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# A NEW VERSION OF L. FEJES TÓTH'S MOMENT THEOREM

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**Abstract**

A version of the celebrated Moment Theorem of László Fejes Tóth is proved where the integrand is based not on the second moment but on another quadratic form.

## 1. Introduction

For any notions related to convexity in this paper, consult P. M. Gruber [21], R. Schneider [22] or T. Bonnesen and W. Fenchel [2]. We call a compact convex set with non-empty interior in $\mathbb{R}^2$ a convex disc. The area (2-dimensional Lebesgue measure) of $X \subset \mathbb{R}^2$ is denoted by $|X|$. We recall that $X$ is Jordan measurable if it is bounded and $|\partial X| = 0$ holds for the boundary $\partial X$. When we speak about Jordan measurable sets in this paper, we always assume that the interior is non-empty. We write $\|\cdot\|$ to denote the Euclidean norm, and $B^2$ to denote the Euclidean unit disc centred at $o$. Moreover, the convex hull of the objects $X_1, \ldots, X_k$ is denoted by $[X_1, \ldots, X_k]$, and the cardinality of the finite set $\Xi$ is denoted by $\#\Xi$.

A core notion for us is the notion of Dirichlet–Voronoi cell. Given $n$ points $y_1, \ldots, y_n \in \mathbb{R}^2$ and a Jordan measurable $C \subset \mathbb{R}^2$, we define the Dirichlet–

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Voronoi cell of $y_j$ associated to $C$ and $\{y_i\}$ by

$$\Pi_j = \big\{y \in C : \, \|y - y_j\| \leqq \|y - y_m\|, \; m = 1, \ldots, n\big\}$$

for $j = 1, \ldots, n$. In addition, for any non-negative function $f$ on $\mathbb{R}^2$, let

$$\Omega\big(f, C, \{y_1, \ldots, y_n\}\big) = \sum_{i=1}^{n} \int_{\Pi_i} f(y - y_i)\, dy. \tag{1}$$

The so called Moment Theorem was first proved on the sphere by L. Fejes Tóth [11], and he himself soon extended it to the plane in [13] for points inside a hexagon as follows. If $C$ is any polygon of at most six sides, $H$ is an $o$-symmetric regular hexagon with $|H| = |C|/n$, and $f$ is a monotone increasing function of $\|x\|$ then for any $\Xi \subset C$ of cardinality at most $n$, we have

$$\Omega(f, C, \Xi) \geqq n \cdot \int_H f(x)\, dx. \tag{2}$$

Recently G. Fejes Tóth [10] has extended the Moment Theorem to the case when $C$ is any convex disc. The Moment Theorem and its analogues have numerous applications in the theory of packing and covering, polytopal approximation, numerical integration, information theory, etc., (see L. Fejes Tóth [13], A. Florian [15] and P. M. Gruber [19], [20] and [21]). Knowing the profound importance, it is not surprising that numerous additional proofs are available (see G. Fejes Tóth [8], A. Florian [14] and P. M. Gruber [17]).

If $f$ is a strictly monotone increasing function of $\|x\|$, then G. Fejes Tóth [9] and P. M. Gruber [18] proved that the typical Dirichlet–Voronoi cell is asymptotically a regular hexagon in any optimal configuration of at most $n$ points for the Moment Theorem (2).

Recently the need in polytopal approximation arose for another version of the Moment Theorem (2) where $f$ is a positive definite quadratic form (see K. J. Böröczky and B. Csikós [6]). Let $q$ be a positive definite quadratic form in two variables. First we search for the shape of the "optimal Dirichlet–Voronoi cell" when $q$ is used in place of $f$ in (1). This hexagon will play the role of regular hexagons in an analogue of (2). We define $\mathcal{X}$ to be the family of all $o$-symmetric hexagons and rectangles inscribed into $B^2$. In addition let

$$I(q) = \min_{H \in \mathcal{X}} \frac{\int_H q(x)\, dx}{|H|^2}.$$

We note that even if $\mathcal{X}$ is not compact, the minimum does exist. According to the Moment Theorem (2) with $n = 1$, if the eigenvalues of $q$ coincide then $I(q) = \frac{5}{18\sqrt{3}}\sqrt{\det q}$, and the optimal hexagons are the regular hexagons inscribed into $B^2$. One of such regular hexagons we denote by $H_q$ in this case.

THEOREM 1.1. *If $\kappa \geqq \tau > 0$ are the eigenvalues of the quadratic form $q$, then*

$$I(q) = \frac{\sqrt{\tau}\left[4\kappa + (4\kappa^2 - 6\tau\kappa + 3\tau^2)^{1/2}\right]}{18\left[2\kappa + (4\kappa^2 - 6\tau\kappa + 3\tau^2)^{1/2}\right]^{1/2}}.$$

*In addition, if $\kappa > \tau$, then the minimum is attained at a unique $H_q \in \mathcal{X}$.*

REMARK. For $\kappa > \tau$, we assume that $q(s,t) = \tau s^2 + \kappa t^2$. Then $H_q$ is symmetric with respect to the coordinate axes, $(1,0)$ is one of its vertices, and two other vertices are $(\cos\varphi, \pm\sin\varphi)$, where $\varphi \in (0, \pi/2)$ is defined by the equation

$$\tan(\varphi/2) = \sqrt{\frac{2\kappa - \sqrt{4\kappa^2 - 6\kappa\tau + 3\tau^2}}{6\kappa - 3\tau}}.$$

Let $q$ be a positive definitive quadratic from. According to Theorem 2.1 in K. J. Böröczky and B. Csikós [6], there exists $\mathrm{div}_q > 0$ such that if $C \subset \mathbb{R}^2$ is a Jordan measurable set with non-empty interior, and $n$ tends to infinity, then

$$\min_{\Xi \subset \mathbb{R}^2,\ \#\Xi \leqq n} \Omega(q, C, \Xi) = \mathrm{div}_q \cdot |C|^2 \cdot n^{-1} + o\left(n^{-1}\right),$$

where $\#$ stands for the cardinality of a finite set. We note that it is not clear whether the minimum on the left hand side is attained by a set $\Xi$ of cardinality $n$. The reason is that if the eigenvalues of $q$ are different then there exist $C$, $\Xi$ and $y$ such that

$$\Omega\left(q, C, \Xi \cup \{y\}\right) > \Omega(q, C, \Xi).$$

For the edge to edge tiling of the plane by translates of $H_q$, each tile is the Dirichlet–Voronoi cell of its centre. Using suitable dilated copies of this tiling, we deduce that

$$\mathrm{div}_q \leqq I(q). \tag{3}$$

Our main result is that if $\tau < \kappa \leqq 2.4\tau$ hold for the eigenvalues of the positive definite form $q$, then the typical Dirichlet–Voronoi cells are close to be homothetic to $H_q$ for asymptotically optimal arrangements. Let us define the corresponding notions. If $\Xi_n \subset \mathbb{R}^2$ is a family of at most $n$ points for $n \geqq 1$, then we say that a sequence $\{\Xi_n\}$ is *asymptotically optimal with respect to* $q$ *and* $C$ if

$$\Omega(q, C, \Xi_n) = \mathrm{div}_q \cdot |C|^2 \cdot n^{-1} + o\bigl(n^{-1}\bigr).$$

In particular $\lim_{n\to\infty} \#\Xi_n/n = 1$.

We say that the planar convex compact sets $K$ and $M$ are $\nu$-close for some $\nu > 0$ if

$$(1+\nu)^{-1}(K - x) \subset M - y \subset (1+\nu)(K - x)$$

hold where $x$ and $y$ are the circumcentres of $K$ and $M$, respectively.

THEOREM 1.2. *Let* $q$ *be a positive definite quadratic form with eigenvalues* $\tau < \kappa \leqq 2.4\tau$, *and let* $C \subset \mathbb{R}^2$ *be Jordan measurable. Then for any asymptotically optimal sequence* $\{\Xi_n\}$ *of configurations of at most* $n$ *points in* $\mathbb{R}^2$ *there exists a sequence* $\{\nu_n\}$ *of positive numbers tending to zero such that* $n - o(n)$ *Dirichlet–Voronoi cells with respect to* $\Xi_n$ *and* $C$ *are hexagons that are* $\nu_n$*-close to the hexagon homothetic to* $H_q$ *with area* $|C|/n$.

Our method does not allow to eliminate the condition $\kappa \leqq 2.4\tau$ in Theorem 1.2, but we believe that the statement holds for any positive definite quadratic form $q$. Now Theorem 1.2 readily yields

COROLLARY 1.3. *For any positive definite quadratic form* $q$ *with eigenvalues* $\tau \leqq \kappa \leqq 2.4\tau$, *we have*

$$\mathrm{div}_q = I(q).$$

Naturally Corollary 1.3 is a consequence of the Moment Theorem (2) if the eigenvalues of $q$ coincide. The elegance of the Moment Theorem is partially due to the fact that it contains no error term if $C$ is a convex disc. If the eigenvalues of $q$ are different, then we can achieve it only for very special $C$'s.

