Planar crossing numbers of graphs embeddable in another surface

Károly J. Böröczky[1]
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
joint work with J. Pach and G. Tóth

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Planar crossing numbers of graphs

Notation

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- $d(v)$ is the degree of a $v \in V(G)$
- $\Delta(G) = \max_{v \in V(G)} d(v)$
- $\sigma(G) = \sum_{v \in V(G)} d(v)^2$

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- $\chi(S)$ is the Euler characteristic of a compact surface S
- c is always some positive absolute constant

Topic

Starting point

Theorem (J. Pach, G. Tóth (2005))

If G can be drawn without crossing on a compact oriented surface S then

$$\operatorname{cr} G \le c^{3-\chi(S)} \cdot \Delta(G) \cdot n.$$

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Main result

Theorem (J. Pach, G. Tóth, K.B. (2006))

The above statement holds for any compact surface S.

Remarks

- The Euler characteristic χ of a compact surface is at most two, and is even if the surface is orientable. Topologically the surface is determined by the Euler characteristic and by orientability. For example the surface with $\chi = 2$ is a sphere, with $\chi = 1$ is a projective plane, and with $\chi = 0$ is either a torus (oriented) or a Klein bottle (non-oriented).

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- The order of the main theorem is optimal apart from the value of the factor.
- To have a non-trivial upper bound on the crossing number, it is not enough to know that the graph G can be drawn crossing free on a compact surface S different from a sphere. We do need say an upper bound on the degrees of the vertices. Specifically if G has e edges then clearly $\mathrm{cr}\,(G) < \binom{e}{2}$. Define G by taking five vertices, and connect any pair of them by $\frac{e}{20}$ vertex-disjoint paths of lengths two. This G can be embedded into S, but the subdivisions of K_5 yield $\mathrm{cr}\,(G) \geq \frac{e^2}{400}$.

Sum of the squared degrees

Theorem (General version)

If G can be drawn without crossing on a compact surface S then

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Remark Let G be connected, and let S be a compact surface of maximal Euler characteristic containing a crossing free drawing of G. Then G defines a CW-cell decomposition of S.

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Proof of the Main Theorem based on the General version:

$$\sum_{v \in V(G)} d(v) \leq 6n - 6\chi(S) \text{ by the Euler relation, therefore}$$

$$\sum_{v \in V(G)} d(v)^2 \leq 12n\Delta(G) \text{ if } n \geq |\chi(S)|.$$

Induction

$\sigma(G) = \sum_{v \in V(G)} d(v)^2$

Theorem (Core of Induction)

If G can be drawn without crossing on a compact surface S then it can be drawn with at most $\tilde{c}\sigma(G)$ crossings on a compact surface $\widetilde{S}$ where $\chi(\widetilde{S}) > \chi(S)$.

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Proof of the General version based on the above statement:
Reverse induction on $\chi = \chi(S)$. Let $c_0 = 16\tilde{c} + 2$, and let $\widetilde{G}$ be the graph adding a vertex for any crossing in the drawing of G on $\widetilde{S}$, hence

$$\sigma(\widetilde{G}) \leq \sigma(G) + 4^2 \cdot \tilde{c}\sigma(G) = (c_0 - 1)\sigma(G).$$

Since the General version holds for $\widetilde{G}$ by induction, we have

$$\mathrm{cr}\, G \leq \mathrm{cr}\, \widetilde{G} + \tilde{c}\sigma(G) \leq c_0^{2-\chi}(c_0 - 1)\sigma(G) + \tilde{c}\sigma(G) \leq c_0^{3-\chi}\sigma(G).$$

Idea of the proof of the "Core" theorem

Rough Plan We find a simple closed curve γ on S

- that is non-separating ($S \setminus \gamma$ is connected), and
- intersects only $\sqrt{\sigma(G)}$ edges of G.

Cutting S along γ yields $\widetilde{S}$, and the free ends of the edges can be reconnected generating only at most $\sigma(G)$ crossings.

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How to find γ? We "enlarge" G into a triangulation G' with at most $\sigma(G)$ vertices whose degrees are at most eight. Then γ is a shortest non-separating cycle in the dual graph H of G'.

