Stability of the functional forms of the Blaschke-Santaló inequality

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Abstract

Stability versions of the functional forms of the Blaschke-Santaló inequality due to Ball, Artstein-Klartag-Milman, Fradelizi-Meyer and Lehec are proved.

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1 Introduction

For general references about convex bodies, see P.M. Gruber [13] or R. Schneider [25], and for a survey on related geometric inequalities, see E. Lutwak [19]. We write 0 to denote the origin of $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ to denote the standard scalar product, $|\cdot|$ to denote the corresponding l_2-norm, and $V(\cdot)$ to denote volume (Lebesgue-measure). Let B^n be the unit Euclidean ball, and let $S^{n-1} = \partial B^n$. A convex body K in $\mathbb{R}^n$ is a compact convex set with non–empty interior. If $z \in \mathrm{int} K$, then the polar of K with respect to z is the convex body

$$K^z = \{x \in \mathbb{R}^n : \langle x - z, y - z \rangle \leq 1 \text{ for any } y \in K\}.$$

From Hahn-Banach's theorem in $\mathbb{R}^n$, $(K^z)^z = K$. According to L.A. Santaló [24] (see also M. Meyer and A. Pajor [20]), there exists a unique $z \in \mathrm{int} K$ minimizing the volume product $V(K)V(K^z)$, which is called the Santaló point of K. In this case z is the centroid of K^z. The Blaschke-Santaló inequality states that if z is the Santaló point (or centroid) of K, then

$$V(K)V(K^z) \leq V(B^n)^2, \tag{1}$$

with equality if and only if K is an ellipsoid. The inequality was proved by W. Blaschke [6] (available also in [7]) for $n \leq 3$, and by L.A. Santaló [24] for all n. The case of equality was characterized by J. Saint-Raymond [23] among o-symmetric convex bodies, and by C.M. Petty [22] among all convex bodies (see also D. Hug [14], E. Lutwak [18], M. Meyer and A. Pajor [20], and M. Meyer and S. Reisner [21] for simpler proofs).

To state functional versions of the Blaschke-Santaló inequality, let us first recall that the usual definition of the Legendre transform of a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $z \in \mathbb{R}^n$ is defined by

$$\mathcal{L}_z \varphi(y) = \sup_{x \in \mathbb{R}^n} \{\langle x - z, y - z \rangle - \varphi(x)\}, \quad \text{for } y \in \mathbb{R}^n$$

and that the function $\mathcal{L}_z \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is always convex and lower semicontinuous. If φ is convex, lower semicontinuous and $\varphi(z) < +\infty$ then $\mathcal{L}_z \mathcal{L}_z \varphi = \varphi$.

Subsequent work by K.M. Ball [2], S. Artstein-Avidan, B. Klartag, V.D. Milman [1], M. Fradelizi, M. Meyer [12] and J. Lehec [16, 17] lead to the functional version of the Blaschke-Santaló inequality (see [2] and [1] for the relation between the functional version and the original Blaschke-Santaló inequality).

Theorem [2, 1, 12, 16, 17] *Let $\varrho : \mathbb{R} \to \mathbb{R}_+$ be a log-concave non-increasing function and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be measurable then*

$$\inf_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} \varrho(\varphi(x))\,dx \int_{\mathbb{R}^n} \varrho(\mathcal{L}_z\varphi(x))\,dx \le \left(\int_{\mathbb{R}^n} \varrho(|x|^2/2)\,dx\right)^2.$$

If ϱ is decreasing there is equality if and only if there exist $a, b, c \in \mathbb{R}$, $a < 0$, $z \in \mathbb{R}^n$ and a positive definite matrix $T : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\varphi(x) = \frac{|T(x+z)|^2}{2} + c \quad \text{for } x \in \mathbb{R}^n,$$

and moreover either $c = 0$, or $\varrho(t) = e^{at+b}$ for $t > -|c|$.

Here we prove a stability version of this inequality.

Theorem 1.1 *Let $\varrho : \mathbb{R} \to \mathbb{R}_+$ be a log-concave and decreasing function with $\int_{\mathbb{R}_+} \varrho < +\infty$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be measurable. Assume that for some $\varepsilon \in (0, \varepsilon_0)$ and for all $z \in \mathbb{R}^n$, the following inequality holds:*

$$\int_{\mathbb{R}^n} \varrho(\varphi(x))\,dx \int_{\mathbb{R}^n} \varrho(\mathcal{L}_z\varphi(x))\,dx > (1-\varepsilon)\left(\int_{\mathbb{R}^n} \varrho(|x|^2/2)\,dx\right)^2$$

1. *If φ is convex, then there exist some $z \in \mathbb{R}^n$, $c \in \mathbb{R}$ and a positive definite matrix $T : \mathbb{R}^n \to \mathbb{R}^n$, such that*

   $$\int_{R(\varepsilon)B^n} \left|\frac{|x|^2}{2} + c - \varphi(Tx+z)\right| dx < \eta\varepsilon^{\frac{1}{129n^2}},$$

   where $\lim_{\varepsilon\to 0} R(\varepsilon) = +\infty$, and $\varepsilon_0, \eta, R(\varepsilon)$ depend on n and ϱ.

2. *If φ is only assumed to be measurable then a weaker version holds: There exists z, c, T as above and $\Psi \subset R(\varepsilon)B^n$ such that*

   $$\int_{R(\varepsilon)B^n\setminus\Psi} \left|\frac{|x|^2}{2} + c - \varphi(Tx+z)\right| dx < \eta\varepsilon^{\frac{1}{129n^2}},$$

   and $V(\Psi \cap RB^n) \le \eta\sqrt{\varepsilon}\,R^n$ for any $R \in [R_0, R(\varepsilon)]$, where $R_0 > 0$ depends only on ϱ.

Remark 1.2 *One cannot expect the L_1-distance between φ and $\frac{|T(x+z)|^2}{2} + c$ to be small on the whole $\mathbb{R}^n$. For instance, if $\varrho(t) = e^{-t}$, and for small $\varepsilon > 0$, $\varphi(x) = |x|^2/2$ if $|x| \le |\log\varepsilon|$, and $\varphi(x) = +\infty$ if $|x| > |\log\varepsilon|$, then, of* *course, for any c and T the function $x \mapsto \frac{|T(x+z)|^2}{2} + c - \varphi(x)$ is not in L_1, but*

$$\int_{\mathbb{R}^n} \varrho(\varphi(x))\,dx \int_{\mathbb{R}^n} \varrho(\mathcal{L}_z\varphi(x))\,dx > (1 - O(\varepsilon|\log\varepsilon|^{n-1})) \left(\int_{\mathbb{R}^n} \varrho(|x|^2/2)\,dx\right)^2$$

for all $z \in \mathbb{R}^n$.

In addition, if φ is only assumed to be measurable, then we may choose it to be infinity on a ball of small enough measure, and set $\varphi(x) = |x|^2/2$ on the complement.

On the other hand, most probably the exponent $\frac{1}{129n^2}$ in Theorem 1.1 can be exchanged into some positive absolute constant.

As a matter of fact, the above functional form of the Blaschke-Santaló inequality deduces from the following more general inequality, which is the result of different contributions as explained below

Theorem [2, 1, 12, 16, 17] *For any measurable $f : \mathbb{R}^n \to \mathbb{R}_+$ with positive integral there exists a particular point $z \in \mathbb{R}^n$ attached to f such that if measurable functions $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ with positive integrals satisfy*

$$f(x)g(y) \leq \varrho^2(\langle x - z, y - z\rangle),$$

for every $x, y \in \mathbb{R}^n$ with $\langle x - z, y - z\rangle > 0$, then

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(x)\,dx \leq \left(\int_{\mathbb{R}^n} \varrho(|x|^2)\,dx\right)^2.$$

Equality holds for this z if and only if there exist $\tilde{\varrho} : \mathbb{R}_+ \to \mathbb{R}_+$, $\xi > 0$ and a positive definite matrix $T : \mathbb{R}^n \to \mathbb{R}^n$, such that $\tilde{\varrho}(e^t)$ is log-concave, and for a.e. $x \in \mathbb{R}^n$ and $s \in \mathbb{R}_+$, we have

$$\varrho(s) = \tilde{\varrho}(s), \quad f(x) = \xi\,\tilde{\varrho}(|T(x - z)|^2) \quad \text{and} \quad g(x) = \xi^{-1}\tilde{\varrho}(|T^{-1}(x - z)|^2).$$

K.M. Ball [2] initiated the study of such inequalities, established the case of even functions f and proved that, in this case, z can be chosen to be the origin. If $\varrho(t) = e^{-t}$, S. Artstein, B. Klartag, V.D. Milman [1] showed that one can choose z to be the mean of f for any f. For any measurable ϱ but for log-concave functions f, M. Fradelizi, M. Meyer [12] constructed the suitable z in the following way. For any $z \in \mathbb{R}^n$, let

$$K_{f,z} = \left\{x \in \mathbb{R}^n : \int_0^{+\infty} r^{n-1} f(z + rx)\,dx \geq 1\right\},$$

which is convex according to K.M. Ball [3]. Actually, [3] only claims that $K_{f,z}$ is convex when f is even, but K.M. Ball first proves that the function

$$x \mapsto \|x\| := \left(\int_0^{+\infty} f(z + rx) r^{n-1} dr \right)^{-1/n}$$

is convex and homogeneous without assuming evenness of f, which is only used to state that the function $\|x\|$ is symmetric (see also J. Bourgain, B. Klartag and V.D. Milman [11], or B. Klartag [15]).