COROLLARY 1.4. *Let* $q$ *be a positive definite quadratic form with eigenvalues* $\tau < \kappa \leqq 2.4\tau$, *and let* $C$ *be a rectangle whose sides are parallel to the principal axes of* $q$. *If* $\Xi \subset C$ *has at most* $n$ *points, and* $H$ *is the dilate of* $H_q$ *with area* $|C|/n$, *then*

$$\Omega(q, C, \Xi) \geqq n \cdot \int_H q(x)\, dx.$$

Let us present the simple argument how Corollary 1.3 leads to Corollary 1.4. We may assume that $\Xi = \{y_1, \ldots, y_n\} \subset \mathrm{int}\, C$, and $o$ is a vertex of $C$. Let $\Gamma$ be the group of congruencies generated by the four reflections through the lines containing the sides of $C$, hence the images $gC$ for $g \in \Gamma$ form an edge to edge tiling of $\mathbb{R}^2$. The common symmetries of $C$ and $q$ yield that

$$\Omega(q, gC, g\Xi) = \Omega(q, C, \Xi) \quad \text{for any } g \in \Gamma.$$

We write $\Pi_i$ to denote the Dirichlet–Voronoi cell of $y_i$ with respect to $\Xi$ and $C$. For any integer $k \geqq 2$, let $\Xi_k$ be the union of all $g\Xi$, $g \in \Gamma$, that lie in $kC$, hence $\#\Xi_k = k^2 n$. It is not hard to see that for any $y_i$ and $g \in \Gamma$ with $g\Xi \subset kC$, the Dirichlet–Voronoi cell of $gy_i$ with respect to $\Xi_k$ and $kC$ is $g\Pi_i$, therefore $\Omega(q, kC, \Xi_k) = k^2\Omega(q, C, \Xi)$. Since Corollary 1.3 yields

$$I(q) \cdot |C|^2 \leqq \liminf_{k\to\infty} k^2 n \cdot \Omega\left(q, C, \frac{1}{k}\,\Xi_k\right) = n\Omega(q, C, \Xi),$$

we conclude Corollary 1.4.

The paper K. J. Böröczky and B. Csikós [6] considers best approximation with respect to the surface area of a smooth convex body in $\mathbb{R}^d$ by circumscribed polytopes of $n$ facets, and proves an asymptotic formula for the surface area difference as $n$ tends to infinity. To state this formula in $\mathbb{R}^d$, [6] assigns a new quadratic form $q^*$ in $d-1$ variables to any positive definite quadratic form $q$ in $d-1$ variables. If $d = 3$ and $\kappa \geqq \tau > 0$ are the eigenvalues of $q$, then $q^*$ is the quadratic form with eigenvalues $\frac{2\kappa+\tau}{\kappa+\tau} \geqq \frac{\kappa+2\tau}{\kappa+\tau}$. In particular the eigenvalues of $q^*$ lie in $[1,2]$, and hence Corollary 1.3 yields the following.

Corollary 1.5. *If $\kappa \geqq \tau > 0$ are the eigenvalues of the quadratic form $q$ in two variables, then*

$$\mathrm{div}_{q^*} = \frac{\sqrt{2\tau+\kappa}}{18(\tau+\kappa)} \cdot \frac{4\tau + 8\kappa + (4\tau^2 - 2\tau\kappa + 7\kappa^2)^{1/2}}{\left[2\tau + 4\kappa + (4\tau^2 - 2\tau\kappa + 7\kappa^2)^{1/2}\right]^{1/2}}.$$

## 2. Proof of Theorem 1.1

For the proof, we used the computer algebra software MuPAD to handle large trigonometric polynomials and rational functions, to simplify and factor them, and to compute their integrals and derivatives. We shall always explain the idea which led us to a formula, but the details of the computation will be omitted when a formula was obtained by the computer.

We may assume that $\tau = 1$ and $q(s,t) = s^2 + \kappa t^2$. We write $P_t$ to denote the point $\big(\cos(t), \sin(t)\big)$. For $a < b < a + \pi$, the triangle $\Delta_{ab} = [o, P_a, P_b]$

can be parameterized by the map

$$(u,v) \mapsto u\left(\cos\frac{a+b}{2}, \sin\frac{a+b}{2}\right) + v\left(-\sin\frac{a+b}{2}, \cos\frac{a+b}{2}\right),$$

where $(u,v)$ is running over the domain $0 \leqq u \leqq \cos\left(\frac{b-a}{2}\right)$, $|v| \leqq u\tan\left(\frac{b-a}{2}\right)$. Thus, the integral of the quadratic form $q$ over the triangle $[o, P_a, P_b]$ can be written as follows

$$\int\limits_{\Delta_{ab}} q = \int\limits_0^{\cos\frac{b-a}{2}} \int\limits_{-\tan\frac{b-a}{2}}^{\tan\frac{b-a}{2}} \left(u\cos\frac{a+b}{2} - v\sin\frac{a+b}{2}\right)^2$$

$$+ \kappa\left(u\sin\frac{a+b}{2} + v\cos\frac{a+b}{2}\right)^2 \mathrm{d}v\,\mathrm{d}u.$$

This integral can be computed explicitly and after some simplification it turns out to be

$$\int\limits_{\Delta_{ab}} q = \frac{1}{12}\sin(b-a)\big(\cos^2 a + \cos^2 b + \cos a\cos b + \kappa(\sin^2 a + \sin^2 b + \sin a \sin b)\big). \tag{4}$$

Observe, that equation (4) is valid also in the case $a = b$, when the triangle degenerates to a segment.

Let $\mathfrak{D}$ be the triangle $\left\{(s_1,s_2) \mid 0 \leqq s_1,\ 0 \leqq s_2,\ s_1+s_2 \leqq \pi\right\}$. The set of the vertices of $\mathfrak{D}$ is $V = \left\{(0,0),(0,\pi),(\pi,0)\right\}$. For $a \in \mathbb{R}$ and $(s_1,s_2) \in \mathfrak{D}$, let $H(a,s_1,s_2)$ denote the convex hull of the vertices $P_{a-s_2}$, $P_a$, $P_{a+s_1}$, $P_{a+\pi-s_2}$, $P_{a+\pi}$, $P_{a+\pi+s_1}$. The map $\mathbb{R}\times(\mathfrak{D}\setminus V) \to \mathcal{X}$, $(a,s_1,s_2) \mapsto H(a,s_1,s_2)$ is a (not injective) parametrization of the set $\mathcal{X}$, so every function on $\mathcal{X}$ can be written as a function of the parameters $a$ and $(s_1,s_2)$ running over $\mathbb{R}$ and $\mathfrak{D}\setminus V$ respectively. The shape of $H(a,s_1,s_2)$ is uniquely determined by the parameters $s_1$ and $s_2$, as for any $a,\tilde{a} \in \mathbb{R}$, $H(\tilde{a},s_1,s_2)$ is a rotated image of $H(a,s_1,s_2)$ about the origin. In particular, the area $A(s_1,s_2)$ of $H(a,s_1,s_2)$ depends only on $s_1$ and $s_2$:

$$A(s_1,s_2) = \sin(s_1) + \sin(s_2) + \sin(s_1+s_2).$$

According to (4), the integral of the quadratic form $q$ over $H(a, s_1, s_2)$ is equal to

$$\int_{H(a,s_1,s_2)} q = \frac{1}{6}\Big[\sin(s_1)\big(\cos^2 a + \cos^2(a+s_1) + \cos a\cos(a+s_1)\big)$$
$$+ \sin(s_2)\big(\cos^2 a + \cos^2(a-s_2) + \cos a\cos(a-s_2)\big)$$
$$+ \sin(s_1+s_2)\big(\cos^2(a+s_1) + \cos^2(a-s_2)$$
$$- \cos(a+s_1)\cos(a-s_2)\big)\Big]$$
$$+ \frac{\kappa}{6}\Big[\sin(s_1)\big(\sin^2 a + \sin^2(a+s_1) + \sin a\sin(a+s_1)\big)$$
$$+ \sin(s_2)\big(\sin^2 a + \sin^2(a-s_2) + \sin a\sin(a-s_2)\big)$$
$$+ \sin(s_1+s_2)\big(\sin^2(a+s_1) + \sin^2(a-s_2)$$
$$- \sin(a+s_1)\sin(a-s_2)\big)\Big].$$

This equation can be transformed into the form

$$\int_{H(a,s_1,s_2)} q = P(s_1,s_2) + \cos(2a)Q(s_1,s_2) + \sin(2a)R(s_1,s_2), \tag{5}$$

where

$$P(s_1,s_2) = \frac{1+\kappa}{24}\big(4\sin(s_1) + 4\sin(s_2) + 4\sin(s_1+s_2) + \sin(2s_1)$$
$$+ \sin(2s_2) - \sin(2s_1+2s_2)\big) > 0,$$
$$Q(s_1,s_2) = \frac{1-\kappa}{24}\big(\sin(s_1) + \sin(s_2) + \sin(3s_1) + \sin(3s_2)$$
$$+ \sin(3s_1+s_2) + \sin(s_1+3s_2)\big),$$
$$R(s_1,s_2) = \frac{1-\kappa}{24}\big(-\cos(s_1) + \cos(s_2) + \cos(3s_1) - \cos(3s_2)$$
$$+ \cos(3s_1+s_2) - \cos(s_1+3s_2)\big).$$

Below we frequently drop the reference to the variables to simplify the formulas.

Our goal is to determine the minimum of

$$F(a, s_1, s_2) = \frac{\int_{H(a,s_1,s_2)} q}{A(s_1, s_2)^2}$$

for $(a, s_1, s_2) \in \mathbb{R} \times (\mathfrak{D} \setminus V)$. By the Cauchy–Schwarz inequality applied in (5), if we fix the shape of the hexagon, i.e., we keep $s_1$ and $s_2$ fixed, then the value of $F(a, s_1, s_2)$ as a function of $a$ oscillates between $\big(P \pm \sqrt{Q^2 + R^2}\,\big)/A^2$. If $R(s_1, s_2) = Q(s_1, s_2) = 0$, then $F$ is constant in $a$, otherwise the minimum $G_s := \big(P - \sqrt{Q^2 + R^2}\,\big)/A^2$, $s = (s_1, s_2)$, is attained when

$$\cos(2a) = \frac{-Q}{\sqrt{Q^2 + R^2}} \quad \text{and} \quad \sin(2a) = \frac{-R}{\sqrt{Q^2 + R^2}}. \tag{6}$$

This means that to find the infimum or minimum of $F(a, s_1, s_2)$, we have to determine the infimum or minimum of the function $G_s : (\mathfrak{D} \setminus V) \to \mathbb{R}$.

