ be measurable. Then (4.2.23) and (4.5.27) in [18] yield that

$$\frac{j\kappa_j}{d\kappa_d}C_{j-1}(K,\beta) = \int_{\mathcal{L}_j^d}\int_{\partial(K|L)} \mathbf{1}_{\beta|L}(z)\,\mathcal{H}^{j-1}(dz)\,\nu_j(dL). \tag{29}$$

By a result of Zalgaller [25] (see [18, p. 93, Note 1]), for almost all $L \in \mathcal{L}_j^d$, $\{x(z)\} = (z+L^\perp)\cap K$ for all $z \in \partial(K|L)$. In this case, $z \in \beta|L$ if and only if $x(z) \in \beta$. Therefore, we get

$$\frac{j\kappa_j}{d\kappa_d}\int_{\partial K}\mathbf{1}_\beta(x)\,C_{j-1}(K,dx) = \int_{\mathcal{L}_j^d}\int_{\partial(K|L)} \mathbf{1}_\beta(x(z))\,\mathcal{H}^{j-1}(dz)\,\nu_j(dL). \tag{30}$$

Monotone convergence implies that (30) holds for any nonnegative measurable function. So far the argument works for any convex body K. Since a ball rolls freely inside K, the curvature measure $C_{j-1}(K,\cdot)$ is absolutely continuous with respect to $\mathcal{H}^{d-1}$ with density $\sigma_{d-j}(x)$; see [10, Corollary 3.4]. This yields the assertion of the lemma.

□

To complete the argument leading to Theorem 1.1, let K be a convex body with a rolling ball. The set of points $x \in \partial K$ where K is differentiable in the generalized sense is denoted by Ξ, and we put $\Xi^c := \partial K \setminus \Xi$. By Aleksandrov's theorem, we have $\mathcal{H}^{d-1}(\Xi^c) = 0$. Since $C_{j-1}(K,\cdot)$ is absolutely continuous with respect to $\mathcal{H}^{d-1}$, (29) implies that

$$\int_{\mathcal{L}_j^d}\mathcal{H}^{j-1}((\Xi^c|L)\cap\partial(K|L))\,\nu_j(dL) = 0. \tag{31}$$

Starting from (12), first we apply (31), then we use the Lebesgue dominated convergence theorem to interchange the limit and $\int_{\mathcal{L}_j^d}\int_{\partial(K|L)}$, based on (14), and finally we apply Lemma 4.1. This gives

$$\begin{aligned}
&\lim_{n\to\infty} n^{\frac{2}{d+1}}[V_j(K)-\mathbb{E}V_j(K(n))]\\
&=\lim_{n\to\infty}\frac{\binom{d}{j}\kappa_d}{2\kappa_j\,\kappa_{d-j}}\int_{\mathcal{L}_j^d}\int_{(\Xi|L)\cap\partial(K|L)}\int_0^{n^{-1/2}}\frac{n^{\frac{2}{d+1}}V(K)\mathbb{P}(z_t\notin K(n)|L)}{V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^{\perp})}\,dt\,\times\\
&\qquad\times\mathcal{H}^{j-1}(dz)\,\nu_j(dL)\\
&=\frac{\binom{d}{j}\kappa_d}{2\kappa_j\,\kappa_{d-j}}\int_{\mathcal{L}_j^d}\int_{(\Xi|L)\cap\partial(K|L)}\lim_{n\to\infty}\int_0^{n^{-1/2}}\frac{n^{\frac{2}{d+1}}V(K)\mathbb{P}(z_t\notin K(n)|L)}{V_{d-1}(M_K^L(z_t)|N_{K|L}(z)^{\perp})}\,dt\,\times\\
&\qquad\times\mathcal{H}^{j-1}(dz)\,\nu_j(dL)\\
&=\frac{d\kappa_d}{j\kappa_j}c_{d,j}V(K)^{\frac{2}{d+1}}\int_{\mathcal{L}_j^d}\int_{(\Xi|L)\cap\partial(K|L)}\sigma_{d-1}(x(z))^{\frac{1}{d+1}}\,\mathcal{H}^{j-1}(dz)\,\nu_j(dL).
\end{aligned}$$

Now we apply again (31) and then Lemma 5.1 to get

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{n}{V(K)}\right)^{\frac{2}{d+1}}[V_j(K)-\mathbb{E}V_j(K(n))]\\
&\qquad=c_{d,j}\frac{d\kappa_d}{j\kappa_j}\int_{\mathcal{L}_j^d}\int_{\partial(K|L)}\sigma_{d-1}(x(z))^{\frac{1}{d+1}}\,\mathcal{H}^{j-1}(dz)\,\nu_j(dL)\\
&\qquad=c_{d,j}\int_{\partial K}\sigma_{d-1}(x)^{\frac{1}{d+1}}\sigma_{d-j}(x)\,\mathcal{H}^{d-1}(dx),
\end{aligned}$$

which completes the proof of Theorem 1.1.

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