Volume product in the plane — lower estimates with stability

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February 5, 2010

Abstract

Let $K \subset \mathbb{R}^2$ be an o-symmetric convex body. Then we have $|K| \cdot |K^*| \geq 8$, with equality if and only if K is a parallelogram. ($| \cdot |$ denotes volume). If $K \subset \mathbb{R}^2$ is a convex body, with $o \in \operatorname{int} K$, then $|K| \cdot |K^*| \geq 27/4$, with equality if and only if K is a triangle and o is its centroid. If $K \subset \mathbb{R}^2$ is a convex body, then we have $|K| \cdot |[(K-K)/2)]^*| \geq 6$, with equality if and only if K is a triangle. These theorems are due to Mahler and Reisner, Mahler and Meyer, and to Eggleston, respectively. We show an analogous theorem: if K has n-fold rotational symmetry about o, then $|K| \cdot |K^*| \geq n^2 \sin^2(\pi/n)$, with equality if and only if K is a regular n-gon of centre o. We will also give stability variants of these four inequalities, both for the body, and for the centre of polarity. For this we use the Banach-Mazur distance (from parallelograms, or triangles), or its analogue with similar copies rather than affine transforms (from regular n-gons), respectively. The stability variants are sharp, up to constant factors. We extend the inequality $|K| \cdot |K^*| \geq n^2 \sin^2(\pi/n)$ to bodies with $o \in \operatorname{int} K$, which contain, and are contained in, two regular n-gons, the vertices of the contained n-gon being incident to the sides of the containing n-gon. Our key lemma is a stability estimate for the area product of two sectors of convex bodies polar to each other. To several of our statements we give several proofs; in particular, we give a new proof for the theorem of Mahler-Reisner.

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RUNNING TITLE: Volume product in the plane

1 Notation

We write o for the origin, $\langle \cdot, \cdot \rangle$ for the scalar product, $\| \cdot \|$ for the Euclidean norm, $[x_1, \ldots, x_k]$ for the convex hull of $\{x_1, \ldots, x_k\}$, and $|\cdot|$ for the volume.

A convex body in $\mathbb{R}^d$ is a compact convex set with non-empty interior. If $o \in \operatorname{int} K$, then its polar (w.r.t. the unit sphere with centre o) is

$$K^* = \{x \in \mathbb{R}^d : \forall y \in K \ \langle x, y \rangle \leq 1\}.$$

If $A : \mathbb{R}^d \to \mathbb{R}^d$ is a non-singular linear map, then $(AK)^* = (A^{-1})^* K^*$, where $(A^{-1})^*$ is the transpose of the inverse of A. It is known (Santaló [50], or Meyer-Pajor [40]), that there exists a unique point $s(K) \in \operatorname{int} K$, called *Santaló point of K*, such that

$$| (K - s(K))^* | = \min \{|(K - z)^*| : z \in \operatorname{int} K\}.$$

Additionally, the origin is the centroid of $(K - s(K))^*$. The uniqueness and the affine invariance of the Santaló point yields that $s(K) = o$ if K is o-symmetric, or if $d = 2$ and K has n-fold symmetry about o for some $n \geq 3$.

For convex bodies $K, L \subset \mathbb{R}^d$, the *Banach-Mazur distance* $\delta_{BM}(K, L)$ is $\min \{\lambda_2/\lambda_1 \mid \lambda_1, \lambda_2 \in (0, \infty), \ \exists \text{ affinity } A, \ \exists x \in \mathbb{R}^d, \ \lambda_1 AK \subset L \subset \lambda_2 AK + x\}$. If we allow for A only similarities, then we obtain the definition of $\delta_{BM}^s(K, L)$. (Clearly, $\delta_{BM}(K, L) \leq \delta_{BM}^s(K, L)$. If both K, L are o-symmetric, or $d = 2$ and both have n-fold rotational symmetry about o, with $n \geq 3$ an integer, then in the definition of $\delta_{BM}(K, L)$, or $\delta_{BM}^s(K, L)$, we may assume $x = o$.) We will write T, P, R_n for a triangle, parallelogram, or regular n-gon, respectively.

2 Introduction

Let $K \subset \mathbb{R}^d$ be a convex body, with $o \in \operatorname{int} K$. Blaschke [5] was the first who considered the so called *volume product* $|K| \cdot |K^*|$ of the body K, and proved that for $d \leq 3$, and o the barycentre of K, its maximum is attained, e.g., if K is an ellipsoid. He was motivated by the investigation of the affine geometry of convex bodies, e.g., of the so called *affine surface area* (a definition cf. in [29], or [8]), that is intimately related to the volume product (cf. [29], [8]). The volume product is invariant under non-singular linear transformations, cf. [L], p. 109. The investigation of the question of the lower estimate

of the volume product was initiated by Mahler [30], [31]. He had in view applications in the geometry of numbers (i.e., investigation of the relation of convex, or more generally, of star-bodies, and lattices, i.e., non-singular linear images of $\mathbb{Z}^d$ in $\mathbb{R}^d$). The volume product, in particular, for o-symmetric K, is a basic quantity, that later has arisen in several branches of mathematics, cf. later in this introduction.

For a while we suppose that K is o-symmetric. Mahler [31], for $d \geq 2$, conjectured the lower bound $4^d/d!$, and proved the lower bound $4^d/(d!)^2$. It is usually credited to Saint Raymond [49] that this conjectured lower bound is attained not only for parallelepipeds and cross-polytopes. However, this had already been observed by Guggenheimer [22] some years earlier, where the way of obtaining all examples of [49] had already been described. These examples are the following. Beginning with $[-1, 1] \subset \mathbb{R}$, we define inductively convex bodies in $\mathbb{R}^d$, from examples in lower dimensions: if $d = d_1 + d_2$ is an arbitrary decomposition of d as a sum of positive integers d_i, then for the already defined bodies in $\mathbb{R}^{d_i}$ we take either their Minkowski sum, or the convex hull of their union. (The Banach spaces with these unit balls are called *Hansen-Lima spaces*.) It is conjectured that the volume product attains its minimum exactly for these bodies. (Although the claim of [22] that its author settled the 3-dimensional case is incorrect.)

Mahler [30] proved the sharp lower bound 8 for $d = 2$. If $K \subset \mathbb{R}^2$ is a convex body, with $o \in \operatorname{int} K$, then he [30] showed $|K| \cdot |K^*| \geq 27/4$, which is sharp. Morerover he [30] showed that, for K a polygon, the lower bound is attained, for the o-symmetric case, or for the case $o \in \operatorname{int} K$, if and only if K is a parallelogram, or a triangle with barycentre at o, respectively. Later Meyer [38] showed that for the case $d = 2$ and $o \in \operatorname{int} K$, the lower bound is attained only for triangles, with barycentre at o. A simpler proof of this is contained in Meyer-Reisner [42], Theorem 15.

The above lower estimate of [31] for $\mathbb{R}^d$, for the o-symmetric case, was sharpened to $2^d \kappa_d/(d! d^d)^{1/2}$ by Dvoretzky-Rogers [13], and to $\kappa_d^2/d^{d/2}$ by Bambah [2]. Then it became clear that the volume product is very important in functional analysis, where it is just the product of the volumes of the unit balls of a finite dimensional Banach space and its dual. This has importance in the so called local theory of Banach spaces, i.e., the asymptotic study of finite dimensional Banach spaces, of high dimension. A number of other geometric characteristics of these Banach spaces have a connection to the volume product. Therefore functional analysts became strongly interested in the subject, which resulted in ever better lower estimates, namely $\operatorname{const}^d \cdot$

$(\log d)^{-d}d^{-d}$ by Gordon-Reisner [20] and later by G. Kuperberg [26], and to $\text{const}^d \cdot d^{-d}$ by Bourgain-Milman [11] (with an unspecified constant). Quite recently $\kappa_d^2/2^d$ was proved by G. Kuperberg [27]. Observe that the quotient of G. Kuperberg's estimate and the conjectured minimum is $(\pi/4 + o(1))^d$.

A class of o-symmetric convex bodies in $\mathbb{R}^d$, for which the lower bound $4^d/d!$ is known, is the class of (non-singular) linear images of convex bodies symmetric with respect to all coordinate hyperplanes (also called *unconditional convex bodies*), cf. Saint Raymond [49], with the equality cases clarified by Meyer [37] and Reisner [47] — these are just the conjectured equality cases; the description of the equality cases was obtained as a consequence of a combinatorial theorem in Bollobás-Reader-Redcliffe [6]. Actually [49] proved this inequality for a larger class of o-symmetric convex bodies. These are the ones, for which the associated norm satisfies the following. There exists a base, such that for the coordinates $x_1, \ldots, x_d$ w.r.t. this base, the projections $(x_1, \ldots, x_d) \rightarrow (x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_d)$, where $1 \leq i \leq d$, are contractions. Moreover, [49] also extended his inequality, for unconditional convex bodies, in the following way. Let $k \geq 2$ be an integer, let an unconditional norm $\| \cdot \|$ on $\mathbb{R}^k$ be given (i.e., the unit ball is unconditional), and let $d_1, ..., d_k \geq 1$ integers. Let $K_i \subset \mathbb{R}^{d_i}$ be o-symmetric convex bodies, which are the unit balls of norms $\| \cdot \|_i$. We consider $\prod_{i=1}^k \mathbb{R}^{d_i}$, with the norm $\|(\|x_i\|_i)\|$, where we consider $\| \cdot \|_i$ as fixed, and $\| \cdot \|$ as variable. Then the volume product of the unit ball of this norm attains its minimum, e.g., for the cases, when $\|(x_i)\|$ equals $\sum_i |x_i|$, or $\max_i |x_i|$.

Mahler's conjecture in the o-symmetric case, together with the conjecture about the equality cases, is also proved for convex polytopes with (at most) $2d + 2$ vertices or facets, for $d \leq 8$, cf. Lopez-Reisner [28].

Mahler's conjecture is also proved for zonoids K in $\mathbb{R}^d$ (i.e., limits in the Hausdorff-metric of finite sums of segments), with centre at o, and with $\text{int } K \neq \emptyset$. This is due to Reisner [45], [46], which papers also proved that the lower bound is attained if and only if K is a parallelepiped. Later, a simpler proof was given by Gordon-Meyer-Reisner [19]. Observe that this settles the case of equality for o-symmetric convex bodies in $\mathbb{R}^2$, since each such body is a zonoid. Both [45], [46] use the connection of the volume product problem with stochastic geometry (geometric probability), as is done also later in Böröczky K. J.-Hug [9], in another context. Manifold other connections to geometric probability are contained in Thompson's book [52], in particular in Ch. VI. [45] also gave an analogue of the last mentioned Saint Raymond's theorem: if each K_i, there considered, is either a zonoid, or the polar of a

zonoid, then $|K| \cdot |K^*| \geq 4^d/d!$. [47] clarified the equality cases in the last mentioned Saint Raymond's theorem: this is the case if and only if $\| \cdot \|$ is a norm of a Hansen-Lima space.

In the o-symmetric case, the upper bound is attained if and only if K is an o-symmetric ellipsoid, which is due to Blaschke [5] ($d \leq 3$) and Santaló [50] (for general d), with the equality case proved in [49]. Ball [2] and Meyer-Pajor [39] pointed out that a proof of the inequality can be given by Steiner symmetrization: namely that Steiner symmetrization does not decrease $|K| \cdot |K^*|$.

A number of simplifications of these proofs has appeared, as well as variants of this problem have been treated. E.g., functional forms of the inverse Blaschke-Santaló inequality (i.e., of the lower estimate of the volume product), cf. Meyer-Reisner [41] (which states in p. 219 that a special case of its Theorem is the Mahler-Meyer theorem), functional forms of the Blaschke-Santaló inequality, cf. Fradelizi-Meyer [17] (which states in pp. 386-387, 393-394 that its results imply the Blaschke-Santaló theorem — with the equality case for o-symmetry). [17] also considers the upper estimate for the volume product for measures other than the Lebesgue measure. As an application to the original volume product problem, [41] gives the following statement. If all non-empty intersections of K with horizontal hyperplanes are positive homothets of a given $(n-1)$-dimensional convex body L, and these intersections have their Santaló points (taken in their affine hull) on a line, then $|K| \cdot |K - s(K)|/(|L| \cdot |L - s(L)|)$ attains its minimum $(n+1)^{n+1}/n^{n+2}$ (that is independent of L), if and only if K is a cone, with base a translate of L. (Examples of such bodies are bodies rotationally symmetric about the x_d-axis.) Further a stability version of the Blaschke-Santaló inequality, for $d \geq 3$, is proved by K. J. Böröczky [8] (stability meant for the Banach-Mazur distance). Cf., e.g., the recent papers [23], [42], [12], [17], [18], [8], and [9], and the references therein.

For the case $o \in \text{int } K$ it is conjectured that $|K| \cdot |K^*| \geq (d+1)^{d+1}/(d!)^2 \sim \text{const} \cdot e^{2d}d^{-d}$, where equality stands only for a simplex with barycentre at o. The lower bound $(d+1)^{d+1}/ \left(d^d(d!)^2 \right)$ is due to Mahler [32], that was sharpened to $\kappa_d^2/(d!)^2$ by Bambah [2], while $\text{const} \cdot (\pi e/2)^d d^{-d}$ has been recently proved by G. Kuperberg [27]. Observe that the quotient of this estimate and the conjectured minimum is $(\pi/(2e) + o(1))^d$. This analogue of Mahler's conjecture, for the asymmetric case, is proved for convex polytopes with at most $d + 3$ vertices or facets, cf. Meyer-Reisner [42], Theorem 10.

