Remarks on the equality case of the Bonnesen inequality

Károly J. Böröczky & Oriol Serra

Archiv der Mathematik
Archives Mathématiques Archives of Mathematics

ISSN 0003-889X
Volume 99
Number 2

Arch. Math. (2012) 99:189-199
DOI 10.1007/s00013-012-0418-7

Springer

Arch. Math. 99 (2012), 189–199
© 2012 Springer Basel AG
0003-889X/12/020189-11
published online August 7, 2012
DOI 10.1007/s00013-012-0418-7

Archiv der Mathematik

Remarks on the equality case of the Bonnesen inequality

Károly J. Böröczky and Oriol Serra

Abstract. An argument is provided for the equality case of the high dimensional Bonnesen inequality for sections. The known equality case of the Bonnesen inequality for projections is presented as a consequence.

1. Introduction. We write μ_d for the d-dimensional Lebesgue measure. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$. For a linear subspace Π of $\mathbb{R}^d$, the orthogonal projection into Π is denoted by p_Π. In the special case when $\Pi = u^\perp$ for a $u \in S^{d-1}$, the orthogonal projection into $u^\perp$ is denoted by π_u. In addition, the convex hull of $x_1, \ldots, x_k$ is denoted by $[x_1, \ldots, x_k]$.

The results in this note belong to the very heart of the Brunn–Minkowski theory, so any of the monographs Bonnesen and Fenchel [3], Gruber [7], and Schneider [10], or the survey paper Gardner [6] provide the sufficient background.

Let A and B be convex bodies (compact convex sets with non-empty interiors) in $\mathbb{R}^d$ for this section. The Brunn–Minkowski inequality states

Theorem 1.1 (Brunn–Minkowski). *If $\alpha, \beta > 0$, then*

$$\mu_d(\alpha\, A + \beta\, B) \geq \left(\alpha\, \mu_d(A)^{\frac{1}{d}} + \beta\, \mu_d(B)^{\frac{1}{d}}\right)^d,$$

with equality if and only if A and B are homothetic.

According to the Hölder inequality, if $M, N > 0$, then

$$\left(\alpha\, M^{\frac{1}{d-1}} + \beta\, N^{\frac{1}{d-1}}\right)^{d-1} \left(\alpha\, \frac{\mu_d(A)}{M} + \beta\, \frac{\mu_d(B)}{N}\right) \geq \left(\alpha\, \mu_d(A)^{\frac{1}{d}} + \beta\, \mu_d(B)^{\frac{1}{d}}\right)^d,$$

with equality if and only if $\frac{\mu_d(A)^{\frac{1}{d}}}{M^{\frac{1}{d-1}}} = \frac{\mu_d(B)^{\frac{1}{d}}}{N^{\frac{1}{d-1}}}$. Therefore the following result due to Bonnesen [2] strengthens the Brunn–Minkowki inequality.

K. J. Böröczky is supported by OTKA grant 75016 and by the EU Marie Curie FP7 IEF grant GEOSUMSETS. O. Serra is supported by the Catalan Research Council under project 2008SGR0258 and the Spanish Research Council under project MTM2008-06620-C03-01.

Theorem 1.2 (Bonnesen I). *If for a linear $(d-1)$-space L in $\mathbb{R}^d$, M and N are the maximal $(d-1)$-volumes of the sections of A and B, respectively, by hyperplanes parallel to L, then*

$$\mu_d(\alpha\, A+\beta\, B)\geq \left(\alpha\, M^{\frac{1}{d-1}}+\beta\, N^{\frac{1}{d-1}}\right)^{d-1}\left(\alpha\,\frac{\mu_d(A)}{M}+\beta\,\frac{\mu_d(B)}{N}\right).$$

Theorem 1.2 has the following consequence about projections (see also Sect. 4).

Theorem 1.3 (Bonnesen II). *For $u\in S^{d-1}$, if $M=\mu_{d-1}(\pi_u A)$ and $N=\mu_{d-1}(\pi_u B)$, then*

$$\mu_d(\alpha\, A+\beta\, B)\geq \left(\alpha\, M^{\frac{1}{d-1}}+\beta\, N^{\frac{1}{d-1}}\right)^{d-1}\left(\alpha\,\frac{\mu_d(A)}{M}+\beta\,\frac{\mu_d(B)}{N}\right).$$

The goal of this note is to characterize the equality cases in Bonnesen's inequalities Theorems 1.2 and 1.3. We use the notations of these theorems. We note that Theorem 1.5, and the two dimensional case of Theorem 1.4 are proved by Freiman et al. [4].

For $u\in S^{d-1}$, we say that a convex body K is obtained from a convex body C by stretching along u, if there exist $\lambda\geq 0$ and $w\in\mathbb{R}^d$ such that $K=C+[w,w+\lambda u]$. In particular $K=C+w$ if $\lambda=0$.

