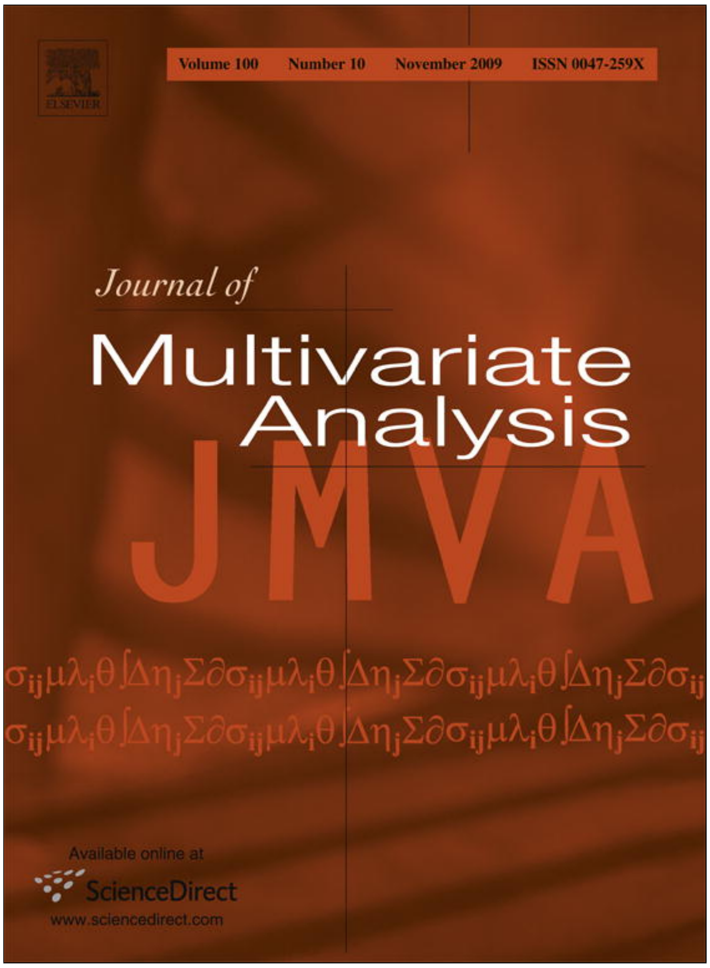
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Mean width of random polytopes in a reasonably smooth convex body

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ABSTRACT

Let K be a convex body in $\mathbb{R}^d$ and let $X_n = (x_1, \ldots, x_n)$ be a random sample of n independent points in K chosen according to the uniform distribution. The convex hull K_n of X_n is a random polytope in K, and we consider its mean width $W(K_n)$. In this article, we assume that K has a rolling ball of radius $\varrho > 0$. First, we extend the asymptotic formula for the expectation of $W(K) - W(K_n)$ which was earlier known only in the case when ∂K has positive Gaussian curvature. In addition, we determine the order of magnitude of the variance of $W(K_n)$, and prove the strong law of large numbers for $W(K_n)$. We note that the strong law of large numbers for any quermassintegral of K was only known earlier for the case when ∂K has positive Gaussian curvature.

1. Introduction and results

The convex hull of n independent, uniformly distributed random points in a given convex body K in $\mathbb{R}^d$ is a type of random polytope that has been studied extensively (basic references can be found in the surveys [1] and [2], see also [3]). As in the seminal papers of Rényi and Sulanke [4,5] (restricted to the planar case), which initiated this line of research, most of the investigations deal with asymptotic results, for n tending to infinity. We note that circumscribed polytopes have also been investigated, among others, by Rényi and Sulanke [6], Kaltenbach [7], and Böröczky and Reitzner [8].

We are interested in asymptotic results on the approximation orders of general convex bodies by random polytopes. We write $g(n) \sim h(n)$ if $\lim_{n\to\infty} \frac{g(n)}{h(n)} = 1$. Let K be a convex body in $\mathbb{R}^d$ with $V(K) = 1$, and let K_n denote the convex hull of n independent, uniformly (according to the Lebesgue measure) distributed random points in K. By $W(\cdot)$ and $V(\cdot)$ we denote, respectively, mean width and volume. Upper and lower bounds for the order of magnitude of the expectation of the mean width difference were determined by Schneider [9]. According to Schneider's theorem there exist constants $\gamma_1, \gamma_2 > 0$ depending on K such that

$$\gamma_1 n^{-2/(d+1)} < W(K) - \mathbb{E}W(K_n) < \gamma_2 n^{-1/d}. \tag{1}$$

The upper bound in (1) is of optimal order for polytopes. This can be verified, for example, with the help of (5). Let C^k_+ denote the set of all convex bodies with boundary of differentiability class C^k and with Gaussian curvature $\kappa(x) > 0$ for all $x \in \partial K$.