M. Fradelizi and M. Meyer [12] proved that there exists a $z \in \mathbb{R}^n$, such that the centre of mass of $K_{f,z}$ is the origin and that this z works. Finally J. Lehec gave a direct and different proof of the general theorem in [17]. He established the existence of a so-called Yao-Yao center for any measurable f and that this point z works also.

We also give a stability version of this more general form of the Blaschke-Santaló inequality.

Theorem 1.3 *If some log-concave functions $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ and $f, g : \mathbb{R}^n \to \mathbb{R}_+$ with positive integrals satisfy that ϱ is non-increasing, the centre of mass of $K_{f,z}$ is the origin for some $z \in \mathbb{R}^n$, and*

$$f(x)g(y) \le \varrho^2(\langle x - z, y - z \rangle)$$

for every $x, y \in \mathbb{R}^n$ with $\langle x - z, y - z \rangle > 0$, if moreover for $\varepsilon > 0$,

$$(1 + \varepsilon) \int_{\mathbb{R}^n} f(x)\, dx \int_{\mathbb{R}^n} g(x)\, dx \ge \left(\int_{\mathbb{R}^n} \varrho(|x|^2)\, dx \right)^2,$$

then there exist $\xi > 0$ and a positive definite matrix $T : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \varrho(|x|^2) - \xi\, f(Tx + z) \right| dx \quad &< \quad \gamma \varepsilon^{\frac{1}{32n^2}} \cdot \int_{\mathbb{R}_+} r^{n-1} \varrho(r^2)\, dr \\ \int_{\mathbb{R}^n} \left| \varrho(|x|^2) - \xi^{-1} g(T^{-1}x + z) \right| dx \quad &< \quad \gamma \varepsilon^{\frac{1}{32n^2}} \cdot \int_{\mathbb{R}_+} r^{n-1} \varrho(r^2)\, dr, \end{aligned}$$

where γ depends only on n.

We strongly believe that the power $\frac{1}{32n^2}$ occurring in Theorem 1.3 can be chosen to be a positive absolute constant.

In this note, the implied constant in $O(\cdot)$ depends only on the dimension n.

2 Stability of the Borell and the Blaschke-Santaló inequalities

C. Borell [9] pointed out the following version of the Prékopa-Leindler inequality:

Theorem 2.1 (Borell) *If $M, F, G : \mathbb{R}_+ \to \mathbb{R}_+$ are integrable functions with positive integrals, and $M(\sqrt{rs}) \geq \sqrt{F(r)G(s)}$ for $r, s \in \mathbb{R}_+$, then*

$$\int_{\mathbb{R}_+} F \cdot \int_{\mathbb{R}_+} G \leq \left(\int_{\mathbb{R}_+} M \right)^2 .$$

Recently the following stability estimate has been obtained in K.M. Ball, K.J. Böröczky [5]. We note that if $M : \mathbb{R}_+ \to \mathbb{R}_+$ is log-concave and non-increasing, then $M(e^t)$ is log-concave on $\mathbb{R}$.

Theorem 2.2 (Ball, Böröczky) *There exists a positive absolute constant c with the following property: If $M, F, G : \mathbb{R}_+ \to \mathbb{R}_+$ are integrable functions with positive integrals such that $M(e^t)$ is log-concave, $M(\sqrt{rs}) \geq \sqrt{F(r)G(s)}$ for $r, s \in \mathbb{R}_+$, and*

$$\left(\int_{\mathbb{R}_+} M \right)^2 \leq (1+\varepsilon) \int_{\mathbb{R}_+} F \cdot \int_{\mathbb{R}_+} G,$$

for some $\varepsilon > 0$, then there exist $a, b > 0$, such that

$$\begin{aligned} \int_{\mathbb{R}_+} |a\, F(bt) - M(t)|\, dt &\leq c \cdot \varepsilon^{\frac{5}{16}} \cdot \int_{\mathbb{R}_+} M(t)\, dt \\ \int_{\mathbb{R}_+} |a^{-1} G(b^{-1}t) - M(t)|\, dt &\leq c \cdot \varepsilon^{\frac{5}{16}} \cdot \int_{\mathbb{R}_+} M(t)\, dt. \end{aligned}$$

For a stability version of the Blaschke-Santaló inequality, we use the Banach-Mazur distance of two convex bodies M and K, which is defined by

$$\delta_{\rm BM}(K, M) = \min\{\ln \lambda :\ K - x \subset \Phi(M) \subset \lambda(K - x) \text{ for } \Phi \in {\rm GL}(n), x \in \mathbb{R}^n\}.$$

Improving on K.J. Böröczky [10], the paper [5] also established the following.

Theorem 2.3 (Ball,Böröczky) *If K is a convex body in $\mathbb{R}^n$, $n \geq 3$, with centroid z, and*

$$V(K)V(K^z) > (1-\varepsilon)V(B^n)^2 \quad \textit{for some } \varepsilon \in (0, \tfrac{1}{2}),$$

then for some $\gamma > 0$ depending only on n, we have

$$\delta_{\rm BM}(K, B^n) < \gamma\, \varepsilon^{\frac{1}{5n}}.$$

We note that according to K.M. Ball [2], Borell's inequality Theorem 2.1 can be used to prove the Blaschke-Santaló inequality. In particular, [5] proves Theorem 2.3 via Theorem 2.2.

3 Proof of Theorem 1.3

Before proving Theorem 1.3, we verify first a simple property of log-concave functions, then show that the centroid is a reasonable centre for the Banach-Mazur distance from ellipsoids.

Proposition 3.1 *If $h, \omega : \mathbb{R} \to \mathbb{R}_+$ are log-concave, ω is even, and*

$$\int_{\mathbb{R}} |r|^{n-1} |h(r) - \omega(r)| \, dr \leq \varepsilon \int_{\mathbb{R}} |r|^{n-1} \omega(r) \, dr$$

for some $\varepsilon \in (0, (250n)^{-(n+1)})$, then $|h(0) - \omega(0)| \leq 250n\varepsilon^{\frac{1}{n+1}} \cdot \omega(0)$.

Proof: We may assume that $\omega(0) = 1$ and $\int_{\mathbb{R}} \omega(r) \, dr = 1$, and hence $\omega(r) \leq 1$ for all r. First, we put forward a few useful facts about the function ω.

Following ideas from K.M. Ball and K.J. Böröczky [4], let us prove first that there exists some $r_0 \geq \frac{1}{2}$ such that $\omega(r) \geq e^{-2|r|}$ if $|r| \leq r_0$, and $\omega(r) \leq e^{-2|r|}$ if $|r| \geq r_0$. For this, notice that since $\int_{\mathbb{R}_+} \omega(r) dr = \frac{1}{2} = \int_{\mathbb{R}_+} e^{-2r} dr$ and $\log \omega$ is concave there exists $r_0 > 0$ satisfying the required property (and r_0 is unique unless $\omega(r) = e^{-2|r|}$ for all r. In this very specific case, we set arbitrarily $r_0 = 1/2$).

Now let us prove that $r_0 \geq 1/2$. We define $\omega^{-1}(t) = \sup\{r \geq 0; \omega(r) \geq t\}$. The hypotheses on ω imply that the support of ω^{-1} is $[0,1]$ and $\int_0^1 \omega^{-1}(t) dt = 1/2$. From Jensen's inequality one deduces that

$$\omega\left(\frac{1}{2}\right) = \omega\left(\int_0^1 \omega^{-1}(t) dt\right) \geq e^{\int_0^1 \log(t) dt} = e^{-1}.$$

Since $\omega(0) = 1$, it follows from the log-concavity of ω that $\omega(r) \geq e^{-2r}$, if $|r| \leq 1/2$. This proves the claim.

In particular, the latter exponential lower bound on ω implies that

$$\omega(r) \geq 1 - 2|r| \quad \text{if} \quad |r| \leq \frac{1}{2}. \tag{2}$$

The fact that the graphs of ω and $r \mapsto e^{-2|r|}$ cross only once on $\mathbb{R}_+$ implies the following useful bound

$$\int_{\mathbb{R}} r^{n-1} \omega(r) \, dr \leq 2 \int_{\mathbb{R}_+} r^{n-1} e^{-2r} \, dr = \frac{(n-1)!}{2^{n-1}} \leq n^{n+1}. \tag{3}$$

Next, we study the function h. Let $a_i = in\varepsilon^{\frac{1}{n+1}}$ for $i \in \mathbb{Z}$. We claim that there exist two ind ices $i \in \{1, \ldots, 5\}$, such that

$$1 - 11n\varepsilon^{\frac{1}{n+1}} \leq h(a_i) \leq 1 + n\varepsilon^{\frac{1}{n+1}}. \tag{4}$$

Suppose that (4) does not hold. Since h is non-decreasing and then non-increasing, there exists $k \in \{1, 2, 3, 4\}$ such that h is monotone on $[a_k, a_{k+1}]$, and $h(a_k)$ and $h(a_{k+1})$ are outside and on the same side of the interval $[1 - 11n\varepsilon^{\frac{1}{n+1}}, 1 + n\varepsilon^{\frac{1}{n+1}}]$. Consequently, for this value of k, either $h(r) < 1 - 11n\varepsilon^{\frac{1}{n+1}}$ for $r \in [a_k, a_{k+1}]$, or $h(r) > 1 + n\varepsilon^{\frac{1}{n+1}}$ for $r \in [a_k, a_{k+1}]$. In any case, using respectively (2) and $\omega \leq 1$, it follows that

$$\int_{a_k}^{a_{k+1}} r^{n-1}|h(r) - \omega(r)|\, dr > \int_{a_k}^{a_{k+1}} r^{n-1} n\varepsilon^{\frac{1}{n+1}}\, dr > n^{n+1}\varepsilon,$$

which from (3) contradicts the condition on h, and hence proves (4).