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Remark H has at most $c^*\sigma(G)$ vertices, and any cell determined by H has at most eight sides

Planar crossing numbers of graphs

Notation, backround and results

Curves on surfaces

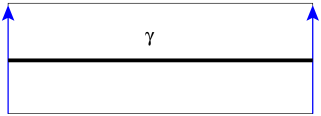
Customizing the graph

Proof of the “Core” Theorem

Curves and orientation

Let γ be a simple closed path on a compact surface S. A small neighbourhood of γ is topologically equivalent either to a strip or to a Möbius strip. In the first case, we say that γ is orientation preserving, and in the second case, it is orientation reversing.

Strip
γ orientation preserving



Möbius strip
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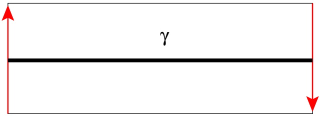

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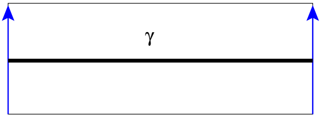

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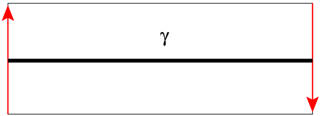
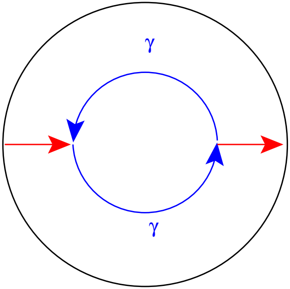
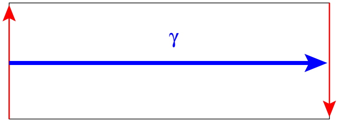


The surface S is orientable if and only if it contains no orientation reversing curve.

Another way to get a Möbius strip





Types of curves on a surface

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- γ is orientation reversing. In this case γ is non-separating. Cutting S along γ, and attaching a disk to the resulting boundary curve, we obtain a compact surface X' with Euler characteristic $\chi(S) + 1$.

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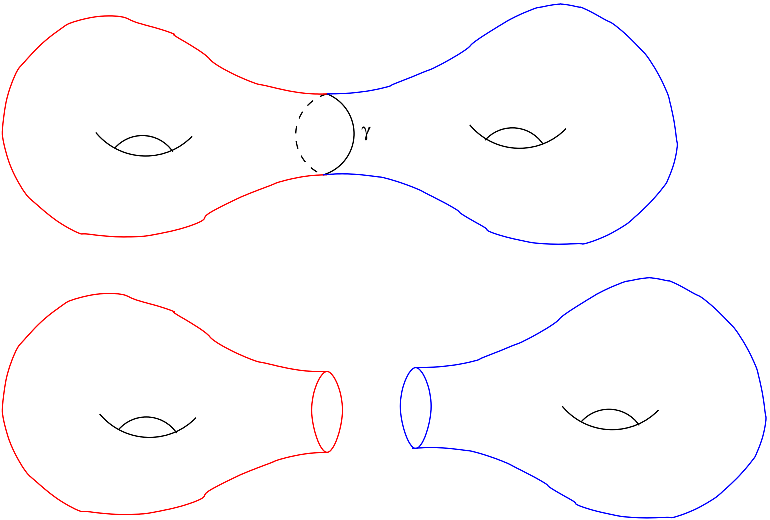
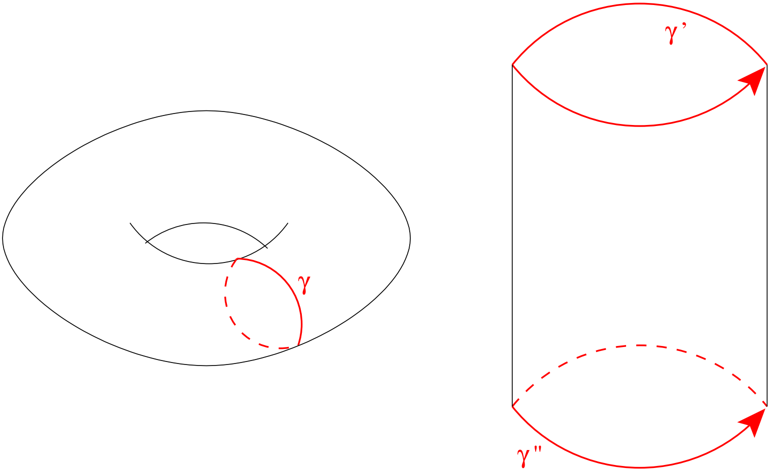
Remark Any surface different from a sphere contains a non-separating curve