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For the case when ∂K is C_+^3, and hence $\kappa(x) > 0$ for all $x \in \partial K$, Schneider, Wieacker [10] proved that

$$W(K) - \mathbb{E}W(K_n) \sim \frac{2\Gamma(\frac{2}{d+1})}{d(d+1)^{\frac{d-1}{d+1}}\kappa_d\kappa_{d-1}^{\frac{2}{d+1}}} \int_{\partial K} \kappa(x)^{\frac{d+2}{d+1}} \mathrm{d}x \cdot \frac{1}{n^{\frac{2}{d+1}}}, \tag{2}$$

where κ_d is the volume of the Euclidean d-dimensional unit ball. Reitzner [11] extended the asymptotic formula (2) to convex bodies with C_+^2 boundary. In the case when the boundary of K is C_+^k for $k \geq 4$, an asymptotic expansion of the expectation was obtained by Gruber [12] and Reitzner [11]. For surveys of further related results, consult the paper Bárány [13], or the monograph Schneider, Weil [14]. In this paper, we further extend the class of convex bodies for which (2) holds. We say that a convex body K has a rolling ball if there exists a $\varrho > 0$ such that any $x \in \partial K$ lies in some ball of radius ϱ contained in K. According to Hug [15], the existence of a rolling ball is equivalent to saying that the exterior unit normal at $x \in \partial K$ is a Lipschitz function of x. In particular, if ∂K is C^2 then K has a rolling ball, which was already observed by W. Blaschke. In this article we shall prove the following theorem.

Theorem 1.1. *The asymptotic formula (2) holds for any convex body K of volume one which has a rolling ball.*

We note that Theorem 1.1 is close to be optimal. Example 2.1 shows that there exists a convex body K with a C^1 boundary such that in fact ∂K is C_+^∞ at all but one point and $\lim_{n\to\infty} n^{\frac{2}{d+1}}(W(K) - \mathbb{E}W(K_n)) = \infty$.

We recall the corresponding results about the expectation of volume for comparison. Bárány and Larman [16] proved that there exist constants $\gamma_1, \gamma_2 > 0$ depending on K such that

$$\gamma_1 n^{-1}(\log n)^{d-1} < V(K) - \mathbb{E}V(K_n) < \gamma_2 n^{-2/(d+1)}. \tag{3}$$

Here, as opposed to (1), the lower bound is optimal for polytopes, and the upper bound is optimal for smooth convex bodies. On the one hand, Bárány and Buchta [17] provided an asymptotic formula for the case when K is a polytope. On the other hand, generalizing a result of Bárány [18] for convex bodies with C_+^3 boundaries, Schütt [19] proved that if $\kappa(x) > 0$ for a set of $x \in \partial K$ of positive $(d-1)$-measure then

$$V(K) - \mathbb{E}V(K_n) \sim c \cdot \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \mathrm{d}x \cdot n^{-2/(d+1)},$$

where the constant c depends only on d. Here the integral above is the so-called affine surface area.

Furthermore, Reitzner [20] proved that the strong law of large numbers holds in the case of random volume approximation of convex bodies with C_+^2 boundary. This result was made possible by the upper bound on the variance of the volume of optimal order obtained in [20]. A matching lower bound on the variance was proved by Bárány and Reitzner in [21] for arbitrary convex bodies. In this article, we prove the analogous estimates on the variance of the mean width for convex bodies with a rolling ball. We note that in the case of random approximation, upper bounds of optimal order on the variance have only been proved for convex bodies that are either polytopes or have C_+^2 boundary (see say Reitzner [22], Buchta [23], Vu [24] and Bárány and Reitzner [21]).

Theorem 1.2. *If K is a d-dimensional convex body of volume one with a rolling ball, then*

$$\gamma_1 n^{-\frac{d+3}{d+1}} < \mathrm{Var}\, W(K_n) < \gamma_2 n^{-\frac{d+3}{d+1}},$$

where the positive constants γ_1, γ_2 depend on K.

The upper bound in Theorem 1.2 yields the strong law of large numbers by standard arguments.

Theorem 1.3. *If K is a d-dimensional convex body of volume one with a rolling ball then*

$$\lim_{n\to\infty} (W(K) - W(K_n)) n^{\frac{2}{d+1}} = \frac{2\Gamma(\frac{2}{d+1})}{d(d+1)^{\frac{d-1}{d+1}}\kappa_d\kappa_{d-1}^{\frac{2}{d+1}}} \int_{\partial K} \kappa(x)^{\frac{d+2}{d+1}} \mathrm{d}x$$

with probability 1.

2. Some general estimates about the mean width of a random polytope

We write $\mathcal{H}^{d-1}$ for the $(d-1)$-dimensional Hausdorff measure. The scalar product is denoted by $\langle \cdot, \cdot \rangle$, the Euclidean unit ball in $\mathbb{R}^d$ centred at the origin is B^d, and ∂B^d is denoted by S^{d-1}.

For any convex body K in $\mathbb{R}^d$, integration with respect to the $(d-1)$-dimensional Hausdorff measure on ∂K is denoted by $\int_{\partial K} \cdots \mathrm{d}x$. We say ∂K is twice differentiable in the generalized sense at an $x \in \partial K$ if there exists a quadratic form Q on $\mathbb{R}^{d-1}$

with the following property: If K is positioned in a way such that $x = o$ and $\mathbb{R}^{d-1}$ is a tangent hyperplane to K from below, then a neighbourhood of o on ∂K is the graph of a convex function f over a $(d-1)$-dimensional ball in $\mathbb{R}^{d-1}$ satisfying

$$f(z) = \frac{1}{2}Q(z) + o(\|z\|^2) \tag{4}$$

as z tends to zero. In this case the generalized Gaussian curvature at x is $\kappa(x) = \det Q$. According to the Alexandrov theorem (see Schneider [25] or Gruber [26]), the boundary ∂K is twice differentiable in the generalized sense almost everywhere.

