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EXTREMAL MEAN WIDTH WHEN COVERING THE 1-SKELETON

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Dedicated to Imre Bárány on occasion of his sixtieth birthday

Abstract

For a given convex body K in $\mathbb{R}^d$, let D_n be the compact convex set of maximal mean width whose 1-skeleton can be covered by n congruent copies of K. Based on the fact that the mean width is proportional to the average perimeter of two-dimensional projections, it is proved that D_n is close to being a segment for large n.

1. *Introduction*

Let us introduce the notation used in this paper. A compact convex set with non-empty interior in $\mathbb{R}^d$ is called a *convex body*, and a two-dimensional compact convex set is called a *convex domain*. We write $V(\cdot)$ to denote the volume (d-dimensional Lebesgue measure) in $\mathbb{R}^d$, and B^d to denote the Euclidean unit ball centred at the origin, and we define $\kappa_d = V(B^d)$. Given a compact convex set C in $\mathbb{R}^d$ and $i = 1, \ldots, d-1$, the mean i-dimensional projection $M_i(C)$ of C is the average i-dimensional measure of the projections of C into linear i-dimensional linear subspaces; $M_1(C)$ is the mean width. We refer to Schneider [8] for the precise definition and main properties of the mean projections, and to Gardner [3] and Santaló [7] for the notions that we need about integral geometry. In addition, the intrinsic i-volume $V_i(C)$ of C is defined by the Steiner formula (see [8]):

$$V(C + \lambda B^d) = \sum_{i=1}^{d} \kappa_{d-i} V_i(C) \cdot \lambda^{d-i}, \qquad \text{for } \lambda \geqslant 0.$$

We note that the mean i-dimensional projection and the intrinsic i-volume are related by

$$M_i(C) = \binom{d}{i}^{-1} \frac{\kappa_{d-i}\kappa_i}{\kappa_d} \cdot V_i(C), \qquad \text{for } i = 1, \ldots, d-1.$$

We present our results in terms of the intrinsic i-volume instead of the mean i-dimensional projection because it is more convenient to work with, since the intrinsic i-volume of an i-dimensional compact convex set is its i-dimensional volume. In addition, the first intrinsic volume of a convex domain is half of its perimeter.

Given a convex body K in $\mathbb{R}^d$, our main goal is to understand extremal coverings by n congruent copies of K with respect to the mean width. Since the mean width measures the one-dimensional size, we require that the n congruent copies of K cover the 1-*skeleton* $\text{skel}_1 D$ of a compact convex set D, namely, the family of points $x \in D$ such that no two-dimensional circular disc centred at x is a subset of C. The notion of a 1-skeleton was introduced by Larman and Rogers [5]. It may not be closed, and it is not continuous with respect to the Hausdorff metric on compact convex sets (while we verify that it satisfies a certain semi-continuity property). Still, the 1-skeleton of any compact convex set is connected, according to [5]. Let us observe that in the planar case, the 1-skeleton is simply the boundary. This fact

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points towards the main idea of the paper; that is, we use the fact that the mean width is proportional to the average perimeter of two-dimensional projections.

We note that coverings of the 1-skeleton of a compact convex set by translates of a convex body K have already been investigated. More precisely, Fejes Tóth, Gritzmann and Wills [2] prove that if n translates of K cover a compact convex set D, then

$$V_1(D) \leqslant (n-1) \cdot \operatorname{diam} K + V_1(K).$$

Moreover, if $d = 2$ and K is the unit circular disc, then they prove that

$$V_1(D) \leqslant 2 \cdot \left(\sqrt{n^2 - 1} + \arcsin \frac{1}{n}\right),$$

where equality holds if and only if the centres are aligned and the distance between any two consecutive centres is $2\sqrt{1 - 1/n^2}$.