Since $e^{-2t} < 1 - t$ and $e^t < 1 + 2t$ for $t \in (0, \frac{1}{2})$, (4) yields that

$$e^{-22n\varepsilon^{\frac{1}{n+1}}} \leq 1 - 11n\varepsilon^{\frac{1}{n+1}} \leq h(a_i), h(a_j) \leq 1 + n\varepsilon^{\frac{1}{n+1}} \leq e^{n\varepsilon^{\frac{1}{n+1}}},$$

thus $h(a_i) < h(a_j)e^{23(a_j - a_i)}$, and $h(0) < h(a_j)e^{23a_j}$ by the log-concavity of h. Using the bounds on $h(a_j)$ and a_j, we get $h(0) < e^{116n\varepsilon^{\frac{1}{n+1}}} < 1 + 250n\varepsilon^{\frac{1}{n+1}}$. On the other hand, the argument leading to (4) yields some integer $m \in [1, 5]$ such that $h(a_{-m}) \geq 1 - 11n\varepsilon^{\frac{1}{n+1}}$. We conclude by the log-concavity of h that

$$h(0) \geq \min\{h(a_{-m}), h(a_j)\} \geq 1 - 11n\varepsilon^{\frac{1}{n+1}}. \quad \square$$

Proposition 3.2 *If the origin 0 is the centroid of a convex body K in $\mathbb{R}^n$, and $E \subset K - w \subset (1+\mu)E$ for an 0-symmetric ellipsoid E and $w \in K$, then*

$$(1 - \mu\sqrt{n+1})E \subset K \subset (1 + 2\mu\sqrt{n+1})E,$$

holds whenever $\mu \in (0, 1/(n+1))$.

Proof: We may assume that $E = B^n$ and $w \neq 0$. Let $w_0 = w/|w|$, and let B^+ be the half-ball $\{x \in B^n : \langle x, w_0\rangle \geq 0\}$. If $\mu < \frac{1}{n+1}$, then $(1+\mu)^{n+1} < e^{\mu(n+1)} < 1 + 2\mu(n+1)$, thus

$$\begin{aligned}
0 &= \int_K \langle x, w\rangle\, dx = V(K)\langle w, w\rangle + \int_{K-w} \langle x, w\rangle\, dx \\
&> V(B^n)\langle w, w\rangle + \int_{B^+} \langle x, w\rangle\, dx - (1+\mu)^{n+1} \int_{B^+} \langle x, w\rangle\, dx \\
&> V(B^n)|w|^2 - 2(n+1)\left(\int_{B^+} \langle x, w_0\rangle\, dx\right)\mu \cdot |w|.
\end{aligned}$$

Therefore

$$\begin{aligned}|w| &\le (n+1)\mu\frac{\int_{B^n}|\langle x,w_0\rangle|\,dx}{V(B^n)} \le (n+1)\mu\left(\frac{\int_{B^n}\langle x,w_0\rangle^2\,dx}{V(B^n)}\right)^{\frac{1}{2}}\\ &= (n+1)\mu\left(\frac{\int_{B^n}|x|^2\,dx}{nV(B^n)}\right)^{\frac{1}{2}} \le \mu\sqrt{n+1}.\end{aligned}$$

Combining this with our hypothesis $w+B^n\subset K\subset w+(1+\mu)B^n$ readily gives the claim. □

Now let us prove Theorem 1.3. It is sufficient to consider the case $\varepsilon\in(0,\varepsilon_0)$ where $\varepsilon_0>0$ depends on n. Replacing also $f(x)$ by $f(x+z)$ and $g(y)$ by $g(y+z)$ we may assume that $z=0$. For suitable $\nu,\mu,\lambda>0$, replacing $\varrho(r)$ by $\nu\varrho(\lambda^2 r)$, $f(x)$ by $\mu\nu f(\lambda x)$ and $g(x)$ by $(\nu/\mu)g(\lambda x)$, we may assume that

$$\int_{\mathbb{R}_+} r^{n-1}\varrho(r^2)\,dr=1 \quad\text{and}\quad \varrho(0)=f(0)=1.$$

Consider the body

$$K_f=\left\{x\in\mathbb{R}^n:\ \int_0^{+\infty} r^{n-1}f(rx)\,dr\ge 1\right\},$$

which is convex since f is log-concave [3]. Its radial function

$$\|x\|_{K_f}^{-1}=\rho_{K_f}(x):=\sup\left\{t\ge 0;\ tx\in K_f\right\},\quad x\in S^{n-1}$$

is equal to $\left(\int_{\mathbb{R}^+} r^{n-1}f(rx)\,dr\right)^{1/n}$. Hence, using polar coordinates shows that

$$\int_{\mathbb{R}^n} f(x)\,dx=nV(K_f). \tag{5}$$

For $x\in\mathbb{R}^n\backslash\{0\}$, let $f_x,g_x:\mathbb{R}\to\mathbb{R}_+$ be defined by $f_x(r)=|r|^{n-1}f(rx)$ and $g_x(r)=|r|^{n-1}g(rx)$. If $\langle x,y\rangle>0$, then the condition on f,g,ϱ yields that $f_x(r)\cdot g_y(s)\le m_{xy}(\sqrt{rs})^2$ for $m_{xy}(r)=r^{n-1}\varrho(r^2\langle x,y\rangle)$ and $r,s\in\mathbb{R}_+$. We deduce by the Borell-Prékopa-Leindler inequality Theorem 2.1 that

$$\int_{\mathbb{R}_+} f_x(r)\,dr\cdot\int_{\mathbb{R}_+} g_y(r)\,dr\le\left(\int_{\mathbb{R}_+} r^{n-1}\varrho(r^2\langle x,y\rangle)\,dr\right)^2=\langle x,y\rangle^{-n},$$

and hence

$$K_g\subset K_f^\circ. \tag{6}$$

The hypothesis of the theorem translated in terms of K_f gives

$$\begin{aligned} n^2V(B^n)^2 &= \left(\int_{\mathbb{R}^n} \varrho(|x|^2)\,dx\right)^2 \leq (1+\varepsilon)\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(x)\,dx \\ &= (1+\varepsilon)n^2V(K_f)V(K_g) \leq (1+\varepsilon)n^2V(K_f)V(K_f^\circ). \end{aligned} \tag{7}$$

From the stability version Theorem 2.3 of the Blaschke-Santaló inequality, for some $\gamma > 0$, $\delta_{\mathrm{BM}}(K_f, B^n) < \gamma\,\varepsilon^{\frac{1}{5n}}$. Thus replacing $f(x)$ by $f(Tx)$ and $g(y)$ by $g(T^{-1}y)$ for a suitable positive definite matrix if necessary, and applying Proposition 3.2, we may assume that

$$B^n \subset K_f \subset (1 + O(\varepsilon^{\frac{1}{5n}}))B^n. \tag{8}$$

Using (6) we get $K_g \subset K_f^\circ \subset B^n$ and (7) yields

$$V(K_g) \geq (1+\varepsilon)^{-1}V(B^n)^2V(K_f)^{-1} \geq (1+O(\varepsilon^{\frac{1}{5n}}))^{-1}V(B^n). \tag{9}$$

For $x \in S^{n-1}$, $\rho_{K_f}(x) = (\int_{\mathbb{R}_+} f_x(r)\,dr)^{\frac{1}{n}} \geq 1$ and $\rho_{K_g}(x) = (\int_{\mathbb{R}_+} g_x(r)\,dr)^{\frac{1}{n}} \leq 1$. We define

$$\begin{aligned} \varphi(x) &:= \int_{\mathbb{R}} f_x(r)\,dr - 2 = \rho_{K_f}(x)^n + \rho_{K_f}(-x)^n - 2 \geq 0, \\ \psi(x) &:= 2 - \int_{\mathbb{R}} g_x(r)\,dr = 2 - \rho_{K_g}(x)^n - \rho_{K_g}(-x)^n \geq 0. \end{aligned}$$

In particular (8) and (9) yield

$$\begin{aligned} \int_{S^{n-1}} \varphi(x)\,dx &= 2n(V(K_f) - V(B^n)) = O(\varepsilon^{\frac{1}{5n}}) \\ \int_{S^{n-1}} \psi(x)\,dx &= 2n(V(B^n) - V(K_g)) = O(\varepsilon^{\frac{1}{5n}}), \end{aligned} \tag{10}$$

where the integration is with respect to the Hausdorff measure on the sphere. To estimate φ pointwize from above, we use the inclusion (8). In order to estimate ψ, we use (10) and the fact that a cap of B^n of height $h \leq 1$ is of volume larger than $h^{\frac{n+1}{2}}V(B^{n-1})/n$ (which forces the convex subset K_g of the unit ball, with almost the same volume, to have a radial function close to 1 pointwize). More precisely, we obtain that there exists $\gamma_0 > 0$ depending only on n, such that

$$\begin{aligned} \varphi(x) &\leq \gamma_0\varepsilon^{\frac{1}{5n}} \quad \text{for any } x \in S^{n-1}, \\ \psi(x) &\leq \gamma_0\varepsilon^{\frac{2}{5n(n+1)}} \quad \text{for any } x \in S^{n-1}. \end{aligned} \tag{11}$$

If ε_0 is chosen small enough (depending on n), then (11) yields that both $\varphi(x) < \frac{1}{2}$ and $\psi(x) < \frac{1}{2}$ for any $x \in S^{n-1}$.