Theorem 1.4. *Equality holds in Theorem 1.2 if and only if either A and B are homothetic, or there exist $v\in S^{d-1}$, homothetic convex bodies A' and B', and a hyperplane H parallel to L, such that $\pi_v(A')=\pi_v(A'\cap H)$, and A and B are obtained from A' and B', respectively, by stretching along v.*

We note that the condition $\pi_v(A')=\pi_v(A'\cap H)$ is equivalent to saying that $A'\subset(A'\cap H)+\mathbb{R}v$. Convex bodies for which there exist such hyperplane H and unit vector v are characterized in Meyer [9].

As we discuss in Sect. 4, the following is a simple consequence of Theorem 1.4 via Steiner symmetrization.

Theorem 1.5 (Freiman, Grynkiewicz, Serra, Stanchescu). *Equality holds in Theorem 1.3 if and only if there exist homothetic convex bodies A' and B' such that A and B are obtained from A' and B', respectively, by stretching along u.*

Our proofs of the two inequalities by Bonnesen and the characterizations of the equality cases are based on the $(d-1)$-dimensional Brunn–Minkowski inequality and its equality case.

As related results, a true discrete analogue of the Bonnesen inequality in the plane is proved by Grynkiewicz and Serra [8], and the equality conditions are clarified by Freiman et al. [5]. In addition, Meyer [9] proves a crucial property of a given convex body's sections of maximal $(d-1)$-volume parallel to a hyperplane.

2. Minkowski linear combinations. In this section we recall some well-known simple but useful observations about Minkowski linear combinations of convex bodies (see Gruber [7] or Schneider [10]). If X is a compact convex set in $\mathbb{R}^d$, then its support function is

$$h_X(v) = \max_{x \in X} \langle v, x \rangle \qquad \text{for } v \in \mathbb{R}^d.$$

Then h_X is a positive homogeneous and convex function on $\mathbb{R}^d$, which determines X uniquely. In addition, if Y is another compact convex set, Π is a linear subspace, and $\alpha, \beta > 0$, then

$$h_{\alpha X + \beta Y} = \alpha\, h_X + \beta\, h_Y \tag{1}$$

$$p_\Pi(\alpha X + \beta Y) = \alpha\, p_\Pi X + \beta\, p_\Pi Y. \tag{2}$$

We note that $v \in S^{d-1}$ is an exterior unit normal vector to a convex body K in $\mathbb{R}^d$ at $x \in K$ if and only if $\langle v, x \rangle = h_K(v)$. The following is a simple but useful consequence of (1).

Claim 2.1. *Let $C = \alpha\, A + \beta\, B$ for convex bodies A, B in $\mathbb{R}^d$ and $\alpha, \beta > 0$, and let $z_0 = \alpha\, x_0 + \beta\, y_0$ for $z_0 \in C, x_0 \in A, y_0 \in B$.*

(i) *If $x_0 \in \partial A$ and $y_0 \in \partial B$ with exterior unit normal vector v, then $z_0 \in \partial C$ with exterior unit normal vector v.*

(ii) *If $z_0 \in \partial C$ with exterior unit normal vector v, then $x_0 \in \partial A$ and $y_0 \in \partial B$ with exterior unit normal vector v.*

Our first application of Claim 2.1 is about planar convex bodies.

Claim 2.2. *Let l be a line in $\mathbb{R}^2$ with $0 \in l$, and let $C = \alpha\, A + \beta\, B$ for convex bodies A, B in $\mathbb{R}^2$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. In addition, we assume that*

(i) *for any $z \in C$, $z + l$ intersects A and B, and $C \cap [z + l] = \alpha(A \cap [z + l]) + \beta(B \cap [z + l])$,*

(ii) *there exists $z \in C$ such that $C \cap [z + l] = A \cap [z + l] = B \cap [z + l]$.*

Then $A = B$.

Proof. Let $l = \mathbb{R}u$ for the unit vector u, and let $v \in u^\perp$ be a unit vector. In this case $\pi_u A = \pi_u B = \pi_u C = [av, bv]$ for some $a < b$. There exist convex functions f, g, φ, ψ on $[a, b]$ such that

$$A = \{tv + su : a \le t \le b \text{ and } -g(t) \le s \le f(t)\}$$

$$B = \{tv + su : a \le t \le b \text{ and } -\psi(t) \le s \le \varphi(t)\}.$$

It follows from condition (i) and from Claim 2.1 that $f'(t) = \varphi'(t)$ and $g'(t) = \psi'(t)$ wherever the derivatives exist, thus there exist constants γ, δ such that $f(t) = \varphi(t) + \gamma$ and $g(t) = \psi(t) + \delta$ for $t \in [a, b]$. However condition (ii) yields that $\gamma = \delta = 0$, therefore $A = B$. $\square$

Let K be a convex body in $\mathbb{R}^d$, and let $u \in S^{d-1}$. For each line l parallel with u and intersecting $\mathrm{int}K$, we translate the segment $l \cap K$ along l into the position where the midpoint of the translated segment lies in $u^\perp$. The closure of the union of these translated segments is the Steiner symmetrical $\mathcal{S}_u K$ of K. For another representation of the Steiner symmetrization, we note that there exist concave functions f and g on $\pi_u(K)$ such that

$$K = \{x + \lambda u : x \in \pi_u K \text{ and } -g(x) \le \lambda \le f(x)\}.$$

Then

$$\mathcal{S}_u K = \left\{ x + \lambda u : x \in \pi_u(K) \text{ and } \frac{-f(x) - g(x)}{2} \le \lambda \le \frac{f(x) + g(x)}{2} \right\}. \quad (3)$$

It follows that $\mathcal{S}_u K$ is a convex body symmetric through $u^\perp$, and $\mu_d(\mathcal{S}_u K) = \mu_d(K)$.