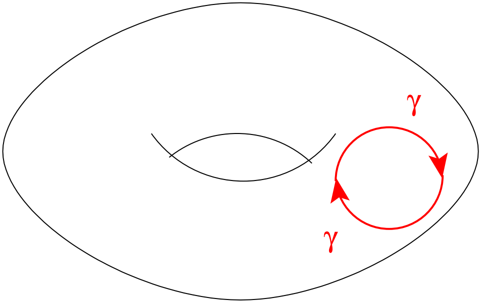
Separating curve



Non-separating and orientation preserving curve

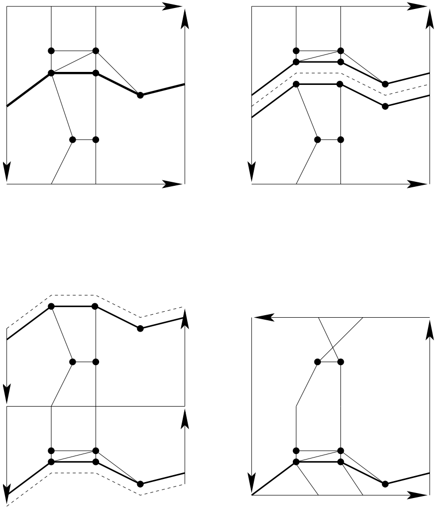


Orientation reversing curve 1



Obtaining the torus from the Euler characteristic -1 surface

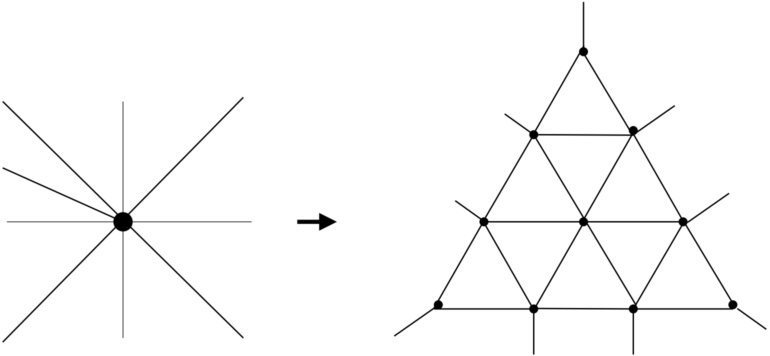
Orientation reversing curve 2



Obtaining the projective plane from the Klein bottle

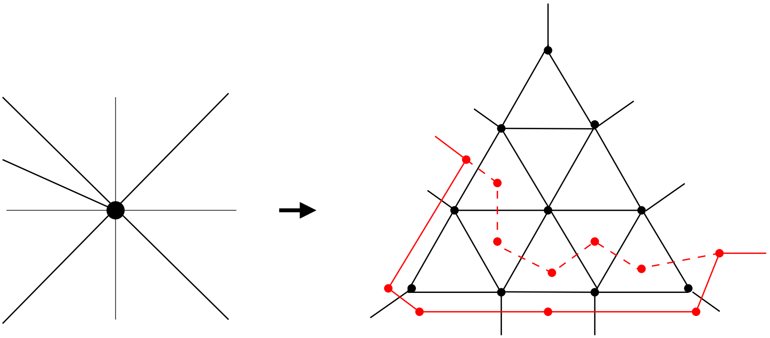
Planar crossing numbers of graphs

Inserting a triangular grid instead of a vertex

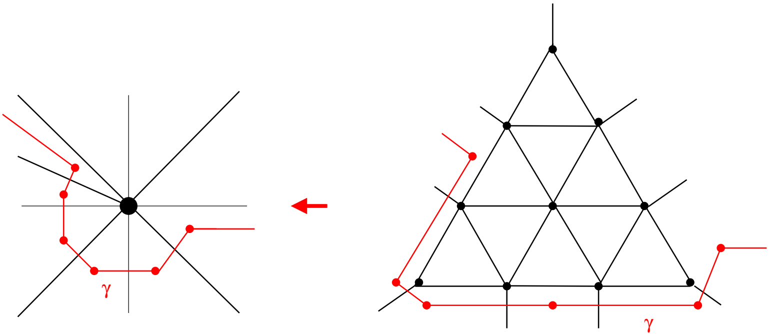


v interchanged into $\approx d(v)^2$ vertices of degrees at most six

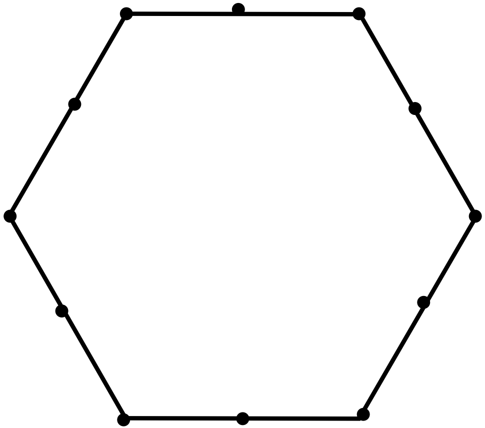
Shortest non-separating dual cycle does not pass through the grid