One has for $|K| \cdot |[K - s(K)]^*|$ the upper estimate κ_d^2, with equality if and

only if K is an ellipsoid, cf. Blaschke [5], Santaló [50] for the inequality, and Petty [44], Meyer-Pajor [40] for the cases of equality. Again, [40] used for the proof, among others, Steiner's symmetrization, but in a more involved manner, than in the o-symmetric case. Actually the same upper estimate holds for $|K| \cdot |[K - b(K)]^*|$, where $b(K)$ is the barycentre of K, and again with equality if and only if K is an ellipsoid, cf. [29], p. 165. Actually, if $s(K)$, or $b(K)$, is o, then $b(K^*)$, or $s(K^*)$, is o, respectively, cf. [29], p. 165, which explains the symmetric role of the Santaló point, and the barycentre.

A general reference to these problems, and their connections to other affine inequalities for convex bodies, is Lutwak [29]. A more recent survey on the volume product is Thompson [53].

For another generalization of the volume product, from the o-symmetric case to the general case, Eggleston [14] proved the following. If $K \subset \mathbb{R}^2$ is a convex body, then $|K| \cdot |[(K - K)/2]^*| \geq 6$, with equality if and only if K is a triangle.

A generalization of this to $\mathbb{R}^d$, however not for polar bodies, but for polars of projection bodies, was given by Zhang [54]; his inequality is $|K|^{d-1} \cdot |(\Pi K)^*)| \geq \binom{2d}{d} d^{-d}$, with equality if and only if K is a simplex. (The projection body $\Pi(K)$ of a convex body $K \subset \mathbb{R}^d$ is the o-symmetric convex body — actually a zonoid — whose support function at a point $u \in S^{d-1}$ is given as the $(d-1)$-volume of the orthogonal projection of K to the linear subspace orthogonal to u. Observe that for $d = 2$ the bodies ΠK and $K - K$ can be obtained from each other by a rotation through $\pi/2$ about the origin, hence their polars have equal areas.) Böröczky, K. J. [7], Theorem 3 proved an almost sharp stability version of this inequality: for S a simplex, $|K|^{d-1} \cdot |(\Pi K)^*)| \leq \binom{2d}{d} d^{-d} (1+\varepsilon)$ implies $\delta_{BM}(K, S) \leq 1 + \text{const}_d \cdot \varepsilon^{1/d}$, while the actual error term cannot be less than $\text{const}_d \cdot \varepsilon^{1/(d-1)}$ ([7], Example 19), which quantity is conjectured to be the exact order of the error term.

For the original question about the lower estimate of $|K| \cdot |[(K - K)/2]^*|$, for $K \subset \mathbb{R}^d$ a convex body, the sharp lower bound is conjectured to be $(d+1)2^d/d! \sim 2^d e^d d^{-d} (1 + o(1))^d$, with equality for K a simplex, cf. [33]. (A calculation, that for K a simplex we have equality, cf. in [36].) This quantity occurs in a number of problems of the theory of packings and coverings, and more generally in density estimates of sytems of convex sets (for the non-symmetric case seemingly even more than the original volume product), cf. e.g., [33], [34], [35] Theorem 5.2, Remark 5.3. Since $|K| \cdot |[(K - K)/2]^*| = [|K|/|(K - K)/2|] \cdot [|(K - K)/2| \cdot |[(K - K)/2]^*|]$, G. Kuperberg's result

and the difference body inequality (Rogers-Shephard, [48]) imply $|K| \cdot |[(K - K)/2]^*| \geq \kappa_d^2/4^d \sim e^d \pi^d 2^{-d} d^{-d} (1 + o(1))^d$. Observe that the quotient of this value and the conjectured value is $(\pi/4 + o(1))^d$.

A question of another character was treated by A. Florian in [15] and [16]. He investigated convex bodies in $\mathbb{R}^2$, contained in the unit circle about o, and showed the sharp estimate $|K| + |K^*| \geq 6$, attained for a square inscribed to the unit circle. He gave as well a stability result in a more special case. See references to earlier results of this type as well in [15] and [16].

We note that F. Barthe and M. Fradelizi in the preprint [3] proved that if K is a convex body and P is a regular polytope in $\mathbb{R}^d$ such that the origin is their centroid, and K has all the symmetries of P — thus the origin is also their Santaló point — then $|K| \cdot |K^*| \geq |P| \cdot |P^*|$.

After essentially finishing our paper we were informed from the paper Nazarov-Petrov-Ryabogin-Zvavitch [43] about the following theorem. For $d \geq 2$ an integer there exist $\varepsilon_d > 0$ and $c_d > 0$ with the following properties. If the Banach-Mazur distance of an o-symmetric convex body $K \subset \mathbb{R}^d$ from the class of parallelepipeds is $1 + \varepsilon \in (1, 1 + \varepsilon_d]$, then the volume product $|K| \cdot |K^*|$ is at least $[4^d/d!](1 + c_d \varepsilon)$. Here the order of the error term is optimal. Together with the paper Böröczky K. J.-Hug [9] (which calls the attention to the fact that the proof in [43] actually gives this stronger, stability variant, cited above; cf. [43], §4), this gives the following. For the case of o-symmetric zonoids K in $\mathbb{R}^d$, with int $K \neq \emptyset$, in particular, for o-symmetric convex bodies in $\mathbb{R}^2$, we have global stability of the parallelotopes, more exactly, the above inequality, without a restriction of the form $0 < \varepsilon \leq \varepsilon_d$. For $\mathbb{R}^2$, this is our Theorem 1, without the specification of the coefficient of ε in the lower estimate. Once more, the order of the error term is optimal.

Since optimality of the order of the above two error terms was not proved in [43] or [9], we show it. Of course, it suffices to deal with the case of zonoids only, for which we give the following example. For $d = 2$ we take $[-1, 1]^2$, and cut off small isosceles right triangles of legs ε at each vertex. For $d \geq 3$ we take the product of this example with $[-1, 1]^{d-2}$, that is an o-symmetric zonoid, say, K. Then $|K| \cdot |K^*| = (4^d/d!) (1 + c_1 \varepsilon + O(\varepsilon^2))$, for some $c_1 > 0$. Clearly $\delta_{BM}(K, [-1, 1]^d) \leq 1 + c_2 \varepsilon + O(\varepsilon^2)$, for some $c_2 > 0$. Now we estimate $\delta_{BM}(K, [-1, 1]^d) = \delta_{BM}(K^*, \text{conv} \{\pm e_i\})$ from below, by $1 + c_3 \varepsilon + O(\varepsilon^2)$, for some $c_3 > 0$ (e_i's are the standard unit vectors). Thus, we have to consider cross-polytopes C_i contained in, and C_o containing K^*, with centres at o. Of

course, it suffices to show

$$|C_i|/|K^*| \leq 1 - c_4\varepsilon + O(\varepsilon^2), \quad \text{for some} \quad c_4 > 0. \tag{1}$$

We may assume that $\text{vert}\, C_i \subset \text{vert}\, K^*$ (vert means the set of vertices). Here $\text{vert}\, K^*$ consists of $\pm e_i$, and still four vertices, close to $(\pm e_1 \pm e_2)/2$. If for some $i \geq 3$ we have $\pm e_i \notin C_i$, then $|C_i| = 0$. If $\pm e_1, \pm e_2 \in C_i$, then (1) holds. Otherwise, e.g., $\pm(1/2, 1/2) \in \text{vert}\, C_i$, and either e.g. $\pm e_1 \in \text{vert}\, C_i$, or $\pm(1/2, -1/2) \in \text{vert}\, C_i$; in both cases $|C_i|/|K^*| = 1/2 + O(\varepsilon)$. So (1) is shown.

A still more recent manuscript Kim-Reisner [25] proved the asymmetric variant of the theorem of [43]. For $d \geq 2$ an integer there exist $\varepsilon'_d > 0$ and $c'_d > 0$ with the following properties. If the Banach-Mazur distance of a convex body $K \subset \mathbb{R}^d$, with $o \in \text{int}\, K$, from the class of simplices is $1 + \varepsilon \in (1, 1 + \varepsilon'_d]$, then the volume product $|K| \cdot |K^*|$ is at least $[(d+1)^{d+1}/(d!)^2](1 + c'_d\varepsilon)$. Again, also here the order of the error term is optimal. (An example is obtained from a regular simplex of edge length 1, and barycentre o, with small regular simplices of edge length ε cut off at each vertex. The argument showing optimality of the order of the error term is like above.)

General information about stability versions of geometric inequalities cf. in Groemer [21].

3   Main statements

For stability versions of the Mahler-Reisner, Mahler-Meyer, Eggleston theorems, we prove the following theorems.

Theorem 1 *Let K be a centrally symmetric convex body in $\mathbb{R}^2$ with $o \in \text{int}\, K$, and P a parallelogram, and*

$$|K| \cdot |K^*| \leq (1 + \varepsilon) \cdot 8, \quad \text{with} \quad \varepsilon > 0.$$

Then $\delta_{BM}(K, P) \leq 1 + 200\varepsilon$. Moreover, if $x \in \mathbb{R}^2$, and $\lambda_i > 0$, and P is a parallelogram, such that $\lambda_1 P + x \subset K \subset \lambda_2 P + x$, and $\lambda_2/\lambda_1 \leq 1 + 200\varepsilon < 2$, then, in the Euclidean norm where $[(\lambda_1 + \lambda_2)/2]P$ is a square of diameter 1, we have that the distance of the centre of $[(\lambda_1 + \lambda_2)/2]P + x$ from o is at most $336 \cdot \sqrt{\varepsilon}$.

Theorem 2 *Let K be a convex body in $\mathbb{R}^2$ with $o \in \mathrm{int}\, K$, and T a triangle, and*

$$|K| \cdot |K^*| \leq (1 + \varepsilon) \cdot 27/4\,, \quad with \;\; \varepsilon > 0\,.$$

Then $\delta_{BM}(K, T) \leq 1 + 900\varepsilon$. Moreover, if $x \in \mathbb{R}^2$, and $\lambda_i > 0$, and T is a triangle, such that $\lambda_1 T + x \subset K \subset \lambda_2 T + x$, and $\lambda_2/\lambda_1 \leq 1 + 900\varepsilon < 4$, then, in the Euclidean norm where $[(\lambda_1 + \lambda_2)/2]T$ is a regular triangle of side 1, we have that the distance of the centre of $[(\lambda_1 + \lambda_2)/2]T + x$ from o is at most $917 \cdot \sqrt{\varepsilon}$.

We note that, for R_n a regular n-gon with centre o,

$$|R_n| \cdot |R_n^*| = (n/2)\sin(2\pi/n) \cdot n \tan(\pi/n) = n^2 \sin^2(\pi/n).$$

We prove the following generalization of the Mahler-Reisner and Mahler-Meyer theorems.

Theorem 3 *Let K_i and K_o be regular n-gons, $n \geq 3$, and let each vertex of K_i lie on a side of K_o, and hence K_i and K_o have a common centroid z. If $K_i \subset K \subset K_o$ for a planar convex body K with $o \in \mathrm{int}\, K$, then*

$$|K| \cdot |K^*| \geq n^2 \sin^2(\pi/n),$$

with equality if and only if $o = z$, and either $K = K_i$ or $K = K_o$.

Let us show how Theorem 3 yields the Mahler-Reisner and Mahler-Meyer theorems. For the o-symmetric case, one considers an (o-symmetric) parallelogram P of maximal area contained in K. Applying a linear map, we may assume that P is a square. Now the Mahler-Reisner theorem follows as $K \subset Q$ for the square Q satisfying that the midpoints of its sides are the vertices of P.

For the Mahler-Meyer theorem, let T be a triangle of maximal area contained in K. Applying a linear map, we may assume that T is regular, and let S be the regular triangle satisfying that the midpoints of the sides of S are the vertices of T. Since $K \subset S$, Theorem 3 yields the Mahler-Meyer theorem.

Another consequence of Theorem 3 is the following.

Corollary 4 *If a convex body K in $\mathbb{R}^2$ has n-fold rotational symmetry about o, where $n \geq 3$, then*

$$|K| \cdot |K^*| \geq n^2 \sin^2(\pi/n),$$

with equality if and only if K is a regular n-gon.

To prove Corollary 4 based on Theorem 3, one just chooses a point $x \in \partial K$ that is the farthest from o, and K_i is the inscribed regular n-gon, of centre o, such that x is one of its vertices, and K_o is the regular n-gon such that the midpoints of the sides of K_o are the vertices of K_i.

Theorem 5 *Let $n \geq 3$ be an integer, K be an n-fold rotationally symmetric convex body in $\mathbb{R}^2$ with $o \in \operatorname{int} K$, and R_n a regular n-gon, and let*

$$|K| \cdot |K^*| \leq (1 + \varepsilon) \cdot n^2 \sin^2(\pi/n), \quad \text{with} \ \ \varepsilon > 0.$$

Then $\delta^s_{BM}(K, R_n) \leq 1 + 18\varepsilon$. Moreover, if $x \in \mathbb{R}^2$, and $\lambda_i > 0$, and R_n is a regular n-gon, such that $\lambda_1 R_n + x \subset K \subset \lambda_2 R_n + x$, and $\lambda_2/\lambda_1 \leq 1 + 18\varepsilon < 1/\cos(\pi/n) \leq 2$, then, in the Euclidean norm where $[(\lambda_1 + \lambda_2)/2]R_n$ is a regular n-gon of diameter 1, we have that the distance of the centre of $[(\lambda_1 + \lambda_2)/2]R_n + x$ from o is at most $263 \cdot \sqrt{\varepsilon}$.