Claim 2.3. *For convex bodies A and B in $\mathbb{R}^d, u \in S^{d-1}$, and $\alpha, \beta > 0$, we have*

$$\alpha\, \mathcal{S}_u A + \beta\, \mathcal{S}_u B \subset \mathcal{S}_u(\alpha\, A + \beta\, B).$$

In addition, if equality holds, and a and b are lines parallel to u intersecting $\operatorname{int} A$ *and* $\operatorname{int} B$, *and there exist parallel supporting hyperplanes at the top endpoints of $a \cap \mathcal{S}_u A$ and $b \cap \mathcal{S}_u B$ to $\mathcal{S}_u A$ and to $\mathcal{S}_u B$, respectively, then there exist parallel supporting hyperplanes at the top endpoints of $a \cap A$ and $b \cap B$, and parallel supporting hyperplanes at the bottom endpoints of $a \cap A$ and $b \cap B$ to A and to B, respectively.*

Proof. Let l be a line parallel with u and intersecting $\operatorname{int}(\alpha\, A + \beta\, B)$, and let z_0 be one of the endpoints of $l \cap (\alpha\, \mathcal{S}_u A + \beta\, \mathcal{S}_u B)$. It follows by Claim 2.1 (ii) that $z_0 = \alpha\, x_0 + \beta\, y_0$, where x_0 and y_0 are boundary points of $\mathcal{S}_u A$ and $\mathcal{S}_u B$, sharing a common exterior unit vector with z_0. Therefore $a = x_0 + \mathbb{R}u$ and $b = y_0 + \mathbb{R}u$ satisfy $l = \alpha\, a + \beta\, b$ and

$$l \cap (\alpha\, \mathcal{S}_u A + \beta\, \mathcal{S}_u B) = \alpha\, (a \cap \mathcal{S}_u A) + \beta\, (b \cap \mathcal{S}_u B).$$

In particular

$$\begin{aligned} \mu_1\left(l \cap \mathcal{S}_u(\alpha\, A + \beta\, B)\right) &= \mu_1\left(l \cap (\alpha\, A + \beta\, B)\right) \\ &\ge \alpha\, \mu_1(a \cap A) + \beta\, \mu_1(b \cap B) \\ &= \alpha\, \mu_1(a \cap \mathcal{S}_u A) + \beta\, \mu_1(b \cap \mathcal{S}_u B) \\ &= \mu_1\left(l \cap [\alpha\, \mathcal{S}_u A + \beta\, \mathcal{S}_u B]\right), \end{aligned} \quad (4)$$

which in turn yields $\alpha\, \mathcal{S}_u A + \beta\, \mathcal{S}_u B \subset \mathcal{S}_u(\alpha\, A + \beta\, B)$.

Assume now that $\alpha\, \mathcal{S}_u A + \beta\, \mathcal{S}_u B = \mathcal{S}_u(\alpha\, A + \beta\, B)$, and hence equality holds in (4) for any line l parallel with u and intersecting $\operatorname{int}(\alpha\, A + \beta\, B)$. It follows that

$$l \cap (\alpha\, A + \beta\, B) = \alpha(a \cap A) + \beta(b \cap B). \quad (5)$$

Writing x_1, y_1, z_1 to denote the top endpoint, and x_2, y_2, z_2 to denote the bottom endpoint of $a \cap A, b \cap B$ and $l \cap (\alpha\, A + \beta\, B)$, we deduce $z_i = \alpha\, x_i + \beta\, y_i$ for $i = 1, 2$, from (5). Therefore Claim 2.1 (ii) completes the argument. □

To introduce another method of symmetrization, let K be a convex body in $\mathbb{R}^d$, and let l be a line. For each hyperplane H orthogonal to l and intersecting $\operatorname{int} K$, consider the $(d-1)$-ball in H with the same $(d-1)$-volume as $H \cap K$ and centred at $H \cap l$. The closure of the union of these $(d-1)$-balls centred on l is a convex body $\mathcal{R}_l K$ by the Brunn–Minkowski inequality, and $\mathcal{R}_l K$ is called the Schwarz-rounding of K. Readily $\mu_d(\mathcal{R}_l K) = \mu_d(K)$. A similar argument to the one for Claim 2.3 (or using the fact that the Schwarz-rounding can be obtained as the limit of repeated Steiner symmetrizations through hyperplanes containing l) yields

Claim 2.4. *For convex bodies A and B in $\mathbb{R}^d$, line l, and $\alpha, \beta > 0$, we have*

$$\alpha\,\mathcal{R}_l A + \beta\,\mathcal{R}_l B \subset \mathcal{R}_l(\alpha\,A + \beta\,B).$$

Schwarz-rounding will be a basic tool for our proof of Theorem 1.4. It was Blaschke who gave a simple proof of the Brunn–Minkowski inequality using Schwarz rounding in [1].

