Polytopal approximation of smooth convex bodies

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How well a polytope of restricted complexity can approximate a smooth convex body in $\mathbb{R}^d$? This natural question has attracted the attention of mathematicians of various background since the middle of the 20th century. In this extended abstract, polytopes are always inscribed, and restricted complexity mostly means restricting the number of vertices of the polytope. In addition distance from the smooth convex body is mostly measured by affine invariant notions like the Banach-Mazur distance or the volume difference.

Concerning notation, we write B^d to denote the Euclidean unit ball of $\mathbb{R}^d$. We recall that the Banach-Mazur distance $\delta_{\rm BM}(K,M)$ of the convex bodies K and M in $\mathbb{R}^d$ is the minimal $\lambda \geq 1$ such that $K-x \subset \Phi(M-y) \subset \lambda(K-x)$ for some $\Phi \in {\rm GL}(d)$ and $x,y \in \mathbb{R}^d$. In the case if K and M are o-symmetric then $x=y=o$ can be assumed.

Let me start with A.M. Macbeath's classical result in [29]. It says that ellipsoids are worst approximable among convex bodies by inscribed polytopes in terms of volume. For any convex body K in $\mathbb{R}^d$ and $n \geq d+1$, let $V(K,n)$ be the maximal volume of polytopes with n vertices inscribed into K. According to [29], if E is an ellipsoid in $\mathbb{R}^d$ with $V(E)=V(K)$ then

$$V(K,n) \leq V(E,n). \tag{1}$$

From now on, problems of approximation by polytopes of "low complexity" and of "high complexity" are discussed separately. In both cases I only present very few results which I feel typical.

1 Polytopes of few vertices

Here the main question is whether an inscribed polytope can reasonably well approximate the convex body at all. As (1) suggests, the convex body is the ball (ellipsoid) in these problems. Few vertices means that the number of vertices is at most exponential in the dimension d for Banach-Mazur distance, and at most $d^{d/2}$ for volume approximation.

Let $P_n \subset B^d$ be a polytope of n vertices. In high dimensions Bárány, Füredi [4], Gluskin [17] and Carl, Pajor [10] obtained independently the following result (all the three papers appeared in 1988!): If $n \geq 2d$ then

$$\sqrt[d]{\frac{V(P_n)}{V(B^d)}} \leq \sqrt{\frac{c\ln\frac{n}{d}}{d}} \tag{2}$$

for some absolute constant $c>0$. We note that if n is at most exponential in d then the estimate of (2) is optimal. Bárány, Füredi [4] also show that to get a polytope P_n with $V(P_n) > \frac{1}{2}\,V(B^d)$, one needs approximately $d^{d/2}$ vertices.

If $d+1 \leq n \leq 2d$ then the estimate $\sqrt[d]{V(P_n)/V(B^d)} \leq \sqrt{c/d}$ resulting from (2) is optimal, as it is shown by the example of the inscribed regular simplex. If $n=d+1$ then Steiner symmetrization (see Steiner [31]) shows that the regular simplex is optimal.

Turning to the Banach-Mazur distance, (2) yields that if $n \geq 2d$ then

$$\delta_{\mathrm{BM}}(P_n, B^d) \geq \sqrt{\frac{d}{c \ln \frac{n}{d}}} \tag{3}$$

This estimate is optimal if n is at most exponential in d. In particular if $\delta_{\mathrm{BM}}(P_n, B^d) \leq 2$ then n is at least exponential in d, and on the other hand this property can be achieved using exponentially many vertices.

Related Problems:

1. I conjecture that (3) also holds for any n with $d + 1 \leq n \leq 2d$. More precisely if $n = d + k$ for $k = 1, \ldots, d$ then

   $$\delta_{\mathrm{BM}}(P_n, B^d) \geq \frac{\tilde{c}d}{\sqrt{k}}$$

   for some absolute constant $\tilde{c} > 0$. This estimate would be optimal as the following (conjecturally optimal) polytopes exhibit. Take the convex hull of k pairwise orthogonal regular simplices of circumradius one and of dimensions either $\lceil \frac{d}{k} \rceil$ or $\lfloor \frac{d}{k} \rfloor$.

2. It is a long standing open problem whether the mean width of P_{d+1} is maximal for the inscribed regular simplex (see Gritzmann, Klee [18] for history, especially for a list of wrong proofs that have been published).

The polytopes conjectured to be extremal in the first problem are known to be extremal in the following cases. If $k = 1$ or $k = 2$ then Steiner symmetrization (see Steiner [31]) yields the results in any dimension (see Böröczky, Jr., Wintsche [9]). In addition the optimality the cross polytope ($k = d$) is known if $d = 3$ (see Fejes Tóth [14]) or $d = 4$ (see Dalla, Larman, Mani-Levitska, Zong [12]).