Let $x \in S^{n-1}$, and hence

$$\int_{\mathbb{R}_+} f_x(r)\, dr \geq 1 \quad \text{and} \quad \int_{\mathbb{R}_+} g_x(r)\, dr \geq 1 - \psi(x) \geq (1 + 2\psi(x))^{-1}.$$

We define $m(r) = r^{n-1}\varrho(r^2)$, which satisfies that $m(e^t)$ is log-concave, and $f_x(r) \cdot g_x(s) \leq m(\sqrt{rs})^2$ for $r, s \in \mathbb{R}_+$. Since

$$\left(\int_{\mathbb{R}_+} m(r)\, dr\right)^2 = 1 \leq (1 + 2\psi(x)) \int_{\mathbb{R}_+} f_x(r)\, dr \cdot \int_{\mathbb{R}_+} g_x(r)\, dr,$$

it follows from Theorem 2.2 that there exists $\alpha(x), \beta(x) > 0$ and an absolute constant $c_0 > 0$ such that

$$\int_{\mathbb{R}_+} |\alpha(x)\, f_x(\beta(x) r) - m(r)|\, dr \leq c_0 \psi(x)^{\frac{5}{16}} \tag{12}$$

$$\int_{\mathbb{R}_+} |\alpha(x)^{-1} g_x(\beta(x)^{-1} r) - m(r)|\, dr \leq c_0 \psi(x)^{\frac{5}{16}}. \tag{13}$$

Using $1 \leq \int_{\mathbb{R}_+} f_x(r)\, dr < 1 + \varphi(x)$ and (12), we deduce that

$$\begin{aligned}
\frac{\alpha(x)}{\beta(x)} &\leq \frac{\alpha(x)}{\beta(x)} \cdot \int_{\mathbb{R}_+} f_x(r)\, dr = \int_{\mathbb{R}_+} \alpha(x)\, f_x(\beta(x) r)\, dr \\
&\leq \int_{\mathbb{R}_+} m(r)\, dr + c_0 \psi(x)^{\frac{5}{16}} = 1 + c_0 \psi(x)^{\frac{5}{16}} \\
\frac{\alpha(x)}{\beta(x)} &\geq \frac{\alpha(x)}{\beta(x)} \cdot (1 - \varphi(x)) \int_{\mathbb{R}_+} f_x(r)\, dr \\
&= (1 - \varphi(x)) \int_{\mathbb{R}_+} \alpha(x)\, f_x(\beta(x) r)\, dr \\
&\geq (1 - \varphi(x)) \left(\int_{\mathbb{R}_+} m(r)\, dr - c_0 \psi(x)^{\frac{5}{16}}\right) \\
&\geq 1 - O\left(\max\{\varphi(x), \psi(x)^{\frac{5}{16}}\}\right).
\end{aligned}$$

For $a(x) = \alpha(x)^{-1}$ and $b(x) = \beta(x)^{-1}$, we have

$$1 - c_0 \psi(x)^{\frac{5}{16}} \leq \frac{a(x)}{b(x)} \leq 1 + O\left(\max\{\varphi(x), \psi(x)^{\frac{5}{16}}\}\right). \tag{14}$$

Since $\varphi(x)$ and $\psi(x)$ are even, (12) can be written in the form

$$\int_{\mathbb{R}} |r|^{n-1}|f(xr) - a(x)b(x)^{n-1}\,\varrho(b(x)^2r^2)|\,dr \le c_0\psi(x)^{\frac{5}{16}}\frac{a(x)}{b(x)}. \tag{15}$$

Thus the hypotheses of Proposition 3.1 are satisfied for the log-concave functions $h(r) = f(rx)$ and $\omega(r) = a(x)b(x)^{n-1}\varrho(b(x)^2r^2)$ because $\int_{\mathbb{R}} |r|^{n-1}\omega(r)dr = \frac{a(x)}{b(x)}$. As $\varrho(0) = f(0) = 1$ we get that (using $n+1 \le 2n$)

$$\left|a(x)b(x)^{n-1} - 1\right| = O\left(\psi(x)^{\frac{5}{32n}}\right). \tag{16}$$

We deduce by comparing (14) and (16) that

$$|a(x) - 1| = O\left(\max\{\varphi(x), \psi(x)^{\frac{5}{32n}}\}\right) \quad \text{and} \quad |b(x) - 1| = O\left(\max\{\varphi(x), \psi(x)^{\frac{5}{32n}}\}\right).$$

We claim that for any $x \in S^{n-1}$, we have

$$\int_{\mathbb{R}_+} r^{n-1}|f(xr) - \varrho(r^2)|\,dr \;\le\; O\left(\max\{\varphi(x), \psi(x)^{\frac{5}{32n}}\}\right) \tag{17}$$

$$\int_{\mathbb{R}_+} r^{n-1}|g(xr) - \varrho(r^2)|\,dr \;\le\; O\left(\max\{\varphi(x), \psi(x)^{\frac{5}{32n}}\}\right). \tag{18}$$

To prove (17), we observe

$$\begin{aligned}\int_{\mathbb{R}_+} r^{n-1}|f(xr) - \varrho(r^2)|\,dr \;&\le\; \int_{\mathbb{R}_+} r^{n-1}|f(xr) - a(x)b(x)^{n-1}\varrho(b(x)^2r^2)|\,dr\\ &+ \int_{\mathbb{R}_+} r^{n-1}a(x)b(x)^{n-1}|\varrho(b(x)^2r^2) - \varrho(r^2)|\,dr\\ &+ \int_{\mathbb{R}_+} r^{n-1}\varrho(r^2)|a(x)b(x)^{n-1} - 1|\,dr.\end{aligned}$$

Here the first term is $O(\psi(x)^{\frac{5}{16}})$ by (15), and the third term is $O(\psi(x)^{\frac{5}{32n}})$ by (16). To bound the second term, we first use (16) to get rid of $a(x)b(x)^{n-1}$. To simplify the notations, we put $M = |b(x)^2 - 1|$. Since $1 - M \le b^2 \le 1 + M$ and ϱ is non-increasing, we obtain

$$|\varrho(b(x)^2r^2) - \varrho(r^2)| \le \varrho((1-M)r^2) - \varrho((1+M)r^2).$$

Thus

$$\begin{aligned}&\int_{\mathbb{R}_+} r^{n-1}|\varrho(b(x)^2r^2) - \varrho(r^2)|\,dr\\ &\le\; \int_{\mathbb{R}_+} r^{n-1}\varrho((1-M)r^2)\,dr - \int_{\mathbb{R}_+} r^{n-1}\varrho((1+M)r^2)\,dr\\ &=\; (1-M)^{-\frac{n}{2}} - (1+M)^{-\frac{n}{2}} = O\left(\max\{\varphi(x), \psi(x)^{\frac{5}{32n}}\}\right),\end{aligned}$$

which in turn yields (17). The proof of (18) is similar.

Now using Hölder's inequality and (10), we deduce that

$$\begin{aligned}\int_{S^{n-1}} \left(\varphi(x) + \psi(x)^{\frac{5}{32n}}\right) dx &\leq \int_{S^{n-1}} \varphi(x)\, dx + O\left(\int_{S^{n-1}} \psi(x)\, dx\right)^{\frac{5}{32n}} \\ &\leq O(\varepsilon^{\frac{1}{32n^2}}).\end{aligned}$$

Therefore integrating (17) and (18) over $x \in S^{n-1}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x) - \varrho(|x|^2)|\, dx &\leq O(\varepsilon^{\frac{1}{32n^2}}) \\ \int_{\mathbb{R}^n} |g(x) - \varrho(|x|^2)|\, dx &\leq O(\varepsilon^{\frac{1}{32n^2}}).\end{aligned}$$

In turn we conclude Theorem 1.3.

4 Proof of Theorem 1.1 (φ convex)

During the proof of Theorem 1.1, $\gamma_1, \gamma_2, \ldots$ denote positive constants that depend only on n. We always assume that $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 > 0$ depends on n and ϱ, and is small enough for the argument to work.