3. Proof of Theorem 1.4. If the conditions stated in Theorem 1.4 hold, then we readily have equality in Theorem 1.2. For the reverse statement, we subdivide the argument into three sections.

3.1. A little preparation. First we introduce some notation. Let $u \in S^{d-1}$ be orthogonal to L, let K be a convex body in $\mathbb{R}^d$, and let Q be the maximal $(d-1)$-volume of the sections of K by hyperplanes parallel to L. For $s \in [0, Q]$, let

$$\begin{aligned} k_-(s) &= \min\{p :\ K \cap (pu + L) \neq \emptyset \text{ and } \mu_{d-1}(K \cap (pu + L)) \geq s\} \\ k_+(s) &= \max\{p :\ K \cap (pu + L) \neq \emptyset \text{ and } \mu_{d-1}(K \cap (pu + L)) \geq s\}. \end{aligned}$$

In addition we define

$$K_-(s) = K \cap (k_-(s)u + L) \quad \text{ and } \quad K_+(s) = K \cap (k_+(s)u + L).$$

We observe that $K \cap (pu + L) \neq \emptyset$ if and only if $p \in [k_-(0), k_+(0)]$, possibly $k_-(Q) = k_+(Q)$, but $k_-(s) < k_+(s)$ if $s < Q$. It follows from the $(d-1)$-dimensional case of the Brunn–Minkowski inequality that

$$\mu_{d-1}(K \cap (pu + L)) \geq s \ \text{ for } s \in (0, Q] \text{ if and only if } p \in [k_-(s), k_+(s)]. \tag{6}$$

We observe that if the "top" and "bottom" sections of K parallel to L are of zero μ_{d-1}-measure, then $\mu_{d-1}(K_+(s)) = \mu_{d-1}(K_-(s)) = s$ for $s \in [0, Q]$. In general, we have

$$\text{if } \ \mu_{d-1}(K_-(s)) > s, \ \text{ then } k_-(s) = k_-(0) \tag{7}$$

$$\text{if } \ \mu_{d-1}(K_+(s)) > s, \ \text{ then } k_+(s) = k_+(0). \tag{8}$$

Calculating the integral of $f(p) = \mu_{d-1}(K \cap (pu + L))$ for $p \in [k_-(0), k_+(0)]$ by calculating the area of the part of $\mathbb{R}^2$ between the graph of f and the first axis using Fubini's theorem, and after that using (6) yield

$$\begin{aligned} \mu_d(K) &= \int_{k_-(0)}^{k_+(0)} \mu_{d-1}(K \cap (pu + L))\, dp \\ &= \int_0^Q \mu_1(\{p \in \mathbb{R} :\ \mu_{d-1}(K \cap (pu + L)) \geq s\})\, ds \\ &= \int_0^Q (k_+(s) - k_-(s))\, ds. \end{aligned} \tag{9}$$

As the final part of our preparation, we discuss the case $k_-(Q) < k_+(Q)$. We have equality in (6) for $s = Q$, therefore the equality case of the Brunn–Minkowski inequality implies that $K \cap (pu + L)$ is a translate of $K_-(Q)$ for $p \in [k_-(Q), k_+(Q)]$. Let $K_+(Q) = K_-(Q) + \lambda v$ for $v \in S^{d-1}$ and $\lambda > 0$. It follows by the convexity of K that $k_-(Q) < k_+(Q)$ implies

$$\{x \in K : [k_-(Q) \leq \langle x, u\rangle \leq k_+(Q)\} = K_-(Q) + [0, \lambda v] \\ \text{and } K \subset K_-(Q) + \mathbb{R}v, \tag{10}$$

and in turn

$$\pi_v(K) = \pi_v(K_-(Q)), \tag{11}$$

and that K is obtained from the convex body

$$K' = \bigcup_{s \in [0,Q]} ((K_+(s) - \lambda v) \cup K_-(s)) \tag{12}$$

by stretching along v.

3.2. A proof of Theorem 1.2. Replacing A and B by $M^{\frac{-1}{d-1}} A$ and $N^{\frac{-1}{d-1}} B$, if necessary, we may assume that

$$M = N = 1. \tag{13}$$

Let $C = \alpha A + \beta B$, and we write $a_-(s), a_+(s), A_-(s), A_+(s)$, or $b_-(s), b_+(s)$, $B_-(s), B_+(s)$, or $c_-(s), c_+(s), C_-(s), C_+(s)$ to denote $k_-(s), k_+(s), K_-(s)$, $K_+(s)$ if $K = A$, or $K = B$, or $K = C$, respectively. We observe that if $t \in (0, 1]$, then

$$\alpha A_+(t) + \beta B_+(t) \subset C \cap ([\alpha a_+(t) + \beta b_+(t)]\, u + L].$$