The following theorem proves the conjecture mentioned in §2, concerning the exact error term in the stability variant of the Zhang projection body inequality, for the case of the plane.

Theorem 6 *Let K be a convex body in $\mathbb{R}^2$ with*

$$|K| \cdot |((K - K)/2)^*| \leq (1 + \varepsilon) \cdot 6, \quad \text{with} \ \ \varepsilon > 0.$$

Then $\delta_{BM}(K, T) \leq 1 + 87\varepsilon$.

Example. 1. We show that the stability statements in Theorems 1, 2, 5, 6, concerning the bodies, are of the exact order of magnitude. For this, let the regular n-gon R_n be inscribed in the unit circle U about o, and let us define K_n as the convex polygon with vertices the vertices of R_n, and $1 + \varepsilon$ times the side-midpoints of R_n, where $\varepsilon \in (0, 1/\cos(\pi/n)]$ (thus $K_n \subset U$). Then $|K_n| \cdot |(K_n)^*| = n^2 \sin^2(\pi/n) + n^2 \sin^2(\pi/n) \cdot (\varepsilon - \varepsilon^2 \cot^2(\pi/n))/(1 + \varepsilon)$. Letting $n = 3$, we have $|K_3| \cdot |((K_3 - K_3)/2)^*| = 6 \cdot (9 + 15\varepsilon + 3\varepsilon^2 - 3\varepsilon^3)/(3 + \varepsilon)^2$. Clearly, $\delta^s_{BM}(R_n, K_n) \leq 1 + \varepsilon$. On the other hand, for suitable

A, x, we have $\lambda_1 A R_n \subset K_n \subset \lambda_2 A R_n + x$ and $\delta_{BM}(R_n, K_n)^2 = (\lambda_2/\lambda_1)^2 \geq |K_n|/|\lambda_1 A R_n| \geq (1+\varepsilon)|R_n|/|R_n|$ (at the last step we have used that $\lambda_1 A R_n \subset U$ is a convex n-gon, hence $|\lambda_1 A R_n| \leq |R_n|$). Hence, $\delta_{BM}(R_n, K_n) \geq \sqrt{1+\varepsilon}$. (For Theorems 1, 2 we use the cases $n = 4, 3$.)

2. For the stability of the centre of polarity (for Theorems 1, 2, 5), we proceed analogously to [25], Proposition 2. An example is a regular n-gon K of centre o, and diameter 1 (with $\lambda_i = 1$). We use the well-known formula (11) from the proof of Lemma 11, for $d = 2$. The inradius of K is at least $1/(2\sqrt{3})$. We let $\|x\| \leq 1/(4\sqrt{3})$, and estimate $(\partial/\partial x_2)^2|(K - x)^*|$ from above by replacing, in the inequality in (11), $h_K(u)$ by $1/(2\sqrt{3})$, and then $(1 - \langle u, x \rangle)^{-4}$ by $\left(1/(4\sqrt{3})\right)^{-4}$. Then, using still $\int_{S^1} u_2^2 du = \pi$, we get

$$(\partial/\partial x_2)|(K - x)^*| = 0 \text{ and } (\partial/\partial x_2)^2|(K - x)^*| \leq 2^8 \cdot 3^3 \cdot \pi.$$

By diam $K = 1$ we have $|K| \leq \pi/4$. Thus we get $(\partial/\partial x_2)^2(|K| \cdot |(K - x)^*|) \leq 2^6 \cdot 3^3 \cdot \pi^2$, and the analogues of these formulas hold for the first and second directional derivatives in any direction. Thus, for $|K| \cdot |(K - x|^*| \geq (1 + \varepsilon) \cdot n^2 \sin^2(\pi/n)$, we have

$$\varepsilon \cdot 27/4 \leq \varepsilon \cdot n^2 \sin^2(\pi/n) \leq |K| \cdot |(K - x)^*| - |K| \cdot |K^*| \leq 2^5 \cdot 3^3 \cdot \pi^2 \|x\|^2,$$

hence, for any x — i.e., without the restriction $\|x\| \leq 1/(4\sqrt{3})$ — we have

$$\|x\| \geq \sqrt{\varepsilon} \cdot \sqrt{2}/(16\pi) \text{ or } \|x\| \geq 1/(4\sqrt{3}).$$

Then the first one of these inequalities holds, if $\varepsilon \in (0, \varepsilon_0]$, where $\sqrt{\varepsilon_0} \cdot \sqrt{2}/(16\pi) = 1/(4\sqrt{3})$, i.e., for $\varepsilon_0 = 8\pi^2/3$. ∎

In the second part of this paper, under preparation, we will show that, for convex n-gons K, the product $|K| \cdot |[K - s(K)]^*|$ is maximal exactly for the affine regular n-gons. Further, we will give stability estimates for the Blaschke-Santaló inequality in the plane, for the o-symmetric case. Here the deviation from the ellipses will be measured by the quotient of the areas of the convex body, and the maximal area inscribed/minimal area circumscribed ellipse of the convex body. If any of these ellipses is the unit circle about o, then even the arithmetic mean of the areas of the body and the polar body is at most π.

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4 Proof of Theorem 3

First we prove a lower bound for the volume product in sectors. The idea of giving lower bounds in sectors separately, and then using the arithmetic-geometric mean inequality, is due to Saint Raymond, [49], proof of Théorème 28. There it is also noted, that this approach settles the two-dimensional o-symmetric case. Our proofs of our Theorems 1, 2, 3, 5 all use this idea.

The particular case $u = u^* = (0,1)$, $v = v^* = (1,0)$ of our following lemma reduces to the two-dimensional case of [49], Théorème 28.

Lemma 7 *Let K be a planar convex body with $o \in \operatorname{int} K$. Let, for some linearly independent $u, v \in \partial K$, and linearly independent $u^*, v^* \in \partial K^*$, the supporting lines to K with exterior normals u^* and v^* intersect K, e.g., at u and v, respectively, and intersect each other at $p \in \mathbb{R}^2$, where $[p,o] \cap [u,v] \neq \emptyset$. Furthermore, let the supporting lines to K^* with exterior normals u and v intersect K^*, e.g., at u^* and v^*, respectively, and intersect each other at $p^* \in \mathbb{R}^2$ with $[p^*, o] \cap [u^*, v^*] \neq \emptyset$. Then, for $C = K \cap [o, u, v, p]$ and $C^* = K^* \cap [o, u^*, v^*, p^*]$, we have*

$$|C| \cdot |C^*| \geq |[o, u, v, p]| \cdot |[o, u^*, v^*]|,$$

with equality if and only if either $C = [o, u, v]$ or $C = [o, u, v, p]$.

Remark. We may assume $C \neq [o, u, v]$. Then, for $p = \lambda u + \mu v$ and $\lambda, \mu > 0$, we have $\lambda + \mu > 1$ and $p^* = \mu u^* + \lambda v^*$. We choose a coordinate system, assuming

$$u = (1,0), \ v = (0,1).$$

Then

$$p = (\lambda, \mu), \ p^* = (1,1), \ u^* = (1, (1-\lambda)/\mu), \ v^* = ((1-\mu)/\lambda, 1),$$

and

$$\begin{cases} |[o, u, v, p]| \cdot |[o, u^*, v^*]| = |[o, u, v]| \cdot |[o, u^*, v^*, p^*]| = \\ (\lambda + \mu)(\lambda + \mu - 1)/(4\lambda\mu) = (2 - \langle u, v^* \rangle - \langle u^*, v \rangle)/4 \,. \end{cases}$$

First we show that Mahler's original proofs ([Mah38]) yield our lemma, except the case of equality.

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First proof. We exclude $C = [o, u, v], [o, u, v, p]$. Let $k \geq 0$ be an integer, and let us suppose that both C and C^* are polygons, such that the total number of their vertices in int $[u, v, p]$, or int $[u^*, v^*, p^*]$, respectively, is at most k. (This case suffices to prove the inequality.) Let C, C^* realize the minimum under these hypotheses. If e.g. C has a vertex $c \in$ int $[u, v, p]$, then we can move c a bit, parallel to the diagonal connecting its neighbours, hence keeping $|C|$ fixed. Then, for C^*, the polar side line will rotate about some of its points. Since the lines of the neighbours of this side intersect outside this side line, by some small rotation $|C^*|$ strictly decreases, a contradiction. Hence we have a situation as for $k = 0$.

For $k = 0$, C has a vertex c, e.g. in relint $[u, p]$, and then $C = [o, u, v, c]$, since else C^* would have a vertex in int $[u^*, v^*, p^*]$. Then $c = (\alpha\lambda + 1 - \alpha, \alpha\mu)$, where $\alpha \in (0, 1)$, and $|C| \cdot |C^*| = (1/4)(1 + (\lambda + \mu - 1)\alpha) \cdot [1 - (1 - \lambda/\mu) - ((1 - \mu)/\lambda - 1)(1 - \alpha\lambda)/(1 - \alpha + \alpha\mu)]$. The fact that this is at least $(\lambda + \mu)(\lambda + \mu - 1)/(4\lambda\mu)$, can be written, after multiplying with the product of the denominators (each of them being positive), and rearranging (using the program package GAP, [51]), as $\lambda(\lambda + \mu - 1)^2 \cdot \alpha(1 - \alpha) \geq 0$. ∎

The second proof follows the lines of Meyer, [37], proof of Théorème I. 2 (more exactly, its two-dimensional case, that gives our lemma for $u = u^* = (0, 1)$, $v = v^* = (1, 0)$).

Second proof. We have

$$1 = \langle u^*, u \rangle = \langle u^*, p \rangle = \langle v^*, p \rangle = \langle v^*, v \rangle = \langle u, p^* \rangle = \langle v, p^* \rangle.$$

For $x \in K \cap [p, u, v]$, the sum of the heights of the triangles $[o, u, v]$ and $[x, u, v]$, belonging to their common side $[u, v]$, is $\langle p^*, x \rangle / \|p^*\|$. Thus the vectors $w := [\|u^* - v^*\|/(2\|p\|)]p$ and $w^* := [\|u - v\|/(2\|p^*\|)]p^*$ satisfy

$$|C| \geq |[o, u, v, x]| = \langle w^*, x \rangle \quad \text{for} \quad x \in K \cap [u, v, p], \quad \text{and} \tag{2}$$

$$|C^*| \geq |[o, u^*, v^*, x^*]| = \langle w, x^* \rangle \quad \text{for} \quad x^* \in K^* \cap [u^*, v^*, p^*]. \tag{3}$$

Since $\langle w^*, p \rangle = |[o, u, v, p]| \geq |C|$, and $\langle w^*, x \rangle < \langle w^*, u \rangle$ for $x \in K \backslash [p, u, v]$, we have $\widetilde{w}^* := |C|^{-1}w^* \in K^* \cap [u^*, v^*, p^*]$, and analogously $\widetilde{w} := |C^*|^{-1}w \in K \cap [u, v, p]$. It follows by applying (2) to $x = \widetilde{w}$, that

$$\begin{cases} |C| \cdot |C^*| \geq \langle w^*, |C^*|\widetilde{w} \rangle = \langle w^*, w \rangle = \\ \langle w^*, p \rangle \cdot \|u^* - v^*\|/(2\|p\|) = |[o, u, v, p]| \cdot |[o, u^*, v^*]|. \end{cases}$$

We also have $\langle w^*, w \rangle = |[o, u, v]| \cdot |[o, u^*, v^*, p^*]|$ by the remark following the statement of this Lemma, hence we have equality in the Lemma if $C = [o, u, v]$ or $C^* = [o, u, v, p]$.

Assume that equality holds in Lemma 7. It follows by (2) and (3) that

$$C = [o, u, v, \widetilde{w}] \quad \text{and} \quad C^* = [o, u^*, v^*, \widetilde{w}^*].$$

In particular C^* has vertices a^* and b^* satisfying

$$\langle a^*, u \rangle = \langle a^*, \widetilde{w} \rangle = 1 \quad \text{and} \quad \langle b^*, v \rangle = \langle b^*, \widetilde{w} \rangle = 1.$$

Checking the vertices of C^*, we have only two choices. Either $a^* = u^*$ and $b^* = v^*$, and hence $C = [o, u, v, p]$, or $a^* = b^* = \widetilde{w}^*$, and hence $C = [o, u, v]$. ∎

The third proof will use an idea of Behrend, [4], proof of (77), pp. 739-740, and of (112), pp. 746-747. Its idea, intuitively, is the following. "If C is close to ouv, then C^* is close to $[ou^*v^*p^*]$, hence $|C^*|$ will be a lot greater than $|ou^*v^*|$. On the other hand, if C is close to $[ouvp]$, then $|C|$ will be a lot greater than $|[ouv]|$."

Third proof. Using the notations of the second proof, we have

$$|C| \geq |[o, u, v, x]|,$$

where now $x \in C \cap [u, v, p]$ is a point farthest from the line $(p^*)^{-1}(1)$, which line passes through u, v. Then there is a supporting line $(x^*)^{-1}(1)$ at x to K, parallel to $(p^*)^{-1}(1)$. Then

$$|C^*| \geq |[o, u^*, v^*, x^*]|,$$

so,

$$|C| \cdot |C^*| \geq |[o, u, v, x]| \cdot |[o, u^*, v^*, x^*]|.$$

Observe that, if x varies in $[u, v, p]$, then $|[o, u, v, x]|$ is proportional to dist $(o, (x^*)^{-1}(1)) = 1/\|x^*\|$. Simultaneously, x^* varies in $[o, p^*] \cap [u^*, v^*, p^*]$, hence $|[o, u^*, v^*, x^*]|$ is proportional to $\|x^*\|$. Hence, $|[o, u, v, x]| \cdot |[o, u^*, v^*, x^*]|$ does not depend on x, so has the same value, as for $x \in [u, v]$, and for $x = p$.