In this paper, we further develop the results of [2]. Since we want to show that a certain compact convex set D is close to a segment, let us consider the radius $r_2(D)$ of the largest two-dimensional circular disc contained in D. We claim that, if s is the diameter of D, then

$$D \subset s + 3r_2(D) \cdot B^d. \tag{1}$$

In order to verify (1), let h be the maximal distance of points of D from s. Then D contains a triangle T such that s is the longest side, and the distance of the opposite vertex from s is h. Therefore (1) follows by

$$\frac{hs}{2} = A(T) \leqslant \frac{3s \cdot r_2(T)}{2}.$$

Before stating the theorem, we quote [2] by observing that the segment s_n of length $n \cdot \operatorname{diam} K$ can be readily covered by n translates of K, and

$$V_1(s_n) = n \cdot \operatorname{diam} K. \tag{2}$$

THEOREM 1.1. *Given $n > c^d$ for some positive absolute constant c, if K is any convex body in $\mathbb{R}^d$, then there exists a compact convex set D_n such that $V_1(D_n)$ is maximal under the condition that the 1-skeleton of D_n can be covered by n congruent copies of K, and D_n satisfies the following conditions:*

(i)
$$r_2(D_n) < \frac{48\sqrt{2\pi\, d}}{n} \cdot \operatorname{diam} K;$$

(ii)
$$V_1(D_n) \leqslant \left(n + \frac{24\pi\, d}{n}\right) \cdot \operatorname{diam} K.$$

REMARKS. (a) Actually, we verify the existence of an optimal D_n for any n.

(b) We also verify that inequalities (i) and (ii) hold if not only the 1-skeleton of D_n, but also the whole of D_n, is covered by $n \geqslant 215d$ congruent copies of K.

We note that the estimates of Theorem 1.1 are essentially optimal (see Example 2.1). During the proof we will use the fact that the unit ball B^d satisfies $V_1(B^d) = d\kappa_d/\kappa_{d-1}$; hence (see Betke, Gritzmann and Wills [1])

$$d \cdot \sqrt{\frac{2\pi}{d+1}} < V_1(B^d) < \sqrt{2\pi d}. \tag{3}$$

The case when the whole of D_n is covered by the congruent copies of K in Theorem 1.1 is much easier to handle; therefore we discuss it first in Section 2, where we also present Example 2.1, showing the optimality of Theorem 1.1. After that, we consider the problem when the 1-skeleton of D_n is covered. We establish in Section 3 that the extremal D_n actually exists. Finally, in Section 4 we show that $r_2(D_n)$ is small, where the most involved part of the argument is to verify that $r_2(D_n)$ stays bounded as n tends to infinity. In Section 5, we briefly summarize the known extremal properties of coverings with respect to other mean projections.

In the proof of Theorem 1.1, we assume that $\operatorname{diam} K = 1$, and we write $K_1, \dots, K_n$ to denote the n congruent copies of K taking part in the covering. We fix a diameter $\tilde{s}$ of the extremal compact convex set D_n, and we write $\tilde{h}$ to denote the maximal distance of the points of D_n from $\tilde{s}$, and $\tilde{T}$ to denote a triangle contained in D_n such that $\tilde{s}$ is a side and the opposite vertex is at a distance $\tilde{h}$ from $\tilde{s}$. Since $V_1(B^d) < \sqrt{2\pi d}$ (see (3)) and D_n is contained in $\tilde{s} + \tilde{h} \cdot B^d$, we deduce that

$$V_1(D_n) \leqslant \operatorname{diam} D_n + \tilde{h} \cdot \sqrt{2\pi d}. \tag{4}$$

On the other hand, (2) yields

$$V_1(D_n) \geqslant n. \tag{5}$$

In addition, we assume that $o \in \tilde{T}$, and we write $\tilde{L}$ to denote the two-dimensional linear subspace spanned by $\tilde{T}$, and $\tilde{C}$ to denote $D_n|\tilde{L}$, where $\cdot|\tilde{L}$ stands for the projection into $\tilde{L}$.