We start with some simplification. We may assume that $\varrho(0) = 1 = \int_{\mathbb{R}_+} \varrho$. Using the same argument on ϱ that the one that we used on ω in the beginning of the proof of Proposition 3.1, there exists some $t_0 \geq 1$ such that $\varrho(t) \leq e^{-t}$ if $t \geq t_0$, and $\varrho(t) \geq e^{-t}$ if $t \in (0, t_0)$. It follows that $\varrho'(0) \geq -1$, and

$$\int_{\mathbb{R}_+} r^{n-1} \varrho(r^2)\, dr \leq \int_{\mathbb{R}_+} r^{n-1} e^{-r^2}\, dr = \frac{\Gamma(n/2)}{2}.$$

For the log-concave function $f(x) = \varrho(\varphi(x))$, we may assume that the origin 0 is the centre of mass of $K_{f,0}$, and hence we only check the condition in Theorem 1.1 at $z = 0$. For $\psi(x) = \mathcal{L}_0 \varphi(x)$, let $g(x) = \varrho(\psi(x))$. It follows from the definition of the Legendre transform that

$$\varphi(x) + \psi(y) \geq \langle x, y \rangle \quad \text{for all } x, y \in \mathbb{R}^n. \tag{19}$$

In particular

$$f(x)g(y) = \varrho(\varphi(x))\varrho(\psi(y)) \leq \varrho^2\left(\frac{\varphi(x) + \psi(y)}{2}\right) \leq \varrho^2\left(\frac{\langle x, y \rangle}{2}\right).$$

Thus we may apply Theorem 1.3, which yields the existence of $\xi > 0$ and a positive definite matrix $T : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| \varrho(|x|^2/2) - \xi\, \varrho(\varphi(Tx)) \right| dx &< \gamma_1 \varepsilon^{\frac{1}{32n^2}} \\
\int_{\mathbb{R}^n} \left| \varrho(|x|^2/2) - \xi^{-1} \varrho(\psi(T^{-1}x)) \right| dx &< \gamma_1 \varepsilon^{\frac{1}{32n^2}},
\end{aligned}$$

where γ_1 depends on n. Since $\mathcal{L}_0(\varphi \circ T) = \psi \circ T^{-1}$, we may assume that T is the identity matrix. We choose $R(\varepsilon)$ in a way such that

$$\varrho(R(\varepsilon)^2) = \varepsilon^{\frac{1}{64n^2}}.$$

As $\varrho(t) \leq e^{-t}$ for $t \geq t_0$, it follows that provided ε_0 is small enough,

$$30 < R(\varepsilon) \leq \sqrt{|\log \varepsilon|}/(8n). \tag{20}$$

Let $c = -\log \xi$ and $\alpha(x) = -\log \varrho(x)$. Hence α is convex and increasing with $\alpha(0) = 0$, $\alpha'(0) \leq 1$, where $\alpha'(x)$ denotes the right-derivative. We deduce

$$\begin{aligned}
\int_{\sqrt{2}R(\varepsilon)B^n} e^{-\alpha(|x|^2/2)} \left| e^{\alpha(|x|^2/2) - \alpha(\varphi(x)) - c} - 1 \right| dx &< \gamma_1 \varepsilon^{\frac{1}{32n^2}} \\
\int_{\sqrt{2}R(\varepsilon)B^n} e^{-\alpha(|x|^2/2)} \left| e^{\alpha(|x|^2/2) - \alpha(\psi(x)) + c} - 1 \right| dx &< \gamma_1 \varepsilon^{\frac{1}{32n^2}},
\end{aligned}$$

which in turn yields by the definition of $R(\varepsilon)$ that

$$\int_{\sqrt{2}R(\varepsilon)B^n} \left| e^{\alpha(|x|^2/2) - \alpha(\varphi(x)) - c} - 1 \right| dx < \gamma_1 \varepsilon^{\frac{1}{64n^2}} \tag{21}$$

$$\int_{\sqrt{2}R(\varepsilon)B^n} \left| e^{\alpha(|x|^2/2) - \alpha(\psi(x)) + c} - 1 \right| dx < \gamma_1 \varepsilon^{\frac{1}{64n^2}}. \tag{22}$$

Next we plan to get rid of the exponential function in (21) and (22). Define $\tilde{\alpha}(x) = \alpha(|x|^2/2)$. Then for all $x \in 1.3R(\varepsilon)B^n$,

$$|\nabla \tilde{\alpha}(x)| = |x|\, \alpha'(|x|^2/2) \leq 1.3R(\varepsilon)\alpha'\big(0.845R^2(\varepsilon)\big).$$

Using, for $s, t \geq 0$, the convexity bound $\alpha'(s) \leq \frac{\alpha((1+t)s) - \alpha(s)}{ts} \leq \frac{\alpha((1+t)s)}{ts}$ together with the relation $\alpha(R(\varepsilon)^2) = |\log \varepsilon|/(64n^2)$, we deduce that the function $\tilde{\alpha}$ satisfies

$$|\nabla \tilde{\alpha}(x)| \leq \gamma_2 |\log \varepsilon| \quad \text{for } x \in 1.3R(\varepsilon)B^n. \tag{23}$$

We claim that the convex function $\tilde{\varphi} = \alpha \circ \varphi$ satisfies

$$|\nabla \tilde{\varphi}(x)| \leq 32\gamma_2 |\log \varepsilon| \quad \text{for } x \in 1.2R(\varepsilon)B^n. \tag{24}$$

Suppose, to the contrary, that there exists $x_0 \in 1.2R(\varepsilon)B^n$ such that the vector $w := \nabla \tilde{\varphi}(x_0)$ satisfies $|w| > 32\gamma_2|\log \varepsilon|$. Since $R(\varepsilon) > 30$, it follows by (23) that

$$|\tilde{\alpha}(x) - \tilde{\alpha}(x_0)| \leq 3\gamma_2 |\log \varepsilon| \ \text{ if } |x - x_0| \leq 3. \tag{25}$$

We define

$$\Xi = \left\{ x \in \mathbb{R}^n : \, |x - x_0| \leq 1 \ \text{ and } \langle w, x - x_0 \rangle \geq \frac{1}{2}|w| \cdot |x - x_0| \right\} \subset 1.3R(\varepsilon)B^n.$$

If $\tilde{\varphi}(x) \leq \tilde{\alpha}(x_0) - c - 4\gamma_2|\log \varepsilon|$ for all $x \in \Xi$, then (25) yields

$$\int_{\sqrt{2}R(\varepsilon)B^n} \left| e^{\alpha(|x|^2/2) - \alpha(\varphi(x)) - c} - 1 \right| dx > \int_{\Xi} |e^{\gamma_2 |\log \varepsilon|} - 1| \, dx > \gamma_1 \varepsilon^{\frac{1}{64n^2}},$$

provided that ε_0 is small enough. This contradiction to (21) provides a $y_0 \in \Xi$ such that $\tilde{\varphi}(y_0) \geq \tilde{\alpha}(x_0) - c - 4\gamma_2|\log \varepsilon|$. For $v = \nabla \tilde{\alpha}(y_0)$, we have

$$\begin{aligned} \langle v, x_0 - y_0 \rangle &\leq \tilde{\varphi}(x_0) - \tilde{\varphi}(y_0) \leq \langle w, x_0 - y_0 \rangle \\ &\leq -\frac{1}{2}|w|\,|x_0 - y_0| \leq -16\gamma_2|\log \varepsilon| \cdot |x_0 - y_0|. \end{aligned}$$

In particular $|v| \geq 16\gamma_2|\log \varepsilon|$. Next let

$$\Xi' = \left\{ x \in \mathbb{R}^n : \, 1 \leq |x - y_0| \leq 2 \ \text{ and } \ \langle v, x - y_0 \rangle \geq \frac{1}{2}|w| \cdot |x - y_0| \right\} \subset 1.3R(\varepsilon)B^n.$$

Combining the above definitions and (25) yields for any $x \in \Xi'$,

$$\begin{aligned} \tilde{\varphi}(x) &\geq \tilde{\varphi}(y_0) + \langle x - y_0, v \rangle \\ &\geq \tilde{\alpha}(x_0) - c - 4\gamma_2|\log \varepsilon| + \frac{1}{2}|w|\,|x - y_0| \\ &\geq \tilde{\alpha}(x_0) - c + 4\gamma_2|\log \varepsilon| \\ &\geq \tilde{\alpha}(x) - c + \gamma_2|\log \varepsilon|. \end{aligned}$$

Consequently,

$$\int_{\sqrt{2}R(\varepsilon)B^n} \left| e^{\alpha(|x|^2/2) - \alpha(\varphi(x)) - c} - 1 \right| dx > \int_{\Xi'} |e^{\gamma_2 |\log \varepsilon|} - 1| \, dx > \gamma_1 \varepsilon^{\frac{1}{64n^2}},$$

provided that ε_0 is small enough. This contradicts (21), hence we may conclude (24).