The non-separating curve for the original graph

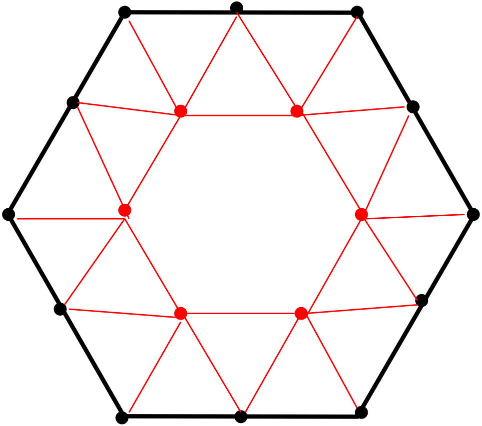


Triangulating a cell



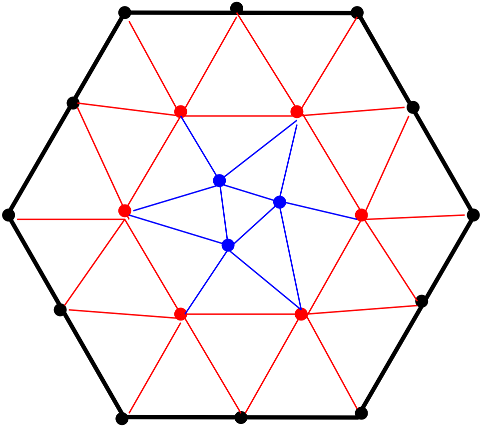
- A large cell

Triangulating a cell



- The first layer

Triangulating a cell



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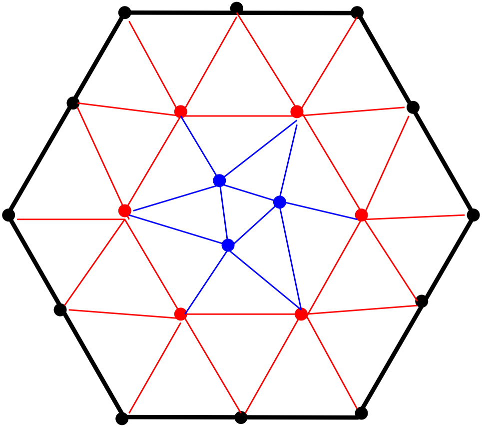

- The number of vertices is multiplied by at most 36.

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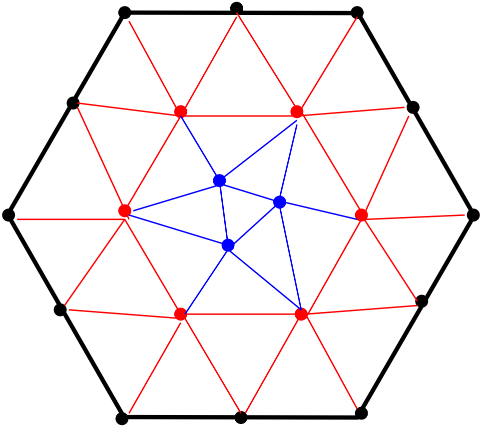

- The number of vertices is multiplied by at most 36.
- The maximal degree of vertices is at most eight.

Planar crossing numbers of graphs

The statement to prove

Theorem

If G can be drawn without crossing on a compact surface S then it can be drawn with at most $\tilde{c}\sigma(G)$ crossings on a compact surface $\widetilde{S}$ where $\chi(\widetilde{S}) > \chi(S)$.

Separating vertices

H^* is a graph drawn on a compact surface such that any cell determined by H^* has at most eight sides.

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- H^* contains a cycle of length at most $8m$ separating u and w.
- There exists a path of length at most $\#V(H^*)/m$ connecting u and w because of the m (internally) vertex disjoint paths between u and w provided by Menger's theorem.

