Periodica Mathematica Hungarica Vol. 53 (1-2), 2006, pp. 1–11

COVERING THE CROSSPOLYTOPE BY EQUAL BALLS

Károly Böröczky Jr. (Budapest), Ildikó Fábián (Budapest) and Gergely Wintsche (Budapest)

[1] Alfréd Rényi Institute of Mathematics, P.O. Box 127, 1364 Budapest, Hungary, E-mail: carlos@renyi.hu

[2] Alfréd Rényi Institute of Mathematics, P.O. Box 127, 1364 Budapest, Hungary, E-mail: fabil@renyi.hu

[3] Faculty of Science, Loránd Eötvös University, Pázmány P. sétány 1/c, 1117 Budapest, Hungary, E-mail: wgerg@ludens.elte.hu

(Received: September 6, 2005; Accepted: January 24, 2006)

Abstract

We determine the minimal radius of $n = 2$, d or $2d$ congruent balls, which cover the d-dimensional crosspolytope.

1. Introduction

Packings of n equal balls into a given shape in E^d have been thoroughly investigated (see P. Brass, W. Moser and J. Pach [4] and K. Böröczky, Jr. [2]), say packings inside larger balls, cubes, and regular tetrahedra and crosspolytope. Here we on ly note that optimal packings of n equal balls into a regular crosspolytope have been determined when $n \leq 2d+1$ (see K. Bezdek [1] and G. Golser [6] for $d = 3$, and K. Böröczky, Jr. and G. Wintsche [3] for all d).

Concerning coverings by n equal balls, the case when a larger ball covered is discussed by C. A. Rogers [10], whose bounds are improved by J-L. Verger-Gaugry [12]. In case of covering the three-dimensional unit cube, G. Kuperberg and W. Kuperberg [8] found the optimal solution if $n = 2, 3, 4, 8$. In addition [8] determined the optimal covering of the unit cube by four balls in E^4. Finally if $d \geq 4$ and $n = 2$ then the solution is due to N. P. Dolbilin and P. I. Sharygin (personal communication). For related results in the planar case, consult P. Brass, W. Moser and J. Pach [4] and K. Böröczky, Jr. [3]. For example, the study of thin coverings

Mathematics subject classification number: 52C17.
Key words and phrases: covering by equal balls, finite coverings.

0031-5303/2006/\$20.00

Akadémiai Kiadó, Budapest
Springer, Dordrecht

of a square by n equal circular discs was initiated by T. Tarnai and Zs. Gáspár [11]. If $n \le 5$ or $n = 7$ then A. Heppes and J. B. M. Melissen [7] solved the problem. For many other values of n, K. J. Nurmela [9] provides thin coverings.

For a k-dimensional simplex S, $k \ge 1$, we write $\sigma(S)$ to denote its centroid; namely, the arithmetic mean of the vertices of S. In particular if S is a segment then $\sigma(S)$ is the midpoint. Given an orthonormal basis $v_1, \dots, v_d$ of E^d, we write O^d to denote the d-dimensional crosspolytope with vertices $\pm v_1, \dots \pm v_d$. Moreover let R_n^d denote the minimal radius of n congruent balls that cover O^d. For the sake of simplicity, the convex hull of $\theta_1, \dots, \theta_k$ is denoted by $[\theta_1, \dots, \theta_k]$ where each θ_i is either a point or a subset of E^d.

In this paper we determine R_n^d for $n = 2, d, 2d$. An interesting feature of the solution for $n = 2$ and $n = d$ that the cases $d = 3$ and $d \ge 4$ are substantially different.

Theorem 1.

(i) $R_2^3 = \frac{\sqrt{11}}{4}$, *and the centres of the balls in an optimal covering are* $\pm\frac{3}{4}\,\sigma(F)$ *for some face* F *of* O^3.

(ii) $R_2^d = \sqrt{1 - \frac{1}{d}}$ *for* $d \ge 4$, *and the centres of the balls in an optimal covering are the centroids of two opposite facets of* O^d.

Theorem 2.

$$R_d^d = \begin{cases} \dfrac{\sqrt{5}}{3}, & \text{if } d = 3; \\[2ex] \sqrt{\dfrac{11}{20}}, & \text{if } d \ge 4. \end{cases}$$

The optimal coverings for Theorem 2 are described below.