For the case of equality we have $C = [o, u, v, x]$ and $C^* = [o, u^*, v^*, x^*]$. We exclude $x \in [u, v]$ and $x = p$. Then x^* varies in relint $(C^* \cap [o, p^*])$, and we get a contradiction as in the second proof. ∎

Proof of Theorem 3. We may assume that o is the Santaló point of K. First we show that $o \in \operatorname{int} K_i$.

We note that as the origin is the centroid of K^*, there exists no line l with $o \in l$ bounding the half planes l^- and l^+ such that the reflected image of $K \cap l^-$ through the line l is strictly contained in $K \cap l^+$. If $n \geq 4$ then the angles of a regular n-gon are at least $\pi/2$, thus $o \in \operatorname{int} K_i$ by the property of the Santaló point above.

If $n = 3$ then we may assume that K is not a parallelogram. In this case for each triangle S cut off by a side s of K_i from K_o, there is a linear transformation A such that the reflected image of AS through the line of As is strictly contained in AK_i (here we use that K is not a parallelogram). Therefore the property of the Santaló point above, applied to AK, yields $o \in \operatorname{int} K_i$.

When indexing the vertices of an n-gon, we identify vertices with indices j and $j \pm n$. Let $x_1, \ldots, x_n$, and $y_1, \ldots, y_n$ denote the vertices of K_i and K_o in counterclockwise order, and $x_1^*, \ldots, x_n^*$, and $y_1^*, \ldots, y_n^*$ denote the vertices of K_i^* and K_o^*, respectively, so that, for $j = 1, \ldots, n$, we have $x_j \in [y_j, y_{j+1}]$, and

$$1 = \langle x_j^*, x_{j-1} \rangle = \langle x_j^*, x_j \rangle = \langle y_j^*, y_{j+1} \rangle = \langle y_j^*, y_j \rangle.$$

In particular, $y_j^* \in [x_j^*, x_{j+1}^*]$. For $j = 1, \ldots, n$, let $C_j = K \cap [o, x_{j-1}, x_j, y_j]$ and $C_j^* = K^* \cap [o, y_{j-1}^*, y_j^*, x_j^*]$. Therefore Lemma 7 yields that

$$|C_j| \cdot |C_j^*| \geq |[o, x_{j-1}, x_j, y_j]| \cdot |[o, y_{j-1}^*, y_j^*]|, \tag{4}$$

with equality if and only if $C_j = [o, x_{j-1}, x_j, y_j]$ or $C_j = [o, x_{j-1}, x_j]$.

By the n-fold rotational symmetry of K_i and K_o around their their common centre, there exist common distances $a = \|x_{j-1} - y_j\|$ and $b = \|x_j - y_j\|$ for $j = 1, \ldots, n$, and hence $a + b$ is the side length of K_o. Since the distance of o from the line of y_j, y_{j+1} is $d_j := \|y_j^*\|^{-1}$ for $j = 1, \ldots, n$, it follows that

$$|C_j| \cdot |C_j^*| = \frac{(ad_{j-1} + bd_j)\sin(2\pi/n)}{4d_{j-1}d_j}.$$

Additionally, we have

$$\frac{n(a+b)^2}{4\tan(\pi/n)} = |K_o| = \frac{(a+b)(d_1 + \ldots + d_n)}{2}.$$

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We deduce by repeated applications of the inequality between the (weighted) arithmetic and geometric means, that

$$
\begin{aligned}
|K| \cdot |K^*| &= \left(\sum_{j=1}^{n} |C_j| \right) \cdot \left(\sum_{j=1}^{n} |C_j^*| \right) \geq n^2 \left(\prod_{j=1}^{n} (|C_j| \cdot |C_j^*|) \right)^{1/n} \quad (5) \\
&= \frac{n^2 \sin(2\pi/n)}{4} \left(\prod_{j=1}^{n} \frac{a d_{j-1} + b d_j}{d_{j-1} d_j} \right)^{1/n} \\
&\geq \frac{n^2(a+b)\sin(2\pi/n)}{4} \left(\prod_{j=1}^{n} d_j \right)^{-1/n} \quad (6) \\
&\geq \frac{n^3(a+b)\sin(2\pi/n)}{4 \sum_{j=1}^{n} d_j} = \frac{n^2 \sin(2\pi/n)\tan(\pi/n)}{2}. \quad (7)
\end{aligned}
$$

Assume that equality holds in Theorem 3. It follows by (6) and (7) that all d_j are equal, thus o is the common centre of K_i and K_o. Further, all C_j have the same area by (5). Therefore the equality conditions in (4) imply that either $K = K_i$ or $K = K_o$. ∎

Remark. In the particular case of Lemma 7, when C is an n-th part of a convex body K with n-fold rotational symmetry about o, we could have referred in the first proof to [MR], to the so called "shadow movement" (although this is more involved than the elementary proof of Mahler used above). That is, we have an ln-gon $K = x_1 \ldots x_{ln}$ (where $l \geq 2$), having n-fold rotational symmetry about o. The movement of the vertices $x_2, x_{2+l}, \ldots x_{2+(n-1)l}$, parallel to the diagonals $x_1 x_3$, etc., preserving the rotational symmetry, and giving a polygon K', of course does not determine a shadow movement. However, we can move only x_2, in the above way, and this determines a shadow movement, giving a polygon K''. (More exactly: only the points of $[x_1, x_2, x_3]$ are moved, in the direction of $x_1 x_3$, the points of the chords parallel to $x_1 x_3$ with the same velocity, so that at any moment the moved chords constitute a triangle with vertices x_1, x_3, and the translate of x_2). Then $|K| = |K'| = |K''|$, and $|(K')^*| = |K^*| + n(|(K'')^*| - |K^*|)$, so $|(K')^*|$ is a linear function of $|(K'')^*|$. Moreover, K'' and K are not affinely equivalent (consider the barycentres of the subpolygons with vertices each l'th vertex of K'', K).

5 Proofs of the stability theorems

The main result in this section is the following stability version of Lemma 7.

Lemma 8 *Let $C, C^*, u, u^*, v, v^*, p, p^*$ be as in Lemma 7, and let $p = \lambda u + \mu v$ for $\lambda, \mu > 0$. If*

$$|C| \cdot |C^*| \leq (1 + \varepsilon)|[o, u, v, p]| \cdot |[o, u^*, v^*]|,$$

for positive $\varepsilon < \min\{\lambda, \mu\}/(\lambda + \mu)$, then for $\gamma := 3[(\lambda + \mu)/(\min\{\lambda, \mu\})](1 + \sqrt{\lambda + \mu})$,

$$either \quad C \subset (1 + \gamma\varepsilon)[o, u, v], \quad or \quad (1 + \gamma\varepsilon)^{-1}[o, u, v, p] \subset C.$$

First proof. We may assume $C \neq [o, u, v]$. We use the notations from the Remark after Lemma 7, and from the second proof of Lemma 7. We have $\widetilde{w} = tp$ and $\widetilde{w}^* = sp^*$ for some $t, s \in (0, 1]$. Since $\langle \widetilde{w}, \widetilde{w}^* \rangle \leq 1$, we have

$$ts(\lambda + \mu) \leq 1. \tag{8}$$

Further, for $\tilde{u}^* := (1, (1 - t\lambda)/(t\mu))$ and $\tilde{v}^* := ((1 - t\mu)/(t\lambda), 1)$,

$$1 = \langle \tilde{u}^*, u \rangle = \langle \tilde{u}^*, \widetilde{w} \rangle = \langle \tilde{v}^*, v \rangle = \langle \tilde{v}^*, \widetilde{w} \rangle.$$

It follows by the second proof of Lemma 7, using the notations $\widetilde{w}, \widetilde{w}^*$ introduced there, that

$$[o, u, v, \widetilde{w}] \subset C \quad \text{and} \quad |C| \leq (1 + \varepsilon)|[o, u, v, \widetilde{w}]|, \quad \text{and} \tag{9}$$
$$[o, u^*, v^*, \widetilde{w}^*] \subset C^* \quad \text{and} \quad |C^*| \leq (1 + \varepsilon)|[o, u^*, v^*, \widetilde{w}^*]|. \tag{10}$$

It follows that if $\langle \tilde{u}^*, x \rangle \geq \langle \tilde{u}^*, u \rangle = 1$ for $x \in C$ then

$$|[x, u, \widetilde{w}]| \leq \varepsilon \cdot |[o, u, v, \widetilde{w}]| = \varepsilon \cdot [(\lambda + \mu)/\mu] \cdot |[o, u, \widetilde{w}]|,$$

and hence $\langle \tilde{u}^*, x \rangle \leq 1 + \varepsilon \cdot (\lambda + \mu)/\mu$. For $\widetilde{\gamma} := (\lambda + \mu)/\min\{\lambda, \mu\}$, we deduce that $C \subset (1 + \widetilde{\gamma} \cdot \varepsilon)[o, u, v, \widetilde{w}]$, and hence $[o, u^*, \tilde{u}^*, v^*, \tilde{v}^*, \widetilde{w}^*] \subset (1 + \widetilde{\gamma} \cdot \varepsilon)C^*$ by polarity, and analogously $C^* \subset (1 + \widetilde{\gamma} \cdot \varepsilon)[o, u^*, v^*, \widetilde{w}^*]$. Since $\varepsilon < \widetilde{\gamma}^{-1}$, we deduce

$$[o, u^*, \tilde{u}^*, v^*, \tilde{v}^*, \widetilde{w}^*] \subset (1 + \widetilde{\gamma} \cdot \varepsilon)^2[o, u^*, v^*, \widetilde{w}^*] \subset (1 + 3\widetilde{\gamma} \cdot \varepsilon)[o, u^*, v^*, \widetilde{w}^*].$$

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For $a := (\lambda - s\lambda, s\lambda + \mu - 1)$, we have $\langle a, v^* \rangle = \langle a, \widetilde{w}^* \rangle = s(\lambda + \mu - 1)$, thus

$$
\begin{aligned}
1 + 3\widetilde{\gamma} \cdot \varepsilon \ \geq \ \frac{\langle a, \widetilde{v}^* \rangle}{\langle a, v^* \rangle} &= \frac{ts(\lambda + \mu - 1) + (1-s)(1-t)}{ts(\lambda + \mu - 1)} \\
&= \ 1 + \left(\frac{1}{s} - 1 \right) \left(\frac{1}{t} - 1 \right) \frac{1}{\lambda + \mu - 1}.
\end{aligned}
$$

It follows by (8) that

$$
\text{either} \ \ 1/s \geq \sqrt{\lambda + \mu}, \ \ \text{or} \ \ 1/t \geq \sqrt{\lambda + \mu}\,.
$$

In the first case, $3\widetilde{\gamma} \cdot (\lambda + \mu - 1)/(\sqrt{\lambda + \mu} - 1) = \gamma$ yields $1/t \leq 1 + \gamma\varepsilon$, and hence $(1 + \gamma\varepsilon)^{-1}[o, u, v, p] \subset C$. On the other hand, if $1/t \geq \sqrt{\lambda + \mu}$, then a similar argument leads to $(1 + \gamma\varepsilon)^{-1}[o, u^*, v^*, p^*] \subset C^*$, and hence $C \subset (1 + \gamma\varepsilon)[o, u, v]$. ∎

The second proof of Lemma 8, where however the constant γ will be different, and which iterates the construction in the proof of Behrend ([4], proof of (77), pp. 739-740, and of (112), pp. 746-747) will be broken up into two parts.

Lemma 9 *Under the hypotheses of Lemma 7, and with $p = \lambda u + \mu v$, for $\lambda, \mu > 0$, we have*

$$
|C| \cdot |C^*| \geq f(\lambda, \mu) + g(\lambda, \mu)\alpha(1 - \alpha)\,,
$$

where

$$
f(\lambda, \mu) := (\lambda + \mu)(\lambda + \mu - 1)/(4\lambda\mu)\,,
$$

$$
g(\lambda, \mu) := (1/4) \cdot (\lambda + \mu - 1)^2 \cdot \min\left\{ 1/[\mu(1 + \lambda/4 + \mu)], 1/[\lambda(1 + \lambda + \mu/4)], 1/(\lambda\mu) \right\}\,,
$$

$$
\alpha := \max\left\{ |[u, v, x]|/|[u, v, p]| \mid x \in C \cap [u, v, p] \right\} \in [0, 1]\,.
$$

Proof. We may suppose $\alpha \in (0, 1)$. Let $x = (x_1, x_2) \in C \setminus [o, u, v]$ realize $\alpha = \max |[u, v, x]|/|[u, v, p]|$. We write $C_i := [o, u, v, x]$, and $C_o := \{(\xi, \eta) \in [o, u, v, p] \mid \xi + \eta \leq x_1 + x_2\}$. Then $C_i \subset C \subset C_o$. Let x divide the chord of $[o, u, v, p]$, parallel to the line uv, and containing x in the ratio $\beta : (1 - \beta)$, where the part of the chord with ratio β has an endpoint in $[u, p]$.