Therefore (6), the analogous relation for $A_-(t)$ and $B_-(t)$, and the $(d-1)$-dimensional case of the Brunn–Minkowski inequality yield that

$$c_+([\alpha + \beta]^{d-1} t) \geq \alpha a_+(t) + \beta b_+(t) \tag{14}$$
$$c_-([\alpha + \beta]^{d-1} t) \leq \alpha a_-(t) + \beta b_-(t). \tag{15}$$

We deduce by (9) that

$$\mu_d(C) \geq \int_0^{(\alpha+\beta)^{d-1}} (c_+(s) - c_-(s))\, ds \tag{16}$$
$$\geq (\alpha + \beta)^{d-1} \int_0^1 [\alpha a_+(t) + \beta b_+(t)] - [\alpha a_-(t) + \beta b_-(t)]\, dt \tag{17}$$
$$= (\alpha + \beta)^{d-1} [\alpha\, \mu_d(A) + \beta\, \mu_d(B)]. \tag{18}$$

3.3. Analyzing the equality case. To simplify the formulae, in addition to (13), we also assume

$$\alpha + \beta = 1. \tag{19}$$

Let us assume that

$$\mu_d(C) = \alpha\, \mu_d(A) + \beta\, \mu_d(B), \tag{20}$$

and hence equality holds in (14) and (15) for $t \in (0,1]$. In particular, there exists no $p > \alpha a_+(t) + \beta b_+(t)$ such that

$$\mu_{d-1}(C \cap [pu + L]) \geq t.$$

Using the $(d-1)$-dimensional Brunn–Minkowski inequality and its equality case, and that $\mu_{d-1}(A_+(t)) = t = \mu_{d-1}(B_+(t))$ if $a_+(t) < a_+(0)$ and $b_+(t) < b_+(0)$, we deduce

$$C_+(t) = \alpha\, A_+(t) + \beta\, B_+(t), \quad \text{and } A_+(t) \text{ and } B_+(t) \text{ are translates,} \tag{21}$$

$$C_-(t) = \alpha\, A_-(t) + \beta\, B_-(t), \quad \text{and } A_-(t) \text{ and } B_-(t) \text{ are translates} \tag{22}$$

for all $t \in (0,1]$. Thus we may assume that

$$A_-(1) = B_-(1) \subset u^\perp. \tag{23}$$

We note that equality holds in (16), as well, therefore

$$\begin{gathered} C_+(1) \text{ and } C_-(1) \text{ are sections of } C \text{ of maximal } (d-1)\text{-volume} \\ \text{among the ones parallel to } L. \end{gathered} \tag{24}$$

For the final part of the argument, we distinguish cases depending on whether the section of maximal $(d-1)$-volume is unique.

Case 1 $a_+(1) = a_-(1)$ and $b_+(1) = b_-(1)$.

First we show that

$$a_+(t) = b_+(t) \quad \text{and} \quad a_-(t) = b_-(t) \quad \text{for } t \in [0,1]. \tag{25}$$

We observe that $a_+(0) = a_+(1)$ is equivalent to saying that the top section of A parallel to L is a section of maximal $(d-1)$-volume. Possibly after reversing u, we may assume that $a_+(0) > a_+(1)$. Let $t_+ \in [0,1)$ be the maximal $t \in [0,1)$ such that $a_+(t) = a_+(0)$, and let $t_- \in [0,1]$ be the maximal $t \in [0,1]$ such that $a_-(t) = a_-(0)$.

Let $\widetilde{A}, \widetilde{B}$, and $\widetilde{C}$ be the Schwarz rounding of A, B, and C with respect to $\mathbb{R}u$. In particular, (13) yields that the maximal $(d-1)$-volumes of the sections of $\widetilde{A}$ and $\widetilde{B}$ parallel to L are 1. It follows from the Bonnesen inequality (18), from the assumption of equality (20), and Claim 2.4 that

$$\begin{aligned} \alpha\, \mu_d(A) + \beta\, \mu_d(B) = \mu_d(C) = \mu_d(\widetilde{C}) &\geq \mu_d(\alpha\, \widetilde{A} + \beta\, \widetilde{B}) \\ &\geq \alpha\, \mu_d(\widetilde{A}) + \beta\, \mu_d(\widetilde{B}) = \alpha\, \mu_d(A) + \beta\, \mu_d(B). \end{aligned}$$

Therefore $\mu_d(\widetilde{C}) = \mu_d(\alpha\, \widetilde{A} + \beta\, \widetilde{B})$, and hence Claim 2.4 yields

$$\widetilde{C} = \alpha\, \widetilde{A} + \beta\, \widetilde{B}.$$

We define $\tilde{a}_-(t), \tilde{a}_+(t), \widetilde{A}_-(t), \widetilde{A}_+(t)$, or $\tilde{b}_-(t), \tilde{b}_+(t), \widetilde{B}_-(t), \widetilde{B}_+(t)$, or $\tilde{c}_-(t)$, $\tilde{c}_+(t), \widetilde{C}_-(t), \widetilde{C}_+(t)$ to denote $k_-(t), k_+(t), K_-(t), K_+(t)$ if $K = \widetilde{A}$, or $K = \widetilde{B}$, or $K = \widetilde{C}$, respectively. We observe that $\tilde{a}_+(t) = a_+(t)$ for $t \in [0,1]$, and $\widetilde{A}_+(t)$ is a $(d-1)$-ball with $\mu_{d-1}(\widetilde{A}_+(t)) = t$ for $t \in [t_+, 1]$, and we have the similar statements for the analogous quantities. Since $\mu_d(\widetilde{C}) = \mu_d(\alpha \widetilde{A} + \beta \widetilde{B})$, the argument in the case of A and B yields the analogues of (21) and (22); namely,

