Typical faces of best approximating three-polytopes

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September 29, 2006

Dedicated to Rolf Schneider on his 65th birthday

Abstract

For a given convex body K in $\mathbb{R}^3$ with C^2 boundary, let P_n^i be an inscribed polytope of maximal volume with at most n vertices, and let $P_{(n)}^c$ be a circumscribed polytope of minimal volume with at most n faces. P.M. Gruber [12] proved that the typical faces of $P_{(n)}^c$ are asymptotically close to regular hexagons in a suitable sense if the Gauß–Kronecker curvature is positive on ∂K. In this paper we extend this result to the case if there is no restriction on the Gauß–Kronecker curvature, moreover we prove that the typical faces of P_n^i are asymptotically close to regular triangles in a suitable sense. In addition writing $P_{(n)}$ and P_n to denote the polytopes with at most n faces or n vertices, respectively, that minimize the symmetric difference metric from K, we prove the analogous statements about $P_{(n)}$ and P_n.

Key words: polytopal approximation, extremal problems
MSC 2000: 52A27, 52A40

1 Introduction

First we introduce some notions that will be used thorough the paper. For functions f and g of positive integers, we write $f(n) = O(g(n))$ if there exists an

[*]Supported by OTKA grants T 042769, 043520 and 049301, and by the EU Marie Curie TOK project DiscConvGeo

absolute constant c such that $|f(n)| \leq c \cdot g(n)$ for all $n \geq 1$, and $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. In addition we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

For a compact convex set C in $\mathbb{R}^3$, we write aff$C$ to denote its affine hull, $V(C)$ to denote its volume (Lebesgue measure), ∂C to denote its boundary and int$C$ to denote its interior. We call C a convex body if int$C \neq \emptyset$, and a convex disc if aff$C$ is a plane. If C is a convex disc then we write $|C|$ to denote its area, and relint$C$ to denote its relative interior. Next let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^3$, let $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x^2}$ be the Euclidean norm of $x \in \mathbb{R}^3$. We write o to denote the origin, and B^2 and B^3 to denote the Euclidean unit disc in $\mathbb{R}^2$ and unit ball in $\mathbb{R}^3$, respectively, centred at o; moreover S^2 to denote the boundary of B^3. For any objects $X_1, \ldots, X_k$, we write $[X_1, \ldots, X_k]$ to denote their convex hull. Concerning additional notions for convex bodies and polytopes in this paper, consult the beautiful monographs R. Schneider [20] and P.M. Gruber [15].

In this paper the distance of two convex bodies K and M in $\mathbb{R}^3$ is mostly measured by their symmetric difference metric $\delta_S(K,M)$; that is, the volume of the symmetric difference $K \Delta M$ of K and M.

Next we fix a convex body K in $\mathbb{R}^3$ with C^2 boundary for the rest of the section. We always integrate on ∂K with respect to the two–dimensional Hausdorff–measure. For any $x \in \partial K$, we write Q_x to denote the second fundamental form at x, hence Q_x is positive semi definite. Its two eigenvalues are the principal curvatures at x, whose product (the determinant of Q_x) is the Gauß–Kronecker curvature $\kappa(x)$ at x. Readily $\kappa(x) \geq 0$ for any $x \in \partial K$.

We define P_n to be a polytope with at most n vertices such that $\delta_S(K,P_n)$ is minimal, and $P_{(n)}$ to be a polytope with at most n faces such that $\delta_S(K,P_{(n)})$ is minimal. In addition let P_n^i be a polytope inscribed into K with at most n vertices and of maximal volume, and let $P_{(n)}^c$ be a polytope circumscribed around K with at most n faces and of minimal volume. A task initiated by L. Fejes Tóth [5] led to the asymptotic formulae

$$\delta_S(K,P_n^i) \sim \frac{1}{4\sqrt{3}} \left(\int_{\partial K} \kappa(x)^{1/4} dx \right)^2 \cdot \frac{1}{n}; \tag{1}$$

$$\delta_S(K,P_{(n)}^c) \sim \frac{5}{36\sqrt{3}} \left(\int_{\partial K} \kappa(x)^{1/4} dx \right)^2 \cdot \frac{1}{n}; \tag{2}$$

$$\delta_S(K,P_n) \sim \left(\frac{1}{12\sqrt{3}} - \frac{1}{16\pi} \right) \left(\int_{\partial K} \kappa(x)^{1/4} dx \right)^2 \cdot \frac{1}{n}; \tag{3}$$

$$\delta_S(K,P_{(n)}) \sim \left(\frac{5}{36\sqrt{3}}-\frac{1}{8\pi}\right)\left(\int_{\partial K}\kappa(x)^{1/4}dx\right)^2\cdot\frac{1}{n}. \qquad (4)$$