For any compact convex set M in $\mathbb{R}^d$, we write h_M to denote its support function; namely, $h_M(u) = \max_{x\in M}\langle u, x\rangle$. In particular, the width of M in the direction $u \in S^{d-1}$ is $w_M(u) = h_M(u) + h_M(-u)$, and the mean width is

$$W(M) = \frac{1}{d\kappa_d}\int_{S^{d-1}} w_M(u)\,\mathrm{d}u = \frac{2}{d\kappa_d}\int_{S^{d-1}} h_M(u)\,\mathrm{d}u.$$

Let K be a convex body in $\mathbb{R}^d$ with volume one. We remark that the implied constant in $O(\cdot)$ in the formulas below depends on K.

We start with examining the expectation of the mean width following ideas set forth in Schneider, Wieacker [10]. For $t \ge 0$ and $u \in S^{d-1}$, let $C(u,t) = \{x \in K : \langle u, x\rangle \ge h_K(u) - t\}$. For $x_1, \ldots, x_n \in K$, we usually write $X_n = (x_1, \ldots, x_n)$ and $K_n = [x_1, \ldots, x_n]$, and we define the function

$$\varphi(t,u,X_n) = \begin{cases} 1 & \text{if } 0 \le t < h_K(u) - h_{K_n}(u) \\ 0 & \text{otherwise}. \end{cases}$$

In particular, for fixed t and u, $\varphi(t,u,X_n) = 1$ if and only if none of $x_1, \ldots, x_n$ lie in $C(u,t)$. We deduce, using the Fubini theorem, that

$$\begin{aligned}\mathbb{E}(W(K) - W(K_n)) &= \frac{2}{d\kappa_d}\int_{K^n}\int_{S^{d-1}} h_K(u) - h_{K_n}(u)\,\mathrm{d}u\,\mathrm{d}X_n \\ &= \frac{2}{d\kappa_d}\int_{K^n}\int_{S^{d-1}}\int_0^{w_K(u)} \varphi(t,u,X_n)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}X_n \\ &= \frac{2}{d\kappa_d}\int_{S^{d-1}}\int_0^{w_K(u)} (1 - V(C(u,t)))^n\,\mathrm{d}t\,\mathrm{d}u.\end{aligned}$$

There exist $\gamma_0, n_0 > 0$ depending on K such that $V(C(u,t)) > \frac{3\ln n}{n}$ for any $n > n_0$, $u \in S^{d-1}$ and $t > \gamma_0(\frac{\ln n}{n})^{\frac{1}{d}}$. Therefore, if $n > n_0$ then

$$\mathbb{E}(W(K) - W(K_n)) = \frac{2}{d\kappa_d}\int_{S^{d-1}}\int_0^{\gamma_0(\frac{\ln n}{n})^{\frac{1}{d}}} (1 - V(C(u,t)))^n\,\mathrm{d}t\,\mathrm{d}u + O(n^{-3}). \tag{5}$$

Now, we are ready to substantiate the remark we made earlier that Theorem 1.1 is of optimal order. We shall accomplish this by way of constructing the following example.

Example 2.1. If K is a convex body in $\mathbb{R}^d$ such that $o \in \partial K$, ∂K is C^∞_+ on $\partial K \setminus o$, and the graph of $f(x) = \|x\|^{\frac{3d+1}{3d}}$ on $\mathbb{R}^{d-1} \cap B^d$ is part of ∂K, then $\mathbb{E}(W(K) - W(K_n)) \ge \gamma\, n^{\frac{-4d}{3d^2+1}}$, where $\gamma > 0$ depends on d and $\frac{4d}{3d^2+1} < \frac{2}{d+1}$.

Proof. We write u_0 to denote opposite of the dth basis vector, and $\gamma_1, \gamma_2, \ldots$ to denote positive constants depending on d. As $f(x) = \|x\|^{1+\alpha}$ for $\alpha = \frac{1}{3d}$, simple calculations show that at all $x - f(x)u_0$ for $x \in \mathbb{R}^{d-1} \cap (B^d \setminus o)$, the exterior unit normal u at $x - f(x)u_0$ to K satisfies $\gamma_1\|x\|^\alpha \le \|u - u_0\| \le \gamma_2\|x\|^\alpha$, and each principal curvature is at least $\gamma_3\|x\|^{\alpha-1} \ge \gamma_4\|u - u_0\|^{\frac{\alpha-1}{\alpha}}$. Let $\Xi(n) = S^{d-1} \cap (u_0 + n^{\frac{-\alpha}{d+\alpha}} B^d)$. In particular if n is large, $u \in \Xi(n)$ and $t \le n^{-\frac{1+\alpha}{d+\alpha}}$, then $C(u,t)$ is contained in a cylinder of height t whose base is of circumradius at most $\sqrt{\gamma_5 t / n^{\frac{1-\alpha}{d+\alpha}}}$. Therefore

$$V(C(u,t)) \le \gamma_6 t^{\frac{d+1}{2}} n^{-\frac{1-\alpha}{d+\alpha}\cdot\frac{d-1}{2}} \le \gamma_6 n^{-1},$$

and in turn (5) yields $\mathbb{E}(W(K) - W(K_n)) \ge \gamma_7 \int_{\Xi(n)} n^{-\frac{1+\alpha}{d+\alpha}}\,\mathrm{d}u \ge \gamma_8 n^{-\frac{d\alpha+1}{d+\alpha}}$. $\square$