Next we prove that

$$\alpha\big(|x|^2/2\big) - \alpha(\varphi(x)) - c > -1 \quad \text{if } x \in 1.1R(\varepsilon)B^n. \tag{26}$$

Otherwise suppose that $x_1 \in 1.1R(\varepsilon)B^n$ and $\tilde{\alpha}(x_1) - \tilde{\varphi}(x_1) - c \leq -1$. If $|x - x_1| \leq (96\gamma_2|\log\varepsilon|)^{-1}$ and ε_0 is small enough, then (23) and (24) imply that $\tilde{\alpha}(x) - \tilde{\varphi}(x) - c \leq -1/3$. Therefore

$$\begin{aligned}\int_{\sqrt{2}R(\varepsilon)B^n} \left|e^{\alpha(|x|^2/2)-\alpha(\varphi(x))-c} - 1\right| dx &\geq \int \left|e^{-\frac{1}{3}} - 1\right| \mathbf{1}_{|x-x_1|\leq(96\gamma_2|\log\varepsilon|)^{-1}}\, dx \\ &\geq \gamma_3|\log\varepsilon|^{-n} > \gamma_1\varepsilon^{\frac{1}{64n^2}},\end{aligned}$$

provided that ε_0 is small enough. This is a contradiction, hence (26) holds.

Since $|t| < 2|e^t - 1|$ if $t \geq -1$, combining (21) and (22) with (26) and its analogue for ψ, we deduce

$$\int_{R(\varepsilon)B^n} \left|\alpha(\varphi(x)) - \alpha(|x|^2/2) + c\right| dx < 2\gamma_1\varepsilon^{\frac{1}{64n^2}} \tag{27}$$

$$\int_{R(\varepsilon)B^n} \left|\alpha(\psi(x)) - \alpha(|x|^2/2) - c\right| dx < 2\gamma_1\varepsilon^{\frac{1}{64n^2}}. \tag{28}$$

For $x \in \mathbb{R}^n$, we define $C(x) = \varphi(x) - \frac{|x|^2}{2}$, $\widetilde{C}(x) = \psi(x) - \frac{|x|^2}{2}$ and

$$F(x) = C(x) + \widetilde{C}(x) \geq 0, \tag{29}$$

where the inequality is a consequence of (19). Summing up (27) and (28), and using the convexity of α in the form $\alpha(b) - \alpha(a) \geq (b-a)\alpha'(a)$ yields that

$$\begin{aligned}4\gamma_1\varepsilon^{\frac{1}{64n^2}} &\geq \int_{R(\varepsilon)B^n} \big(\alpha(\varphi(x)) - \alpha(|x|^2/2) + \alpha(\psi(x)) - \alpha(|x|^2/2)\big)\, dx \\ &\geq \int_{R(\varepsilon)B^n} \alpha'(|x|^2/2)\big(\varphi(x) - |x|^2/2 + \psi(x) - |x|^2/2\big)\, dx \\ &= \int_{R(\varepsilon)B^n} \alpha'(|x|^2/2)F(x)\, dx\end{aligned}$$

This is the point where α influences the estimates. Using (29), we get that

$$\int_{R(\varepsilon)B^n} F(x)\, dx < \frac{4\gamma_1}{\alpha'(0)} \cdot \varepsilon^{\frac{1}{64n^2}}. \tag{30}$$

Observe that with our notation, (19) reads as $C(y)+\widetilde{C}(x)+|x-y|^2/2 \geq 0$ or equivalently $C(x) \leq C(y)+F(x)+|x-y|^2/2$. Since F takes non-negative values, we get that for all $x, y \in \mathbb{R}^n$,

$$|C(x)-C(y)| \leq F(x)+F(y)+\frac{|x-y|^2}{2}\cdot \tag{31}$$

For $t \in \mathbb{R}$, we write $\lceil t \rceil$ for the smallest integer not smaller than t, which satisfies $\lceil t \rceil + 1 \leq 2t$ if $\lceil t \rceil \geq 3$. Set

$$k = \left\lceil \sqrt{\frac{4V(B^n)}{2^{n+1}}} \left(\int_{R(\varepsilon)B^n} F(z)\, dz \right)^{-\frac{1}{2}} \cdot R(\varepsilon)^{\frac{n+2}{2}} \right\rceil, \tag{32}$$

which is at least 3 if ε_0 is chosen small enough by (20) and (30). Let us denote $\sigma := V(R(\varepsilon)B^n)^{-1} \int_{R(\varepsilon)B^n} C(y)\, dy$. Taking advantage of (31), we get that

$$\begin{aligned}
\int_{R(\varepsilon)B^n} |C(x)-\sigma|\, dx &\leq V(R(\varepsilon)B^n)^{-1} \int_{R(\varepsilon)B^n} \int_{R(\varepsilon)B^n} |C(x)-C(y)|\, dxdy \\
&\leq V(R(\varepsilon)B^n)^{-1} \sum_{i=1}^{k} \int_{R(\varepsilon)B^n} \int_{R(\varepsilon)B^n} \\
&\quad \left| C\left(\tfrac{i}{k}\, x + (1-\tfrac{i}{k})y\right) - C\left(\tfrac{i-1}{k}\, x + (1-\tfrac{i-1}{k})y\right) \right| \, dxdy \\
&\leq \sum_{i=0}^{k} \frac{2}{V(R(\varepsilon)B^n)} \int_{R(\varepsilon)B^n} \int_{R(\varepsilon)B^n} F\left(\tfrac{i}{k}\, x + (1-\tfrac{i}{k})y\right)\, dxdy \\
&\quad + \frac{1}{V(R(\varepsilon)B^n)} \sum_{i=1}^{k} \int_{R(\varepsilon)B^n} \int_{R(\varepsilon)B^n} \frac{|x-y|^2}{k^2}\, dxdy.
\end{aligned} \tag{33}$$

For $i \in \{0, \ldots, k\}$ in (33), we claim that

$$\int_{R(\varepsilon)B^n} \int_{R(\varepsilon)B^n} F\left(\tfrac{i}{k}\, x + (1-\tfrac{i}{k})y\right)\, dxdy \leq 2^n V(R(\varepsilon)B^n) \int_{R(\varepsilon)B^n} F(z)\, dz. \tag{34}$$

If $i \geq k/2$, then for fixed y, using the substitution $z = \frac{i}{k}\, x + (1-\frac{i}{k})y$, we have

$$\begin{aligned}
\int_{R(\varepsilon)B^n} \int_{R(\varepsilon)B^n} F\left(\tfrac{i}{k}\, x + (1-\tfrac{i}{k})y\right)\, dxdy &= \frac{k^n}{i^n} \int_{R(\varepsilon)B^n} \int_{\frac{i}{k}\, R(\varepsilon)B^n + (1-\frac{i}{k})y} F(z)\, dzdy \\
&\leq 2^n V(R(\varepsilon)B^n) \int_{R(\varepsilon)B^n} F(z)\, dz.
\end{aligned}$$

If $i < k/2$, then for fixed y, we obtain (34) using the substitution $z = \frac{i}{k}x + (1 - \frac{i}{k})y$ for fixed x.

In (33), we use the rough estimate $|x - y| \le 2R(\varepsilon)$, and obtain by (34), (32) and $k + 1 \le 2k$ that

$$\begin{aligned}\int_{R(\varepsilon)B^n} |C(x) - \sigma|\, dx &\le (k+1)2^{n+1} \int_{R(\varepsilon)B^n} F(z)\, dz + \frac{4V(B^n)R(\varepsilon)^{n+2}}{k} \\ &\le 3c_0 \left(\int_{R(\varepsilon)B^n} F(z)\, dz \right)^{\frac{1}{2}} R(\varepsilon)^{\frac{n+2}{2}}\end{aligned}$$

where $c_0 > 0$ is an absolute constant such that $\sqrt{2^{n+1}4V(B^n)} < c_0$ for $n \ge 2$. Since $R(\varepsilon) \le \sqrt{|\log \varepsilon|}$ by (20), we deduce by the definition of $C(x)$ and (30) that

$$\int_{R(\varepsilon)B^n} |\varphi(x) - \tfrac{|x|^2}{2} - \sigma|\, dx < 3c_0 \sqrt{\frac{4\gamma_1}{\alpha'(0)}} \cdot \varepsilon^{\frac{1}{128n^2}} \cdot |\log \varepsilon|^{\frac{n+2}{4}},$$

completing the proof the first part of Theorem 1.1.

5 Proof of Theorem 1.1 (φ measurable)

In this section, $\eta_1, \eta_2, \ldots$ denote positive constants that depend only on n and ϱ. Since $\int_{\mathbb{R}^n} \varrho(\mathcal{L}_z\varphi(x))\, dx > 0$ for all z, the function $\mathcal{L}_z\varphi$ cannot be identically infinite. Hence we may consider the lower convex hull $\varphi_* = \mathcal{L}_z\mathcal{L}_z\varphi$ of φ. It follows that $\mathcal{L}_z\varphi_* = \mathcal{L}_z\varphi$. We may assume as in the proof of Theorem 1.1 that $\varrho(0) = 1 = \int_{\mathbb{R}_+} \varrho$, and hence $\varrho'(0) \ge -1$. Let again $\alpha(t) = -\log \varrho(t)$, which is convex, increasing, and satisfies $\alpha(0) = 0$ and $0 < \alpha'(0) \le 1$, where $\alpha'(x)$ denotes the right-derivative.