We iterate this construction. Let $y, z \in C$, and $\overline{y}, \overline{z} \in C_o$ lie on the other sides of the lines ux, vx than o, and let them realize $\max |[u, x, y]|, \max |[v, x, z]|$ and $\max |[u, x, \overline{y}]|, \max |[v, x, \overline{z}]|$ under these conditions. We define $\gamma :=$

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$|[u, x, y]/|[u, x, \overline{y}]| \in [0, 1]$ and $\delta := |[v, x, z]|/|[v, x, \overline{z}]| \in [0, 1]$. Let $C_i' := C_i \cup [u, x, y] \cup [v, x, z]$, and let C_o' be the intersection of C_o and the support half-planes of C at y, z, with boundaries parallel to the lines ux, vx. Then $C_i \subset C_i' \subset C \subset C_o' \subset C_o$. So for their polars (in the angular domain $u^* o v^*$) we have $(C_o)^* \subset (C_o')^* \subset C^*$. Hence,

$$\begin{cases} |C| \cdot |C^*| \geq |C_i'| \cdot |(C_o')^*| \geq \\ |C_i| \cdot |(C_o)^*| + |C_i' \setminus C_i| \cdot |(C_o)^*| + |C_i| \cdot |(C_o')^* \setminus (C_o)^*| = \\ |C_i| \cdot |(C_o)^*| + (|T_y| + |T_z|) \cdot |(C_o)^*| + |C_i| \cdot (|(T^*)_y| + |(T^*)_z|) \,, \end{cases}$$

where $T_y := [u, x, y]$, $T_z := [v, x, z]$, and the triangles $(T^*)_y, (T^*)_z$ have as their vertices the polars of the three first, or three last consecutive side lines of C_o' in the open angular domain $u^* o v^*$, taken in the positive orientation, respectively.

First we estimate $|T_y| \cdot |(C_o)^*| + |C_i| \cdot |(T^*)_y|$ from below. We have

$$|C_i| = [1 + (\lambda + \mu - 1)\alpha]/2 \,,$$

$$|(C_o)^*| = (1/2) \cdot [1/ \left(1 + (\lambda + \mu - 1)\alpha\right)] \cdot (\lambda + \mu)(\lambda + \mu - 1)/(\lambda\mu) \,,$$

$$|T_y| = \gamma\beta \left((\lambda + \mu - 1)/2\right) \alpha(1 - \alpha) \,.$$

By using the program package GAP, [51],

$$\begin{cases} |(T^*)_y| = (1/2) \cdot (\lambda + \mu - 1)^2 \cdot (1 - \gamma)\beta\alpha(1 - \alpha)/ \\ \left[\mu \cdot [1 + (\lambda + \mu - 1)\alpha] \cdot [\beta(1 - \alpha - \gamma\alpha + \gamma\alpha^2) + \right. \\ \left. \gamma\beta\alpha(1 - \alpha)\lambda + \alpha(1 + \gamma\beta - \gamma\alpha\beta)\mu]\right] \,. \end{cases}$$

Here the denominator is a product of three factors, all being positive. (For the third factor observe that the coefficients of λ, or μ are non-negative or positive, respectively, and the constant term is minimal for $\gamma = 1$, and is then non-negative.) The second factor of the denominator will cancel with $|C_i|$, and its third factor will be estimated from above as follows. The coefficients of λ, or μ, in it are estimated from above by setting $\beta = \gamma = 1$, and then $\alpha = 1/2$, or $\alpha = 1$, obtaining $1/4$, or 1, respectively, and the constant term is estimated from above by setting $\gamma = 0$, $\beta = 1$, and then $\alpha = 0$, obtaining 1.

Hence, minimizing for $\gamma \in [0, 1]$,

$$\begin{cases} |T_y| \cdot |(C_o)^*| + |C_i| \cdot |(T^*)_y| \geq \\ (1/4) \cdot ((\lambda + \mu - 1)^2/\mu) \cdot \min \{1/(1 + \lambda/4 + \mu), 1/\lambda\} \cdot \beta\alpha(1 - \alpha) \end{cases}$$

(the first term being estimated from below by setting $\alpha = 1$ in the denominator of the second factor of $|(C_o)^*|$). Changing the roles of λ, μ, of $\beta, 1-\beta$, and of γ, δ, we obtain similarly

$$
\begin{cases}
|T_z| \cdot |(C_o)^*| + |C_i| \cdot |(T^*)_z| \geq \\
(1/4) \cdot ((\lambda + \mu - 1)^2/\lambda) \cdot \min\{1/(1 + \lambda + \mu/4), 1/\mu\} \cdot (1 - \beta)\alpha(1 - \alpha).
\end{cases}
$$

Hence,

$$
\begin{cases}
|C| \cdot |C^*| \geq |C_i| \cdot |(C_o)^*| + (|T_y| + |T_z|) \cdot |(C_o)^*| + |C_i| \cdot (|(T^*)_y| + |(T^*)_z|) \\
\geq f(\lambda, \mu) + g(\lambda, \mu) \cdot \alpha(1 - \alpha).
\end{cases}
$$

∎

Corollary 10 *Under the hypotheses of Lemma 9, let*

$$
|C| \cdot |C^*| \leq (1 + \varepsilon) \cdot f(\lambda, \mu),
$$

where $\varepsilon \in (0, g(\lambda, \mu)/(4f(\lambda, \mu)))$. Further let $\alpha_{\pm} := \left[1 \pm \sqrt{1 - (4f(\lambda, \mu)/g(\lambda, \mu))\varepsilon}\right]/2$, and $\alpha_+ + (1 - \alpha_+)\min\{(1 - \lambda)/\mu, (1 - \mu)/\lambda\} > 0$. Then

$$
\text{either } \ C \subset [1 + (\lambda + \mu - 1)\alpha_-] \cdot [o, u, v],
$$
$$
\text{or } \ C \supset [\alpha_+ + (1 - \alpha_+) \cdot \min\{(1 - \lambda)/\mu, (1 - \mu)/\lambda\}] \cdot [o, u, v, p].
$$

Proof. By hypotheses and Lemma 9, for α from Lemma 9,

$$
f(\lambda, \mu) \cdot (1 + \varepsilon) \geq |C| \cdot |C^*| \geq f(\lambda, \mu) + g(\lambda, \mu)\alpha(1 - \alpha),
$$

hence

$$
\alpha^2 - \alpha + (f(\lambda, \mu)/g(\lambda, \mu))\varepsilon \geq 0,
$$

i.e., $\alpha \leq \alpha_-$, or $\alpha \geq \alpha_+$, where $\alpha_{\pm} \in \mathbb{R}$ and $\alpha_- < \alpha_+$.

Let $x \in C \cap [u, v, p]$, with $|[u, v, x]|$ maximal. Then C lies below the line $l := \{y \mid y \text{ lies above the line } uv, \text{ and } |[u, v, y]| = \alpha \cdot |[u, v, p]|\}$. If $\alpha \leq \alpha_-$, then C lies below the line l_-, defined analogously to l, but using α_- rather than α. If $\alpha \geq \alpha_+$, then $C \supset [o, u, v, x]$, hence $C \supset [o, u, v, \nu x]$, where νx lies on the line l_+, defined analogously to l, but using α_+ rather than α. Hence C contains the quadrangle obtained from $[o, u, v, \nu x]$, by replacing its side lines $u(\nu x), v(\nu x)$ by lines through νx, parallel to up, vp. We further

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diminish this last quadrangle by translating its side lines parallel to up, or vp so, that they should contain the points of intersection of the sides vp, or up with the line l_+, respectively. The formulas in the corollary then follow by simple calculations. ■

Remark. It is probable that with more work one could sharpen the stability estimates in the second proof of Lemma 8, iterating further the construction of inscribed/circumscribed polygons (defining, in an analogous manner, some closer approximations $C_i \subset C_i' \subset C_i'' \subset C \subset C_o'' \subset C_o' \subset C_o$, etc.). However, this way does not seem to be suitable to give estimates, which are sharp, up to a quantity $o(\varepsilon)$.

The first inequality in the next lemma is related to [25], Proposition 1, but is formulated with constants according to our particular needs in this paper. The second inequality in our next lemma is related to an opposite inequality as in Proposition 2 of [25], but the idea of the proof is similar.

Lemma 11 *Let $d \geq 2$ be an integer, $K_0 \subset \mathbb{R}^d$ be a convex body, and let $0 < \varepsilon_1 \leq \varepsilon_1(K_0) := \min\{1/2, 2^{-2d-1}(\kappa_{d-1}/(d\kappa_d^2)) \cdot |K_0|/(\mathrm{diam}\, K_0)^d\}$. Let $K \subset \mathbb{R}^d$ be a convex body, and let $(1 - \varepsilon_1)K_0 + a \subset K \subset (1 + \varepsilon_1)K_0 + b$, where $a, b \in \mathbb{R}^d$. Then*

$$\|s(K) - s(K_0)\| \leq c_1(K_0) \cdot \varepsilon_1,$$

where

$$c_1(K_0) := (\mathrm{diam}\, K_0)^{(d+1)^2} |K_0|^{-d-2} \cdot d(d\kappa_d/\kappa_{d-1})^{d+2}.$$

If moreover, $\varepsilon_2 > 0$, and $|K_0| \cdot |(K_0 - s(K_0))^*| \leq |K| \cdot |(K - s(K))^*|$, and $c \in \mathrm{int}\, K$, and $|K| \cdot |(K - c)^*| \leq |K_0| \cdot |(K_0 - s(K_0))^*| + \varepsilon_2 \leq \kappa_d^2$, then

$$\|c - s(K_0)\| \leq c_1(K_0) \cdot \varepsilon_1 + c_2(K_0) \cdot \sqrt{\varepsilon_2},$$

where

$$c_2(K_0) := \sqrt{(\mathrm{diam}\, K_0)^{d+2}/|K|} \cdot \sqrt{2^{d+3}/((d+1)\kappa_d)}.$$

Proof. We will suppose that the point of homothety of $(1-\varepsilon_1)K_0 + a$ and $(1+\varepsilon_1)K_0 + b$, that is in the first body, is o (this does not change $K_0 - s(K_0)$, $K - s(K)$, $K - c$; namely, we consider c as "fixed to K"). Thus $a = b = o$ can be supposed.

We have

$$
\begin{cases}
(\partial/\partial x_d)|(K-x)^*| = \int_{S^{d-1}} u_d\, (h_K(u) - \langle u, x\rangle)^{-d-1}\, du\,, \\
(\partial/\partial x_d)^2|(K-x)^*| = (d+1) \int_{S^{d-1}} u_d^2\, (h_K(u) - \langle u, x\rangle)^{-d-2}\, du \qquad (11) \\
\ge (d+1)(\operatorname{diam} K)^{-d-2}\kappa_d\,,
\end{cases}
$$

where $u = (u_1, \ldots, u_d)$, h_K is the support function of K, and κ_d the volume of the unit ball in $\mathbb{R}^d$. The analogues of these formulas hold for the first and second directional derivatives in any direction.

First we estimate $\|s(K) - s(K_0)\|$ from above. We may assume that $s(K) - s(K_0) = (0, \ldots, 0, \delta)$, where $\delta > 0$.

We begin by showing that $s(K) \in \operatorname{int}((1 - \varepsilon_1)K_0)$, and even estimate $\operatorname{dist}(s(K), \operatorname{bd}[(1 - \varepsilon_1)K_0])$ from below. Let $\eta := \operatorname{dist}(s(K), \operatorname{bd} K) \le \operatorname{dist}(s(K), \operatorname{bd}[(1 + \varepsilon_1)K_0])$. Then $(K - s(K))^*$ contains $(\operatorname{diam} K)^{-1}B^d$, and a point at distance η^{-1} from o (with B^d the unit ball about o). Therefore

$$
\kappa_d^2 \ge |K| \cdot |(K - s(K))^*| \ge |K| \cdot (\operatorname{diam} K)^{-d+1}(\kappa_{d-1}/d)\eta^{-1}\,. \qquad (12)
$$

Hence, by $\varepsilon_1 \le 1/2$,

$$
\begin{cases}
\eta_0 := 2^{-2d+1}\left(\kappa_{d-1}/(d\kappa_d^2)\right) \cdot |K_0|/[(\operatorname{diam} K_0)^{d-1}] \le \\
\left(\kappa_{d-1}/(d\kappa_d^2)\right) \cdot |K|/(\operatorname{diam} K)^{d-1} \le \eta \le \\
\operatorname{dist}(s(K), \operatorname{bd}[(1 + \varepsilon_1)K_0]) \le \\
\operatorname{dist}\left(s(K), \operatorname{bd}[(1 - \varepsilon_1)K_0 + 2\varepsilon_1 \cdot \operatorname{diam} K_0 \cdot B^d]\right)\,.
\end{cases}
$$

Thus, for $\varepsilon_1 \le \eta_0/(4 \cdot \operatorname{diam} K_0)$,

$$
s(K) \in \operatorname{int}[(1 - \varepsilon_1)K_0] \quad\text{and}\quad \eta_0/2 \le \operatorname{dist}(s(K), \operatorname{bd}[(1 - \varepsilon_1)K_0])\,.
$$

Then, using convexity of the function t^{-d-1} for $t > 0$, and (12) for K_0, rather than K,

$$
\begin{cases}
0 = \int_{S^{d-1}} u_d\, (h_K(u) - \langle u, s(K)\rangle)^{-d-1}\, du \ge \\
\int_{S^{d-1}} u_d[h_{K_0}(u) + \operatorname{sg} u_d \cdot \varepsilon_1 h_{K_0}(u) - \langle u, s(K_0)\rangle - \delta u_d]^{-d-1} du \ge \\
\int_{S^{d-1}} u_d\, (h_{K_0}(u) - \langle u, s(K_0)\rangle)^{-d-1}\, du + \\
(d+1) \cdot \int_{S^{d-1}} u_d\, (\delta u_d - \varepsilon_1 \cdot \operatorname{sg} u_d \cdot h_{K_0}(u)) \times \\
(h_{K_0}(u) - \langle u, s(K_0)\rangle)^{-d-2}\, du \ge \\
\delta(d+1)(\operatorname{diam} K_0)^{-d-2} \int_{S^{d-1}} u_d^2 du - \\
\varepsilon_1(d+1) \cdot \operatorname{diam} K_0 \cdot [(\kappa_{d-1}/(d\kappa_d^2))\, |K_0|/ \\
(\operatorname{diam} K_0)^{-d+1}]^{-d-2} \int_{S^{d-1}} |u_d| du\,.
\end{cases}
\qquad (13)
$$

Here, $\int_{S^{d-1}} u_d^2 du = \kappa_d$, and $\int_{S^{d-1}} |u_d| du \leq \int_{S^{d-1}} du$, and comparing the first and last terms of (13), we get the first inequality of the Lemma.