Let us describe the optimal coverings for Theorem 2: In the three-dimensional case, the optimal coverings are in bijective correspondence with any family e_1, e_2, e_3 of pairwise non-intersecting edges of O^3; more precisely, the centres of the balls are $\frac{2}{3}\,\sigma(e_i)$, $i = 1, 2, 3$. If $d \ge 4$ then the optimal coverings by d equal balls are in a bijective correspondence with any family $e_1, \dots, e_d$ of edges of O^d such that no two edges intersect, no two are parallel, and no three are edges of an octahedron. Given any such family $e_1, \dots, e_d$, the centre z_i of the ith ball in the optimal covering can be determined as follows: Writing v, w, x, y to denote the vertices of O^d such that $e_i = [v, w]$, and the edges $[-v, x]$ and $[-w, y]$ are among $e_1, \dots, e_d$, we have

$$z_i = \frac{7}{20}\,v + \frac{7}{20}\,w + \frac{1}{20}\,x + \frac{1}{20}\,y.$$

In particular if $d \le 7$ then the optimal coverings of O^d by d equal balls are unique up to congruency.

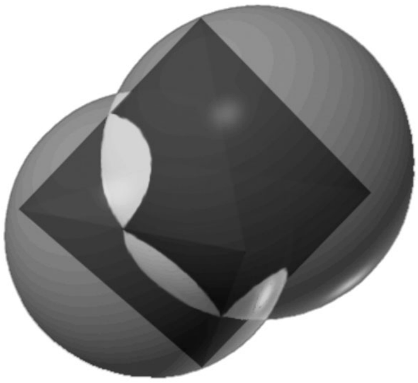

FIGURE 1. *The case of two balls*

THEOREM 3. *$R_d^{2d} = 1/2$, and the centres of the balls in an optim al covering are $\pm v_1/2, \ldots, \pm v_d/2$.*

2. The case of two balls

PROOF OF THEOREM 1. Let us consider the three-dimensional case first. Readily two balls of radius $\sqrt{11}/4$ and of centre $\pm(1/4, 1/4, 1/4)$ cover O^3. Next we assume that two balls with common radius $R \leq \sqrt{11}/4$ cover O^3. Neither of the balls cover four vertices of O^3 because that ball would cover two opposite vertices with distance 2. Thus the balls cover three-three neighbouring vertices of O^3, for instance v_1, v_2, v_3 and $-v_1, -v_2, -v_3$. Now one of the balls contains the midpoint m of the edge $[v_1, -v_2]$, say the one that contains v_1, v_2, v_3. Since the tetrahedron $[m, v_1, v_2, v_3]$ is of circumradius $\sqrt{11}/4$, and of circumcentre $(1/4, 1/4, 1/4)$, we deduce Theorem 1 (i).

If $d \geq 4$ then $\sqrt{1 - 1/d}$ is the common circumradius of the facets of O^d, and two balls of radius $\sqrt{1 - 1/d}$ centred at the centroids of two opposite facets of O^d cover O^d. On the other hand, if two balls of common radius $R \leq \sqrt{1 - 1/d}$ cover O^d then we may assume that one of the balls contains $v_1, \ldots, v_d$, and the other contains $-v_1, \ldots, -v_d$. In turn, we conclude Theorem 1. $\square$

3. The case of three balls in E^3

First we make some observations concerning an edge e of O^3 where we denote the endpoints of e by v and w. More precisely, the following points lie on the boundary of the ball of centre $\frac{2}{3}\,\sigma(e) = \frac{1}{3}v + \frac{1}{3}w$ and of radius $\frac{\sqrt{5}}{3}$: v, w ; the point $\frac{2}{3}\,v + \frac{1}{3}(-w)$ of the edge $[v, -w]$; the point $\frac{1}{3}\,v + \frac{2}{3}\,z$ of the edge $[v, z]$ where

FIGURE 2. *The case of three balls*

$z \neq \pm v, \pm w$ is a vertex of O^3; and finally $\sigma(F)$ for any face F of O^3 where F contains v and does not contain e.