$$\begin{aligned}\widetilde{C}_+(t) = \alpha \widetilde{A}_+(t) + \beta \widetilde{B}_+(t), &\quad \text{and } \widetilde{A}_+(t) \text{ and } \widetilde{B}_+(t) \text{ are}\\ &(d-1)\text{-balls with } (d-1)\text{-volume } t \text{ for } t \in [t_+, 1], \end{aligned} \tag{26}$$

$$\begin{aligned}\widetilde{C}_-(t) = \alpha \widetilde{A}_-(t) + \beta \widetilde{B}_-(t), &\quad \text{and } \widetilde{A}_-(t) \text{ and } \widetilde{B}_-(t) \text{ are}\\ &(d-1)\text{-balls with } (d-1)\text{-volume } t \text{ for } t \in [t_-, 1]. \end{aligned} \tag{27}$$

Let Π be any two-dimensional linear subspace containing u. In particular, $\Pi \cap \widetilde{A} = p_\Pi \widetilde{A}, \Pi \cap \widetilde{B} = p_\Pi \widetilde{B}$ and $\Pi \cap \widetilde{C} = p_\Pi \widetilde{C}$, and hence (2) implies

$$\Pi \cap \widetilde{C} = \alpha(\Pi \cap \widetilde{A}) + \beta(\Pi \cap \widetilde{B}). \tag{28}$$

We plan to apply Claim 2.2 to $\Pi \cap \widetilde{A}, \Pi \cap \widetilde{B}$ and $\Pi \cap \widetilde{C}$ with $l = \mathbb{R}u$. Let $v \in S^{d-1} \cap u^\perp \cap \Pi$. We observe that for $t_+ < t \leq 1$, the radii of $\widetilde{A}_+(t), \widetilde{B}_+(t)$ and $\widetilde{C}_+(t)$ coincide by (26), and if $x \in \widetilde{A}_+(t), y \in \widetilde{B}_+(t), z \in \widetilde{C}_+(t)$ are relative boundary points with exterior normal v, then there exists a common exterior unit normal vector to $\widetilde{A}$ at x and to $\widetilde{B}$ at y by Claim 2.1. Combining this with the analogous properties of $\widetilde{A}_-(t), \widetilde{B}_-(t)$, and $\widetilde{C}_-(t)$ implies condition (i) of Claim 2.2. In addition if $z_0 \in \widetilde{C}_+(1)$ is a relative boundary point with exterior normal v, then (23) yields that $z_0 + l$ intersects all of $\widetilde{A}_+(1), \widetilde{B}_+(1)$, and $\widetilde{C}_+(1)$ in $\{z_0\}$. Therefore we may apply Claim 2.2 and deduce that $\Pi \cap \widetilde{A} = \Pi \cap \widetilde{B}$. Therefore $\widetilde{A} = \widetilde{B}$, which in turn yields (25).

Next we claim that

$$h_A(w) = h_B(w) \qquad \text{for } w \in S^{n-1}. \tag{29}$$

We may assume that $w \neq \pm u$, and let Π be two-dimensional linear subspace spanned by u and w. Again let $v \in S^{d-1} \cap u^\perp \cap \Pi$. We plan to apply Claim 2.2 to $p_\Pi A, p_\Pi B$ and $p_\Pi C$ with $l = \mathbb{R}v$. We deduce condition (i) by (21), (22) and (25), and condition (ii) by (23). Therefore $p_\Pi A = p_\Pi B$, and hence $h_A(w) = h_{p_\Pi A}(w) = h_{p_\Pi B}(w) = h_B(w)$.

Finally (29) yields that $A = B$.

Case 2 Either $a_+(1) > a_-(1)$, or $b_+(1) > b_-(1)$.

We may assume that $a_+(1) - a_-(1) \geq b_+(1) - b_-(1)$, and hence $a_+(1) > a_-(1)$. It follows that $A_+(1) = A_-(1) + \lambda v$ for suitable $v \in S^{d-1}$ and $\lambda > 0$. It follows by (11) that

$$\pi_v A = \pi_v A_-(1). \tag{30}$$

If $b_+(1) > b_-(1)$, then $B_+(1) = B_-(1) + \tau w$ for $w \in S^{d-1}$ and $\tau > 0$. If $b_+(1) = b_-(1)$, then we set $\tau = 0$ and $w = v$, and still have $B_+(1) = B_-(1) + \tau w$. It follows from (21) and (22) that $C_+(1) = C_-(1) + \alpha\lambda v + \beta\tau w$.