We turn to the second inequality. We have

$$\|c - s(K_0)\| \leq \|c - s(K)\| + \|s(K) - s(K_0)\| \leq \|c - s(K)\| + c_1(K_0)\varepsilon_1 , \quad (14)$$

and

$$|K| \cdot |(K - c)^*| \leq |K_0| \cdot |(K_0 - s(K_0))^*| + \varepsilon_2 \leq |K| \cdot |(K - s(K))^*| + \varepsilon_2 .$$

We use (11) on the line of $s(K), c$, which gives

$$|K|(d+1)(\mathrm{diam}\, K)^{-d-2}\kappa_d \cdot \|c - s(K)\|^2/2 \leq \varepsilon_2 . \quad (15)$$

(14) and (15) give the second inequality of the Lemma. ∎

Proof of Theorem 5. 1. First we estimate $\delta_{BM}^s(K, R_n)$ from above. Here we may assume that o is the Santaló point of K, i.e., its centre of rotational symmetry. As explained in §3, there exist regular n-gons K_i and K_o centred at the origin, such that $K_i \subset K \subset K_o$, and the midpoints of the sides of K_o are the vertices of K_i. Assuming that the unit circular disc about o is the incircle of K_o, we have $K_o^* = K_i$. Now the radii from o to the vertices of K_i divide K_o into n congruent deltoids $\widetilde{C}_1, \ldots, \widetilde{C}_n$ whose common vertex is the origin. In particular $\widetilde{C}_j^* := \widetilde{C}_j \cap K_i$ is the corresponding triangular sector of K_i, $j = 1, \ldots, n$. For the congruent sectors $C_j = \widetilde{C}_j \cap K$ of K, and the congruent sectors $C_j^* = \widetilde{C}_j \cap K^*$ of K^*, $j = 1, \ldots, n$, we have

$$(1 + \varepsilon)n^2|\widetilde{C}_1| \cdot |\widetilde{C}_1^*| = (1 + \varepsilon)|K_i| \cdot |K_o| \geq |K| \cdot |K^*| = n^2|C_1| \cdot |C_1^*|.$$

We observe that $\widetilde{C}_1^* = [o, u, v]$ and $\widetilde{C}_1 = [o, u, v, p]$, where $p = \lambda u + \lambda v$ for $\lambda = [\cos(\pi/n)]^{-2}/2$, and

$$|C_1| \cdot |C_1^*| \leq (1 + \varepsilon)|[o, u, v, p]| \cdot |[o, u^*, v^*]|.$$

We deduce by Lemma 8 that either $C_1 \subset (1 + \gamma\varepsilon)\widetilde{C}_1^*$, or $(1 + \gamma\varepsilon)^{-1}\widetilde{C}_1 \subset C_1$, where $\gamma := 6(1 + \sqrt{2\lambda}) \leq 18$. Therefore the rotational symmetry yields that either $K \subset (1 + 18\varepsilon)K_i$, or $(1 + 18\varepsilon)^{-1}K_o \subset K$.

2. Now we turn to the proof of the stability of the centre of polarity. The point x is the point of homothety of $\lambda_1 R_n + x$ and $\lambda_2 R_n + x$, and $x \in \lambda_1 R_n + x$; we will suppose $x = o$. Simultaneously, we have to replace K^* with $(K - c)^*$,

for some $c \in \text{int}\, K$ ("fixed to K"). Let $K_{0,n} := [(\lambda_1 + \lambda_2)/2]R_n$. Then $\lambda_1 R_n \subset K \subset \lambda_2 R_n$ and $\lambda_2/\lambda_1 \leq 1 + 18\varepsilon$ imply

$$\begin{cases} K_{0,n}(1 - 9\varepsilon) \subset K_{0,n}/[(1 + \lambda_2/\lambda_1)/2] \subset K \\ \subset K_0/[(\lambda_1/\lambda_2 + 1)/2] \subset K_{0,n}(1 + 9\varepsilon)\,. \end{cases} \tag{16}$$

Note that by hypothesis $\varepsilon < 1/18$, so here

$$1 - 9\varepsilon > 0\,. \tag{17}$$

Restricting Lemma 11 to $d = 2$, we have $\varepsilon_1(K_0) = [1/(32\pi^2)] \cdot |K_0|/$ $(\text{diam}\, K_0)^2$, and $c_1(K_0) = 2\pi^4 \cdot (\text{diam}\, K_0)^9 |K_0|^{-4}$, and $c_2(K_0) = 4\sqrt{2/(3\pi)} \cdot$ $(\text{diam}\, K_0)^2 |K_0|^{-1/2}$. Here $\text{diam}\, K_0 = 1$, hence $\min_n |K_{0,n}|$ is attained for $n = 3$, so $\min_n \varepsilon_1(K_{0,n}) = \sqrt{3}/(128\pi^2)$, and $\max_n c_1(K_{0,n}) = 512\pi^4/9$, and $\max_n c_2(K_{0,n}) = 8\sqrt{2/\pi}3^{-3/4}$.

We apply Lemma 11 for $d = 2$, replacing there $\varepsilon_1(K_0)$ by $\min_n \varepsilon_1(K_{0,n})$, and $c_1(K_0)$ by $\max_n c_1(K_{0,n})$, and $c_2(K_0)$ by $\max_n c_2(K_{0,n})$. By (16) we may choose $\varepsilon_1 = 9\varepsilon$. Also, $\varepsilon_2 \leq \pi^2\varepsilon$. So

$$\|c - s(K_0)\| \leq (512\pi^4/9) \cdot 9\varepsilon + 8\sqrt{2/\pi}3^{-3/4} \cdot \pi\sqrt{\varepsilon}\,, \tag{18}$$

for

$$0 < \varepsilon \leq \varepsilon^* := [\sqrt{3}/(128\pi^2)]/9 = 0.0001523... < 1/18\,. \tag{19}$$

However, we will use (18) only for $0 < \varepsilon \leq \varepsilon^{**}$, for some $\varepsilon^{**} \in (0, \varepsilon^*]$, to be chosen later.

First let $0 < \varepsilon \leq \varepsilon^{**}$. Then (18) gives

$$\|c - s(K_0)\| \leq \left(512\pi^4\sqrt{\varepsilon^{**}} + 8\sqrt{2\pi}3^{-3/4}\right) \cdot \sqrt{\varepsilon}\,. \tag{20}$$

Second let $\varepsilon \geq \varepsilon^{**}$. Then by $\lambda_2/\lambda_1 \leq 2$ we have

$$\begin{cases} \|c - s(K_0)\| \leq \text{diam}\,(\lambda_2 R_n) = \lambda_2/[(\lambda_1 + \lambda_2)/2] \\ \leq 4/3 \leq [4/(3\sqrt{\varepsilon^{**}})] \cdot \sqrt{\varepsilon}\,. \end{cases} \tag{21}$$

By (20) and (21), we have

$$\|c - s(K_0)\| \leq \left(\max\left\{512\pi^4\sqrt{\varepsilon^{**}} + 8\sqrt{2\pi}3^{-3/4}, 4/(3\sqrt{\varepsilon^{**}})\right\}\right) \cdot \sqrt{\varepsilon}\,. \tag{22}$$

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Now we minimize the coefficient of $\sqrt{\varepsilon}$ in (22), that is a function of ε^{**}. This minimum occurs when the two terms under the maximum sign are equal, that occurs for $\varepsilon^{**} = 0.0000258...$, and its value is $262.30682...$. (Observe that $0 < \varepsilon^{**} < \varepsilon^{*}$). ∎

For the proofs of Theorems 1 and 2, we need a simple stability version of the inequality between the arithmetic and geometric means. If $n \geq 2$ and $0 < a_1 \leq \ldots \leq a_n$, then

$$
\begin{aligned}
\frac{a_1 + \ldots + a_n}{n \cdot (a_1 \cdot \ldots \cdot a_n)^{1/n}} &= \frac{(\sqrt{a_n} - \sqrt{a_1})^2 + 2\sqrt{a_1 a_n} + \sum_{1 < j < n} a_j}{n \cdot (a_1 \cdot \ldots \cdot a_n)^{1/n}} \\
&\geq \frac{(\sqrt{a_n} - \sqrt{a_1})^2 + n \cdot (a_1 \cdot \ldots \cdot a_n)^{1/n}}{n \cdot (a_1 \cdot \ldots \cdot a_n)^{1/n}} \\
&\geq 1 + \frac{1}{n}\left(1 - \sqrt{\frac{a_1}{a_n}}\right)^2 .
\end{aligned}
$$

It follows that

$$
\begin{cases}
\text{if } \varepsilon \geq 0 \text{ and } (a_1 + \ldots + a_n)/[n \cdot (a_1 \cdot \ldots \cdot a_n)^{1/n})] \leq 1 + \varepsilon, \\
\text{then } a_j/a_k \geq 1 - 2\sqrt{n\varepsilon} \text{ for any } 1 \leq j, k \leq n.
\end{cases} \tag{23}
$$

Proof of Theorem 1. 1. First we estimate $\delta_{BM}(K, P)$ from above. Here we may assume that o is the Santaló point of K, i.e., its centre of symmetry. As explained in §3, we may assume that $K_i \subset K \subset K_o$, where K_o is a square of side length two centered at o, and the midpoints of the sides of K_o are the vertices of K_i. In particular, K_i and K_o are polar to each other. It also follows that $\delta_{BM}(K, P) \leq 2$, and hence if $\varepsilon \geq 0.005$, then we are done. Therefore we assume that $\varepsilon < 0.005$.

Now K_o can be dissected into four unit squares $S_o^1 := [0, 1] \times [0, 1]$, $S_o^2 := [-1, 0] \times [0, 1]$, $-S_o^1$ and $-S_o^2$. We write $S_i^j = S_o^j \cap K_i$, $C_j = S_o^j \cap K$ and $C_j^* = S_o^j \cap K^*$ for $j = 1, 2$, and hence Lemma 7 implies $|C_j| \cdot |C_j^*| \geq |S_i^1| \cdot |S_o^1|$ for $j = 1, 2$. We deduce by the hypothesis $|K| \cdot |K^*| \leq (1+\varepsilon) \cdot 8$ and Lemma 7 that

$$
\begin{cases}
(1 + \varepsilon) \cdot |S_i^1| \cdot |S_o^1| \geq [(|C_1| + |C_2|)/2] \cdot [(|C_1^*| + |C_2^*|)/2] \\
\geq \sqrt{|C_1| \cdot |C_2| \cdot |C_1^*| \cdot |C_2^*|}, \text{ and } |C_j| \cdot |C_j^*| \geq |S_i^1| \cdot |S_o^1|.
\end{cases} \tag{24}
$$

In particular,

$$
|C_j| \cdot |C_j^*| \leq (1 + \varepsilon)^2 \cdot |S_i^1| \cdot |S_o^1| \leq (1 + 2.005\varepsilon) \cdot |S_i^1| \cdot |S_o^1| \text{ for } j = 1, 2.
$$

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To apply Lemma 8, we have $\lambda = \mu = 1$ and $\gamma = 6(1 + \sqrt{2}) < 15$ both in the cases of C_1 and C_2. Therefore, for each of $j = 1, 2$, either $C_j \subset (1 + \gamma \cdot 2.005\varepsilon)S_i^j$, or $(1 + \gamma \cdot 2.005\varepsilon)^{-1}S_o^j \subset C_j$. If both of C_1 and C_2 satisfies either the first, or the second condition, then $\delta_{BM}(K, P) \leq 1 + 31\varepsilon$, and we are done. Therefore we suppose that $C_1 \subset (1 + \gamma \cdot 2.005\varepsilon)S_i^1$, and $(1 + \gamma \cdot 2.005\varepsilon)^{-1}S_o^2 \subset C_2$, and seek a contradiction. We have $|C_1| \leq (1 + \gamma \cdot 2.005\varepsilon)^2/2$, and since the diagonal of S_o^2 not containing o is a subset of C_2, we also have $|C_2| \geq (1 + \gamma \cdot 2.005\varepsilon)^{-1}$. It follows by $\varepsilon < 0.005$ that $|C_1| < (1 - 2\sqrt{2}\varepsilon)|C_2|$. On the other hand, (23) applied in (24) leads to $|C_1| \geq (1 - 2\sqrt{2}\varepsilon)|C_2|$, a contradiction.