The central angles corresponding to the consecutive sides of the possibly degenerated hexagon $H(a, s_1, s_2)$ are $s_1$, $s_2$ and $s_3 = \pi - s_1 - s_2$. The permutation group $S_3$ of these three angles acts on the triangle $\mathfrak{D}$ as the group of all affine symmetries of $\mathfrak{D}$. It is clear from the geometrical meaning of the function $G_s$ that it is invariant under this action.

Let us introduce two new parameters $t_1 = \tan(s_1/2)$ and $t_2 = \tan(s_2/2)$. If $(s_1, s_2)$ is running over $\mathfrak{D}$, then $(t_1, t_2)$ is changing in the set

$$\mathfrak{D}' = \big\{ (t_1, t_2) \mid 0 \leqq t_1,\ 0 \leqq t_2,\ t_1 t_2 \leqq 1 \big\} \cup \big\{ (\infty, 0), (0, \infty) \big\}.$$

Denote by $G : \mathfrak{D}' \setminus \{(0,0)\} \to \mathbb{R}$ the function which expresses $G_s$ in terms of the new parameters. $G$ is related to $G_s$ by the identity

$$G\big(\tan(s_1/2,), \tan(s_2/2)\big) = G_s, \quad s = (s_1, s_2).$$

The advantage of this reparametrization is that any trigonometric polynomial of the variables $s_1$ and $s_2$ can be written as a rational function of $t_1$ and $t_2$ due to the identities $\sin(x) = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$ and $\cos(x) = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$. In particular, we have

$$P = \frac{(2/3)(1+\kappa)(t_1 + t_2)\big(2t_1t_2 + t_1^2 + t_2^2 - t_1^2t_2^2 + 1\big)}{(t_1^2+1)^2(t_2^2+1)^2},$$

$$Q = (2/3)(1-\kappa)(t_1+t_2)$$
$$\times \frac{\big(2t_1t_2 - 2t_1^2 - 2t_2^2 - 10t_1^2t_2^2 + t_1^4 + t_2^4 + 2t_1^3t_2 + 2t_1t_2^3 + 2t_1^3t_2^3 + 3t_1^4t_2^4 + 1\big)}{(t_1^2+1)^3(t_2^2+1)^3},$$

$$R = \frac{(4/3)(1-\kappa)\big(-2t_1t_2 - t_1^2 - t_2^2 - 3t_1^2t_2^2 + 2t_1^3t_2^3 + 1\big)(t_2-t_1)(t_2+t_1)}{(t_1^2+1)^3(t_2^2+1)^3},$$

$$A = \frac{4(t_1+t_2)}{(t_1^2+1)(t_2^2+1)}.$$

Substituting these expressions into the formula defining $G$, we obtain

$$G = \frac{1+\kappa}{24}\cdot S + \frac{1-\kappa}{24}\sqrt{T}, \tag{7}$$

where

$$S = \frac{\big(2t_1t_2 + t_1^2 + t_2^2 - t_1^2t_2^2 + 1\big)}{t_1+t_2}, \tag{8}$$

$$T = \frac{1}{(t_1+t_2)^2}\big[-4t_1t_2 - 2t_1^2 - 2t_2^2 + t_1^4 + t_2^4 + 4t_1t_2^3 + 4t_1^3t_2$$
$$+ 12t_1^2t_2^2 - 2t_1^2t_2^4 - 20t_1^3t_2^3 - 2t_1^4t_2^2 + 9t_1^4t_2^4 + 1\big].$$

The following lemma describes those (possibly degenerated) hexagons for which $\int_{H(a,s_1,s_2)} q$ does not depend on $a$.

LEMMA 2.1. *For a point $(s_1,s_2)\in\mathfrak{D}$, $Q(s_1,s_2)=R(s_1,s_2)=0$ if and only if one of the following cases are fulfilled*

1. $\kappa = 1$;
2. *$H(a,s_1,s_2)$ is a segment, i.e., $s_1=s_2=0$ or $\{s_1,s_2\}=\{0,\pi\}$;*
3. *$H(a,s_1,s_2)$ is a square, i.e., $s_1=s_2=\pi/2$ or $\{s_1,s_2\}=\{0,\pi/2\}$;*
4. *$H(a,s_1,s_2)$ is a regular hexagon, i.e., $s_1=s_2=\pi/3$.*

PROOF. We shall solve the system of equations $P=Q=0$ for the unknown parameters $t_1$ and $t_2$. Since $(1-\kappa)(t_1+t_2)$ is a common factor of $P$ and $Q$, the system is solved by any $t_1$, $t_2$ when $[\kappa=1]$ and it is also solved by any $(t_1,t_2)\in\mathfrak{D}'$ satisfying $t_1+t_2=0$. The straight line $t_1+t_2=0$ cuts the domain $\mathfrak{D}'$ at the origin so the second case gives only one solution, $[s_1=s_2=t_1=t_2=0]$.

Suppose now that $\kappa \neq 1$ and $t_1 + t_2 \neq 0$. Then we are to find the intersection points of the algebraic curves

$$
\begin{aligned}
(9) \qquad 0 &= 2t_1t_2 - 2t_1^2 - 2t_2^2 - 10t_1^2t_2^2 + t_1^4 + t_2^4 + 2t_1^3t_2 + 2t_1t_2^3 \\
&\quad + 2t_1^3t_2^3 + 3t_1^4t_2^4 + 1, \\
0 &= \big(-2t_1t_2 - t_1^2 - t_2^2 - 3t_1^2t_2^2 + 2t_1^3t_2^3 + 1\big)\,(t_2 - t_1).
\end{aligned}
$$

If $t_1 = t_2$, then the second equation is fulfilled, and the first equation can be factored as

$$0 = (t_1 - 1)(t_1 + 1)\big(3t_1^2 - 1\big)\left(t_1^2 + 1\right).$$

Thus, the geometrically relevant solutions obtained in this case are$[t_1 = t_2 = 1,\ s_1 = s_2 = \pi/2]$ and $\left[t_1 = t_2 = 1/\sqrt{3},\ s_1 = s_2 = \pi/3\right]$.

If $t_1 \neq t_2$, then the second equation of (9) can be divided by$(t_2 - t_1)$. The degree of the remaining equations can be reduced if we rewrite the equations in terms of the elementary symmetric polynomials$\sigma_1 = t_1 + t_2$ and $\sigma_2 = t_1t_2$. The obtained equations are the following:

$$
\begin{aligned}
(10) \qquad 0 &= 6\sigma_2 - 2\sigma_1^2 - 12\sigma_2^2 + \sigma_1^4 + 2\sigma_2^3 + 3\sigma_2^4 - \sigma_1^2\sigma_2 + 1, \\
0 &= -\sigma_1^2 - 3\sigma_2^2 + 2\sigma_2^3 + 1.
\end{aligned}
$$

Expressing $\sigma_1^2$ from the second equation and substituting the result into the first equation we obtain a polynomial equation for$\sigma_2$ which has the following factorization:

$$0 = \sigma_2(\sigma_2 + 1)(\sigma_2 - 1)^4.$$

Solutions for which $[\sigma_2 = 0,\ \sigma_1 = \pm 1]$ correspond to cases when $H(a, s_1, s_2)$ is a square. The cases $[\sigma_2 = t_1t_2 = -1]$ and $[\sigma_2 = 1,\ \sigma_1 = 0]$ give no geometrically relevant solution. □

LEMMA 2.2. *We have*

$$I(q) \leqq \frac{4\kappa + \sqrt{4\kappa^2 - 6\kappa + 3}}{18\sqrt{2\kappa + \sqrt{4\kappa^2 - 6\kappa + 3}}}.$$

PROOF. It is enough to show that the right hand side is in the range of $G$. However, if we evaluate $G$ at $t_1 = t_2 = \sqrt{\frac{2\kappa - \sqrt{4\kappa^2 - 6\kappa + 3}}{6\kappa - 3}}$ we get exactly the right hand side. □

LEMMA 2.3. *The following function is strictly monotone increasing in* $\kappa \geqq 1$*.*

$$\frac{4\kappa + \sqrt{4\kappa^2 - 6\kappa + 3}}{18\sqrt{2\kappa + \sqrt{4\kappa^2 - 6\kappa + 3}}} \cdot \frac{1}{\sqrt{\kappa}}$$

PROOF. We write $h$ to denote the function above, and define $B = 2\kappa$ and $C = 2\kappa + \sqrt{4\kappa^2 - 6\kappa + 3}$. In particular $C > B$, and $h$ is proportional to $\frac{B+C}{\sqrt{BC}} = \frac{1+(C/B)}{\sqrt{C/B}}$, which is a strictly monotone increasing function of $C/B$ if $C > B$. Since $C/B = 1 + \frac{1}{2}\sqrt{3(1-\kappa^{-1})^2 + 1}$ is a strictly increasing function of $\kappa \geqq 1$, we conclude the lemma. □

LEMMA 2.4. *The infimum $I(q)$ of $G_s$ is a minimum, attained at an inner point of the triangle $\mathfrak{D}$.*

PROOF. Since $\mathfrak{D}$ is compact and $G_s$ is continuously defined on $\mathfrak{D} \setminus V$, to prove that the infimum is attained somewhere, it is enough to show that $\lim G_s = \infty$ as $s = (s_1, s_2) \in \mathfrak{D} \setminus V$ tends to one of the vertices of $\mathfrak{D}$. By the $S_3$ invariance of $G_s$, it is enough to check this for the vertex $(0,0)$. However, equations (7) and (8) imply immediately that $G_s = G(t_1, t_2)$ is asymptotically equal to $1/\big(12(t_1 + t_2)\big)$ as $(t_1, t_2) \in \mathfrak{D}'$ tends to the origin.