LEMMA 1. *If e_1, e_2, e_3 are pairwise non-intersecting edges of O^3 then the balls of radius $\frac{\sqrt{5}}{3}$ and of centres $\frac{2}{3}\,\sigma(e_i)$, $i = 1, 2, 3$, cover O^3.*

PROOF OF LEMMA 1. We write B_i to denote the ball of centre $\frac{2}{3}\,\sigma(e_i)$ and of radius $\frac{\sqrt{5}}{3}$. Since each B_i contains the origin, it is sufficient to show that B_1, B_2, B_3 cover any face F of O^3. The argument below is based on the observations preceding Lemma 1.

If some e_i is a side of F then the opposite vertex v of F is the endpoint of some e_j. Now simple argument shows that B_j contains $\frac{1}{3}(F - v) + v$, and B_i contains the rest of F (that is a trapezoid).

Therefore we assume that the vertices w_1, w_2, w_3 of F are endpoints of e_1, e_2, e_3, respectively. We may also assume that $[e_i, w_{i+1}]$ is a face of O^3 for $i = 1, 2, 3$ and $w_4 = w_1$. Now B_i covers the trapezoid whose vertices are $\sigma(F)$, $\frac{1}{3}\,w_i + \frac{2}{3}\,w_{i+1}$, w_i and $\frac{2}{3}\,w_i + \frac{1}{3}\,w_{i-1}$ for $i = 1, 2, 3$ and $w_0 = w_3$. Since these trapezoids cover F, we conclude Lemma 1. □

LEMMA 2. *Three balls with radii smaller than $\frac{\sqrt{5}}{3}$ cannot cover O^3.*

PROOF OF LEMMA 2. We suppose that three balls B_1, B_2, B_3 with radii smaller than $\sqrt{5}/3$ cover O^3, and seek a contradiction. None of the balls covers two opposite vertices of O^3, or some three vertices of O^3 because the distance of opposite vertices is 2, and the circumradius of any face is $\sqrt{2/3} > \sqrt{5}/3$. It follows that every ball covers two-two neighbouring vertices; say, B_1 contains $v_1, -v_2$, B_2 contains $v_2, -v_3$ and B_3 contains $v_3, -v_1$. Since both v_1 and v_2 are of distance at least $2\sqrt{5}/3$ from $x = (-1/3, -2/3, 0)$, and both $-v_2$ and $-v_3$ are of distance at least

$2\sqrt{5}/3$ from $y = (0, 2/3, 1/3)$, neither B_1 nor B_2 contains x and y. Therefore B_3 covers the tetrahedron $[v_3, -v_1, x, y]$. However the circumradius of this tetrahedron is $\sqrt{5}/3$, which is a contradiction. In turn we conclude Lemma 2. □

Lemmae 1 and 2 prove Theorem 2 if $d = 3$, where the uniqueness of the optimal arrangement follows from the proof of Lemma 2. □

4. The case of d balls in E^d, $d \geq 4$

First we construct a covering of O^d by d balls of radius $\sqrt{\frac{11}{20}}$. We write $v_{i+kd} = v_i$ for any integer k, and define

$$C_i := \frac{1}{20}v_{i-1} + \frac{7}{20}v_i - \frac{7}{20}v_{i+1} - \frac{1}{20}v_{i+2} \quad \text{for } i = 1 \dots d;$$

$$\widetilde{B}_i := \text{ the ball with centre } C_i \text{ and with rad ius } \sqrt{\frac{11}{20}} \quad \text{for } i = 1 \dots d.$$

In order to show that $\widetilde{B}_1, \dots, \widetilde{B}_d$ cover O^d, we define $C(F) \in \operatorname{relint} F$ for every face F of O^d. If F is a vertex then readily $C(F) = F$.

1. If $\dim F = 1$ then

$$C(F) := \begin{cases} \frac{7}{10}v_i + \frac{3}{10}v_{i+1}, & \text{if } F = [v_i, v_{i+1}]; \\ \frac{3}{10}(-v_i) + \frac{7}{10}(-v_{i+1}), & \text{if } F = [-v_i, -v_{i+1}]; \\ \sigma(F) & \text{otherwise.} \end{cases}$$