2. The stability of the centre of polarity is deduced from Lemma 11 like in Theorem 5, by supposing $x = 0$. Simultaneously, we have to replace K^* with $(K - c)^*$, for some $c \in \text{int } K$ ("fixed to K"). Let $K_0 := [(\lambda_1 + \lambda_2)/2]P$. Now $\varepsilon_1(K_0) = 1/(64\pi^2)$, and also $c_1(K_0)$, $c_2(K_0)$, ε_2 are numerical constants. We only note that by hypothesis $\varepsilon < 0.005$, and then we use the sharper estimate $\delta_{BM}(K, P) \leq 1 + 31\varepsilon$, and we have, analogously to (17), $1 - (31/2)\varepsilon > 0$, and analogously to (19), $\varepsilon^* := [\varepsilon_1(K_0)]/(31/2) < 0.005$. The optimal choice of ε^{**} is ε^*. The distance to be estimated from above is at most $335.10941... \cdot \sqrt{\varepsilon}$. ∎

Proof of Theorem 2. 1. First we estimate $\delta_{BM}(K, T)$ from above. We may assume that K is not a parallelogram, and o is the Santaló point of K. As it is explained in §3, we may assume that $K_i \subset K \subset K_o$, where K_i and K_o are regular triangles, the midpoints of the sides of K_o are the vertices of K_i. It also follows that $\delta_{BM}(K, T) \leq 4$, and hence if $\varepsilon \geq 1/300$, then we are done. Therefore we assume that $\varepsilon < 1/300$.

We use the notation and ideas of the proof Theorem 3. In particular $o \in \text{int } K_i$. We may assume that that the circumradius of K_i is 1, and hence $d_1 + d_2 + d_3 = 3$, and $a = b = \sqrt{3}$.

Since $|K| \cdot |K^*| \leq (1 + \varepsilon) \cdot |K_o| \cdot |K_o^*|$, and we used the inequality between arithmetic and geometric means for $|C_1|, |C_2|, |C_3|$ in (5), and for d_1, d_2, d_3 in the step from (6) to (7), for $j, k = 1, 2, 3$, we deduce by (23) that

$$|C_j|/|C_k| \geq 1 - 2\sqrt{3}\varepsilon \geq 4/5 \qquad (25)$$

$$d_j/d_k \geq 1 - 2\sqrt{3}\varepsilon \geq 4/5. \qquad (26)$$

Since $d_1 + d_2 + d_3 = 3$, we have

$$d_j \geq 3/(1 + 5/4 + 5/4) = 6/7, \qquad (27)$$
$$d_j \leq 3/(1 + 4/5 + 4/5) = 15/13. \qquad (28)$$

Like in the proof of Theorem 1, by hypothesis, and by Lemma 7,

$$\begin{cases} (1 + \varepsilon)\left(\prod_{j=1}^3(|[o, x_{j-1}, x_j, y_j]| \cdot |[o, y_{j-1}^*, y_j^*]|)\right)^{1/3} \geq \\ \left(\prod_{j=1}^3(|C_j| \cdot |C_j^*|)\right)^{1/3}, \quad \text{and} \quad |C_j| \cdot |C_j^*| \geq |[o, x_{j-1}, x_j, y_j]| \cdot |[o, y_{j-1}^*, y_j^*]|. \end{cases}$$

Hence, for each $j = 1, 2, 3$, we have

$$|C_j| \cdot |C_j^*| \leq (1 + 3.1\varepsilon)|[o, x_{j-1}, x_j, y_j]| \cdot |[o, y_{j-1}^*, y_j^*]|. \qquad (29)$$

Let $j = 1, 2, 3$. To apply Lemma 8, we define $\lambda_j, \mu_j > 0$ by

$$y_j = \lambda_j x_{j-1} + \mu_j x_j.$$

Since $\lambda_j/\mu_j = |[o, x_j, y_j]|/|[o, x_{j-1}, y_j]| = d_j/d_{j-1}$, (26) implies

$$\frac{\lambda_j + \mu_j}{\min\{\lambda_j, \mu_j\}} \leq 1 + 5/4 = 9/4.$$

Now the distances of y_j, or o from the line of x_{j-1}, x_j are $3/2$, or $\|x_{j+1}^*\|^{-1} = 3/2 - d_{j+1} \geq 9/26$, by (28), respectively, and hence

$$\begin{cases} \lambda_j + \mu_j = \langle x_j^*, y_j \rangle = \langle x_j^*, x_j \rangle + \|x_j^*\| \cdot \langle \|x_j^*\|^{-1} x_j^*, y_j - x_j \rangle \\ \leq 1 + (3/2)/(9/26) = 16/3. \end{cases}$$

We define $\gamma_j := 3[(\lambda_j + \mu_j)/\min\{\lambda_j, \mu_j\}](1 + \sqrt{\lambda_j + \mu_j})$, and hence

$$3.1\gamma_j \leq 3.1 \cdot 3 \cdot (9/4) \cdot (1 + 4/\sqrt{3}) < 70.$$

In particular, it follows by Lemma 8 and (29) that

either $\quad (1 + 70\varepsilon)^{-1}[o, x_{j-1}, x_j, y_j] \subset C_j, \quad$ or $\quad C_j \subset (1 + 70\varepsilon)[o, x_{j-1}, x_j].$

We note that $1 + 70\varepsilon \leq 5/4$ and $\|x_{j-1} - x_j\| = \sqrt{3}$. If $(1 + 70\varepsilon)^{-1}[o, x_{j-1}, x_j, y_j] \subset C_j$, then (28) yields

$$\begin{cases} |C_j| \geq |[o, x_{j-1}, x_j, (4/5)y_j]| = \\ (4/5) \cdot (\sqrt{3}/2) \cdot (3/2 - d_{j+1} + 3/2) \geq (2\sqrt{3}/5) \cdot 48/26 > 1.27. \end{cases} \qquad (30)$$

30

On the other hand, if $C_j \subset (1 + 70\varepsilon)[o, x_{j-1}, x_j]$, then (27) yields

$$
\begin{cases}
|C_j| \leq (5/4)^2 \cdot |[o, x_{j-1}, x_j]| = (5/4)^2 \cdot (\sqrt{3}/2) \cdot (3/2 - d_{j+1}) \leq \\
(5/4)^2 \cdot (\sqrt{3}/2) \cdot (9/14) < 0.87.
\end{cases} \tag{31}
$$

Comparing (25), (30) and (31) shows that either $(1+70\varepsilon)^{-1}[o, x_{j-1}, x_j, y_j]$ $\subset C_j$ for all $j = 1, 2, 3$, or $C_j \subset (1 + 70\varepsilon)[o, x_{j-1}, x_j]$ for all $j = 1, 2, 3$. Therefore either $(1 + 70\varepsilon)^{-1} K_o \subset K$, or $K \subset (1 + 70\varepsilon) K_i$, and hence the Banach-Mazur distance of K from the triangles is at most $1 + 70\varepsilon$.

2. The stability of the centre of polarity is deduced from Lemma 11 like in Theorem 5 and Theorem 2, by supposing $x = o$. Simultaneously, we have to replace K^* with $(K - c)^*$, for some $c \in \mathrm{int}\, K$ ("fixed to K"). Let $K_0 := [(\lambda_1 + \lambda_2)/2]T$. Now $\varepsilon_1(K_0) = \sqrt{3}/(128\pi^2)$. We only note that by hypothesis $\varepsilon < 1/300$, and then we use the sharper estimate $\delta_{BM}(K, T) \leq 1 + 70\varepsilon$, and we have, analogously to (17), $1 - (70/2)\varepsilon > 0$, and analogously to (19), $\varepsilon^* := [\varepsilon_1(K_0)]/(70/2) < 1/300$. The optimal choice of ε^{**} is ε^*. The distance to be estimated from above is at most $916.69531... \cdot \sqrt{\varepsilon}$. ∎

We turn to the proof of Theorem 6. We proceed analogously as in Lemma 9 and Corollary 10. Again, the proof of Lemma 12 will use an idea of Behrend, [4], proof of (77), pp. 739-740, and of (112), pp. 746-747.

As in the proof of Theorem 2, we assume that $K_i \subset K \subset K_o$, where $K_o = [a, b, c]$, $K_i = [a', b', c']$ are regular triangles, $a' = (b + c)/2$, $b' = (c + a)/2$, $c' = (a + b)/2$. Now we assume $\|a - b\| = 2$. We let $\alpha_1 := \max\{|[x, b', c']|/|[a, b', c']| \mid x \in K \cap [a, b', c']\}$, $\alpha_2 := \max\{|[x, c', a']|/|[b, c', a']| \mid x \in K \cap [b, c', a']\}$, $\alpha_3 := \max\{|[x, a', b']|/|[c, a', b']| \mid x \in K \cap [c, a', b']\}$. Then $\alpha_i \in [0, 1]$, and we let $\alpha := (\alpha_1 + \alpha_2 + \alpha_3)/3 \in [0, 1]$.

Lemma 12 *With the above notations, we have*

$$
|K| \cdot |[(K - K)/2]^*| \geq 6 + (3/2)\alpha(1 - \alpha).
$$

Proof. The supporting lines of K, parallel to and different from the side lines of K_o, contain points a'', b'', c'' of K, with a'' lying in the triangle $b'ac'$, etc. We let $K_i' := [a', c'', b', a'', c', b'']$, and K_o' the hexagon bounded by the supporting lines of K parallel to the sides of K_o. We have

$$
K_i' \subset K \subset K_o'.
$$

Hence,
$$|K| \cdot |[(K-K)/2]^*| \geq |K_i'| \cdot |[(K_o - K_o)/2)]^*| \,. \tag{32}$$

Here
$$|K_i'| = (\sqrt{3}/4)(1 + \alpha_1 + \alpha_2 + \alpha_3) \,, \tag{33}$$

$$\begin{cases} |[(K_o' - K_o')/2]^*| = 2(4/\sqrt{3})^2[(1+\alpha_1)^{-1}(1+\alpha_2)^{-1} + \\ (1+\alpha_2)^{-1}(1+\alpha_3)^{-1} + (1+\alpha_3)^{-1}(1+\alpha_1)^{-1}]\sin(\pi/3)/2 \,. \end{cases} \tag{34}$$

Now, (32), (33), (34), and the arithmetic-geometric mean inequality imply

$$\begin{cases} |K_i'| \cdot |[(K_o - K_o)/2]^*| = 2(1+3\alpha)(3+3\alpha)\times \\ (1+\alpha_1)^{-1}(1+\alpha_2)^{-1}(1+\alpha_3)^{-1} \geq 6(1+3\alpha)(1+\alpha)^{-2} \,. \end{cases} \tag{35}$$

It suffices to show that the last quantity in (35) is at least $6 + (3/2)\alpha(1-\alpha)$. However, if we replace here $3/2$ by some $c \geq 0$, this claimed inequality becomes equivalent to

$$\alpha(1-\alpha)\left(1 - (c/6)(1+\alpha)^2\right) \geq 0 \,,$$

that is satisfied for $c = 3/2$. ∎

Proof of Theorem 6. We will use the notations in Lemma 12 and its proof. By hypotheses and Lemma 12,

$$6 \cdot (1+\varepsilon) \geq |K| \cdot |[(K-K)/2]^*| \geq 6 + (3/2)\alpha(1-\alpha) \,,$$

hence
$$\alpha^2 - \alpha + 4\varepsilon \geq 0 \,,$$

i.e., $\alpha \leq \alpha_-$, or $\alpha \geq \alpha_+$, where $\alpha_\pm$ are the roots of the last polynomial. They are real, with $\alpha_- < \alpha_+$, for

$$\varepsilon \in [0, 1/16) \,,$$

which last inequality will be supposed preliminarily.

For $\alpha \leq \alpha_-$ we have

$$\delta_{BM}(K, T) \leq 1 + \alpha_1 + \alpha_2 + \alpha_3 = 1 + 3\alpha_- \,. \tag{36}$$

Now let $\alpha \geq \alpha_+$. We proceed analogously, as in the proof of Corollary 10. We write $\beta_i := 1 - \alpha_i \in [0, 1]$, and $\beta := 1 - \alpha \in [0, 1]$. Then $\beta = (\sum \beta_i)/3 \leq$

α_-, hence $\beta_i \leq \sum \beta_i \leq 3\alpha_-$. We have $K \cap [a', b', c] \supset [a', b', c'']$. We diminish this last triangle by retaining its side line $a'b'$, and replacing its sides $[a', c''], [b', c'']$ by sides containing c'', and parallel to $[a', c], [b', c]$. We further diminish this last triangle by retaining its side line $a'b'$, and translating its side lines parallel to $[a', c], [b', c]$, so that they should contain $c''(b'), c''(a')$, where $[c''(a'), c''(b')] \ni c''$ is a chord of $[a', b', c]$ parallel to $[a', b']$, with $c''(a') \in [c, a']$ and $c''(b') \in [c, b']$. Of course this is only possible for $\beta_3 \leq 3\alpha_- \leq 1/2$; therefore we preliminarily suppose also

$$\alpha_- \leq 1/6, \quad \text{or, equivalently,} \quad \varepsilon \in [0, 1/28.8] \ (\subset [0, 1/16)) \, .$$

In this case the vertex c''' of the last triangle opposite its side on $[a', b']$ depends only on β_3: it lies on the angle bisector of the triangle $[a', c, b']$ at c, and $\|c''' - c\| = \beta_3\sqrt{3}$. Lastly we replace c''' by c'''', which is constructed analogously as c''', but replacing at the beginning β_3 by $3\alpha_- \ (\geq \beta_3)$. Analogously we define the points a'''', b''''. Then $[a'''', b'''', c''''] \subset [a', c''', b', a'''', c', b''''] \subset K$, hence

$$\delta_{BM}(K, T) \leq 1/(1 - (9/2)\alpha_-) \, . \tag{37}$$

Here we have $1 - (9/2)\alpha_- \geq 1/4$, i.e., $\alpha_- \leq 1/6$, thus $a'''', b'''', c'''' \notin \text{int} [a', b', c']$.