Under the assumption that $\kappa(x)>0$ for all $x\in\partial K$, (1) is due to P.M. Gruber [8], (2) is due to P.M. Gruber [9], moreover (3) and (4) follow from combining M. Ludwig [17] and K. Böröczky, Jr. and M. Ludwig [3]. The cases when possibly $\kappa(x)=0$ are due to K. Böröczky, Jr. [2].

The goal of this paper is to continue a task initiated by P.M. Gruber [11] and [12]; namely, to describe the typical faces of the extremal polytopes above. For $\varepsilon>0$ and convex discs C and M, we say that C is ε–close to M if there exist $x\in C$ and $y\in M$ with

$$(1+\varepsilon)^{-1}\cdot(C-x)\subset M-y\subset(1+\varepsilon)\cdot(C-x).$$

For any $x\in\partial K$, we write $u(x)$ to denote the exterior unit normal to ∂K at x. Let $\rho(x)\geq 0$ be a continuous function on ∂K such that $\rho(x)>0$ if $\kappa(x)>0$, and let M_n be a sequence of polytopes such that $o\in \mathrm{int}M_n$ for each n, and the number $f(n)$ of faces of M_n tends to infinity with n. A face F of M_n is called *proper* if there exits a unique point $x_F\in\partial K$ such that $u(x_F)$ is an exterior normal also to F, and in addition Q_{x_F} is positive definite. Given $k\geq 3$, we say that the typical faces of M_n are asymptotically regular k–gons with respect to the density function ρ if the following properties hold. There exists $\nu(n)>0$ with $\lim_{n\to\infty}\nu(n)=0$ such that for all but $\nu(n)$ percent of the faces F of M_n, F is a proper k–gon, and F is $\nu(n)$–close to some k–gon which is regular with respect to Q_{x_F} and is of area $\frac{\int_{\partial K}\rho(x)dx}{f(n)\cdot\rho(x_F)}$.

In Theorems 1.1 and 1.2, K is any convex body in $\mathbb{R}^3$ with C^2 boundary, and $P_{(n)}$, $P^c_{(n)}$, P_n, P^i_n are defined as above. If $\kappa(x)>0$ for all $x\in\partial K$ then Theorem 1.1 for $P^c_{(n)}$ is due to P.M. Gruber [12].

THEOREM 1.1 *The typical faces of both $P^c_{(n)}$ and $P_{(n)}$ are asymptotically regular hexagons with respect to the density function $\kappa(x)^{1/4}$.*

Remark: Both $P^c_{(n)}$ and $P_{(n)}$ have exactly n faces. In addition each face of $P^c_{(n)}$ touches K in its centroid, and if F is a face of $P_{(n)}$ then $|F\cap K|=\frac{1}{2}|F|$.

THEOREM 1.2 *The typical faces of both P^i_n and P_n are asymptotically regular triangles with respect to the density function $\kappa(x)^{1/4}$.*

Remark: Both P_n^i and P_n have $2n - o(n)$ faces, and each vertex of P_n^i lies in ∂K. In addition there exists $\mu(n) > 0$ with $\lim_{n\to\infty} \mu(n) = 0$ such that for all but $\mu(n)$ percent of the faces F of P_n, we have $\frac{1-\mu(n)}{2}\,|F| < |F \cap K| < \frac{1+\mu(n)}{2}\,|F|$.

Let us discuss some other results that follow from the methods of the proofs of Theorems 1.1 and 1.2. Given a convex body C in $\mathbb{R}^3$, its support function h_C is defined by $h_C(u) = \max_{x\in C}\langle x, u\rangle$ for $u \in \mathbb{R}^3$. If M is another convex body then the L_1–metric of C and M is