To prove that the minimum is attained inside the triangle $\mathfrak{D}$, we have to show that the minimum of the restriction of $G_s$ onto any of the sides of $\mathfrak{D}$ is larger than the global minimum of $G_s$. Referring again to the $S_3$-invariance of $G_s$ it is enough to consider one of the sides, say the side $0 < s_1 < \pi$, $s_2 = 0$, which side is characterized also by $t_1 > 0 = t_2$. However if $t_2 = 0$ then

$$G = \frac{1+\kappa}{24}\left(t_1 + \frac{1}{t_1}\right) + \frac{1-\kappa}{24}\left|t_1 - \frac{1}{t_1}\right|.$$

This function is symmetric in $t_1$ and $1/t_1$, so we may assume without loss of generality that $t_1 \geqq 1$. In that case $G = (1/12)(t_1 + \kappa/t_1)$, from which we can see, that the minimum of the restriction of $G_s$ onto the sides of the triangle $\mathfrak{D}$ is $\sqrt{\kappa}/6$. Lemma 2.3 and letting $\kappa$ tend to $\infty$ yield

$$\frac{4\kappa + \sqrt{4\kappa^2 - 6\kappa + 3}}{18\sqrt{2\kappa + \sqrt{4\kappa^2 - 6\kappa + 3}}} < \frac{\sqrt{\kappa}}{6},$$

therefore Lemma 2.2 completes the proof. □

The following statement is a direct consequence of L. Fejes Tóth's Moment Theorem (2), but we include the simple argument for the sake of completeness.

LEMMA 2.5. *For $\kappa = 1$, $I(q) = 5\sqrt{3}/54$ and the minimum is attained by regular hexagons.*

PROOF. The peculiarity of the case $\kappa = 1$ is that in this case the coefficient of $\sqrt{T}$ in $G$ is 0, therefore $G$ is smooth. By Lemma 2.4, the point $(t_1, t_2) \in \mathfrak{D}'$ at which $G = (1+\kappa)S/24$ attains its minimum must satisfy

$$\frac{\partial S}{\partial t_1}(t_1, t_2) = \frac{\left(2t_1t_2 + t_1^2 - 1\right)(1+t_2)(1-t_2)}{12(t_1+t_2)^2} = 0,$$

$$\frac{\partial S}{\partial t_2}(t_1, t_2) = \frac{\left(2t_1t_2 + t_2^2 - 1\right)(1+t_1)(1-t_1)}{12(t_1+t_2)^2} = 0.$$

It is easy to list all the solutions of this system of equations:

$$\left[\{t_1, t_2\} = \{\pm 1, 0\}\right], \quad \left[t_1 = t_2 \in \left\{\pm 1, \pm 1/\sqrt{3}\right\}\right], \quad \left[t_1 = -t_2 \in \{\pm i, \pm 1\}\right].$$

The only solution which belongs to the interior of the domain $\mathfrak{D}'$ is $t_1 = t_2 = 1/\sqrt{3}$, and these parameters correspond to the regular hexagons. □

LEMMA 2.6. *If $\kappa > 1$, then the regular hexagon does not minimize $G$.*

PROOF. Computations in the proof of the previous lemma show that the derivative of $S$ vanishes at $t_1 = t_2 = 1/\sqrt{3}$. At this point, the derivative of $T$ vanishes as well, since $T \geqq 0$ everywhere and $T = 0$ at $t_1 = t_2 = 1/\sqrt{3}$. On the other hand,

$$G\left(1/\sqrt{3}, 1/\sqrt{3}\right) = (1+\kappa)\,\frac{5\sqrt{3}}{108} \quad \text{and} \quad \frac{\partial^2 T}{(\partial t_1)^2}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{8}{3},$$

and consequently,

$$\begin{aligned} G\left(1/\sqrt{3}+\varepsilon, 1/\sqrt{3}\right) &= (1+\kappa)\,\frac{5\sqrt{3}}{108} + O(\varepsilon^2) + \frac{1-\kappa}{24}\sqrt{(4/3)\varepsilon^2 + O(\varepsilon^3)} \\ &= (1+\kappa)\frac{5\sqrt{3}}{108} + \frac{1-\kappa}{12\sqrt{3}}|\varepsilon| + O(\varepsilon^2). \end{aligned}$$

This means, that if $|\varepsilon| > 0$ is sufficiently small, then $G\left(1/\sqrt{3}+\varepsilon, 1/\sqrt{3}\right)$ is smaller than $G\left(1/\sqrt{3}, 1/\sqrt{3}\right)$. □

Now we are ready to complete the proof of Theorem 1.1. Case $\kappa = 1$ is verified in Lemma 2.5. Suppose $\kappa > 1$. By Lemma 2.4 we know that $G$ attains its minimum at an inner point of the domain $\mathfrak{D}'$. Those points in $\mathfrak{D}'$ at which $G$ is not differentiable are characterized by the equation $Q = R = 0$. By Lemma 2.1, there is only one such point in the interior of $\mathfrak{D}'$, the point

$t_1 = t_2 = 1/\sqrt{3}$, which corresponds to the regular hexagon. According to Lemma 2.6, the regular hexagon does not minimize the function $G$, so we conclude that $G$ attains its minimum at point(s) at which $G$ is differentiable.

At a point $(t_1, t_2)$ at which $G$ is minimal, we have

$$\frac{(1+\kappa)}{24}\frac{\partial S}{\partial t_1} + \frac{(1-\kappa)}{48\sqrt{T}}\frac{\partial T}{\partial t_1} = 0,$$

$$\frac{(1+\kappa)}{24}\frac{\partial S}{\partial t_2} + \frac{(1-\kappa)}{48\sqrt{T}}\frac{\partial T}{\partial t_2} = 0.$$

Combining these equations we obtain

$$\frac{\partial S}{\partial t_1}\frac{\partial T}{\partial t_2} - \frac{\partial S}{\partial t_2}\frac{\partial T}{\partial t_1} = 0. \tag{11}$$

Substituting the explicit form of $S$ and $T$ into (11) and factoring the left hand side we obtain.

$$16\frac{\left(2t_1t_2 + t_2^2 - 1\right)\left(2t_1t_2 + t_1^2 - 1\right)(t_1 - t_2)(t_1t_2 - 1)t_1t_2}{(t_1 + t_2)^4} = 0$$

Using the parameter $t_3 = \tan\left((\pi - s_1 - s_2)/2\right)$, we can rewrite this equation in the form

$$\frac{16}{t_1 + t_2}t_1t_2t_3(t_1 - t_2)(t_2 - t_3)(t_3 - t_1) = 0.$$

Equation $t_1t_2t_3 = 0$ characterizes the boundary points of $\mathfrak{D}'$, thus by Lemma 2.4, four sides of the extremal hexagon are equal. As the geometric role of $t_1$, $t_2$ and $t_3$ is symmetric, we may assume without loss of generality that $t_1 = t_2$. The common value of them, which will be denoted by $t$ is in the open interval $(0, 1)$. The restriction of $G$ onto the diagonal $t_1 = t_2$ has the form

$$G(t,t) = \frac{(1+\kappa)(1 + 4t^2 - t^4) + (1-\kappa)|3t^2 - 1|(1 - t^2)}{48t}$$

$$= \begin{cases} \left((1-2\kappa)t^4 + 4\kappa t^2 + 1\right)/(24t) & \text{if } t \in \left(0, 1/\sqrt{3}\,\right], \\ \left((\kappa - 2)t^4 + 4t^2 + \kappa\right)/(24t) & \text{if } t \in \left[1/\sqrt{3}, 1\right). \end{cases}$$

The derivative of the function $G_1(t) = \left((1-2\kappa)t^4 + 4\kappa t^2 + 1\right)/(24t)$ has two positive real roots, $\tau_1 = \sqrt{\frac{2\kappa - \sqrt{4\kappa^2 - 6\kappa + 3}}{6\kappa - 3}}$ and $\tau_2 = \sqrt{\frac{2\kappa + \sqrt{4\kappa^2 - 6\kappa + 3}}{6\kappa - 3}}$. Since the roots of $G_1'$ are of multiplicity one and $G_1(t)$ tends to $+\infty$ as $t$ tends to 0, the function $G_1$ decreases on the intervals $(0,\tau_1)$ and $(\tau_2,\infty)$, and increases on the interval $(\tau_1,\tau_2)$. Evaluating the derivative of $G_1$ at $1/\sqrt{3}$ we obtain $G_1'\big(1/\sqrt{3}\big) = (\kappa-1)/12 > 0$, consequently, $\tau_1 < 1/\sqrt{3} < \tau_2$ and

$$\min_{0<t<1/\sqrt{3}} G(t,t) = \min_{0<t<1/\sqrt{3}} G_1 = G_1(\tau_1) = \frac{4\kappa+\sqrt{4\kappa^2-6\kappa+3}}{18\sqrt{2\kappa+\sqrt{4\kappa^2-6\kappa+3}}}.$$