Now we get back to making preparations for proving the upper bound on the variance of $W(K_n)$. According to the Efron–Stein jackknife inequality (see Reitzner [20]), we have that

$$\operatorname{Var} W(K_n) \le (n+1)\mathbb{E}(W(K_{n+1}) - W(K_n))^2. \tag{6}$$

We write $f \ll g$ if $f \le \gamma g$ for a constant $\gamma > 0$ depending only on K. For $t \ge 0$, $u \in S^{d-1}$ and $x_1, \ldots, x_{n+1} \in K$, let $X_{n+1} = (x_1, \ldots, x_{n+1})$, $K_{n+1} = [x_1, \ldots, x_{n+1}]$ and $K_n = [x_1, \ldots, x_n]$. Further, we define the function

$$\bar{\varphi}(t,u,X_{n+1}) = \begin{cases} 1 & \text{if } h_{K_n}(u) \le t \le h_{K_{n+1}}(u) \\ 0 & \text{otherwise}. \end{cases}$$

We set the volume of the empty set to be zero. It follows from the Efron–Stein jackknife inequality and the Fubini theorem that

$$\begin{aligned}\operatorname{Var} W(K_n) &\ll n\int_{K^{n+1}}\left(\int_{S^{d-1}} h_{K_{n+1}}(u)-h_{K_n}(u)\,\mathrm{d}u\right)^2 \mathrm{d}X_{n+1}\\ &= n\int_{K^{n+1}}\int_{S^{d-1}}\int_{S^{d-1}}(h_{K_{n+1}}(u)-h_{K_n}(u))(h_{K_{n+1}}(v)-h_{K_n}(v))\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}X_{n+1}\\ &= n\int_{K^{n+1}}\int_{S^{d-1}}\int_{S^{d-1}}\int_0^{w_K(v)}\int_0^{w_K(u)}\bar{\varphi}(t,u,X_{n+1})\bar{\varphi}(s,v,X_{n+1})\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}X_{n+1}\\ &= n\int_{S^{d-1}}\int_{S^{d-1}}\int_0^{w_K(v)}\int_0^{w_K(u)} V(C(u,t)\cap C(v,s))(1-V(C(u,t)\cup C(v,s)))^n \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}u.\end{aligned}$$

For any $u \in S^{d-1}$ and $s, t \geq 0$, let

$$\Sigma(u,t;s) = \{v \in S^{d-1} : C(u,t)\cap C(v,s) \neq \emptyset\},$$

and for $v \in \Sigma(u,t;s)$, let

$$V_+(u,t;v,s) = \max\{V(C(u,t)), V(C(v,s))\}.$$

Therefore our estimate of the variance yields that if $n > n_0$, then

$$\operatorname{Var} W(K_n) \ll n\int_{S^{d-1}}\int_0^{\gamma_0\left(\frac{\ln n}{n}\right)^{\frac{1}{d}}}\int_0^t\int_{\Sigma(u,t;s)} V_+(u,t;v,s)(1-V_+(u,t;v,s))^n \mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u + O(n^{-2}). \tag{7}$$

3. Proof of Theorem 1.1

Let K be a convex body in $\mathbb{R}^d$ with a rolling ball of radius $\varrho > 0$. We write u_x to denote the exterior unit normal at $x \in \partial K$. In particular, if f is measurable on S^{d-1} then by formula (2.5.30) in [25]

$$\int_{S^{d-1}} f(u)\,\mathrm{d}u = \int_{\partial K} f(u_x)\kappa(x)\,\mathrm{d}x. \tag{8}$$

Let $x \in \partial K$. The existence of the rolling ball yields that

$$V(C(u_x,t)) \geq \frac{2\kappa_{d-1}\varrho^{\frac{d-1}{2}}t^{\frac{d+1}{2}}}{d+1} \quad \text{for } t \in [0,\varrho]. \tag{9}$$

In addition, if $\kappa(x)$ exists and positive then we deduce by (4) that

$$\lim_{t\to 0} t^{-\frac{d+1}{2}} V(C(u_x,t)) = \frac{2^{\frac{d+1}{2}}\kappa_{d-1}}{(d+1)\kappa(x)^{\frac{1}{2}}}. \tag{10}$$

We will need the following asymptotic formula using the gamma function (see Artin [27]). First we note that for $\alpha > 0$, the representation of the beta function by the gamma function and the Stirling formula imply

$$\lim_{n\to\infty} n^\alpha \int_0^1 \tau^{\alpha-1}(1-\tau)^n \mathrm{d}\tau = \lim_{n\to\infty} n^\alpha \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} = \Gamma(\alpha).$$