We deduce by (10) and (24) that

$$\begin{aligned} C_-(1) + [0, \alpha\lambda v] + [0, \beta\tau w] &= \alpha\,(A_-(1) + [0, \lambda v]) + \beta\,(B_-(1) + [0, \tau w]) \\ &\subset \{z \in C : [c_-(1) \leq \langle z, u\rangle \leq c_+(1)\} \\ &= C_-(1) + [0, \alpha\lambda v + \beta\tau w]. \end{aligned}$$

Since $\langle v, u\rangle > 0$ and $\langle w, u\rangle > 0$, we conclude that $v = w$ also if $b_+(1) > b_-(1)$.

We deduce by (10) that

$$A \subset A_-(1) + \mathbb{R}v \quad \text{ and } \quad C \subset C_-(1) + \mathbb{R}v, \tag{31}$$

and claim that

$$B \subset B_-(1) + \mathbb{R}v. \tag{32}$$

If $\tau > 0$, then (32) also follows from (10). If $\tau = 0$, then we should prove that $y_0 + \mathbb{R}v$ is a supporting line to B for any relative boundary point y_0 of $B_-(1)$. Now $x_0 = y_0$ is a relative boundary point of $A_-(1) = B_-(1)$ [see (23)], hence $z_0 = \alpha\, x_0 + \beta\, y_0 = y_0$ is a relative boundary point of $C_-(1) = \alpha\, A_-(1) + \beta\, B_-(1) = B_-(1)$ [compare (22)]. Thus (31) yields that there exists a supporting hyperplane H containing $z_0 + \mathbb{R}v$ at z_0 to C, and in turn Claim 2.1 (ii) implies that H is a supporting hyperplane at y_0 to B. We conclude (32).

We define

$$\begin{aligned} A' &= \bigcup_{t \in [0,1]} ((A_+(t) - \lambda v) \cup A_-(t)) \\ B' &= \bigcup_{t \in [0,1]} ((B_+(t) - \tau v) \cup B_-(t)) \\ C' &= \bigcup_{t \in [0,1]} ((C_+(t) - \alpha\lambda v - \beta\tau v) \cup C_-(t)). \end{aligned}$$

We deduce by $C = A + B$, (31), and (32) that $C' = A' + B'$. In addition

$$\begin{aligned} \mu_d(C') &= \mu_d(C) - \mu_{d-1}(C_-(1)) \cdot \langle (\alpha\lambda + \beta\tau)v, u\rangle \\ &= \alpha\mu_d(A) + \beta\mu_d(B) - \alpha\mu_{d-1}(A_-(1)) \cdot \langle \lambda v, u\rangle \\ &\quad - \beta\mu_{d-1}(A_-(1)) \cdot \langle \tau v, u\rangle \\ &= \alpha\mu_d(A') + \beta\mu_d(B'). \end{aligned}$$

Since both A' and B' have a unique section parallel to L of maximal $(d-1)$-dimensional volume, we deduce by Case 1 that $A' = B'$. We conclude Theorem 1.4 by (30). □

4. Proof of Theorem 1.5. In this section, we assume

$$M = \mu_{d-1}(\pi_u A) = \mu_{d-1}(\pi_u B) = N = 1 \;\text{ and }\; \alpha + \beta = 1. \tag{33}$$

If the convex bodies A' and B' are homothetic, then (33) yields that A' and B' are translates. If in addition A and B are obtained from $A'B'$, respectively, by stretching along u, then readily

$$\mu_d(\alpha A + \beta B) = \alpha\mu_d(A) + \beta\mu_d(B).$$

For the reverse direction, first we explain how Theorem 1.2 yields Theorem 1.3 it via Steiner symmetrization. Let $\widetilde{A}, \widetilde{B}$, and $\widetilde{C}$ be the Steiner symmetrials of A, B, and $C = \alpha A + \beta B$. In particular $\alpha\widetilde{A} + \beta\widetilde{B} \subset \widetilde{C}$ according to Claim 2.3. We also observe that $\pi_u A = u^\perp \cap \widetilde{A}$ and $\pi_u B = u^\perp \cap \widetilde{B}$ are sections of maximal $(d-1)$-measure of $\widetilde{A}$ and $\widetilde{B}$, respectively, parallel to $L = u^\perp$. Therefore Theorem 1.2 and the conditions (33) yield

$$\mu_d(\alpha A+\beta B)=\mu_d(\widetilde{C})\geq\mu_d(\alpha\widetilde{A}+\beta\widetilde{B}) \geq \alpha\mu_d(\widetilde{A})+\beta\mu_d(\widetilde{B})=\alpha\mu_d(A)+\beta\mu_d(B).$$

Next we assume that $\mu_d(\alpha A + \beta B) = \alpha\mu_d(A) + \beta\mu_d(B)$, and hence

$$\widetilde{C} = \alpha\widetilde{A} + \beta\widetilde{B} \tag{34}$$

$$\mu_d(\alpha\widetilde{A} + \beta\widetilde{B}) = \alpha\mu_d(\widetilde{A}) + \beta\mu_d(\widetilde{B}). \tag{35}$$