The derivative of the function $G_2(t) = \big((\kappa-2)t^4+4t^2+\kappa\big)/(24t)$ has exactly two positive real roots $\tau_3 = \sqrt{\frac{\kappa}{2+\sqrt{3\kappa^2-6\kappa+4}}}$ and $\tau_4 = \sqrt{\frac{\kappa}{2-\sqrt{3\kappa^2-6\kappa+4}}}$ if $\kappa < 2$. As $\kappa$ tends to 2 from below, $\tau_4$ is running out to infinity and for $\kappa > 2$ it becomes pure imaginary. Thus, in the range $\kappa \geqq 2$, $\tau_3$ is the only positive real root of $G_2'$. As $\tau_3$ and $\tau_4$ are simple roots of $G_2'$, and $\lim_{t\to 0} G_2(t) = +\infty$, $G_2$ decreases on the interval $(0,\tau_3)$ and starts increasing at $\tau_3$. If $\kappa \geqq 2$, then it remains increasing on the whole interval $(\tau_3,\infty)$, if $\kappa < 2$, then it increases only on the interval $[\tau_3,\tau_4]$ and it becomes decreasing again on the interval $[\tau_4,\infty)$. The values of $G_2'$ at $1/\sqrt{3}$ and at 1 have opposite sign as $G_2'\big(1/\sqrt{3}\big) = -G_2'(1) = (1-\kappa)/12 < 0$, thus we have $1/\sqrt{3} < \tau_3 < 1 (< \tau_4)$, and therefore

$$\min_{1/\sqrt{3}\leqq t<1} G(t,t) = \min_{1/\sqrt{3}\leqq t<1} G_2 = G_2(\tau_3) = \frac{\sqrt{\kappa}}{18}\frac{4+\sqrt{3\kappa^2-6\kappa+4}}{\sqrt{2+\sqrt{3\kappa^2-6\kappa+4}}}.$$

The first part of the theorem will follow if we show that $G_1(\tau_1) < G_2(\tau_3)$. Let $C = 2\kappa+\sqrt{4\kappa^2-6\kappa+3}$ as in Lemma 2.3, and let

$$\tilde{C} = 2\kappa+\sqrt{4\kappa^2-6\kappa^3+3\kappa^4}.$$

Since $(4\kappa^2-6\kappa^3+3\kappa^4)-(4\kappa^2-6\kappa+3) = 3(\kappa+1)(\kappa-1)^3 > 0$, we have $\tilde{C} > C$. We also have $\sqrt{\tilde{C}C} > 2\kappa$, since both $C$ and $\tilde{C}$ are greater than $2\kappa$. These imply

$$G_2(\tau_3)-G_1(\tau_1) = \frac{2\kappa+\tilde{C}}{18\sqrt{\tilde{C}}} - \frac{2\kappa+C}{18\sqrt{C}} = \frac{\big(\sqrt{\tilde{C}C}-2\kappa\big)\big(\sqrt{\tilde{C}}-\sqrt{C}\big)}{18\sqrt{\tilde{C}C}} > 0,$$

as we wanted to show.

Let us summarize what we know about an extremal hexagon. It has two consecutive sides of equal length. If the central angles belonging to these sides are $s_1 = s_2 = s$, then

$$\tan(s/2) = \tau_1 = \sqrt{\frac{2\kappa - \sqrt{4\kappa^2 - 6\kappa + 3}}{6\kappa - 3}}.$$

This information determines the shape of the hexagon uniquely. To find which rotation of this hexagon gives the minimum, we refer to equation (6). Since $R(s_1, s_2)$ is skew symmetric, $R(s,s) = 0$. Inequality $\tan(s/2) = \tau_1 < 1/\sqrt{3}$ yields that $s < \pi/3$, $\sin(3s) > 0$, $\sin(s) + \sin(4s) > 0$ and $Q(s,s) < 0$. In conclusion, the common vertex $P_a$ of the equal sides of the extremal hexagon must satisfy $\cos(2a) = 1$, and $\sin(2a) = 0$, in other words, $P_a$ must lie on the first coordinate axis. This completes the proof of Theorem 1.1. □

## 3. Proof of Theorem 1.2

### 3.1. An equivalent formulation and some properties of the second moment

We actually prove the following statement equivalent to Theorem 1.2. Let $q$ be a positive definite quadratic form with eigenvalues $\tau < \kappa \leqq 2.4\tau$, and let $C \subset \mathbb{R}^2$ be Jordan measurable. For any $\nu > 0$, and for any asymptotically optimal sequence $\{\Xi_n\}$ of configurations of at most $n$ points in $\mathbb{R}^2$, the number $f(n)$ of Dirichlet–Voronoi cells with respect to $\Xi_n$ and $C$ that are hexagons and $\nu$-close to the hexagon homothetic to $H_q$ with area $|C|/n$ satisfies

$$f(n) = n - o(n). \tag{12}$$

To show that (12) yields Theorem 1.2, set $n_0 = 1$. It follows by induction on $k = 1, 2, \dots$ and (12) that there exists integer $n_k > n_{k-1}$ such that if $n > n_k$ and $f_k(n)$ is the number of Dirichlet–Voronoi cells with respect to $\Xi_n$ and $C$ that are hexagons and $1/k$-close to the hexagon homothetic to $H_q$ with area $|C|/n$ then $f_k(n) > n - n/k$. Therefore we may choose $\nu_n = 1/k$ in Theorem 1.2 if $n_k < n \leqq n_{k+1}$.

Let us prepare for the proof of (12). First we discuss some estimates on the second moment over triangles. Let $R = [o, b, c]$ be a triangle that has angle $\alpha$ at $o$. Then direct computation yields

$$\int_R \|x\|^2 \, dx = |R|^2 \cot\alpha + \frac{1}{6} \cdot \|c - b\|^2 |R|. \tag{13}$$

For $\alpha \in \left(0, \frac{\pi}{2}\right)$, we define

$$\gamma(\alpha) = \cot\alpha + \frac{1}{3}\tan\alpha.$$

In particular if the angle of $R$ at $b$ is $\frac{\pi}{2}$ then $\int_R \|x\|^2\,dx = \gamma(\alpha)|R|^2$. The importance of $\gamma$ stems from the following estimate.

LEMMA 3.1. *Let the triangles $S = [a, v, w]$ and $\widetilde{S} = [\tilde{a}, v, w]$ intersect in the common side $[v, w]$ and have the same area, and let $w \in [a, \tilde{a}, v]$. Writing $\alpha$ and $\tilde{\alpha}$ to denote the angle of $S$ and $\widetilde{S}$ at $a$ and $\tilde{a}$, respectively, we have*

$$\int_S \|y-a\|^2\,dy + \int_{\widetilde{S}} \|y-\tilde{a}\|^2\,dy \geqq 2\gamma\left(\frac{\alpha+\tilde{\alpha}}{2}\right)|S|^2. \tag{14}$$

PROOF. We deduce by (13) that (14) is equivalent to

$$|S|(\cot\alpha + \cot\tilde{\alpha}) + \frac{\|v-w\|^2}{3} \geqq 2\left(\cot\frac{\alpha+\tilde{\alpha}}{2} + \frac{1}{3}\tan\frac{\alpha+\tilde{\alpha}}{2}\right)|S|. \tag{15}$$

To prove (15), we may assume $\angle(v, w, \tilde{a}) \leqq \angle(v, w, a)$, hence $\angle(v, w, a)$ is obtuse according to $w \in [a, \tilde{a}, v]$. Let $a_0 = \tilde{a} + 2(w - \tilde{a})$, let $\alpha_0 = \angle(v, a_0, w)$, and let $S_0 = [v, a_0, w]$ (see Figure 1). In particular $\alpha \leqq \alpha_0$, $\alpha_0 + \tilde{\alpha} < \pi$ and $|S_0| = |S|$. For the triangle $T = (S_0 - a_0) \cup (\tilde{a} - \widetilde{S})$, its angle at $o$ is $\alpha_0 +$

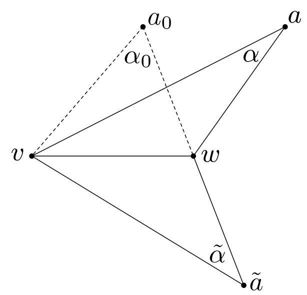


*Fig. 1*

$\tilde{\alpha}$, and the opposite side $\sigma$ is of length $2\|v - w\|$. Let $o'$ be the orthogonal projection of $o$ to the perpendicular bisector of the segment $\sigma$, and let $\zeta$ be the angle of the triangle $[o', \sigma]$ at $o'$, and hence $\zeta > \alpha_0 + \tilde{\alpha}$. Writing $h$ to denote the distance of $o'$ from $\sigma$, we have

$$\|v-w\|^2 = |T| \cdot \|v-w\|/h = 2|S|\tan\frac{\zeta}{2} \geqq 2|S|\tan\frac{\alpha_0+\tilde{\alpha}}{2}.$$

In addition even if either $\alpha_0$ or $\tilde{\alpha}$ is obtuse, we still have

$$\cot\alpha_0 + \cot\tilde{\alpha} = 2\cot\frac{\alpha_0+\tilde{\alpha}}{2}\cdot\frac{\left(\tan\frac{\alpha_0+\tilde{\alpha}}{2}\right)^2\left[1+\left(\tan\frac{\alpha_0-\tilde{\alpha}}{2}\right)^2\right]}{\left(\tan\frac{\alpha_0+\tilde{\alpha}}{2}\right)^2-\left(\tan\frac{\alpha_0-\tilde{\alpha}}{2}\right)^2} \geqq 2\cot\frac{\alpha_0+\tilde{\alpha}}{2},$$

therefore

$$|S|(\cot\alpha_0+\cot\tilde{\alpha}) + \frac{\|v-w\|^2}{3} \geqq 2\left(\cot\frac{\alpha_0+\tilde{\alpha}}{2}+\frac{1}{3}\tan\frac{\alpha_0+\tilde{\alpha}}{2}\right)|S|.$$

Differentiation with respect to $\alpha$ shows that the function

$$f(\alpha) = \cot\alpha - 2\left(\cot\frac{\alpha+\tilde{\alpha}}{2}+\frac{1}{3}\tan\frac{\alpha+\tilde{\alpha}}{2}\right)$$

is decreasing on $(0,\alpha_0]$. In turn we conclude (15). □

### 3.2. The graph of skew edges

We may assume that $\tau = 1$ and $q(s,t) = s^2+\kappa t^2$,

- no four points of $\Xi_n$ lie on a circle,
- no three points of $\Xi_n$ determine a right angle.