2. *When the whole of D_n is covered*

Each $K_i|\tilde{L}$ has diameter at most one, and hence is of perimeter at most π according to the Cauchy formula. In turn, the isometric inequality yields that the area of $K_i|\tilde{L}$ is at most $\pi/4$. As the projections $K_1|\tilde{L}, \dots, K_n|\tilde{L}$ cover $\tilde{T}$, we deduce that $\operatorname{diam} D_n \leqslant \pi n/2\tilde{h}$ and $\tilde{h} \leqslant \sqrt{\pi n/2}$. Substituting these estimates into (4) results in

$$V_1(D_n) \leqslant \frac{\pi}{2\,\tilde{h}} \cdot n + \pi\sqrt{d} \cdot \sqrt{n}.$$

Therefore inequality (5) yields $\tilde{h} \leqslant 2$ for $n \geqslant 215d$.

Next we claim that

$$\operatorname{diam} D_n \leqslant \left(1 - \frac{\tilde{h}^2}{48}\right) \cdot n. \tag{6}$$

We may assume that

$$\operatorname{diam} D_n \geqslant \left(1 - \frac{2^2}{48}\right) \cdot n \geqslant \frac{11n}{12};$$

hence $\tilde{s}$ contains a segment s' of length at least $11n/24$ such that any line in $\tilde{L}$ orthogonally intersecting s' intersects D_n in a segment of length at least $\tilde{h}/2$. We write σ and σ' to denote the two arcs of $\partial\tilde{C}$ connecting the endpoints of $\tilde{s}$. If K_i intersects σ, then let p_i and q_i be the furthest points of $\sigma \cap K_i$, and otherwise let $p_i = q_i$ be any point of σ. We define the points $p_i', q_i' \in \sigma'$ analogously, and claim that, if the projection of K_i into $\operatorname{aff} \tilde{s}$ is contained in s', then

$$d(p_i, q_i) + d(p_i', q_i') \leqslant 2 - \frac{\tilde{h}^2}{8}. \tag{7}$$

We may assume that p_i, q_i, p_i' and q_i' are vertices of a quadrilateral Q. Since Q has an angle at least $\pi/2$, and its diagonals are at most one, we deduce that either p_iq_i or $p_i'q_i'$ is of length at most

$$\sqrt{1 - \tilde{h}^2/4}.$$

In turn, the claim (7) readily follows. Now the number of the K_i whose projection into $\operatorname{aff} \tilde{s}$ is contained in s' is at least $11n/24 - 2 \geqslant n/3$; therefore,

$$\operatorname{diam} D_n \leqslant \frac{1}{2}\left(\sum_{i=1}^{n} d(p_i, q_i) + d(p_i', q_i')\right) \leqslant \frac{2n}{3} + \frac{n}{3} \cdot \left(1 - \frac{\tilde{h}^2}{16}\right) = \left(1 - \frac{\tilde{h}^2}{48}\right) \cdot n,$$

completing the proof of (6). In turn, (4) yields

$$V_1(D_n) \leqslant \left(1 - \frac{\tilde{h}^2}{48}\right) \cdot n + \tilde{h} \cdot \sqrt{2\pi d}. \tag{8}$$

Optimizing the upper bound in $\tilde{h}$ leads to Theorem 1.1(ii), and part (i) is a direct consequence of $r_2(D_n) \leqslant \tilde{h}$ and $V_1(D_n) \geqslant n$. □

Let us show that the estimates of Theorem 1.1 are essentially optimal.

EXAMPLE 2.1. (a) There exists a convex body K in $\mathbb{R}^d$ such that the optimal D_n in Theorem 1.1 is a segment of length $n \operatorname{diam} K$ for large n.

We consider an isosceles triangle in $\mathbb{R}^d$ with equal angles $\pi/6$, and we define K to be the convex body resulting from rotating the triangle around its longest side in $\mathbb{R}^d$. Assuming that this longest side is one, the inequality (8) in the proof of Theorem 1.1 can be replaced by

$$V_1(D_n) \leqslant n - \tilde{h} \cdot cn + \tilde{h} \cdot \sqrt{2\pi d}$$

for suitable positive absolute constant c. Therefore $\tilde{h} = 0$ for large n.