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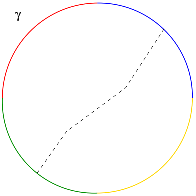
Recall $\#V(H) \leq$c$^*\sigma(G)$.

Aiming at $k \leq c\sqrt{\sigma(G)}$

Divide γ into four paths (arcs) of length $k/4$



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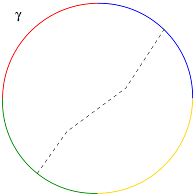

Claim Any path connecting the red and the gold arcs, or the blue and the green arcs is of length at least $k/4$.

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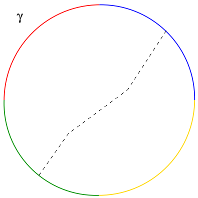
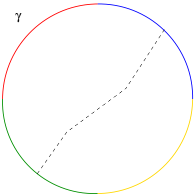


Claim Any path connecting the red and the gold arcs, or the blue and the green arcs is of length at least $k/4$.
m= minimal number of vertices whose deletion separates the red and the gold arcs.

Proving $k \leq c\sqrt{\sigma(G)}$



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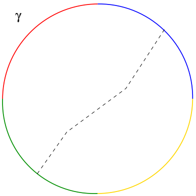

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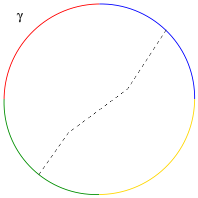

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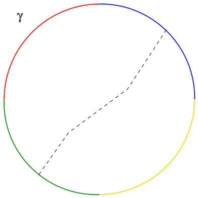

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- $\exists$ a path of length at most $32c^*\sigma(G)/k$ connecting the red and gold arcs $\Rightarrow$

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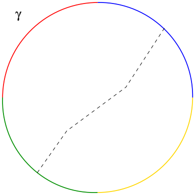

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- $k/4 \leq 32c^*\sigma(G)/k$

Economic non-separating curve for G

Corollary

For any graph G drawn on some compact surface S different from a sphere, there exists a non-separating curve on S that cuts only $c\sqrt{\sigma(G)}$ edges.

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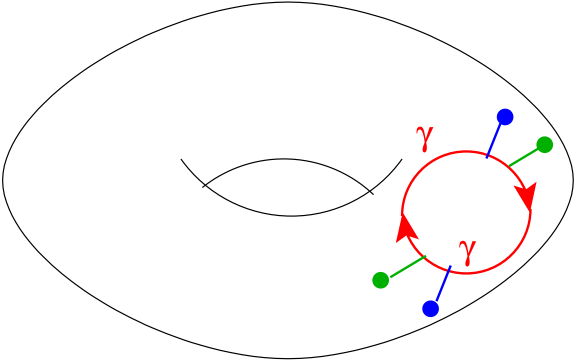
Remark The bound cannot be improved in general.

γ is orientation reversing



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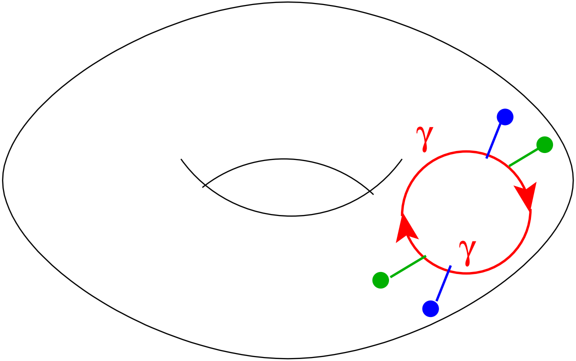

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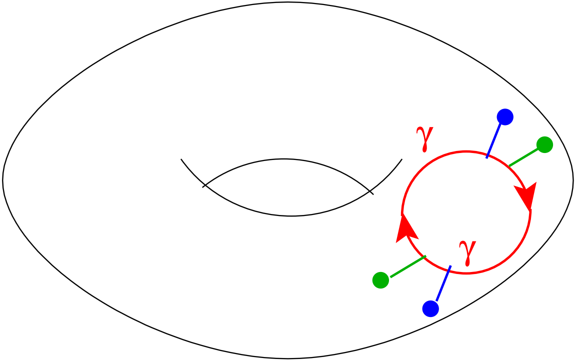
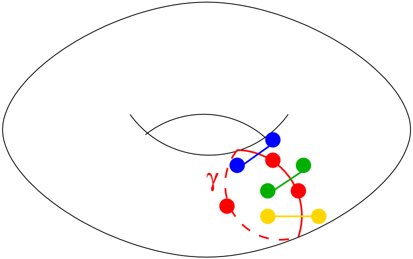
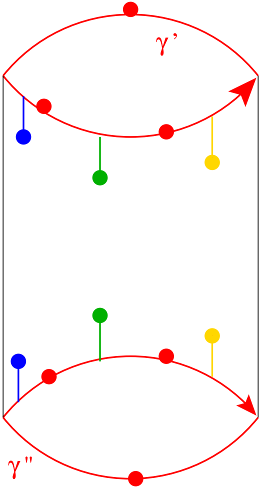