We write $\widetilde{\mathcal{D}}_n$ to denote the Dirichlet–Voronoi tiling of $C$ induced by $\Xi_n$, and hence for any vertex $v$ of $\widetilde{\mathcal{D}}_n$ in $\operatorname{int} C$, $v$ is of degree three, and $v$ does not lie on the line connecting two closest points of $\Xi_n$ to $v$. For $\eta = \sqrt[4]{I(q)|C|^2}$, we claim that if $n$ is large and

$$(16) \qquad x+\frac{\eta}{\sqrt[4]{n}}B^2 \subset C \quad \text{then} \quad \Xi_n \cap \left(x+\frac{2\eta}{\sqrt[4]{n}}B^2\right) \neq \emptyset.$$

Otherwise if $z \in x+\frac{\eta}{\sqrt[4]{n}}B^2$ lies in the Dirichlet–Voronoi cell of $a \in \Xi_n$ then $q(z-a) \geqq \|z-a\|^2 \geqq \frac{\eta^2}{\sqrt{n}}$, thus

$$\Omega(q,C,\Xi_n) \geqq \pi\cdot\eta^4 n^{-1} = \pi\cdot I(q)|C|^2 n^{-1},$$

which contradicts the asymptotic optimality of $\Xi_n$ for large $n$.

Next we claim that if $n$ is large, and $\Pi$ is a Dirichlet–Voronoi cell for $\Xi_n$ and $C$ such that $x+\frac{6\eta}{\sqrt[4]{n}}B^2 \subset C$ for some $x\in\Pi$ then

$$(17) \qquad \Pi \subset \operatorname{int} C \text{ is a convex polygon and } \operatorname{diam}\Pi \leqq 4\eta n^{-\frac{1}{4}}.$$

Let $a \in \Xi_n$ correspond to $\Pi$, and hence $\|x-a\| \leqq \frac{2\eta}{\sqrt[4]{n}}$ by (16). If there exists $z \in \Pi$ with $\|z-a\| > \frac{2\eta}{\sqrt[4]{n}}$ then let $z' \in [z,a]$ satisfy $\frac{2\eta}{\sqrt[4]{n}} < \|z'-a\| < \frac{3\eta}{\sqrt[4]{n}}$. We have $z' + \frac{\eta}{\sqrt[4]{n}} B^2 \subset a + \frac{4\eta}{\sqrt[4]{n}} B^2 \subset C$, thus (16) applied to $z'$ contradicts that $z \in \Pi$. Therefore $\Pi \subset a + \frac{2\eta}{\sqrt[4]{n}} B^2$, concluding the proof (17).

For large $n$, we may choose a finite set $\{W_{i,n}\}$ of squares of side length $n^{-\frac{1}{8}}$ lying in $C$ such that for any $W_{i,n}$, its distance from $\partial C$ and from any other $W_{j,n}$ is at least $6\eta n^{-\frac{1}{4}}$,

- $\partial W_{i,n}$ contains no vertex of $\widetilde{\mathcal{D}}_n$,
- no edge of $\widetilde{\mathcal{D}}_n$ contains a vertex of $W_{i,n}$,

moreover the union $W_n$ of all $W_{i,n}$ satisfies

$$\lim_{n\to\infty} |W_n|/|C| = 1. \tag{18}$$

In particular if $\Pi \in \widetilde{\mathcal{D}}_n$ satisfies $\Pi \cap W_{i,n} \neq \emptyset$ then

$$\Pi \text{ satisfies (17), and hence } \Pi \cap W_{j,n} = \emptyset \text{ for } j \neq i. \tag{19}$$

We define $\mathcal{D}_n$ to be the Dirichlet–Voronoi cell complex with respect to $\Xi_n$ and $W_n$, and hence the tiles of $\mathcal{D}_n$ are the non-empty intersections of the tiles of $\widetilde{\mathcal{D}}_n$ with $W_n$. For any Dirichlet–Voronoi cell $\Pi$ of $\mathcal{D}_n$, we write $a(\Pi)$ to denote the corresponding point of $\Xi_n$. In particular, vertices of $W_n$ have degree two as vertices of $\mathcal{D}_n$. If $v$ is a vertex of $\mathcal{D}_n$ different from the vertices of $W_n$ then $v$ is of degree three, and if, in addition, $v \in \operatorname{int} W_n$ and it is a common vertex of the Dirichlet–Voronoi cells $\Pi$ and $\widetilde{\Pi}$ of $\mathcal{D}_n$ then $v \notin \left[a(\Pi), a(\widetilde{\Pi})\right]$.

Next for any triple $(\Pi, e, v)$ where $\Pi$ is a Dirichlet–Voronoi cell of $\mathcal{D}_n$, $e$ is a side of $\Pi$ intersecting $\operatorname{int} W_n$, and $v$ is an endpoint of $e$, we define a *prescheme* $S$ associated to $\mathcal{D}_n$ to be $S = \left[a(\Pi), v, w\right]$ where $w$ is the closest point of $e$ to $a(\Pi)$. We write $a(S) = a(\Pi)$, $w(S) = w$ and $v(S) = v$. We note that possibly $a(S) \notin W_n$, and either $S$ is a segment with $v(S) = w(S)$, or $S$ is a triangle whose angle at $w(S)$ is at least $\pi/2$.

We call a prescheme $S$ a *scheme* associated to $\mathcal{D}_n$ if it is a triangle, and either $v(S) \in \operatorname{int} W_n$ or $w(S) \in \operatorname{int} W_n$. Let $\Sigma_n$ denote the family of schemes associated to $\mathcal{D}_n$, and hence (18) and (19) yield

$$\lim_{n\to\infty} |\cup \Sigma_n|/|C| = 1. \tag{20}$$

We observe that for any scheme $S$ associated to $\mathcal{D}_n$, the reflected image $\widetilde{S}$ of $S$ through the line passing through $v(S)$ and $w(S)$ is a scheme, as well, with $v(S)=v(\widetilde{S})$ and $w(S)=w(\widetilde{S})\in\left[a(S),a(\widetilde{S}),v(S)\right]$. Moreover the conditions on $\Xi_n$ at the beginning of this section ensure that if $w(S)\in\operatorname{int}W_n$ and it is a vertex of $\mathcal{D}_n$ then $\left[w(S),v(S)\right]\cap\left[a(S),a(\widetilde{S})\right]=\emptyset$, and the angle of $S$ at $w(S)$ is obtuse.

We call an edge $e$ of $\mathcal{D}_n$ a *skew edge* of $\mathcal{D}_n$ if $e=S\cap\widetilde{S}$ for some schemes $S$ and $\widetilde{S}$ associated to $\mathcal{D}_n$ such that $e\cap\left[a(S),a(\widetilde{S})\right]=\emptyset$ and $w(S)=w(\widetilde{S})\in\operatorname{int}W_n$ (see Figure 2). In this case $e=\left[w(S),v(S)\right]$, and we call $w(S)=$

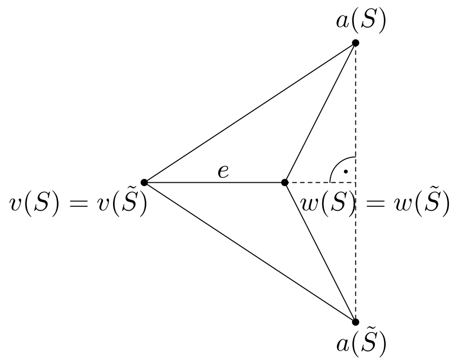


*Fig. 2*

$w(\widetilde{S})$ the initial endpoint of $e$, and $v(S)=v(\widetilde{S})$ the terminal endpoint of $e$. We note that possibly $v(S)\in\partial W_n$, and $e=\Pi\cap\widetilde{\Pi}$ for the Dirichlet–Voronoi cells $\Pi$ and $\widetilde{\Pi}$ of $\mathcal{D}_n$ with $a(\Pi)=a(S)$ and $a(\widetilde{\Pi})=a(\widetilde{S})$.

We define a related planar graph $G_n$. Its vertex set consists of the vertices of $\mathcal{D}_n$ that lie either in $\operatorname{int}W_n$ or are endpoints of the skew edges of $\mathcal{D}_n$. The edges of $G_n$ are the skew edges of $\mathcal{D}_n$. In particular the vertices of $G_n$ are the points of the form $v(S)$ as $S$ runs through schemes associated to $\mathcal{D}_n$. It follows by applying the Euler theorem in each $W_{i,n}$ that the number of vertices of $\mathcal{D}_n$ in $W_{i,n}$ is two more than twice the number of cells of $\mathcal{D}_n$ lying in $W_{i,n}$. Since none of the four vertices of $W_{i,n}$ is a vertex of $G_n$, (19) yields that

(21) the number of vertices of $G_n$ is at most $2n$.

### 3.3. The stars of the vertices

Next we define the *star* $\operatorname{St}(v)$ for any vertex $v$ of $G_n$ to be the family of all schemes $S$ with $v(S)=v$. Let $G_{i,n}$, $i=1,\dots,k(n)$, be the connected components of $G_n$. For each $G_{i,n}$, let $m_{i,n}$ be the number of vertices of $G_{i,n}$, and let $\Psi_{i,n}$ be the union of all $\operatorname{St}(v)$ where $v$ is a vertex of $G_{i,n}$.