$$\delta_1(C,M) = \int_{S^2} |h_C(u) - h_M(u)|\,du.$$

In particular if $M \subset C$ then $\delta_1(C,M)$ is proportional to the difference of the mean widths of C and M. The paper S. Glasauer and P.M. Gruber [6] introduced an ingenious method to translate a result about polytopal approximation with respect to δ_S into a "dual" result with respect to δ_1. The paper [6] discussed only the case when $\kappa(x) > 0$ for all $x \in \partial K$ (see also M. Ludwig [17]), but this restriction is not necessary (see K. Böröczky, Jr. [2]). During the argument one takes the dual of some polytope. Therefore it is not enough to know the shape of a typical face but also its position with respect to ∂K in the case of volume approximation (see the Remarks above). In summary the analogues of Theorems 1.1 and 1.2 also hold if the extremal polytopes were not defined in terms of δ_S but in terms of δ_1, and the only difference is that the density function is $\kappa(x)^{3/4}$ in the case of δ_1. Actually if $\kappa(x) > 0$ for all $x \in \partial K$ then the statement about inscribed polytopes and the L_1–metric is due to P.M. Gruber [12].

Finally the Hausdorff metric $\delta_H(C,M)$ of two convex bodies C and M is the minimal d such that any point of C is of distance at most d from M, and any point of M is of distance at most d from C. Then the analogues of Theorems 1.1 and 1.2 also hold if the extremal polytopes were not defined in terms of δ_S but in terms of δ_H, and the only difference is that the density function is $\kappa(x)^{1/2}$ in the case of δ_H. If $\kappa(x) > 0$ for all $x \in \partial K$ then all the statements about the Hausdorff metric are due to P.M. Gruber [11].

Next we discuss uniform distribution of the faces of the extremal polytopes. We may assume that $o \in \text{int}K$, and we write $r_{\partial K}$ to denote radial projection onto ∂K. Let M_n be the extremal polytope with n vertices or n faces in any of the extremal problems above, let $\mathcal{F}_n$ denote the family of faces of M_n, and let $\rho(x)$ be the corresponding density function on ∂K. We say that the radial projection of the faces of M_n are uniformly distributed on ∂K with respect to $\rho(x)$ if for any Jordan measurable $X \subset \partial K$, we have

$$\frac{\int_X \rho(x)\,dx}{\int_{\partial K} \rho(x)\,dx} = \lim_{n\to\infty} \frac{\#\{F \in \mathcal{F}_n : r_{\partial K}(F) \subset X\}}{\#\mathcal{F}_n} \tag{5}$$
$$= \lim_{n\to\infty} \frac{\#\{F \in \mathcal{F}_n : r_{\partial K}(F) \cap X \neq \emptyset\}}{\#\mathcal{F}_n}.$$

This formula (in an analogous form) was proved first by S. Glasauer and R. Schneider [7] if the metric is δ_H and $\kappa(x) > 0$ for all $x \in \partial K$. The cases if the metric is δ_S or δ_1 and $\kappa(x) > 0$ for all $x \in \partial K$ are due to S. Glasauer and P.M. Gruber [6]. Finally the restriction that $\kappa(x) > 0$ for all $x \in \partial K$ was removed by K. Böröczky, Jr. [2]. We note that replacing $\mathcal{F}_n$ in (5) by the family $\mathcal{V}_n$ of the vertices of M_n, the resulting formula holds in all cases, as well.

Let us discuss the proofs of Theorems 1.1 and 1.2. Applying the method developed by P.M. Gruber [8] and [9], the proofs of the asymptotic formulae (1) to (4) are based on the moment theorem of L. Fejes Tóth [5] and its variants. Therefore stability versions of these statements lead to information on the typical faces of the extremal polytopes. For $P^c_{(n)}$ the original moment theorem of L. Fejes Tóth [5] forms the core of the proof. In this case P.M. Gruber [12] and G. Fejes Tóth [4] provided the necessary stability versions (see Section 3.1). Concerning the variants of the moment theorem used for P^i_n, P_n and $P_{(n)}$, the stability versions are proved in Section 3. We note that the error estimates are of optimal order in all stability statements in Section 3.

For P^i_n and P_n the proof of Theorem 1.2 is not substantially simpler if we assume that the Gauß-Kronecker curvature is positive everywhere on ∂K. The reason is that one only deals with triangular faces. However in the case of $P_{(n)}$ and $P^c_{(n)}$ it is essential that the average number of sides of the "typical faces" is at most six. If the Gauß-Kronecker curvature is positive then one can simply use that the statement holds for all faces of a three-polytope. Otherwise one needs a more careful approach. More precisely if a planar polygon is tiled by small enough polygons then the average number of sides of the tiles is at most six according to Lemma 4.1.