- Connect the free ends of the edges cut by γ inside the attached disk $\Rightarrow$
- At most $\binom{k}{2} \leq c\sigma(G)$ crossings are generated

γ is orientation preserving - Preparation





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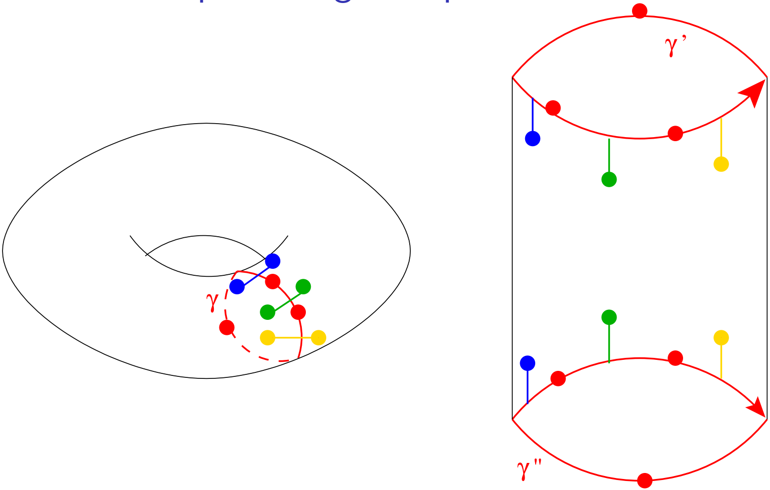
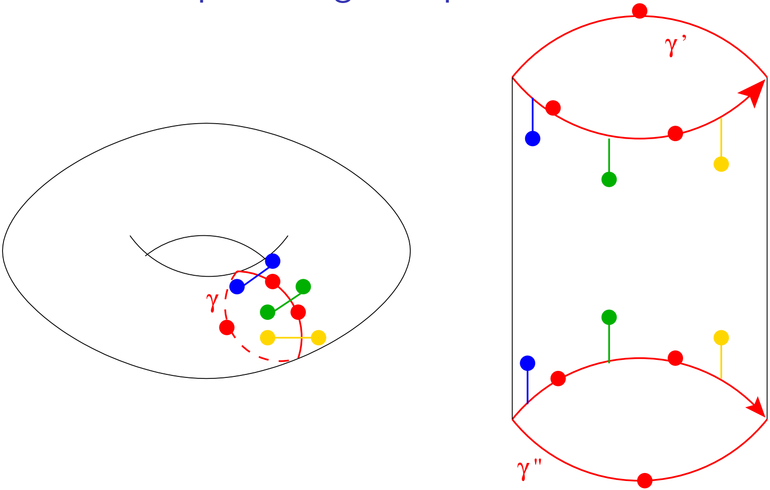


$m=$ minimal number of vertices separating γ' and γ''.

γ is orientation preserving - Preparation



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- $\exists$ a cycle on $\widetilde{S}$ of length at most $8m$ separating γ' and γ'' $\Rightarrow$

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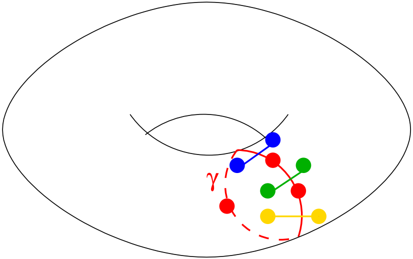