For $t \in \mathbb{R}$, we also introduce

$$\alpha_*(t) = \begin{cases} \alpha(t) & \text{if } t \ge 0 \\ \alpha'(0) \cdot t & \text{if } t \le 0. \end{cases}$$

As we shall see shortly, we can replace α by α_* in the inequalities. Observe first that $\alpha_* \le \alpha$ and that

$$\alpha_*'(t) \ge \alpha'(0) = \alpha_*'(0) \quad \text{for all } t \in \mathbb{R}.$$

Let $\varrho_*(t) = e^{-\alpha_*(t)}$. As $\varrho_*(t) \geq \varrho(t)$, $\varphi_*(x) \leq \varphi(x)$ and $\mathcal{L}_z\varphi_* = \mathcal{L}_z\varphi$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \varrho_*(\varphi_*(x))\,dx \int_{\mathbb{R}^n} \varrho_*(\mathcal{L}_z\varphi_*(x))\,dx &\geq \int_{\mathbb{R}^n} \varrho_*(\varphi(x))\,dx \int_{\mathbb{R}^n} \varrho_*(\mathcal{L}_z\varphi(x))\,dx \\
&\geq (1-\varepsilon)\left(\int_{\mathbb{R}^n} \varrho(|x|^2/2)\,dx\right)^2 \\
&= (1-\varepsilon)\left(\int_{\mathbb{R}^n} \varrho_*(|x|^2/2)\,dx\right)^2 \qquad (35)
\end{aligned}$$

for any z. We may assume that the origin 0 is the centre of mass of $K_{f,0}$ for the log-concave function $f = \varrho_* \circ \varphi_*$. Therefore

$$\int_{\mathbb{R}^n} \varrho_*(\varphi_*(x))\,dx \int_{\mathbb{R}^n} \varrho_*(\mathcal{L}_0\varphi_*(x))\,dx \leq \left(\int_{\mathbb{R}^n} \varrho_*(|x|^2/2)\,dx\right)^2. \qquad (36)$$

We have proved in the course of the argument for Theorem 1.1 that possibly after a positive definite linear transformation, there exists $\sigma \in \mathbb{R}$ such that

$$\int_{R_*(\varepsilon)B^n} \left|\varphi_*(x) - \sigma - \frac{|x|^2}{2}\right| dx < \eta_1 \varepsilon^{\frac{1}{129n^2}} \qquad (37)$$

where

$$\alpha_*(R_*(\varepsilon)^2) = \frac{|\log\varepsilon|}{64n^2}. \qquad (38)$$

In particular $\lim_{\varepsilon\to 0} R_*(\varepsilon) = +\infty$ and $30 < R_*(\varepsilon) \leq \sqrt{|\log\varepsilon|}/(8n)$. Set $R(\varepsilon) := \frac{1}{2}\, R_*(\varepsilon)$.

Proposition 5.1 *If $\varepsilon > 0$ is small enough, then*

$$\left|\varphi_*(x) - \sigma - \frac{|x|^2}{2}\right| < \eta_2 \varepsilon^{\frac{2}{129n^2(n+2)}} < 1 \qquad \textit{for all } x \in \tfrac{5}{3}\, R(\varepsilon)\, B^n.$$

Proof: Let us denote by c the convex function $\varphi_* - \sigma$ and $f(x) = c(x) - |x|^2/2$. Set $\delta = \eta_1 \varepsilon^{1/(129n^2)}$. Assume that ε is small enough so that $\delta < 1$. Our starting point is (37) which reads as $\int_{R_*(\varepsilon)B^n} |f| \leq \delta$.

Let $r \in (0,1)$ and $x \in \mathbb{R}^n$ with $|x| \leq R_*(\varepsilon) - 1$. If $v(x)$ is a subgradient of c at x, we get by convexity of c that for all y,

$$f(y) \geq f(x) + \langle v(x) - x, y - x\rangle - \frac{|y-x|^2}{2}.$$

Since the ball $B(x,r)$ of center x and radius r is included in $B(0,R_*(\varepsilon)) = R_*(\varepsilon)\,B^n$, we deduce that

$$\begin{aligned}\delta &\geq \int_{B(x,r)} |f| \geq \int_{B(x,r)} f(y)\,dy \\ &\geq f(x)\,V(B(x,r)) - \int_{B(x,r)} \frac{|y-x|^2}{2}\,dy = v_n r^n f(x) - c_n r^{n+2}\end{aligned}$$

for suitable quantities v_n, c_n depending only on n. Rearranging, $f(x) \leq (\delta + c_n r^{n+2})/(v_n r^n)$. Choosing $r = \delta^{1/(n+2)} < 1$, we obtain that for all x with $|x| \leq R_*(\varepsilon) - 1$,

$$f(x) \leq d_n \delta^{\frac{2}{n+2}}. \tag{39}$$

In order to establish the proposition, it remains to prove a similar lower bound on f. Consider $x \in \mathbb{R}^n$ with $|x| \leq \mathbb{R}_*(\varepsilon) - 2$. Let $r \in (0,1)$ to be specified later. Consider a point $y \in B(x,r) \setminus \{x\}$. It can be written as $y = x + su$ with $s \in (0,r]$ and $u \in \mathbb{R}^n$, $|u| = 1$. By convexity,

$$\begin{aligned}c(y) &\leq \frac{s}{r} c(x+ru) + \left(1 - \frac{s}{r}\right) c(x) \\ &= \frac{s}{r}\left(f(x+ru) + \frac{|x+ru|^2}{2}\right) + \left(1 - \frac{s}{r}\right)\left(f(x) + \frac{|x|^2}{2}\right).\end{aligned}$$

Rearranging the squares and using the upper bound (39) gives

$$\begin{aligned}-f(y) &= \frac{|y|^2}{2} - c(y) \geq -\frac{1}{2}s(r-s) - \frac{s}{r} f(x+ru) - \left(1-\frac{s}{r}\right) f(x) \\ &\geq -\frac{1}{2}s(r-s) - \frac{s}{r} d_n \delta^{\frac{2}{n+2}} - \left(1-\frac{s}{r}\right) f(x).\end{aligned}$$

Integrating in $y = x + su$ in spherical coordinates of origin x, and changing variables $s = rt$, $t \in (0,1]$ gives for suitable positive numbers depending only on the dimension

$$\begin{aligned}\delta &\geq \int_{R_*(\varepsilon)B^n} |f| \geq \int_{B(x,r)} |f| \geq \int_{B(x,r)} -f(y)\,dy \\ &\geq -\frac{1}{2}\int_0^r s(r-s) nV(B^n) s^{n-1} ds - d_n \delta^{\frac{2}{n+2}} \int_0^r \frac{s}{r} nV(B^n) s^{n-1} ds \\ &\quad -f(x) \int_0^r \left(1 - \frac{s}{r}\right) nV(B^n) s^{n-1} ds \\ &= -c_n r^{n+2} - d'_n \delta^{\frac{2}{n+2}} r^n - c'_n f(x) r^n.\end{aligned}$$

Choosing $r = \delta^{1/(n+2)} < 1$ and rearranging yields $f(x) \geq -c''_n \delta^{\frac{2}{n+2}}$, provided $|x| \leq R_*(\varepsilon) - 2$. Since $R_*(\varepsilon) > 30$, the claim follows. □

We now estimate how close the weight function $\varrho_* \circ \varphi_*$ is to be a constant function on $R(\varepsilon)B^n$. We claim that

$$\frac{\varrho_* \circ \varphi_*(x)}{\varrho_* \circ \varphi_*(y)} \geq \varepsilon^{\frac{1}{16n^2}} \quad \text{for all } x, y \in R(\varepsilon)B^n. \tag{40}$$

Since the function $\alpha_* = -\log \rho_*$ is increasing, we deduce from Proposition 5.1 that

$$|\alpha_*(\varphi(x)) - \alpha_*(\varphi(y))| \leq \alpha_* \left(\frac{R(\varepsilon)^2}{2} + 1 + \sigma \right) - \alpha_*(\sigma - 1) \quad \text{for } x, y \in R(\varepsilon)B^n.$$

Therefore it is sufficient to prove that

$$\Omega := \alpha_* \left(\frac{R(\varepsilon)^2}{2} + 1 + \sigma \right) - \alpha_*(\sigma - 1) \leq \frac{|\log \varepsilon|}{16n^2}. \tag{41}$$

It follows by (27) and (28) that

$$\int_{R_*(\varepsilon)B^n} \Big(\alpha_*\big(\varphi_*(x)\big) + \alpha_*\big(\mathcal{L}_0\varphi_*(x)\big) - 2\alpha_*\big(|x|^2/2\big) \Big)\, dx < \eta_8 \varepsilon^{\frac{1}{64n^2}}.$$

We note that by definition $\varphi_*(x) + \mathcal{L}_0\varphi_*(x) \geq |x|^2$ for all $x \in \mathbb{R}^n$. Hence the monotonicity of α_* yields

$$\int_{R_*(\varepsilon)B^n} \Big(\alpha_*\big(\varphi_*(x)\big) + \alpha_*\big(|x|^2 - \varphi_*(x)\big) - 2\alpha_*\big(|x|^2/2\big) \Big)\, dx < \eta_8 \varepsilon^{\frac{1}{64n^2}}. \tag{42}$$