Now if $\frac{(\alpha+1)\ln n}{n} \leq \tau < 1$, then $(1-\tau)^n < \mathrm{e}^{-n\tau} \leq n^{-(\alpha+1)}$. Therefore, if $f(n) \in (0,1)$ satisfies $f(n) \geq \frac{(\alpha+1)\ln n}{n}$ for large n, then

$$\int_0^{f(n)} \tau^{\alpha-1}(1-\tau)^n \mathrm{d}\tau \sim \Gamma(\alpha)n^{-\alpha}$$

as n tends to infinity. For $\beta \geq 0$ and $\omega > 0$, it follows using the substitution $\tau = \omega t^{\frac{d+1}{2}}$ that

$$\int_0^{g(n)} t^\beta(1-\omega t^{\frac{d+1}{2}})^n \mathrm{d}t \sim \frac{2}{(d+1)\omega^{\frac{2(\beta+1)}{d+1}}}\cdot\Gamma\left(\frac{2(\beta+1)}{d+1}\right) n^{-\frac{2(\beta+1)}{d+1}}, \tag{11}$$

assuming that $g(n) \in (0, \omega^{-\frac{2}{d+1}})$ for all n, and $g(n) \geq \left(\frac{(\alpha+1)\ln n}{\omega n}\right)^{\frac{2}{d+1}}$ for large n, where $\alpha = \frac{2(\beta+1)}{d+1}$.

Proof of Theorem 1.1. For the n_0 coming from (5), we define

$$\theta_n(u) = n^{\frac{2}{d+1}} \frac{2}{d\kappa_d} \int_0^{\gamma_0 \left(\frac{\ln n}{n}\right)^{\frac{1}{d}}} (1 - V(C(u,t)))^n \, dt$$

for $n > n_0$ and $u \in S^{d-1}$. According to (5), we have

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}(W(K) - W(K_n)) = \lim_{n\to\infty} \int_{\partial K} \theta_n(u_x)\kappa(x)\, dx. \tag{12}$$

Since for large n, $\theta_n(u) < \gamma$ for some γ depending only on K by (9) and (11) (with $\beta = 0$) for any $u \in S^{d-1}$, and $\kappa(x) \le \varrho^{-(d-1)}$ for any $x \in \partial K$, we may apply the Lebesgue dominated convergence theorem.

Let $x \in \partial K$ such that $\kappa(x)$ exists and positive. Now for any $\varepsilon \in (0, 1)$, (10) yields that there exists a $t_\varepsilon > 0$ such that

$$(1-\varepsilon) \cdot \frac{2^{\frac{d+1}{2}} \kappa_{d-1}}{(d+1)\kappa(x)^{\frac{1}{2}}} \cdot t^{\frac{d+1}{2}} \le V(C(u_x, t)) \le (1+\varepsilon) \cdot \frac{2^{\frac{d+1}{2}} \kappa_{d-1}}{(d+1)\kappa(x)^{\frac{1}{2}}} \cdot t^{\frac{d+1}{2}}$$

for $t \in (0, t_\varepsilon)$. Therefore (11) (with $\beta = 0$) implies

$$\lim_{n\to\infty} \theta_n(u_x) = \frac{2\kappa(x)^{\frac{1}{d+1}} \Gamma\left(\frac{2}{d+1}\right)}{(d+1)^{\frac{d-1}{d+1}} d\kappa_d \kappa_{d-1}^{\frac{2}{d+1}}}.$$

In turn, we conclude Theorem 1.1 by (12). $\square$

4. Proof of the upper bound in Theorem 1.2

To prove Theorem 1.2, we observe that if $a \in (0, 1)$ then

$$\frac{(1-\frac{a}{2})^n}{(1-a)^n} > \left(1 + \frac{a}{2}\right)^n > \frac{an}{2},$$

which in turn yields

$$a(1-a)^n < \frac{2}{n}\left(1 - \frac{a}{2}\right)^n. \tag{13}$$

Since our estimate on the variance depends on (7), we estimate the size of $\Sigma(u, t; s)$ for $u \in S^{d-1}$. The existence of the rolling ball of radius ϱ at $x \in C(u, t) \cap \partial K$ shows that $\|u_x - u\| \le \sqrt{\frac{2t}{\varrho}}$ for $t \le \varrho$. In particular, let $0 < s \le t \le \varrho$. If $v \in \Sigma(u, t; s)$ then $\|v - u\| < 4\varrho^{-\frac{1}{2}} t^{\frac{1}{2}}$, and hence the $(d-1)$-measure of $\Sigma(u, t; s)$ is at most $\gamma t^{\frac{d-1}{2}}$ for some $\gamma > 0$ depending on d. We set $\gamma^* = \frac{\kappa_{d-1}\varrho^{\frac{d-1}{2}}}{d+1}$, and simplify (7) by applying first (13) and (9), and secondly the formula (11) to obtain

$$\begin{aligned} \operatorname{Var} W(K_n) &\ll n \int_{S^{d-1}} \int_0^{\gamma_0\left(\frac{\ln n}{n}\right)^{\frac{1}{d}}} \int_0^t \frac{t^{\frac{d-1}{2}}}{n} \left(1 - \gamma^* t^{\frac{d+1}{2}}\right)^n ds\, dt\, du + O(n^{-2}) \\ &\ll \int_0^{\gamma_0\left(\frac{\ln n}{n}\right)^{\frac{1}{d}}} t^{\frac{d+1}{2}} \left(1 - \gamma^* t^{\frac{d+1}{2}}\right)^n dt + O(n^{-2}) \ll n^{-\frac{d+3}{d+1}}. \end{aligned}$$