(b) If $K = B^d$, then

$$r_2(D_n) > \frac{\sqrt{d}}{50n} \qquad \text{and} \qquad V_1(D_n) > 2n + \frac{d}{6n} \quad \text{for large } n.$$

It follows by (1) and $18V_1(B^d) < 50\sqrt{d}$ that it is sufficient to verify the inequality for $V_1(D_n)$. Let Z_n be the right cylinder with base a $(d-1)$-ball of radius $\varrho = V_1(B^{d-1})/(2n)$ and height $2n(1 - 2\varrho^2/3)$. Then Z_n can be covered by n unit balls for large n, and

$$\begin{aligned} V_1(Z_n) &= 2n \cdot \left(1 - \frac{2\varrho^2}{3}\right) + \varrho \cdot V_1(B^{d-1}) \\ &= 2n + \frac{1}{6n} \cdot V_1(B^{d-1})^2, \end{aligned}$$

where $V_1(B^{d-1}) > \sqrt{d}$ according to (3).

3. *Existence of the extremal set when the* 1*-skeleton is covered*

When the 1-skeleton of D_n is covered, the proof of Theorem 1.1 is mostly based on the following simple property: if L is a two-dimensional linear subspace and D is a compact convex set, then $\partial(D|L)$ is contained in the projection of $\operatorname{skel}_1 D$ into L. In particular,

$$\partial(D_n|L) \subset K_1|L \cup \ldots \cup K_n|L, \tag{9}$$

which in turn yields that

$$\operatorname{diam} D_n \leqslant n. \tag{10}$$

Now the existence of the extremal D_n readily follows from (10), the Blaschke selection theorem and the following claim.

LEMMA 3.1. *Given a sequence of compact convex sets* $\{C_k\}$ *that tends to a compact convex set* C*, if* $x \in \operatorname{skel}_1 C$ *and* $\varepsilon > 0$*, then* $(x + \varepsilon B^d) \cap \operatorname{skel}_1 C_k \neq \emptyset$ *holds for large* k.

REMARK. The analogous statement holds for the m-skeleton (see Section 5 for the definition).

Proof of Lemma 3.1. We suppose that Lemma 3.1 does not hold for certain $\varepsilon > 0$, and seek a contradiction. We choose a hyperplane H in such a way that x is an extremal point of $H \cap C$.

Let x_k be the closest point of C_k to x, and let H_k be the hyperplane that contains x_k, and is parallel to H. We deduce that, if k is large, then $d(x, x_k) < \varepsilon/2$, and x_k is not the midpoint of any segment in $H_k \cap C_k$ that has length at least ε/d. For such a k, we write x_k in the form

$$\sum_{i=0}^{m} \lambda_i y_i,$$

where $m \leqslant d$, $y_0, \ldots, y_m$ are extremal points of $H_k \cap C_k$, and the coefficients satisfy $\lambda_0 \geqslant \ldots \geqslant \lambda_m \geqslant 0$ and $\sum_{i=0}^{m} \lambda_i = 1$. In particular, $\lambda_0 \geqslant 1/(d+1)$. Since $y_0 \in \operatorname{skel}_1 C_k$, we have $d(x, y_0) \geqslant \varepsilon$; hence $d(x_k, y_0) > \varepsilon/2$. In addition, the point

$$p = \frac{1}{1-\lambda_0} \sum_{i=1}^{m} \lambda_i y_i$$

of $H_k \cap C_k$ satisfies

$$p - x_k = \frac{-\lambda_0}{1-\lambda_0}(y_0 - x_k);$$

therefore the segment $y_0 p$ contains a segment with midpoint x_k length at least ε/d. This is absurd, which in turn yields Lemma 3.1. □

4. *The proof of Theorem* 1.1

The main part of the argument is to show that $\tilde{h}$ is bounded. Here we heavily use the non-trivial fact that the 1-skeleton is connected, according to [5].