2. If $\dim F = 2$ then

$$C(F) := \begin{cases} \frac{4}{10}v_i + \frac{3}{10}v_{i+1} + \frac{3}{10}v_k, & \text{if } F = [v_i, v_{i+1}, v_k], \\ & \quad k \not\equiv i-1,\ i,\ i+1 \mod d; \\ \frac{4}{10}v_i + \frac{3}{10}v_{i+1} + \frac{3}{10}(-v_k), & \text{if } F = [v_i, v_{i+1}, -v_k], \\ & \quad k \not\equiv i,\ i+1 \mod d; \\ \frac{3}{10}(-v_k) + \frac{3}{10}(-v_i) + \frac{4}{10}(-v_{i+1}), & \text{if } F = [-v_k, -v_i, -v_{i+1}], \\ & \quad k \not\equiv i,\ i+1,\ i+2 \mod d; \\ \frac{3}{10}(-v_i) + \frac{4}{10}(-v_{i+1}) + \frac{3}{10}v_k, & \text{if } F = [-v_i, -v_{i+1}, v_k], \\ & \quad k \not\equiv i,\ i+1 \mod d; \\ \sigma(F) & \text{otherwise.} \end{cases}$$

3. If $\dim F \geq 3$, then $C(F) := \sigma(F)$.

$C(F)$ is defined in a way that the following statement holds:

Proposition. *If v is a vertex of some face F of O^d and $v \in \widetilde{B}_i$ then $C(F) \in \widetilde{B}_i$.*

Proof of the Proposition. We may assume that $i = 2$. Now the centre of $\widetilde{B}_2$ is $C_2 = \left(\frac{1}{20}, \frac{7}{20}, -\frac{7}{20}, -\frac{1}{20}, 0, \ldots, 0\right)$, and $\widetilde{B}_2$ covers the edge $[v_2, -v_3]$ of O^d. Let Ω be the orthogonal linear transformation satisfying $\Omega(v_j) = -v_{5-j}$. Then $\Omega(C_2) = C_2$, and for any face F of O^d that contains $-v_3$, $\Omega(F)$ is a face that contains v_2. Therefore it is sufficient to check the distance of $C(F)$ and C_2 for any face F containing v_2:

1. If $\dim F = 1$, then

$$\|C([v_2, v_3]) - C_2\|^2 = \frac{11}{20};$$

$$\|C([v_2, \pm v_i]) - C_2\|^2 \leq \|(0, 1/2, 0, 1/2, 0, \ldots, 0) - C_2\|^2 = \frac{9}{20} \quad \text{for } i \neq 3.$$

2. Let $\dim F = 2$. If $F = [v_2, v_3, v_k]$ for $k \not\equiv 1, 2, 3 \bmod d$ then

$$\|C(F) - C_2\|^2 \leq \|(0, 4/10, 3/10, 3/10, 0, \ldots, 0) - C_2\|^2 = \frac{11}{20}.$$

If $F = [v_1, v_2, v_k]$ for $k \not\equiv 0, 1, 2 \bmod d$ then

$$\|C(F) - C_2\|^2 \leq \|(4/10, 3/10, 3/10, 0, \ldots, 0) - C_2\|^2 = \frac{11}{20}.$$

If $F = [v_i, v_{i+1}, v_2]$ for $i \not\equiv 1, 2, 3 \bmod d$ then

$$\|C(F) - C_2\|^2 \leq \|(0, 3/10, 0, 4/10, 3/10, 0, \ldots, 0) - C_2\|^2 = \frac{8.4}{20}.$$

If $F = [v_i, v_{i+1}, -v_k]$ for $i = 1, 2$ and $k \not\equiv i, i+1 \bmod d$ then

$$\|C(F) - C_2\|^2 \leq \|(-3/10, 4/10, 3/10, 0, \ldots, 0) - C_2\|^2 = \frac{11}{20}.$$

If $F = [-v_i, -v_{i+1}, v_2]$ for $i \not\equiv 1, 2 \bmod d$ then

$$\|C(F) - C_2\|^2 \leq \|(-4/10, 3/10, 0, \ldots, 0, -3/10) - C_2\|^2 = \frac{8.4}{20}.$$

Finally if $C(F) = \sigma(F)$ then

$$\|C(F) - C_2\|^2 \leq \|(1/3, 1/3, 1/3, 0, 0, 0, \ldots, 0) - C_2\|^2 = \frac{11}{20}.$$

3. If $\dim F \geq 3$ then

$$\|C(F) - C_2\|^2 \leq \|(-1/4, -1/4, 1/4, 1/4, 0, \ldots, 0) - C_2\|^2 = \frac{11}{20}.$$