We observe that $m_{i,n} = 1$ if and only if $G_{i,n}$ is an isolated vertex $v$ of $G_n$. In this case, $\cup\Psi_{i,n} = \cup\,\mathrm{St}\,(v)$ is a triangle with circumcentre $v \in \mathrm{int}\,W_n$. Writing $r$ to denote the circumradius of $\cup\,\mathrm{St}\,(v)$, we define

$$H(v) = \frac{1}{r} \bigcup_{S\in \mathrm{St}\,(v)} \left\{ S - a(S), a(S) - S \right\},$$

which is a hexagon satisfying $H(v) \in \mathcal{X}$ (see Figure 3). Therefore if $v$ is an

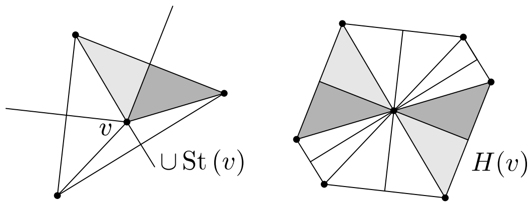


*Fig. 3*

isolated vertex of $G_n$ and $\mathrm{St}\,(v) = \Psi_{i,n}$ then

$$\frac{\sum_{S\in\Psi_{i,n}} \int_S q\big(x - a(S)\big)\, dx}{|\cup \Psi_{i,n}|^2} = \frac{\frac{1}{2}\int_{H(v)} q(x)\, dx}{\frac{1}{4}\big|H(v)\big|^2} \geqq 2I(q). \tag{22}$$

Our main goal is to show that an even better estimate holds for $\Psi_{i,n}$ if either $m_{i,n} \geqq 2$ or $m_{i,n} = 1$ and $H(v)$ is "far" from $H_q$. To make this idea more specific, let us introduce some notation and constants. Choose three non-neighbouring vertices of $H_q$, and let $T_q$ be their convex hull, hence $|T_q| = \frac{1}{2}|H_q|$. In particular if $v$ is an isolated vertex of $G_n$ and $\cup\,\mathrm{St}\,(v)$ is homothetic either to $T_q$ or to $-T_q$ then $H(v) = H_q$. We note that for $\mu \in \left(0, \frac{1}{2}\right)$, if a triangle $T$ is $\mu/4$-close to $\lambda T_q$ for some $\lambda > 0$, and satisfies

$$\big(1 + (\mu/4)\big)^{-1} A(\lambda T_q) \leqq A(T) \leqq \big(1 + (\mu/4)\big)\, A(\lambda T_q),$$

then $T$ is $\mu$-close to $T_q$. Now there exists some $\mu \in (0, \frac{1}{2})$ depending only on $q$ and $\nu$ with the following property. If $\Pi \subset \mathrm{int}\,W_n$ is a Dirichlet–Voronoi cell of $\mathcal{D}_n$ whose vertices are isolated vertices of $G_n$, and there exists $\lambda > 0$ such that for each vertex $v$ of $\Pi$, $\cup\,\mathrm{St}\,(v)$ is $\mu$-close either to $\lambda T_q$ or to $-\lambda T_q$, then $\Pi$ is a hexagon that is $\nu$-close to $\lambda H_q$. Since $H_q$ is the unique extremal hexagon in $\mathcal{X}$ according to Theorem 1.1, there exists some $\delta$ with the following properties:

$$1 < 1 + \delta < \frac{1.01533}{1.01532},$$

and if $v$ is an isolated vertex of $G_n$, and $\cup\operatorname{St}(v)$ is not $\mu/4$-close to any homothetic copy of either $T_q$ or $-T_q$, then

$$\frac{\sum_{S\in\operatorname{St}(v)}\int_S q\big(x-a(S)\big)\,dx}{\big|\cup\operatorname{St}(v)\big|^2}=2\cdot\frac{\int_{H(v)}q(x)\,dx}{\big|H(v)\big|^2}\geqq(1+\delta)2I(q). \tag{23}$$

The core statement to prove (12) (and in turn Theorem 1.2) is the following estimate. If either $m_{i,n}\geqq 2$, or $m_{i,n}=1$ and $\cup\Psi_{i,n}$ is not $\mu/4$-close to any homothetic copy of either $T_q$ or $-T_q$ then

$$\sum_{S\in\Psi_{i,n}}\int_S q\big(x-a(S)\big)\,dx\geqq(1+\delta)\cdot\frac{2I(q)}{m_{i,n}}\cdot|\cup\Psi_{i,n}|^2. \tag{24}$$

If $m_{i,n}=1$ then (24) follows by (23).

### 3.4. The proof of (24) if $m_{i,n}\geqq 2$

Our plan is to apply a linear map that transforms $q$ into the Euclidean form, and to show (24) using estimates on the second moment. Unfortunately the estimates in this case hold only if there is a restriction on $\kappa$. We define the linear transformation $\Phi$ by $\Phi(s,t)=(s,t\sqrt{\kappa})$ for $(s,t)\in\mathbb{R}^2$. In particular $q(x)=\|\Phi x\|^2$ and $\det\Phi=\sqrt{\kappa}$. We set $G^*_{i,n}=\Phi G_{i,n}$, $\Xi^*_n=\Phi\Xi_n$ and $W^*_n=\Phi W_n$. For any $S\in\Psi_{i,n}$, we call $\Phi S$ a scheme for $G^*_{i,n}$, and define $a(\Phi S)=\Phi a(S)$, $v(\Phi S)=\Phi v(S)$ and $w(\Phi S)=\Phi w(S)$. If $e$ is an edge of $G_{i,n}$ with initial endpoint $w$ and terminal endpoint $v$, then we say that $\Phi w$ is the initial endpoint and $\Phi v$ is the terminal endpoint of $\Phi e$.

We observe that $\Phi\mathcal{D}_n$ is typically not the the family of Dirichlet–Voronoi cells with respect to $\Xi^*_n$ and $W^*_n$. Let us see what properties of the schemes prevail after applying $\Phi$.

Let $S$ and $\widetilde{S}$ be two schemes for $G^*_{i,n}$ with $v(S)=v(\widetilde{S})$ and $w(S)=w(\widetilde{S})$. We call these triangles twins, and observe that they satisfy

$$|S|=|\widetilde{S}|\quad\text{and}\quad w(S)\in\big[v(S),a(S),a(\widetilde{S})\big], \tag{25}$$

even if $S$ and $\widetilde{S}$ are typically not congruent. In addition we define

$$\alpha(S)=\alpha(\widetilde{S})=\frac{\angle\big(v(S),a(S),w(S)\big)+\angle\big(v(S),a(\widetilde{S}),w(S)\big)}{2}<\frac{\pi}{2}.$$

In particular, Lemma 3.1 yields

$$\int_S \|y - a(S))\|^2\,dy + \int_{\widetilde S} \|y - a(\widetilde S))\|^2\,dy \geqq \gamma\big(\alpha(S)\big)\,|S|^2 + \gamma\big(\alpha(\widetilde S)\big)\,|\widetilde S|^2. \tag{26}$$

Let us define the *star of a vertex* $v$ *of* $G^*_{i,n}$. If $v \in \operatorname{int} W^*_n$ then the star $\mathrm{St}^*\, v$ at $v$ is simply the set of schemes $S$ for $G^*_{i,n}$ with $v(S) = v$. Next let $v \in \partial W^*_n$. So far, $v$ is the vertex of exactly two schemes $S$ and $\widetilde S$ for $G^*_{i,n}$, and these two schemes are twins intersecting in an edge of$G^*_{i,n}$. For technical purposes, also in this case, we need six schemes at $v$ whose angles at $v$ add up to $2\pi$, hence we define four "degenerate schemes". We choose four segments $S_1$, $S_2$, $S_3$, $S_4$ such that $v$ is an endpoint of each, and each intersects $S \cup \widetilde S$ in $v$. Let $\beta$ be the sum of the angles of $S$ and $\widetilde S$ at $v$ (see Figure 4). For

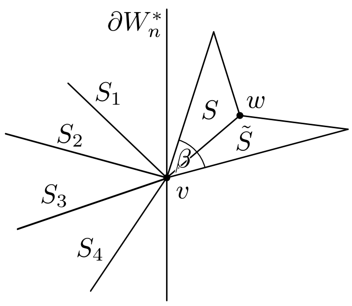


*Fig. 4*

$i = 1, 2, 3, 4$, we define $v(S_i) = v$, $a(S_i)$ to be the other endpoint of $S_i$, and $w(S_i)$ to be the midpoint of $S_i$. In addition, we define the angle of $S_i$ at $v(S_i)$ to be $\frac{2\pi-\beta}{4} = \frac{\pi}{2} - \frac{\beta}{4}$, at $w(S_i)$ to be $\frac{\pi}{2}$, and at $a(S_i)$ to be $\alpha(S_i) = \frac{\beta}{4}$. We call $S_1$ and $S_2$ twins as well as $S_3$ and $S_4$. We set integrals over any $S_i$ to be zero, and define $\mathrm{St}^*(v) = \{S_1, S_2, S_3, S_4, S, \widetilde S\}$.