First, we associate a geometric graph G to the covering *via* constructing a sequence of four graphs. Let G_1 be the graph on $K_1, \ldots, K_n$ as vertices such that a pair $\{K_i, K_j\}$ is an edge if and only if $i \neq j$, and K_i and K_j intersect. Since the 1-skeleton is connected, G is a connected graph. Let G_2 be a spanning tree of G_1, namely, a minimal connected graph on $K_1, \ldots, K_n$ such that each edge of G_2 is an edge of G_1. We number the $n-1$ edges of G_2, and associate a point $v_k \in K_i \cap K_j$ to the kth edge $\{K_i, K_j\}$ for $k = 1, \ldots, d-1$. Since coincidences may occur, let $\{v_1, \ldots, v_m\}$, for $m \leqslant n-1$, be the family of different points among $v_1, \ldots, v_{n-1}$.

Next we define the geometric graph G_3 on $v_1, \ldots, v_m$ in such a way that the segment $v_k v_l$ represents an edge if and only if $k \neq l$, so v_k and v_l are contained in the same K_i. Then G_3 is connected, as well, and we let G be a spanning tree of G_3. We write σ to denote the union of the edges of G, which is a connected set consisting of segments. In addition, the total length $|\sigma|$ of σ is at most $n-2$, and

$$\operatorname{skel}_1 D_n \subset \sigma + B^d.$$

We think about the first intrinsic volume as a mean perimeter of two-dimensional projections. Let $\mu_{d,i}$ denote the unique invariant measure on the Grassmanian $\mathrm{Gr}(d,i)$ of linear i-spaces in $\mathbb{R}^d$ such that $\mu_{d,i}(\mathrm{Gr}(d,i)) = 1$. Then

$$V_1(D_n) = \frac{V_1(B^d)}{\pi} \cdot \int_{\mathrm{Gr}(d,2)} V_1(D_n|L)\, d\mu_{d,2}(L),$$
$$|\sigma| = \frac{V_1(B^d)}{\pi} \cdot \int_{\mathrm{Gr}(d,2)} \big|\sigma|L\big|\, d\mu_{d,2}(L).$$

We note that if L is any linear two-space, then

$$\partial(D_n|L) \subset \sigma|L + B^d.$$

PROPOSITION 4.1. *Given a convex domain C in $\mathbb{R}^2$, let γ be a connected union of finitely many segments such that $\partial C \subset \gamma + B^2$. Then*

(i) $V_1(C) \leqslant |\gamma| + \pi$;

(ii) $\frac{33}{32} V_1(C) \leqslant |\gamma| + \pi$ *if* $r_2(C) \geqslant 19$.

Proof. If s is a segment and l is a linear subspace of dimension one, then the length of $s|l$ is just the integral of the function $\#(s \cap (x + l^\perp))$ over l. Therefore the Cauchy formula yields

$$V_1(C) = \frac{1}{2}P(C) = \frac{\pi}{2} \cdot \int_{\mathrm{Gr}(2,1)} \int_{l^\perp} \frac{1}{2} \cdot \#(\partial C \cap (x + l))\, dx\, d\mu_{2,1}(l),$$

$$|\gamma| = \frac{\pi}{2} \cdot \int_{\mathrm{Gr}(2,1)} \int_{l^\perp} \#(\gamma \cap (x + l))\, dx\, d\mu_{2,1}(l),$$

where the integrals readily make sense. (Actually, these formulae are well-known in geometry; see, say, [7].) For any line l passing through the origin, let a and $\tilde{a}$ denote the length of $C|l$ and that of $C|l^\perp$, respectively. Since the projections of γ into l and $l^\perp$ are of length at least $a - 2$ and $\tilde{a} - 2$, respectively, we deduce that

$$a + \tilde{a} \leqslant \int_{l^\perp} \#(\gamma \cap (x + l))\; dx + \int_l \#(\gamma \cap (x + l^\perp))\, dx + 4,$$

which in turn yields statement (i). Therefore, let $r_2(C) > 19$, and we claim that

$$\frac{33}{32}(a + \tilde{a}) \leqslant \int_{l^\perp} \#(\gamma \cap (x + l))\, dx + \int_l \#(\gamma \cap (x + l^\perp))\, dx + 4. \tag{11}$$