5. Proof of the lower bound in Theorem 1.2

The idea of the proof is similar to the one in Reitzner [22]; namely, $\operatorname{Var} W(K_n)$ is at least the sum of the variances inside "independent caps". First we separate the part of ∂K where reasonably sized caps are contained in touching balls of fixed radius. Next we verify the technical estimates (15) and (18), which lead to the estimates (20) and (21) ensuring the independence of the caps in the final argument. In addition, we need Lemma 5.1 to estimate the "variance inside a cap".

For any polytope P and vertex z of P, we write $N_P(z)$ to denote the exterior normal cone to z. We recall the Alexandrov theorem (see Schneider [25] or Gruber [26]) that the boundary ∂K of a convex body K is twice differentiable in the generalized sense almost everywhere with respect to $\mathcal{H}^{d-1}$. We deduce by (8) that the $(d-1)$-measure of the points $x \in \partial K$ with $\kappa(x) > 0$ is positive. Therefore there exists $R > 0$ and a $\Xi' \subset \partial K$ with $\mathcal{H}^{d-1}(\Xi') > 0$ such that each principal curvature at all $x \in \Xi'$ is at least $\frac{2}{R}$. For any $x \in \Xi'$ there exists a maximal $\sigma_x \in (0, \frac{\varrho}{8d^2}]$ such that $C(u_x, \sigma_x) \subset x - Ru_x + RB^d$, and σ_x, being lower semi-continuous, is a measurable function of $x \in \Xi'$. Therefore there exists $\sigma \in (0, \frac{\varrho}{8d^2}]$ such that if Ξ

denotes the family of $x \in \partial K$ such that $C(u_x, \sigma) \subset x - Ru_x + RB^d$ then

$$\mathcal{H}^{d-1}(\Xi) > 0. \tag{14}$$

For $u \in S^{d-1}$ and $t > 0$, we define $H(u,t) = \{z : \langle z, u\rangle = h_K(u) - t\}$.

Let $x \in \Xi$, and let $t \in (0, \sigma)$. The existence of the rolling ball and the definition of Ξ imply

$$(x - tu_x + \sqrt{\varrho t}B^d) \cap H(u_x, t) \subset H(u_x, t) \cap K \subset x - tu_x + \sqrt{2Rt}B^d \cap K. \tag{15}$$

Let $w_1, \ldots, w_d$ be the vertices of a regular $(d-1)$-simplex in $H(u_x, t)$ whose circumcentre is $x - tu_x$, and whose circumradius is $\sqrt{\varrho t}$, and hence

$$\left(x - tu_x + \frac{\sqrt{\varrho t}}{d}B^d\right) \cap H(u_x, t) \subset [w_1, \ldots, w_d] \subset K.$$

In addition, we set $w_0 = x$, and for $j = 0, \ldots, d$,

$$\Delta_j(x,t) = w_j + \frac{1}{4d}([w_0, \ldots, w_d] - w_j).$$

In particular, (15) yields

$$V(\Delta_j(x,t)) \gg t^{\frac{d+1}{2}} \quad \text{for } j = 0, \ldots, d. \tag{16}$$

If $z_j \in \Delta_j(x,t)$, $j = 0, \ldots, d$, and $v \in u_x^{\perp}$ then

$$\begin{aligned} &\sqrt{\varrho t}/(2d) \le h_{[z_1,\ldots,z_d]}(v) - \langle z_0, v\rangle \le 2\sqrt{\varrho t} \\ &t/2 \le \langle z_0, u_x\rangle - h_{[z_1,\ldots,z_d]}(u_x) \le t, \end{aligned} \tag{17}$$

and hence on the one hand, the tangent of the angle of u_x and any $u \in N_{[z_0,\ldots,z_d]}(z_0)$ is at most $\frac{2d\sqrt{t}}{\sqrt{\varrho}}$, and on the other hand, if the tangent of the angle of u_x and any $u \in S^{d-1}$ is at most $\frac{\sqrt{t}}{4\sqrt{\varrho}}$ then $u \in N_{[z_0,\ldots,z_d]}(z_0)$. Therefore defining

$$\Sigma_1(x,t) = S^{d-1} \cap \left(u_x + \frac{\sqrt{t}}{8\sqrt{\varrho}}B^d\right),$$

$$\Sigma_2(x,t) = S^{d-1} \cap \left(u_x + \frac{2d\sqrt{t}}{\sqrt{\varrho}}B^d\right),$$

we have

$$\Sigma_1(x,t) \subset S^{d-1} \cap N_{[z_0,\ldots,z_d]}(z_0) \subset \Sigma_2(x,t). \tag{18}$$