Next, we bound from below the three terms appearing inside the above integral, when the variable is in the smaller domain $\frac{5}{3}\, R(\varepsilon)B^n \backslash (\frac{4}{3}\, R(\varepsilon)B^n)$. Observe that if $x \in \frac{5}{3}\, R(\varepsilon)B^n \backslash (\frac{4}{3}\, R(\varepsilon)B^n)$, then by Proposition 5.1

$$\varphi_*(x) \geq \frac{(\frac{4}{3}\, R(\varepsilon))^2}{2} + \sigma - 1 \geq \frac{R(\varepsilon)^2}{2} + \sigma + 1.$$

Since α_* is convex, increasing and verifies $\alpha_*(0) = 0$, $\alpha_*'(0) = \alpha'(0)$ we get that

$$\alpha_*(\varphi_*(x)) \geq \alpha_* \left(\frac{R(\varepsilon)^2}{2} + \sigma + 1 \right) = \Omega + \alpha_*(\sigma - 1) \geq \Omega + \alpha'(0)(\sigma - 1).$$

Still assuming that $x \in \frac{5}{3}\, R(\varepsilon)B^n \backslash (\frac{4}{3}\, R(\varepsilon)B^n)$ and taking advantage of Proposition 5.1, we obtain that

$$\alpha_*(|x|^2 - \varphi_*(x)) \geq \alpha_*(-\sigma - 1) \geq \alpha'(0)(-\sigma - 1).$$

Eventually, Equation (38) gives for $x \in \frac{5}{3}\, R(\varepsilon)B^n \backslash (\frac{4}{3}\, R(\varepsilon)B^n)$,

$$\alpha_*(|x|^2/2) \leq \alpha_*\big((5R(\varepsilon)/3)^2/2\big) \leq \alpha_*(R_*(\varepsilon)^2) = \frac{|\log\varepsilon|}{64n^2}.$$

Since the integrand in (42) is always non-negative, the above three inequalities together with (42) easily yield that

$$\int_{\frac{5}{3}\, R(\varepsilon)B^n \backslash (\frac{4}{3}\, R(\varepsilon)B^n)} \left(\Omega - 2\alpha'(0) - \frac{|\log\varepsilon|}{32n^2} \right) dx < \eta_8 \varepsilon^{\frac{1}{64n^2}}.$$

From this, we conclude that (41) and thus (40) hold if ε is small enough.

Next, we define the set

$$\Psi =: \left\{ x \in R(\varepsilon)\, B^n : \varphi(x) > \varphi_*(x) + \varepsilon^{\frac{1}{128n^2}} \right\}.$$

Since the inequality $\varphi(x) \geq \varphi_*(x)$ is true for all $x \in \mathbb{R}^n$, it follows from equation (37) and the bound $R(\varepsilon) < \sqrt{|\log\varepsilon|}$ that

$$\int_{R(\varepsilon)B^n \backslash \Psi} \left| \frac{|x|^2}{2} + \sigma - \varphi(x) \right| dx < \eta_{10} \varepsilon^{\frac{1}{129n^2}}.$$

Therefore our final task is to provide a suitable upper bound on the volume of the set Ψ.

Let $R_0 > 0$ be defined by $\alpha'(0) \cdot (\frac{R_0^2}{2} - 2) = 1$. Since $\alpha'(0) \in (0, 1]$, $R_0 \geq \sqrt{6}$. From now on we consider $R \in [R_0, R(\varepsilon)]$. Since $\alpha_*'(t) \geq \alpha'(0)$ for $t \in \mathbb{R}$, we have for all $x \in \Psi$

$$\begin{aligned}
\varrho_*(\varphi(x)) &= e^{-\alpha_*(\varphi(x))} \leq e^{-\alpha_*\left(\varphi_*(x) + \varepsilon^{\frac{1}{128n^2}}\right)} \\
&\leq e^{-\alpha'(0)\varepsilon^{\frac{1}{128n^2}}} \varrho_*(\varphi_*(x)) \leq \left(1 - \alpha'(0)\varepsilon^{\frac{1}{127n^2}}\right) \varrho_*(\varphi_*(x)),
\end{aligned}$$

where the last inequality is valid if ε is small enough. This improves on the trivial estimate $\varrho_* \circ \varphi \leq \varrho_* \circ \varphi_*$. Let us see how the improvement passes to integrals:

$$\begin{aligned}
&\int_{R\, B^n} \varrho_* \circ \varphi = \int_{\Psi \cap R\, B^n} \varrho_* \circ \varphi + \int_{R\, B^n \backslash \Psi} \varrho_* \circ \varphi \\
&\leq \left(1 - \alpha'(0)\varepsilon^{\frac{1}{127n^2}}\right) \int_{\Psi \cap R\, B^n} \varrho_* \circ \varphi_* + \int_{R\, B^n \backslash \Psi} \varrho_* \circ \varphi_* \\
&= \int_{R\, B^n} \varrho_* \circ \varphi_* - \alpha'(0)\varepsilon^{\frac{1}{127n^2}} \int_{\Psi \cap R\, B^n} \varrho_* \circ \varphi_*.
\end{aligned}$$

However, (40) readily gives

$$\int_{\Psi\cap R\,B^n} \varrho_* \circ \varphi_* \geq \varepsilon^{\frac{1}{16n^2}} \frac{V(\Psi\cap R\,B^n)}{V(R\,B^n)} \int_{R\,B^n} \varrho_* \circ \varphi_*.$$

Hence, combining this with the former estimate, we deduce that

$$\int_{R\,B^n} \varrho_* \circ \varphi \;\; \leq \;\; \left(1 - \frac{\varepsilon^{\frac{1}{16n^2}} V(\Psi\cap R\,B^n)}{V(R\,B^n)} \cdot \alpha'(0)\varepsilon^{\frac{1}{127n^2}}\right) \int_{R\,B^n} \varrho_* \circ \varphi_*.$$

Our goal is to draw information on the volume of Ψ from the above inequality and the almost equality in the functional Blaschke-Santaló inequality. But this requires a similar inequality involving integrals on the whole space. Building on the latter estimate,

$$\begin{aligned}
\int_{\mathbb{R}^n} \varrho_* \circ \varphi &= \int_{R\,B^n} \varrho_* \circ \varphi + \int_{\mathbb{R}^n\setminus R\,B^n} \varrho_* \circ \varphi \\
&\leq \left(1 - \varepsilon^{\frac{1}{8n^2}} \alpha'(0) \frac{V(\Psi\cap R\,B^n)}{V(R\,B^n)}\right) \int_{R\,B^n} \varrho_* \circ \varphi_* + \int_{\mathbb{R}^n\setminus R\,B^n} \varrho_* \circ \varphi_* \\
&= \int_{\mathbb{R}^n} \varrho_* \circ \varphi_* - \varepsilon^{\frac{1}{8n^2}} \alpha'(0) \frac{V(\Psi\cap R\,B^n)}{V(R\,B^n)} \int_{R\,B^n} \varrho_* \circ \varphi_*. \qquad (43)
\end{aligned}$$

If $|x| = R_0$, then Proposition 5.1 and the properties of α_* yield

$$\begin{aligned}
&\alpha_*(\varphi_*(x)) - \alpha_*(\varphi_*(0)) \geq \alpha_* \left(\frac{R_0^2}{2} + \sigma - 1\right) - \alpha_*(\sigma+1) \\
&\geq \alpha_*'(\sigma+1) \left(\frac{R_0^2}{2} - 2\right) = \frac{\alpha_*'(\sigma+1)}{\alpha'(0)} \geq 1,
\end{aligned}$$

thus the log-concave function $\varrho_* \circ \varphi_*$ verifies $\varrho_*(\varphi_*(x)) \leq e^{-1} \varrho_*(\varphi_*(0))$ whenever $|x| = R_0$. Then, elementary estimates for one-dimensional log-concave functions (applied on all radii) give

$$\frac{\int_{R_0\,B^n} \varrho_* \circ \varphi_*}{\int_{\mathbb{R}^n} \varrho_* \circ \varphi_*} \geq \frac{\int_{R_0\,B^n} e^{-|x|/R_0}\,dx}{\int_{\mathbb{R}^n} e^{-|x|/R_0}\,dx}.$$

Since the latter ratio depends only on n, we consider it as a constant. Hence we deduce from (43) that for $R \in [R_0, R(\varepsilon)]$

$$\frac{\int_{\mathbb{R}^n} \varrho_* \circ \varphi}{\int_{\mathbb{R}^n} \varrho_* \circ \varphi_*} \leq 1 - \frac{\eta_{11} \varepsilon^{\frac{1}{8n^2}} V(\Psi\cap R\,B^n)}{V(R\,B^n)}.$$

On the other hand, (35) and (36) give $\frac{\int_{\mathbb{R}^n} \varrho_* \circ \varphi}{\int_{\mathbb{R}^n} \varrho_* \circ \varphi_*} \geq 1-\varepsilon$. Comparing the latter two estimates leads to

$$V(\Psi \cap R\, B^n) \leq \eta_{11}^{-1} \varepsilon^{1-\frac{1}{8n^2}} V(R\, B^n).$$

The proof of Theorem 1.1 is therefore complete.

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