We may assume that $a \geqslant \tilde{a}$, and that l is the first coordinate axis. For any integer k, we define the interval I_k on l to be the closed interval $[2k - 1, 2k + 1]$ if k is odd, and the open interval $(2k - 1, 2k + 1)$ if k is even. We say that a vertical line is *proper* if it does not go through a self-intersection point of γ, or an endpoint of a segment in γ. In addition, an interval I_k is called *saturated* if k is even and any vertical proper line intersecting I_k intersects γ in at least two points. If the number of saturated intervals is at least $a/32$, then (11) readily follows; hence we assume that the number of saturated intervals is at most $a/32$.

Now there exists an interval s of length at least $a/2$ contained in $C|l$ such that, if a vertical line intersects s, then it intersects C in a segment of length at least $r_2(C)$. Since there exist at least $a/8$ intervals I_k with even k that intersect s, we can find at least $a/16 + 1$ among them that are not saturated. If I_k is one of these at least $a/16 + 1$ intervals, then we associate a vertical proper line l_k to I_k which intersects I_k, and intersects γ only in one point. We call l_k a *witness*. Let l_{k_1} and l_{k_2} be two consecutive witnesses; hence there exists an I_m with odd m that lies between l_{k_1} and l_{k_2}. Next, we consider the points $p, q \in \partial C$ whose first coordinate is $2m$, and let $p', q' \in \gamma$ be the points whose distances from p and q, respectively, are at most one. Then the projection of the segment $p'q'$ into $l^\perp$ is of length at least $r_2(C) - 2 \geqslant 17$, and there is a polygonal path in γ that connects p' and q', and does not intersect l_{k_1} and l_{k_2}. Therefore

$$\int_{l^\perp} \#(\gamma \cap (x + l))\, dx > 17 \cdot \frac{a}{16}.$$

In turn, we deduce that (11) holds, and Proposition 4.1 as well. □

Since the diameter of the o-symmetric set $\frac{1}{2}(D_n - D_n)$ is $|\tilde{s}|$, we deduce that $V_1(D_n) \leqslant (|\tilde{s}|/2) \cdot V_1(B^d)$. We assume that $\tilde{h} \geqslant 60$; hence $r_2(\tilde{T}) \geqslant 20$ according to (1). Let Σ be the family of linear two-spaces such that $\tilde{s}|L$ is of length at least $|\tilde{s}|/2$, and $r_2(D_n|L) \geqslant 19$. Then $\mu_{d,2}(\Sigma) > c_1^d$ for some positive absolute constant c_1, and, if $L \in \Sigma$, then

$$V_1(D_n|L) \geqslant \frac{|\tilde{s}|}{2} \geqslant \frac{V_1(D_n)}{V_1(B^d)}.$$

We deduce by Proposition 4.1 and by $V_1(B^d) < \sqrt{2\pi d}$ (see (3)) that

$$V_1(D_n) + \frac{c_1^d}{32\,\pi} \cdot V_1(D_n) < |\sigma| + V_1(B^d) \leqslant n - 2 + \sqrt{2\pi d}.$$

Thus $V_1(D_n) \geqslant n$ yields that $n < c_2^d$ for some positive absolute constant c_2; or, in other words, if $n \geqslant c_2^d$, then $\tilde{h} \leqslant 60$.