In turn, we conlude the Proposition. □

To any sequence $F_0 \subset F_1 \subset \ldots \subset F_d$ of faces of O^d with $\dim F_i = i$, we assign the simplex $[C(F_0), C(F_1), \ldots, C(F_d)]$. The family of all these simplices tile O^d because $C(F) \in \operatorname{relint} F$ for every face F of O^d. Therefore the Proposition yields

LEMMA 3. *For $d \geq 4$, some d balls of radius $\sqrt{\frac{11}{20}}$ cover O^d, hence, $R_d^d \leq \sqrt{\frac{11}{20}}$.*

Next we show that the bound of Lemma 3 on R_d^d is optimal.

LEMMA 4. *If the balls $B_1, \ldots, B_d$ with common radius R cover the edges of O^d then $R \geq \sqrt{\frac{11}{20}}$.*

Moreover if $R = \sqrt{\frac{11}{20}}$ then there exist edges $e_1, \ldots, e_d$ of O^d such that no two edges intersect, no two are parallel, and no three are edges of an octahedron, and the centre z_i of B_i can be determined as follows: Writing v, w, x, y to denote the vertices of O^d such that $e_i = [v, w]$, and the edges $[-v, x]$ and $[-w, y]$ are among $e_1, \ldots, e_d$, we have

$$z_i = \frac{7}{20}v + \frac{7}{20}w + \frac{1}{20}x + \frac{1}{20}y.$$

PROOF OF LEMMA 4. We assume that $R \leq \sqrt{\frac{11}{20}}$, and prove Lemma 4 through several steps.

(1) *There exist pairwise disjoint edges $e_1 \ldots, e_d$ such that $e_i \subset B_i$ for $i = 1, \ldots, d$.*

The reason is that the distance of any two opposite vertices of O^d is 2, and the circumradius of any two–face is $\sqrt{\frac{2}{3}} > \sqrt{\frac{11}{20}}$.

(2) *Let v, w, x and y be pairwise different vertices of O^d such that $[v, w]$ is an edge, and $x, y \neq -v, -w$. Writing $z = \frac{1}{3}v + \frac{2}{3}w$, the circumradii of the triangles $[z, v, -w]$, $[z, -v, w]$, $[z, -w, x]$ and $[z, x, y]$ are larger than $\sqrt{\frac{11}{20}}$.*

(3) *$e_i \neq -e_j$ for any $i \neq j$.*

Otherwise, assuming that $e_i = [v_1, -v_2]$ and $e_j = [-v_1, v_2]$, the point $z = \frac{2}{3}v_1 + \frac{1}{3}v_2$ is not covered by any B_k according to (2).

(4) *Let G be the graph on $e_1, \ldots, e_d$ such that $\{e_i, e_j\}$ is and edge of G if and only if $e_i \cap -e_j \neq \emptyset$. It follows from (1) and (3) that G is the union of disjoint cycles.*

Assume that $e_i \cap -e_j \neq \emptyset$. If $e_i = [v, w]$ and $e_j = [-v, x]$ for some vertices v, w, x of O^d then we call $[w, -v]$ and $[v, x]$ bridge edges between e_i and e_j.

(5) *If f is a bridge between e_i and e_j then $f \subset B_i \cup B_j$.*

Let $e_i = [v, w]$, $e_j = [-v, x]$ and $f = [w, -v]$ for some vertices v, w, x of O^d. Then the point $z = \frac{2}{3}w + \frac{1}{3}(-v)$ satisfies that $z \notin B_k$ for $k \neq j$ by (2), hence $[z, -v] \subset B_j$. Moreover for any $p \in [z, w]$, we have that $\| -w - p\| > 2\sqrt{\frac{11}{20}}$, and $[p, u, y]$ contains a congruent copy of $[z, u, y]$ for any two vertices $x, y \neq \pm v, \pm w$ of O^d. We deduce $[z, w] \cap B_k = \emptyset$ for $k \neq i, j$ by (2), therefore $[z, w] \subset B_i \cup B_j$.