The second problem has been solved by Linhart [26] if $d = 3$. His argument is based on the spherical Moment Theorem of Fejes Tóth [14]. Actually the spherical Moment Theorem of Fejes Tóth also yields the following results in the three dimensional case. If $n = 6$ or $n = 12$ then the optimal P_n with respect to volume approximation and the Banach-Mazur distance is the regular octahedron and icosahedron, respectively.

It follows by (2) that the volume of a convex body cannot be well approximated by polytopes of polynomial many vertices in d. However there exists algorithm polynomial in d that estimates well the volume with high probability according to Dyer, Frieze, Kannan [13]. The high degree in [13] has been brought down in a series of papers, culminating in an essentially degree four bound of Lovász, Vempala [27]. In addition A.R. Barron [2] and G. Cheang, A.R. Barron [11] (see also Artstein-Avidan, Friedland, Milman [1]) construct a non-convex body X with linear complexity in d such that $\frac{1}{2} B^d \subset X \subset B^d$.

2 Best approximation with many vertices

Let K be a convex body in $\mathbb{R}^d$. We discuss approximation of K by polytopes of say n vertices where n tends to infinity. For much broader surveys on the subject, consult P.M. Gruber [22] and [25].

We note that the Gauss-Kronecker curvature $\kappa(x)$ can be defined at most points $x \in \partial K$, hence the affine surface area

$$A(K) = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, dx$$

is well-defined (see Schütt, Werner [32]). In addition a flag of a polytope P in $\mathbb{R}^d$ is a sequence $F_0 \subset \ldots \subset F_{d-1}$ where F_i is an i-face of P. Using random polytopes, if $A(K) > 0$ and n is large then Bárány [3] proved the existence of a polytope $P \subset K$ with at most n flags such that

$$V(K \backslash P) \leq \gamma(d) A(K)^{\frac{d+1}{d-1}} n^{\frac{-2}{d-1}}$$

where $\gamma(d) > 0$ depends only on d. This estimate is optimal up to the value of $\gamma(d)$ according to Böröczky, Jr. [6]. If ∂K is C^2 and $P_n \subset K$ is a polytope with n vertices that has maximal volume then we even have the asymptotic formula

$$V(K \backslash P) \sim \tfrac{\mathrm{del}_{d-1}}{2}\, A(K)^{\frac{d+1}{d-1}} n^{\frac{-2}{d-1}} \tag{4}$$

as n tends to infinity. Here $\mathrm{del}_2 = \frac{1}{2\sqrt{3}}$ (see Gruber [19]), and $\mathrm{del}_d \sim \frac{d}{2\pi e}$ as d tends to infinity (see P. Mankiewicz, C. Schütt [30]). The formula (4) was conjectured by Fejes Tóth [15] if $d = 3$, and proved by Gruber [21] if $\kappa(x)$ is positive for any d. The restriction $\kappa(x) > 0$ was removed by Böröczky, Jr. [5]. Generalizing results in Glasauer, Gruber [16], [5] also showed that the vertices of P_n are uniformly distributed on ∂K with respect to the density function $\kappa(x)^{\frac{1}{d+1}}$. In the three dimensional case, following Gruber [23], Böröczky, Tick, Wintsche [8] proved that the typical faces of P_n are asymptotically regular triangles in a suitable sense. Now if ∂C is C^3 with positive curvature then Böröczky, Jr. [7] even estimated the error term in (4), which estimate was substantially improved by Gruber [24].

Next let ∂K be C^2, and let P_n be a polytope with n vertices such that $\delta_{\mathrm{BM}}(K, P_n)$ is minimal. Combining ideas in Gruber [20] and Böröczky, Jr. [5], one can prove the following. Writing u_x to denote the exterior unit normal at $x \in \partial K$, K can be translated in a way such that $o \in \mathrm{int} K$, and

$$\delta_{\mathrm{BM}}(K, P_n) - 1 \sim \frac{1}{2} \left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{2}{d-1}} \left(\int_{\partial K} \frac{\kappa(x)^{\frac{1}{2}}}{\langle x, u_x\rangle^{\frac{d-1}{2}}}\, dx\right)^{\frac{2}{d-1}} n^{\frac{-2}{d-1}} \tag{5}$$

as n tends to infinity where κ_m is the volume of the unit m-ball, and ϑ_m is the minimal density of coverings of $\mathbb{R}^m$ by unit balls. Here the integral in the parentheses is the so called centro-affine surface area.

Related Problems:

1. Prove (4) or (5) if $d \geq 3$ and ∂K is not C^2 but still $A(K) > 0$.

2. Prove the analogue of (4) or (5) if not the number of vertices is restricted but the number of k-faces where $1 \leq k \leq d-2$ and $d \geq 4$.

The first problem was solved in the plane by Ludwig [28]. For the second problem, if the number of facets is restricted ($k = d-1$) or $d = 3$ and $k = 1$ then the analoguous results are known.

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