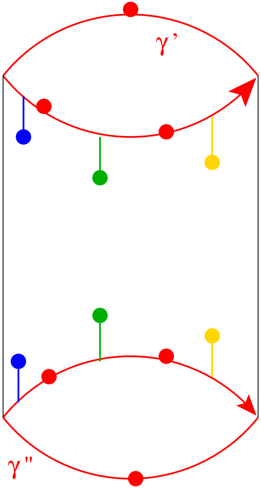

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- $\exists$ a cycle on $\widetilde{S}$ of length at most $8m$ separating γ' and γ'' $\Rightarrow$
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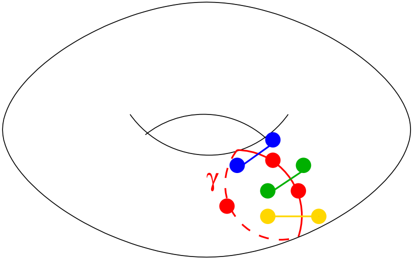
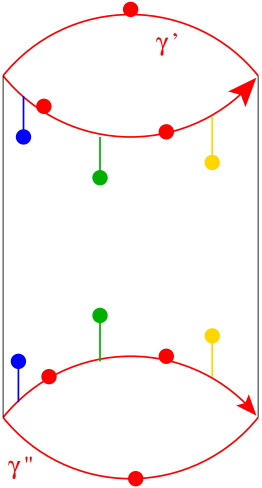
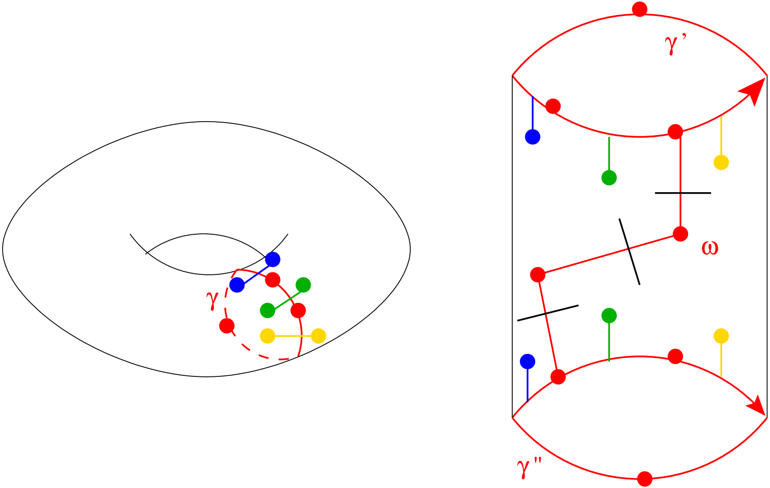




m= minimal number of vertices separating γ' and γ''.

- $\exists$ a cycle on $\widetilde{S}$ of length at most $8m$ separating γ' and γ'' $\Rightarrow$
- $8m \geq k$ $\Rightarrow$
- $\exists$ a path ω of length at most $c\sigma(G)/k$ connecting γ' and γ''

γ is orientation preserving - Final touch



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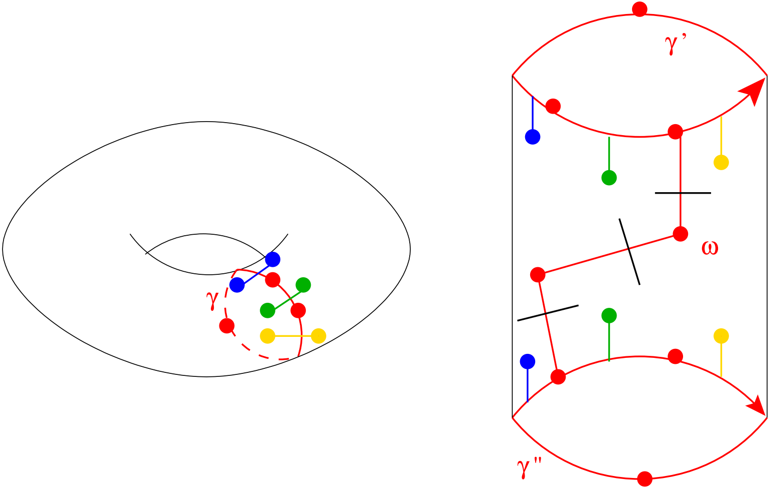

- For each edge cut by γ, connect the free end on γ' to the free end on γ'' along ω