(6) *G contains no three-cycle.*

The argument resembles the proof of Lemma 2. Indirectly we suppose that G has a three-cycle, say $e_1 = [v_1, -v_2]$, $e_2 = [v_2, -v_3]$ and $e_3 = [v_3, -v_1]$. We define $z = \frac{2}{3}v_1 + \frac{1}{3}v_2$ and $z' = -\frac{2}{3}v_1 - \frac{1}{3}v_3$. Then $z, z' \notin B_1$, and $z, z' \notin B_k$ for $k \geq 3$ by (2), hence $z, z', v_2, -v_3$ are contained in B_2. Since the tetrahedron $[z, z', v_2, -v_3]$ is of circumradius $\frac{\sqrt{5}}{3} > \sqrt{\frac{11}{20}}$, we have arrived at a contradiction. In turn we conclude (6).

(7) *Given some $i = 1, \ldots, d$, let $f_{i1}, f_{i2}, f_{i3}, f_{i4}$ be the bridge edges meeting e_i. For any ball B of radius R that contains e_i, we define*

$$\Phi_i(B) = \sum_{j=1}^{4} |B \cap f_{ij}|$$

where $|\cdot|$ stands for the length of a segment. Then

$$\Phi_i(B) \leq 2\sqrt{2}.$$

Moreover if $\Phi_i(B) = 2\sqrt{2}$ then $R = \sqrt{\frac{11}{20}}$, and writing v, w, x, y to denote the vertices of O^d such that $e_i = [v, w]$, and $[-v, x]$ and $[-w, y]$ are among $e_1, \ldots, e_d$, we have

$$z_i = \frac{7}{20}\,v + \frac{7}{20}\,w + \frac{1}{20}\,x + \frac{1}{20}\,y.$$

According to (6), we may assume that $e_i = [v_2, v_3]$, and the edges intersecting $-e_i$ among $e_1, \ldots, e_d$ are $[v_1, -v_2]$ and $[v_4, -v_3]$. In particular the bridge edges are $f_{i1} = [v_2, v_1]$, $f_{i2} = [v_2, -v_3]$, $f_{i3} = [v_3, -v_2]$ and $f_{i4} = [v_3, v_4]$. We define the orthogonal linear map $\varphi : E^d \to E^d$ by

$$\varphi(x_1, x_2, x_3, x_4, x_5, \ldots, x_d) = (x_4, x_3, x_2, x_1, -x_5, \ldots, -x_d).$$

Then $\varphi(e_i) = e_i$ and $\varphi(f_{ij}) = f_{i(5-j)}$ for $j = 1, \ldots, 4$. Let L be the linear two–space spanned by $v_1 + v_4$ and $v_2 + v_3$, hence φ is a reflection through L. For any object X in E^d, we write $X' = \frac{1}{2}(X + \varphi(X))$. In particular if X is a point then X' is the orthogonal projection of X into L.

We may assume that $R = \sqrt{\frac{11}{20}}$, and $\Phi_i(B)$ is maximal among the balls of radius R containing e_i. We write w to denote the centre of B, hence the ball B' is of radius R and of centre w'. Moreover B' contains e_i, and for any $j = 1, \ldots, 4$,

$$|B' \cap f_{ij}| \geq \frac{1}{2}\left(\left|\varphi\left(B \cap f_{i(5-j)}\right)\right| + |B \cap f_{ij}|\right).$$

In particular $\Phi_i(B')$ is also maximal. Moreover w' is of the same distance from v_2 and v_3, and is of the form

$$w' = (\alpha, \beta, \beta, \alpha, 0, \dots, 0)$$

for some real α and β. Writing p and q to denote the points where $\partial B'$ intersects the relative interior of the bridge edges f_{i1} and f_{i2}, we have

$$\Phi_i(B') = 2(\|p - v_2\| + \|q - v_2\|).$$

First we assume that v_2 (hence also v_3) lies in $\mathrm{int}B'$, hence

$$\|w' - v_2'\| < \sqrt{\frac{1}{20}}.$$

Since $\Phi_i(B')$ is maximal, there exists no w'' in a small neighbourhood of w' in L such that $\|p - w''\| < \sqrt{\frac{11}{20}}$ and $\|q - w''\| < \sqrt{\frac{11}{20}}$. Therefore $w' \in [p', q']$, and

$$\|w' - q'\| = \sqrt{\frac{1}{20}}.$$

It follows that the angle $\angle p'q'v_2' = \angle w'q'v_2'$ is acute. Now readily $\|q' - v_2'\| < \frac{1}{\sqrt{5}}$, hence $\|p' - v_2'\| < \frac{\sqrt{2}}{\sqrt{5}}$ as $\angle p'v_2'q' = \frac{\pi}{4}$. We conclude that

$$\Phi_i(B') = 2\left(\sqrt{2}\|p' - v_2'\| + \|q' - v_2'\|\right) < \frac{6}{\sqrt{5}} < 2\sqrt{2}.$$