Combining (35) and Theorem 1.4 shows that there exist homothetic convex bodies $\widetilde{A}'$ and $\widetilde{B}'$, and a $v \in S^{d-1}$ such that $\widetilde{A}$ and $\widetilde{B}$ are obtained from $\widetilde{A}'$ and $\widetilde{B}'$, respectively, by stretching along v. Since $\widetilde{A}$ and $\widetilde{B}$ are symmetric through $u^\perp$, we deduce that $v = \pm u$. Therefore we may assume that $\widetilde{A}'$ and $\widetilde{B}'$ are also symmetric through $u^\perp$. We deduce by the conditions (33) that actually $\widetilde{A}'$ and $\widetilde{B}'$ are translates, therefore $\widetilde{A}' = \widetilde{B}'$ can be assumed. Therefore there exists a non-negative convex function φ on $\pi_u A = \pi_u B$, and $a, b \geq 0$, such that

$$\begin{aligned}
\widetilde{A}' = \widetilde{B}' &= \{x + \lambda u : x \in \pi_u A \text{ and } -\varphi(x) \leq \lambda \leq \varphi(x)\}\\
\widetilde{A} &= \{x + \lambda u : x \in \pi_u A \text{ and } -\varphi(x) - a \leq \lambda \leq \varphi(x) + a\}\\
\widetilde{B} &= \{x + \lambda u : x \in \pi_u A \text{ and } -\varphi(x) - b \leq \lambda \leq \varphi(x) + b\}.
\end{aligned}$$

We deduce by (3) that there exist functions θ and ψ on $\pi_u A$ such that

$$\begin{aligned}
A &= \{x + \lambda u : x \in \pi_u A \text{ and } \theta(x) - \varphi(x) - a \leq \lambda \leq \theta(x) + \varphi(x) + a\}\\
B &= \{x + \lambda u : x \in \pi_u A \text{ and } \psi(x) - \varphi(x) - b \leq \lambda \leq \psi(x) + \varphi(x) + b\}.
\end{aligned}$$

It follows that $\theta(x) + \varphi(x) + a, -(\theta(x) - \varphi(x) - a), \psi(x) + \varphi(x) + b$ and $\psi(x) - \varphi(x) - b$ are convex. Since convex functions on a compact set are Lipschitz, both φ and θ are almost everywhere differentiable on $\pi_u A$. For each $x \in \pi_u \mathrm{int}\, A$, there are parallel supporting hyperplanes to $\widetilde{A}$ at $x + (\varphi(x) + a)u$, and to $\widetilde{B}$ at $x + (\varphi(x) + b)u$, thus (34) and Claim 2.3 imply that

$$(\theta(x) + \varphi(x) + a)' = (\psi(x) + \varphi(x) + b)' \qquad \text{for almost all } x \in \pi_u A.$$

Therefore there exists some $\omega \in \mathbb{R}$ such that $\psi(x) = \theta(x) + \omega$ for $x \in \pi_u A$. By possibly interchanging the roles of A and B, we may assume that $\omega \geq 0$. In particular defining

$$A' = B' = \{x + \lambda u : x \in \pi_u A \text{ and } \theta(x) - \varphi(x) \leq \lambda \leq \theta(x) + \varphi(x)\},$$

both A and B are obtained from $A' = B'$ by stretching along u. $\square$

Acknowledgement. We are grateful to the anonymous referee for correcting mistakes and improving the presentation.

References

[1] W. Blaschke, Kreis und Kugel, Leipzig, Veit, 1916.

[2] T. Bonnesen, Les problèmes de isopérimètres et des isédiphanes, Imprimerie Gauthier–Villars et C^{ie}, Paris, 1929.

[3] T. Bonnesen and W. Fenchel, Theory of Convex Bodies. BCS Associates, 1987.

[4] G.A. Freiman et al., Inverse additive problems for Minkowski sumsets II. J. Geom. Anal. doi:10.1007/s12220-011-9251-7.

[5] G.A. Freiman et al., Inverse additive problems for Minkowski sumsets I. Collectanea Matematica. doi:10.1007/s13348-012-0060-5.

[6] R.J. Gardner, The Brunn–Minkowski inequality, Bull. Amer. Math. Soc. (N.S.) **39** (2002), 355–405.

[7] P.M. Gruber, Convex and Discrete Geometry, Springer, Berlin, 2007.

[8] D. Grynkiewicz and O. Serra Properties of two dimensional sets with small sumset. Journal of Combinatorial Theory, Series A **117** (2010), 164–188.

[9] M. Meyer, Two maximal volume hyperplane sections of a convex body generally intersect, Period. Math. Hungar. **36** (1998), 191–197.

[10] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge University Press, 1993.

Károly J. Böröczky
Alfréd Rényi Institute of Mathematics,
Hungarian Academy of Sciences, Budapest, Hungary
e-mail: carlos@renyi.hu

Oriol Serra
Universitat Politècnica de Catalunya,
Barcelona Tech., Catalonia, Spain
e-mail: oserra@ma4.upc.edu

Received: 23 March 2012