In the final part of the proof, we use the notation set up in Section 2 for $\tilde{C}$. If $\tilde{h} \geqslant 2$, then among the sets $K_i|\tilde{L}$ there exist at least $n/3$ that do not intersect either σ or σ'; hence these $K_i|\tilde{L}$ satisfy $d(p_i, q_i) + d(p_i', q_i') \leqslant 1$. In particular, $\operatorname{diam} D_n \leqslant 5n/6$. On the other hand, we deduce by $V_1(D_n) \geqslant n$ and by (4) that $\operatorname{diam} D_n \geqslant n - 60\sqrt{2\pi d}$, which leads to a contradiction if n is large. Therefore $\tilde{h} < 2$. Now the argument used in Section 2 completes the proof of Theorem 1.1. □

5. *Extremal properties of other mean projections*

The m-skeleton of a compact convex set D is formed by the points of D which are not centres of any $(m+1)$-dimensional ball that is contained in D (see Larman and Rogers [6]). In this section, we summarize what is known about the supremum of $V_m(D)$ for some $2 \leqslant m \leqslant d-1$, where the m-skeleton of D is covered by n congruent copies of a given convex body K. Actually, we may also write 'maximum' in place of 'supremum', which fact can be verified analogously to the case of the 1-skeleton.

The properties below are either due to Gritzmann [4], or based on ideas in [4] (see also [2]).

- *If the m-skeleton of the compact convex set D is covered by n congruent copies of the given convex body K, then*

$$V_m(D) \leqslant n \cdot V_m(K). \tag{12}$$

The inequality (12) is a consequence of the facts that $V_m(K)$ is proportional to the mean m-dimensional projection, and that the projection of D into an m-dimensional linear subspace coincides with the projection of the m-skeleton of D.

- *For any n, $2 \leqslant m \leqslant d-1$ and convex body K in $\mathbb{R}^d$, there exists a covering of some convex compact set D by n congruent copies of K such that*

$$V_m(D) \geqslant n \cdot \frac{V_m(K)}{2em}.$$

We may assume that $\operatorname{diam} K = 1$, and that the segment ou is a diameter of K. We recall that K/u is the projection of K into $u^{\perp}$, and we observe that for any $y \in ((m-1)/m)(K/u)$, the line $y + \operatorname{lin} u$ intersects K in a segment of length at least $1/m$. Thus the covering is given by the translates $(i/m)u + K$, $i = 1, \ldots, n$, and D is the intersection of union of these translates and the infinite cylinder $\operatorname{lin} u + ((m-1)/m)(K/u)$. Finally,

$$V_m(K) < 2\,V_{m-1}(K/u) \qquad \text{and} \qquad V_m(D) > \frac{n}{m}\left(\frac{m-1}{m}\right)^{m-1} V_{m-1}(K/u)$$

yield $V_m(D) \geqslant n \cdot V_m(K)/2em$.

- *Given a convex body K and $2 \leqslant m \leqslant d-1$, let D_n be the convex compact set of maximal intrinsic m-volume that can be covered by n congruent copies of K. Then $r_{m+1}(D_n)$ stays bounded as n tends to infinity.*

The main observation is that D_n has an m-dimensional section with content at least $cV_m(D_n)$, where $c > 0$ depends only on m and d. It follows that

$$V_{m+1}(D_n) \geqslant \frac{c}{m+1} V_m(D_n)\, r_{m+1}(D_n),$$

which in turn yields the boundedness of $r_{m+1}(D_n)$.

If D_n is a convex compact set of maximal intrinsic m-volume whose m-skeleton can be covered by n congruent copies of K, then we do not know whether for any K, $r_{m+1}(D_n)$ stays bounded as n tends to infinity.

• *If $2 \leqslant m \leqslant d-1$, then the asymptotic structure of the optimal covering by n congruent copies of K for large n seems to depend on K.*

We discuss now only the case $d = 3$ and $m = 2$. If K is a right cylinder of height two over the unit disc, then most probably the optimal compact convex sets are right cylinders of height $2n$ over a unit disc for large n. However, if K is a pyramid over a regular hexagon and the height of K is small, then most probably the optimal compact convex sets for large n are two-dimensional with large two-dimensional inradius, and the optimal arrangement is based on the tiling of the plane by regular hexagons.

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