Therefore let v_2 and v_3 lie on $\partial B'$, hence

$$\|w' - v_2\| = \|w' - v_3\| = 2\alpha^2 + (1-\beta)^2 + \beta^2 = \sqrt{\frac{11}{20}},$$

which can be written in the form

$$\alpha^2 + 9\left(\frac{1}{6} - \frac{1}{3}\beta\right)^2 = \frac{1}{40}.$$

Since the orthogonal projection of w' into the line passing through v_2 and v_1 is the midpoint of the segment $[v_2, p]$, we deduce that

$$\Phi_i(B') = 4\left(\left\langle w' - v_2, \frac{v_1 - v_2}{\sqrt{2}}\right\rangle + \left\langle w' - v_2, \frac{-v_3 - v_2}{\sqrt{2}}\right\rangle\right) =$$

$$= 2\sqrt{2}(\alpha - 3\beta + 2) = 2\sqrt{2}\left(\alpha + 9\left(\frac{1}{6} - \frac{1}{3}\beta\right) + \frac{1}{2}\right).$$

The maximality of $\Phi_i(B')$ and the equality case of the inequality between quadratic and arithmetic mean yield that $\alpha = \frac{1}{6} - \frac{1}{3}\beta$. Therefore $\alpha = \frac{1}{20}$, $\beta = \frac{7}{20}$ and $\Phi_i(B') = 2\sqrt{2}$. This proves (7) without the characterization of the equality case.

Let us assume that $\Phi_i(B) = 2\sqrt{2}$. The argument above shows that $R = \sqrt{\frac{11}{20}}$, and

$$w' = \left(\sqrt{1/20}, \sqrt{7/20}, \sqrt{7/20}, \sqrt{1/20}, 0, \ldots, 0\right),$$

hence w is of the form

$$w = \left(\sqrt{1/20} + a, \sqrt{7/20} + b, \sqrt{7/20} - b, \sqrt{1/20} - a, c_5, \ldots, c_d\right)$$

for some real $a, b, c_5, \ldots, c_d$. Since

$$\|w - v_2\|^2 + \|w - v_3\|^2 = \|w' - v_2\|^2 + \|w' - v_3\|^2 + 4a^2 + 4b^2 + 2c_5^2 + \ldots + 2c_d^2,$$

we conclude that $w = w'$.

(8) *Completing the proof of Lemma 4.*

First we suppose that $R < \sqrt{\frac{11}{20}}$. Given a bridge edge f between e_i and e_j with $e_i \cap -e_j \neq \emptyset$, let $\psi(f)$ be the sum of the lengths of the intersections of f with B_i and B_j. Then the sum of $\psi(f)$ over all the $2d$ bridge edges f is at least $d \cdot 2\sqrt{2}$ according to (5), and less than $d \cdot 2\sqrt{2}$ according to (7). This contradiction yields that $R \geq \sqrt{\frac{11}{20}}$. The characterization of the possible covering for $R = \sqrt{\frac{11}{20}}$ follows by (7). □

Now Theorem 2 for $d \geq 4$ readily follows from Lemma 3 and Lemma 4. □

5. The case of 2d balls

Proof of Theorem 3. First we observe that if F is a face of O^d, and v is a vertex of F then the distance of $\frac{1}{2}v$ and $\sigma(F)$ is $1/2$. Therefore the balls with centres $\pm\frac{1}{2}v_i$, $i = 1, \ldots, d$, and radius $1/2$ cover O^d.

Next let $2d$ balls of common radius $R \leq 1/2$ cover O^d. Then any ball contains exactly one vertex of O^d, and one of the balls contains the origin. Thus $R = 1/2$, and considering the centroids of the faces yields that the centres are $\pm\frac{1}{2}v_i$, $i = 1, \ldots, d$. □

Acknowledgment. Research of K. Böröczky Jr. is supported in part by OTKA grants 43520 and 42769.

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