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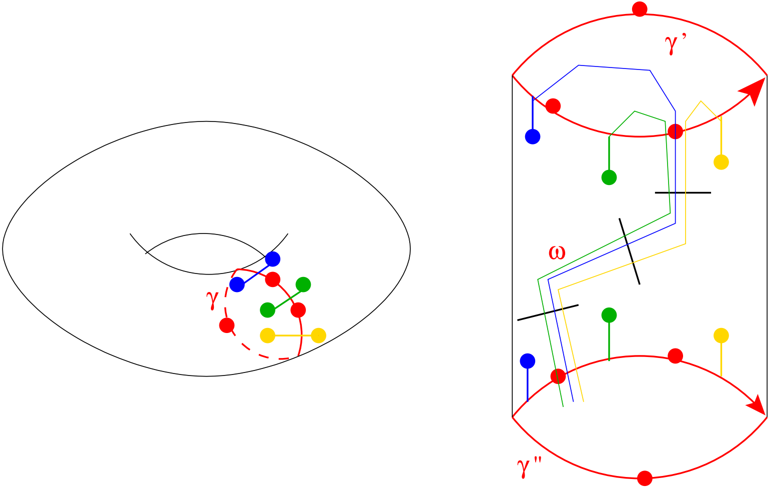

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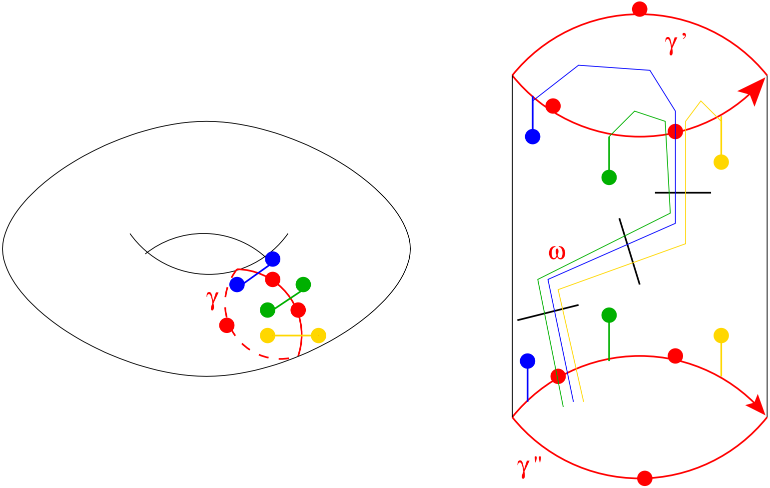


- For each edge cut by γ, connect the free end on γ' to the free end on γ'' along ω
- At most $c\sigma(G)$ crossings are generated

Good News

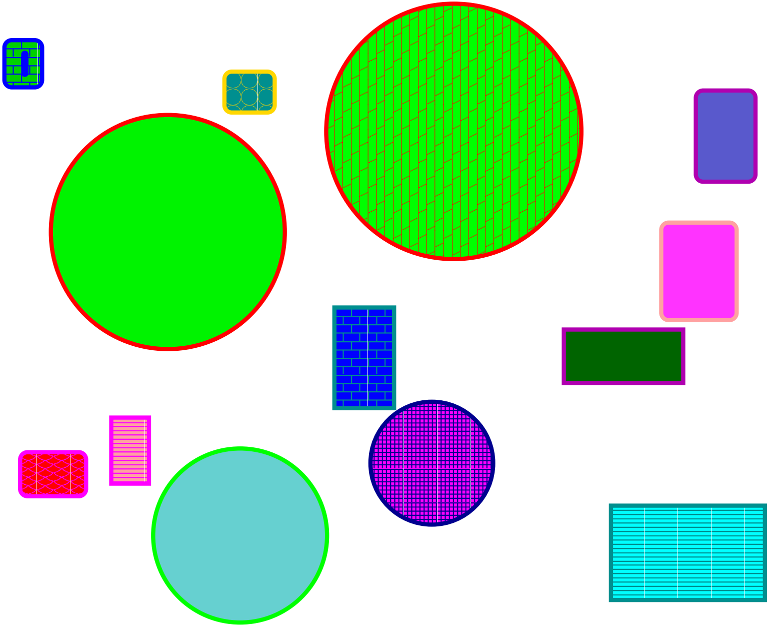
PROOF is OVER

Good News

PROOF is OVER

TIME to WAKE UP

A gift from Csenge



Summary

Theorem (J. Pach, G. Tóth, K.B. (2006))

If G can be drawn without crossing on a compact surface S then

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Theorem (D.R. Wood, J.A. Telle (2006))

For every graph M there is constant $C = C(M)$ such that every M-minor free graph G satisfies

$$\mathrm{cr}\, G \leq C \cdot \Delta(G)^2 \cdot n.$$