For $j = 1, 2$, we consider the dual cones

$$\Sigma_j^*(x,t) = \{y \in \mathbb{R}^d : \langle y, v\rangle \le 0 \text{ for all } v \in \Sigma_j(x,t)\},$$

which satisfy

$$\Sigma_2^*(x,t) \subset \{t(y - z_0) : t \ge 0 \text{ and } y \in [z_0, z_1, \ldots, z_d]\} \subset \Sigma_1^*(x,t). \tag{19}$$

Let $\gamma = 2^9 d^2 R/\varrho$, $\sigma_0 = \sigma/\gamma$ and $\tilde{\gamma} = 2\sqrt{R\gamma}$. If $x \in \Xi$ and $t \in (0, \sigma_0)$ then it follows by (15) that

$$C(x, \gamma t) \subset x + \tilde{\gamma}\sqrt{t}B^d. \tag{20}$$

Next, if $z_0 \in \Delta_0(x,t)$ and $y \in H(u_x, \gamma t) \cap K$ then (15) yields that the tangent of the angle of $-u_x$ and $y - z_0$ is at most $\frac{2\sqrt{2R\gamma t}}{(\gamma/2)t} = \frac{\sqrt{\varrho}}{4d\sqrt{t}}$, therefore $y - z_0 \in \Sigma_2^*(x,t)$. In particular,

$$K \setminus C(x, \gamma t) \subset z_0 + \Sigma_2^*(x,t). \tag{21}$$

If A is an event in some probability space then we write $I(A)$ to denote the indicator function. In addition for $x \in \Xi$, $t \in (0, \sigma_0)$, and $z_i \in \Delta_i(x,t)$, $i = 0, \ldots, d$, writing $F = [z_1, \ldots, z_d]$ we define

$$\overline{W}_F(z_0) = \frac{2}{d\kappa_d}\int_{\Sigma_2(x,t)} h_{[z_0,F]}(u)\,\mathrm{d}u.$$

Naturally, $\overline{W}_F(z_0)$ depends on x and t as well but it will always be clear from the context what x and t are.

Lemma 5.1. *If Z is a random point chosen uniformly from $\Delta_0(x,t)$ for $x \in \Xi$ and $t \in (0,\sigma_0)$, and $z_i \in \Delta_i(x,t)$ for $i = 1,\ldots,d$; then*

$$\operatorname{Var} \overline{W}_{[z_1,\ldots,z_d]}(Z) \gg t^{d+1}.$$

Proof. We define $F = [z_1,\ldots,z_d]$. Let w be the centroid of the facet of $\Delta_0(x,t)$ opposite to x, let $w_1 = \frac{2}{3}x + \frac{1}{3}w$ and $w_2 = \frac{1}{3}x + \frac{2}{3}w$. In addition, we define (compare (19))

$$\Psi_1 = (w_1 - \Sigma_2^*(x,t)) \cap \Delta_0(x,t),$$
$$\Psi_2 = (w_2 + \Sigma_2^*(x,t)) \cap \Delta_0(x,t).$$

In particular, there exists some $\gamma_0 > 0$ depending on K such that

$$V(\Psi_j) \geq \gamma_0 V(\Delta_0(x,t)). \tag{22}$$

Moreover, for any $Z_1 \in \Psi_1$ and $Z_2 \in \Psi_2$, if $v \in S^{d-1}$ then

$$h_{[Z_1,F]}(v) - h_{[Z_2,F]}(v) \geq 0$$

by $[Z_2,z_1,\ldots,z_d] \subset [Z_1,z_1,\ldots,z_d]$, and if even $v \in \Sigma_1(x,t)$ (compare (18)) then

$$h_{[Z_1,F]}(v) - h_{[Z_2,F]}(v) = \langle v, Z_1\rangle - \langle v, Z_2\rangle \geq \langle v, w_1\rangle - \langle v, w_2\rangle \gg t.$$

Therefore if $Z_1 \in \Psi_1$ and $Z_2 \in \Psi_2$ then

$$\overline{W}_F(Z_1) - \overline{W}_F(Z_2) \gg t \cdot \mathcal{H}^{d-1}(\Sigma_1(x,t)) \gg t^{\frac{d+1}{2}}.$$

In turn, we deduce (compare (22))

$$\begin{aligned}\operatorname{Var} \overline{W}_F(Z) &= \frac{1}{2}\mathbb{E}(\overline{W}_F(Z_1) - \overline{W}_F(Z_2))^2 \\ &\geq \frac{1}{2}\mathbb{E}[(\overline{W}_F(Z_1) - \overline{W}_F(Z_2))^2 I(Z_1 \in \Psi_1, Z_2 \in \Psi_2)] \\ &\gg t^{d+1}\mathbb{E}[I(Z_1 \in \Psi_1, Z_2 \in \Psi_2)] \gg t^{d+1}. \quad \square\end{aligned}$$