Finally we define $\Psi^*_{i,n}$ to be the union of all $\mathrm{St}^*(v)$ as $v$ runs through the vertices of $G^*_{i,n}$, $i = 1, \ldots, k(n)$. Since the substitution $y = \Phi x$ yields

$$\frac{\sum_{S\in\Psi_{i,n}} \int_S q\big(x - a(S)\big)\,dx}{|\cup \Psi_{i,n}|^2} = \sqrt{\kappa} \cdot \frac{\sum_{S\in\Psi^*_{i,n}} \int_S \|y - a(S)\|^2\,dy}{\big|\cup \Psi^*_{i,n}\big|^2},$$

and Lemma 2.3 yields

$$2\,I(q) < 1.01532 \cdot \sqrt{\kappa} \cdot \frac{\gamma\left(\frac{\pi}{6}\right)}{6} \quad \text{for } \kappa \in [1, 2.4],$$

our goal (24) follows if

$$\sum_{S\in\Psi^*_{i,n}} \int_S \left\| y - a(S) \right\|^2 dy > \frac{1.01533\gamma\left(\frac{\pi}{6}\right)}{6m_{i,n}} \cdot \left| \cup \Psi^*_{i,n} \right|^2. \tag{27}$$

Applying (26) and the Cauchy–Schwarz inequality leads to

$$\frac{\sum_{S\in\Psi^*_{i,n}} \int_S \left\| y - a(S) \right\|^2 dy}{\left| \cup \Psi^*_{i,n} \right|^2} \geqq \frac{\sum_{S\in\Psi^*_{i,n}} \gamma\left(\alpha(S)\right) |S|^2}{\left| \cup \Psi^*_{i,n} \right|^2}$$

$$\geqq \left( \sum_{S\in\Psi^*_{i,n}} \gamma\left(\alpha(S)\right)^{-1} \right)^{-1}.$$

We write $e_{i,n}$ to denote the number of edges of $G^*_{i,n}$, hence

$$\#\Psi^*_{i,n} = 6m_{i,n} - 2e_{i,n}.$$

We observe that the sum of the angles at the vertices of $G^*_{i,n}$ of the elements of $\Psi^*_{i,n}$ is $m_{i,n}2\pi$. The contribution of a twin $S$ and $\widetilde{S}$ of schemes of $\Psi^*_{i,n}$ to this sum is $2\pi - \alpha(S) - \alpha(\widetilde{S})$ if $S \cap \widetilde{S}$ is an edge of $G^*_{i,n}$, and $\pi - \alpha(S) - \alpha(\widetilde{S})$ otherwise. Since $e_{i,n}$ twins intersect in an edge of $G^*_{i,n}$, we have

$$\sum_{S\in\Psi^*_{i,n}} \alpha(S) = m_{i,n}\pi.$$

First assume $m_{i,n} \geqq 3$, hence $e_{i,n} \geqq m_{i,n} - 1$ as $G^*_{i,n}$ is connected. Since $\alpha\gamma(\alpha)$ is increasing and $\gamma(\alpha)^{-1}$ is concave on $(0, \frac{\pi}{2})$ (see Propositions 3.4 and 3.5, respectively, in K. J. Böröczky, P. Tick, G. Wintsche [7]), we have

$$6m_{i,n}\left( \sum_{S\in\Psi^*_{i,n}} \gamma\left(\alpha(S)\right)^{-1} \right)^{-1} \geqq \frac{6m_{i,n}}{6m_{i,n} - 2e_{i,n}} \cdot \gamma\left( \frac{m_{i,n}\pi}{6m_{i,n} - 2e_{i,n}} \right)$$

$$\geqq \frac{6m_{i,n}}{4m_{i,n} + 2} \cdot \gamma\left( \frac{m_{i,n}\pi}{4m_{i,n} + 2} \right) \geqq \frac{9}{7} \cdot \gamma\left( \frac{3\pi}{14} \right).$$

Numerical evaluation shows $\frac{9}{7}\gamma\big(\frac{3\pi}{14}\big) > 1.01533\,\gamma\big(\frac{\pi}{6}\big)$, which yields (27) in this case.

Finally we assume $m_{i,n} = 2$, hence $\#\Psi^*_{i,n} = 10$. In this case $G^*_{i,n}$ has a unique edge, which is the intersection of say $S_1, S_2 \in \Psi^*_{i,n}$. Now $\mathrm{St}^*\big(v(S_1)\big)$ has four more elements, which we denote by $S_3$, $S_4$, $S_5$, $S_6$. In addition $\mathrm{St}^*\big(w(S_1)\big)$ has four elements denoted by $S_7$, $S_8$, $S_9$, $S_{10}$. Since the sum of the angles of $S_1$ and $S_2$ at $w(S_1)$ is at least $\pi$, there exists some $\varphi \geqq 0$ such that

$$\alpha(S_1) + \alpha(S_2) + \alpha(S_3) + \alpha(S_4) + \alpha(S_5) + \alpha(S_6) = \pi - \varphi;$$
$$\alpha(S_7) + \alpha(S_8) + \alpha(S_9) + \alpha(S_{10}) = \pi + \varphi.$$

It follows that

$$6m_{i,n}\Bigg(\sum_{S\in\Psi^*_{i,n}} \gamma\big(\alpha(S)\big)^{-1}\Bigg)^{-1} \geqq 12\left(6\gamma\left(\frac{\pi-\varphi}{6}\right)^{-1} + 4\gamma\left(\frac{\pi+\varphi}{4}\right)^{-1}\right)^{-1}.$$

As the derivative of $\gamma(\alpha)^{-1}$ is decreasing, the function

$$6\gamma\left(\frac{\pi-\varphi}{6}\right)^{-1} + 4\gamma\left(\frac{\pi+\varphi}{4}\right)^{-1}$$

is decreasing in $\varphi \in [0, \pi)$. Therefore

$$6m_{i,n}\Bigg(\sum_{S\in\Psi^*_{i,n}} \gamma\big(\alpha(S)\big)^{-1}\Bigg)^{-1} \geqq 12\left(6\gamma\left(\frac{\pi}{6}\right)^{-1} + 4\gamma\left(\frac{\pi}{4}\right)^{-1}\right)^{-1}$$
$$> 1.01533\gamma\left(\frac{\pi}{6}\right),$$

where the last inequality follows by numerical evaluation. We conclude (27), and in turn (24).

### 3.5. Proof of (12) based on (22) and (24)

As the sequence $\{\Xi_n\}$ is asymptotically optimal, (3) yields

(28) $$\Omega(q, C, \Xi_n) \leqq I(q)\cdot|C|^2 n^{-1} + o\big(n^{-1}\big).$$

Let us number the components $G_{i,n}$ in a way such that $i \leqq g(n)$ if and only if $G_{i,n}$ is an isolated vertex, and $\cup\Psi_{i,n}$ is $\mu/4$-close to some homothetic copy of either $T_q$ or $-T_q$. In particular $m_{i,n} = 1$ if $i \leqq g(n)$.

It follows by (22), (24), and the Cauchy–Schwarz inequality that

(29)

$$\Omega(q, C, \Xi_n) \geqq \sum_{i=1}^{k(n)} \sum_{S\in\Psi_{i,n}} \int_S q\big(x - a(S)\big)\, dx$$

$$\geqq 2I(q) \sum_{i=1}^{k(n)} \frac{|\cup \Psi_{i,n}|^2}{m_{i,n}} + \delta\, 2I(q) \sum_{i>g(n)} \frac{|\cup \Psi_{i,n}|^2}{m_{i,n}}$$

$$\geqq 2I(q) \cdot \frac{\big(\sum_{i=1}^{k(n)} |\cup \Psi_{i,n}|\big)^2}{\sum_{i=1}^{k(n)} m_{i,n}} + \delta\, 2I(q) \cdot \frac{\big(\sum_{i>g(n)} |\cup \Psi_{i,n}|\big)^2}{\sum_{i>g(n)} m_{i,n}}.$$

Since $\sum_{i=1}^{k(n)} m_{i,n} \leqq 2n$ according to (21), the estimates (20) and (28) yield

$$\sum_{i>g(n)} |\cup \Psi_{i,n}| = o(1).$$

In turn we deduce by (20) that

$$\sum_{i=1}^{g(n)} |\cup \Psi_{i,n}| = |C| - o(1). \tag{30}$$

Let $A = \big(\sum_{i=1}^{g(n)} |\cup \Psi_{i,n}|\big)/g(n)$, and for each $i = 1, \ldots, g(n)$, let $t_i = |\cup \Psi_{i,n}| - A$ (here we drop the reference to $n$). In particular $\sum_{i=1}^{g(n)} t_i = 0$. Since $m_{i,n} = 1$ if $i \leqq g(n)$, we deduce by (28), (29) and (30) that

$$\frac{I(q)\cdot |C|^2}{n} + o\big(n^{-1}\big) \geqq 2I(q) \sum_{i=1}^{g(n)} (A + t_i)^2 = 2I(q) g(n) A^2 + 2I(q) \sum_{i=1}^{g(n)} t_i^2$$

$$\geqq \frac{2I(q)\cdot |C|^2}{g(n)} + o\big(n^{-1}\big) + 2I(q) \sum_{i=1}^{g(n)} t_i^2.$$

In particular $g(n) = 2n - o(n)$. We renumber the $t_i$, $i = 1, \ldots, g(n)$, in a way such that $|t_i| \leqq \frac{\mu}{8} \cdot \frac{|C|}{2n}$ if and only if $i \leqq h(n)$, hence $h(n) = 2n - o(n)$, as well. Since $A = \frac{|C|}{2n} + o\left(n^{-1}\right)$, it follows that for large $n$ and any $i \leqq h(n)$, the area of the triangle $\cup\Psi_{i,n}$ is between $\left(1+(\mu/4)\right)^{-1}\frac{|C|}{2n}$ and $\left(1+(\mu/4)\right)\frac{|C|}{2n}$, therefore $\cup\Psi_{i,n}$ is $\mu$-close to the homothet of either $T_q$ or $-T_q$ of area $\frac{|C|}{2n}$.

Since each vertex of $\mathcal{D}_n$ but the vertices of $W_{i,n}$ is of degree three, the number of Dirichlet–Voronoi cells whose all vertices are some $G_{i,n}$ for $i \leqq h(n)$ is $n - o(n)$. All these Dirichlet–Voronoi cells are $\nu$-close to the homothet of $H_q$ with area $\frac{|C|}{n}$ by the definition of $\mu$. All but $o(n)$ of them lies in $\operatorname{int} W_n$, hence $n - o(n)$ of these Dirichlet–Voronoi cells are Dirichlet–Voronoi cells for $\Xi_n$ with respect to $C$. We conclude (12), and in turn Theorem 1.2.

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