Now, (36) and (37) give

$$\begin{cases} \delta_{BM}(K, T) \leq \max\{1 + 3\alpha_-, \ 1/(1 - (9/2)\alpha_-)\} = \\ 1/(1 - (9/2)\alpha_-) = 1 + [(9/2)\alpha_-]/[1 - (9/2)\alpha_-] \leq \\ 1 + [(9/2)\alpha_-]/[1 - (9/2)(1/6)] = 1 + 18\alpha_- \, . \end{cases} \tag{38}$$

By convexity of the respective function,

$$\alpha_- = (1 - \sqrt{1 - 16\varepsilon})/2 \leq (24/5)\varepsilon, \quad \text{for} \quad \varepsilon \in [0, 1/28.8] \, . \tag{39}$$

Thus, by (38) and (39),

$$\delta_{BM}(K, T) \leq 1 + 18\alpha_- \leq 1 + 86.4\varepsilon \, . \tag{40}$$

There remained the case $\varepsilon \geq 1/28.8$. Then

$$\delta_{BM}(K, T) \leq 4 \leq 1 + 86.4\varepsilon \, . \tag{41}$$

Lastly, (40) and (41) together prove the theorem. ■

6 A short proof of the inequality of Mahler-Reisner

Theorem 13 *(Mahler-Reisner [30], [46]). If K is an o-symmetric convex body in $\mathbb{R}^2$, then*

$$|K| \cdot |K^*| \geq 8\,,$$

with equality if and only if K is a parallelogram.

In the proof of this theorem we use the results of [42], more exactly, the proof of their Theorem 15. Actually we will make only slight modifications in its proof.

Proof of Theorem 13. Like in [42], proof of their Theorem 15, we may suppose a diameter of K is $[(-1,0),\ (1,0)]$, where $K \subset \mathbb{R}^2$ has a minimal volume product among 0-symmetric convex bodies. Let

$$K = \{(x,y) \mid x \in [-1,\ 1],\ -f(-x) \leq y \leq f(x)\}\,,$$

where

$$\begin{cases} f(x) \ \text{ is a concave function on } \ [-1,\ 1], \ \text{ with} \\ f(-1) = f(1) = 0, \ \text{ and } \ f(x) > 0 \ \text{ for } \ x \in (-1,\ 1)\,. \end{cases} \tag{42}$$

If the graph of f consists of two segments, we are done. If not, then, by Lemma 14 of [42], there are functions g, h, both satisfying (42) above, both not proportional to f, such that $f = (g+h)/2$.

Let $t \in [-1,\ 1]$. Let $f_t := f + t(h-g)/2$. Then the area of the convex body

$$K_t := \{(x,y) \mid x \in [-1,\ 1],\ -f_t(-x) \leq y \leq f_t(x)\}$$

is a linear function of t. By Theorem 1 of [42] the reciprocal of $\varphi(t) := |[K_t - s(K_t)]^*|$ is a convex function of t. Hence $\min \varphi$ is attained either for $t = -1$ or for $t = 1$. Since $K = K_0$ has minimum volume product, φ is constant. Then, by Proposition 7 of [42], K_1 is an affine image of K, by an affinity of the form $(x,y) \to (x, ux + vy + w)$. By [42], p. 140, Remark to Proposition 7, we have $h(x) = vf(x) + ux + w$. Putting here $x = \pm 1$, we see $u = w = 0$. Hence, h is proportional to f, a contradiction. ∎

References

[1] Bambah, R. P., Polar reciprocal convex bodies, *Proc. Cambr. Phil. Soc.* **51** (1955), 377-378, MR **16**,1145

[2] Ball, K. M., PhD Thesis, Univ. of Cambridge, 1988

[3] Barthe, F., Fradelizi, M. The volume product of convex bodies with many symmetries, preprint

[4] Behrend, F., Über einige Affininvarianten konvexer Bereiche, *Math. Ann.* **113** (1937), 713-747. Zbl. **15.**367

[5] Blaschke, W., Über affine Geometrie VII: Neue Extremeigenschaften von Ellipse und Ellipsoid, *Ber. über die Verhandl. der Königl. Sächs. Gesellschaft der Wiss. zu Leipzig, Math.-Phys. Klasse* **69**, (1917), 306-318. Jahresberichte Fortschr. Math. **46.**1112

[6] Bollobás, B., Leader, I., Radcliffe, A. J., Reverse Kleitman inequalities, *Proc. London Math. Soc. (3)*, **58** (1989), 153-168, MR **90f:**52017

[7] Böröczky, K. J., The stability of the Rogers-Shephard inequality and of some related inequalities, *Adv. in Math.*, **190** (2005), 47-76 MR **2005i:**52013

[8] Böröczky, K. J., Stability of the Blaschke-Santaló and the affine isoperimetric inequality, *Adv. in Math.*, accepted

[9] Böröczky, K. J., D. Hug, Stability of the reverse Blaschke-Santaló inequality for zonoids and applications, *Adv. in Appl. Math.*, accepted

[10] Böröczky, K. J., E. Makai, Jr., The volume product in the plane II, Upper estimates, the polygonal case, and stability, in preparation

[11] Bourgain, J., Milman, V., New volume ratio properties for convex symmetric bodies in R^n, *Invent. Math.* **88** (1987), 319-340, MR **88f:**52013

[12] Campi, S., Gronchi, P., On volume product inequalities for convex sets, *Proc. Amer. Math. Soc.* **134** (2006), 2393-2402, MR **2007a:**52010

[13] Dvoretzky, A., Rogers, C. A., Absolute and unconditional convergence in normed linear spaces, *Proc. Nat. Acad. Sci Wash.* **36** (1950), 192-197, MR **11,**525

[14] Eggleston, H. G., Note on a conjecture of Santaló, *Mathematika* **8** (1961), 63-65, MR **24#A**1070

[15] Florian, A., On the area sum of a convex set and its polar reciprocal, *Math. Pannonica* **7** (1996), 171-176, MR **97i:**52007

[16] Florian, A., A stability theorem for the area sum of a convex polygon and its polar reciprocal, *Math. Pannonica* **9** (1998), 3-15, MR **2000c:**52010

[17] Fradelizi, M., Meyer, M., Some functional forms of the Blaschke-Santaló inequality, *Math. Z.* **256** (2007), 379-395, MR **2008c:**52013

[18] Fradelizi, M., Meyer, M., Functional inequalities related to Mahler's conjecture, *Monatsh. Math.* accepted

[19] Gordon, Y., Meyer, M., Reisner, S., Zonoids with minimal volume product. A new proof, *Proc. Amer. Math. Soc.* **104** (1988), 273-276, MR **89i:**52015

[20] Gordon, Y., Reisner, S., Some aspects of volume estimates to various parameters in Banach spaces, In: *Proc. Res. Workshop on Banach space theory* (Iowa City, Iowa, 1981), (Ed. Bor-Luh Lin), Univ. Iowa, Iowa City, IA, 1982, 23-53, MR **85g:**46020

[21] Groemer, H., Stability of geometric inequalities, In: *Handbook of Convex Geometry* (eds. P. M. Gruber, J. M. Wills), North Holland, Amsterdam, 1993, 125-150, MR **94i:**52011

[22] Guggenheimer, H., Correction to "Polar reciprocal convex bodies", Isr. J. Math. 14 (1973), 309-316, *Isr. J. Math.,* **29** (1978), 312, MR **57,**7383

[23] Hug, D., Contributions to affine surface area, Manuscr. Math. *Manuscr. Math.*, **91** (1996), 283-301, MR **98d:**52009

[24] John, F., Extremum problems with inequalities as subsidiary conditions, In: *Studies and essays presented to R. Courant on his 60th birthday*, Interscience, New York, N. Y., 1948, 187-204, MR **10,**719

[25] Kim, J., Reisner, S., Local minimality of the volume-product at the simplex, manuscript, http://arxiv.org/pdf/1001.0217v1

[26] Kuperberg. G., A low-technology estimate in convex geometry, *Internat. Math. Res. Notices*, 1992 (9), 181-183, arXiv:math/9804023, MR **93h:**52010

[27] Kuperberg. G., From the Mahler conjecture to Gauss linking integrals, *Geom. Funct. Anal.* **18** (2008), 870-892, arXiv:math/0610.5904, MR **2009i:**52005

[28] Lopez, M. A., Reisner, S., A special case of Mahler's conjecture, *Discr. Comput. Geom.* **20** (1998), 163-177, MR **99h:**52009

[29] Lutwak, E., Selected affine isoperimetric inequalities, In: *Handbook of Convex Geometry* (Eds. P. M. Gruber, J. M. Wills), North Holland, Amsterdam etc. 1993, 151-176, MR **94h:**52014

[30] Mahler, K., Ein Minimalproblem für konvexe Polygone, *Mathematika B (Zutphen)* **7** (1938), 118-127, Zbl. **20.**50

[31] Mahler, K., Ein Übertragungsprinzip für konvexe Körper, *Časopis Pěst. Mat. Fys.* **68** (1939), 93-102, MR **1,**202

[32] Mahler, K, *Lectures on the geometry of numbers*, Univ. of Colorado, 1950

[33] Makai, Jr., E., On the thinnest non-separable lattice of convex bodies, *Stud. Sci. Math. Hungar.* **13** (1978), 19-27, MR **1983a:**52016

[34] Makai, Jr., E., Five-neighbour packing of convex plates, (Proc. Conf. Intuitive Geom., Siófok, 1985) *Coll. Math. Soc. J. Bolyai* **48**, *Intuitive Geom.* (Eds. K. Böröczky, G. Fejes Tóth), J. Bolyai Math. Soc., Elsevier, Amsterdam, 1987, 373-381, MR **88k:**52019

[35] Makai, Jr., E., Martini, H., The cross-section body, plane sections of convex bodies and approximation of convex bodies, I, *Geom. Dedicata* **63** (1996), 267-296, MR **99d:**52005

[36] Makai, Jr., E., Martini, H., Any non-transilluminable lattice of balls in $\mathbb{R}^4$ has a density at least $2\pi^2/256$, manuscript in preparation

[37] Meyer, M., Une caractérisation volumique de certains espaces normés de dimension finie, *Isr. J. Math.* **55** (1986), 317-326, MR **88f:**52017

[38] Meyer, M., Convex bodies with minimal volume product in $\mathbb{R}^2$, *Monatsh. Math.* **112** (1991), 297-301, MR **92k:**52015

[39] Meyer, M., Pajor, A., On Santaló's inequality, in *Geometric Aspects of Functional Analysis (1987-88),* eds. J. Lindenstrauss and V. D. Milman, Lect. Notes in Math. 1376, Springer, Berlin, 1989, 261-263 MR **90h:**52012

[40] Meyer, M., Pajor, A., On the Blaschke-Santaló inequality, *Arch. Math. (Basel)* **55** (1990), 82-93, MR **92b:**52013

[41] Meyer, M., Reisner, S., Inequalities involving integrals of polar-conjugate concave functions, *Mh. Math.* **125** (1998), 219-227, MR **99c:**52004

[42] Meyer, M., Reisner, S., Shadow systems and volumes of polar convex bodies, *Mathematika* **53** (2006), 129-148, arXiv:math/0606305, MR **2008d:**52010

[43] Nazarov, F., Petrov, F., Ryabogin, D., Zvavitch, A., A remark on the Mahler conjecture: local minimality of the unit cube, accepted in *Duke Math. J.*, http://arxiv.org/pdf/0905.0867vl

[44] Petty, C. M., Affine isoperimetric problems, *Ann. N. Y. Acad. Sci.* **440** (1985), 113-127, MR **87a:**52014

[45] Reisner, S., Random polytopes and the volume-product of symmetric convex bodies, *Math. Scand.* **57** (1985), 386-392, MR **87g:**52011

[46] Reisner, S., Zonoids with minimal volume product, *Math. Z.* **192** (1986), 339-346, MR **87g:**52022

[47] Reisner, S., Minimal volume-product in Banach spaces with a 1-unconditional basis, *J. London Math. Soc. (2)* **36** (1987), 126-136, MR **88h:**46029

[48] Rogers, C. A., Shephard, G. C., Convex bodies associated with a given convex body, *J. London Math. Soc.* **33** (1958), 270-281, MR **21#**318

[49] Saint Raymond, J., Sur le volume des corps convexes symétriques, *Sém. d'Initiation à l'Analyse 20e Année*, 1980-1981, Exp. **11** (Univ. Paris VI, Paris, 1981), MR **84j:**46033

[50] Santaló, L. A., An affine invariant for convex bodies of n-dimensional space (Spanish), *Portugal. Math.* **8** (1949), 155-161, MR **12,**526

[51] M. Schönert et al., *GAP — Groups, Algorithms and Programming*, Lehrstuhl D für Math., Rheinisch-Westfälische Hochschule, Aachen, 5th ed. (1995).

[52] Thompson, A. C. *Minkowski geometry*, Encycl. of Math. and its Appls., 63, Cambridge Univ. Press, Cambridge, 1996 MR **97f:**52001

[53] Thompson, A. C. Dualität, Volumenprodukt und Mahlersche Ungleichungen *Math. Semesterberichte* **54** (2007), 141-153, Zbl **1155.**52006

[54] Zhang, G. Y., Restricted chord projection and affine inequalities, *Geom. Dedicata* **39** (1991), 213-222, MR **92f:**52017