It is sufficient to prove the lower bound in Theorem 1.2 for large enough n. We fix

$$t_n = n^{-\frac{2}{d+1}}, \tag{23}$$

and hence $V(C(u_x,t_n)) \approx 1/n$ for all $x \in \Xi$. We choose a maximal family of points $y_1,\ldots,y_m \in \Xi$ such that for $i \neq j$, we have (compare (20))

$$\|y_i - y_j\| \geq 2\tilde{\gamma}\sqrt{t_n}.$$

In particular, (14) yields

$$m \gg n^{\frac{d-1}{d+1}}. \tag{24}$$

For $j = 1,\ldots,m$, let A_j denote the event that each $\Delta_i(y_j,t_n)$, $i = 0,\ldots,d$ contains exactly one random point out of $x_1,\ldots,x_n$, and $C(y_j,\gamma t_n)$ contains no other random point (compare (21)). We note that there exist positive α,β depending only on K such that for $i = 0,\ldots,d$, we have

$$V(\Delta_i(y_j,t_n)) \geq \alpha/n \quad \text{and} \quad V(C(y_j,\gamma t_n)) \leq \beta/n.$$

Thus for $j = 1,\ldots,m$, we have

$$\mathbb{P}\{A_j\} \geq \binom{n}{d+1}\left(\frac{\alpha}{n}\right)^{d+1}\left(1 - \frac{\beta}{n}\right)^{n-d-1} \gg 1. \tag{25}$$

If A_j holds then we write Z_j to denote the random point in $\Delta_0(y_j,t_n)$, and F_j to denote the convex hull of the random points in $\Delta_i(y_j,t_n)$ for $i = 1,\ldots,d$. Hence for any $u \in \Sigma_2(y_j,t_n)$, (21) yields

$$h_{K_n}(u) = h_{[Z_j,F_j]}(u) \tag{26}$$

given A_j. In particular, if $1 \leq i < j \leq m$ and A_i, A_j hold, then $\overline{W}_{F_i}(Z_i)$ and $\overline{W}_{F_j}(Z_j)$ are independent according to (18).

Next, we introduce the sigma algebra $\mathcal{F}$ that keeps track of everything except the location of $Z_j \in \Delta_0(y_j, t_n)$ for which A_j occurs. We decompose the variance by conditioning on $\mathcal{F}$:

$$\begin{aligned}\operatorname{Var} W(K_n) &= \mathbb{E} \operatorname{Var}(W(K_n) \mid \mathcal{F}) + \operatorname{Var} \mathbb{E}(W(K_n) \mid \mathcal{F}) \\ &\geq \mathbb{E}(\operatorname{Var} W(K_n) \mid \mathcal{F}).\end{aligned}$$

The independence structure mentioned above implies that

$$\begin{aligned}\operatorname{Var}(W(K_n) \mid \mathcal{F}) &= \sum_{I(A_j)=1} \operatorname{Var}_{Z_j} W(K_n) \\ &= \sum_{I(A_j)=1} \operatorname{Var}_{Z_j} \overline{W}_{F_j}(Z_j),\end{aligned}$$

where the variance is taken with respect to the random point $Z_j \in \Delta_0(y_j, t_n)$, and we sum over all $j = 1, \dots, m$ with $I(A_j) = 1$. Combining this with Lemma 5.1, (23), (24) and with (25) implies

$$\begin{aligned}\operatorname{Var} W(K_n) &\gg \mathbb{E}\left(\sum_{I(A_j)=1} t_n^{d+1}\right) \gg n^{-2}\mathbb{E}\left(\sum_{j=1}^{m} I(A_j)\right) \\ &\gg n^{-2} m \gg n^{-\frac{d+3}{d+1}}.\end{aligned}$$

6. Proof of Theorem 1.3

First, we deduce by Chebyshev's inequality that

$$\begin{aligned}\mathbb{P}\left(|W(K) - W(K_n) - \mathbb{E}(W(K) - W(K_n))|\, n^{\frac{2}{d+1}} \geq \varepsilon\right) &\leq \varepsilon^{-2} n^{\frac{4}{d+1}} \operatorname{Var} W(K_n) \\ &\ll n^{-\frac{d-1}{d+1}}.\end{aligned}$$

Since the sum $\sum_{k=2}^{\infty} n_k^{-\frac{d-1}{d+1}}$ is finite for $n_k = k^4$, the sum of the probabilities

$$\mathbb{P}\left(\left|W(K) - W(K_{n_k}) - \mathbb{E}(W(K) - W(K_{n_k}))\right| n_k^{\frac{2}{d+1}} \geq \varepsilon\right)$$

for $k \geq 2$ is finite as well. Therefore the Borel–Cantelli lemma and Theorem 1.1 yield that

$$\lim_{k\to\infty} (W(K) - W(K_{n_k})) n_k^{\frac{2}{d+1}} = \frac{2\Gamma\left(\frac{2}{d+1}\right)}{d(d+1)^{\frac{d-1}{d+1}} \kappa_d \kappa_{d-1}^{\frac{2}{d+1}}} \int_{\partial K} \kappa(x)^{\frac{d+2}{d+1}} \, dx \tag{27}$$

with probability 1. Now, $W(K) - W(K_n)$ is decreasing, and hence

$$(W(K) - W(K_{n_{k-1}})) n_{k-1}^{\frac{2}{d+1}} \leq (W(K) - W(K_n)) n^{\frac{2}{d+1}} \leq (W(K) - W(K_{n_k})) n_k^{\frac{2}{d+1}}$$

holds for $n_{k-1} \leq n \leq n_k$. As $\lim_{k\to\infty} \frac{n_k}{n_{k-1}} = 1$, the subsequence limit (27) yields Theorem 1.3.

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