Detailed proof of the theorems is only presented in the case of $P_{(n)}$. For $P^c_{(n)}$, P^i_n and P_n, we only sketch the necessary changes in the argument.

2 A transfer lemma

As usual in polytopal approximation, we plan to transfer the original problem in $\mathbb{R}^3$ into a planar problem where certain integral expressions based on the second moment are investigated. A useful tool is the Taylor formula that we use in the following form: Let f be a convex C^2 function on a convex disc $\widetilde{C} \subset \mathbb{R}^2$ satisfying $o \in \text{relint}\widetilde{C}$. For $y \in \widetilde{C}$, we write l_y to denote the linear form representing the derivative of f at y, and q_y to denote the quadratic form representing the second derivative of f at y. Now if $a, y \in \widetilde{C}$ then there exists $t \in (0,1)$ satisfying

$$f(y) = f(a) + l_a(y-a) + \tfrac{1}{2}\, q_{a+t\,(y-a)}(y-a). \tag{6}$$

We write $p_{\mathbb{R}^2}$ to denote orthogonal projection into $\mathbb{R}^2$. Let C and C' be convex discs with $C' \subset$ relintC and $C \subset \text{relint}\widetilde{C}$. In addition let P be a polytope with $\widetilde{C} \subset p_{\mathbb{R}^2}(P)$, and let φ be the convex piecewise linear function defined on C whose graph is part of ∂P. We write $F_1, \ldots, F_k$ to denote the faces of P whose relative interiors intersect the graph of φ above C, and assume that relintF_i intersects the graph of φ above C' if and only if $i \leq k'$. Moreover we define

$$\Pi_i = C \cap p_{\mathbb{R}^2}(F_i),\ i = 1, \ldots, k.$$

We also assume that for any F_i, $i = 1, \ldots, k$, there exists an $a_i \in \widetilde{C}$ such that the exterior unit normal to F_i coincides with the exterior unit normal to the graph of f at $(a_i, f(a_i))$. In particular affF_i is the graph of the function $\varphi_i(y) = f(a_i) + l_{a_i}(y-a_i) + \alpha_i$ of $y \in \mathbb{R}^2$ for some $\alpha_i \in \mathbb{R}$. In addition the Taylor formula (6) yields the existence of a continuous function $g_i(y-a_i)$ of $y \in \widetilde{C}$ such that $f(y) = f(a_i) + l_{a_i}(y-a_i) + g_i(y-a_i)$, and for any $y \in \widetilde{C}$ there exists $z \in \widetilde{C}$ with $g_i(y-a_i) = \frac{1}{2}\, q_z(y-a_i)$. We observe that

$$g_i(y-a_i) - \alpha_i \leq g_j(y-a_j) - \alpha_j \quad \text{for } y \in \Pi_i,\ i, j = 1, \ldots, k. \tag{7}$$

Moreover the α_i satisfy the following conditions.

$$\text{If } K \subset P \text{ then} \qquad \alpha_i \leq 0 \qquad \text{for } i = 1, \ldots, k, \tag{8}$$

$$\text{if } P \subset K \text{ then} \quad g_{a_i}(y-a_i) \leq \alpha_i \quad \text{for } i = 1, \ldots, k \text{ and } y \in \Pi_i. \tag{9}$$

Let us assume that $l_o = 0$ and the graph of f is part of the boundary of a C^2 convex body K. In particular the second fundamental form is

$$Q_x = q_0 \quad \text{at } x = (o, f(o)) \in \partial K,$$

and we also assume that Q_x is positive definite. If $u(x) = (o,-1)$ is the unit exterior normal to ∂K at x, moreover $x_i = (a_i, f(a_i)) \in \partial K$ and $z_i = (a_i, \varphi_i(a_i)) \in \mathrm{aff}\, F_i$ for $i = 1,\ldots,k$ then

$$\alpha_i = \langle u(x), x_i - z_i\rangle,\ i = 1,\ldots,k. \tag{10}$$

Our goal is to investigate $\Omega = \{(1-t)f(y) + t\varphi(y) : y \in C \text{ and } t \in [0,1]\}$, which is the part of $K\Delta P$ near C, and satisfies

$$V(\Omega) = \sum_{i=1}^{k} \int_{\Pi_i} |g_i(y-a_i) - \alpha_i|\, dy. \tag{11}$$

LEMMA 2.1 *Let* $\varepsilon \in (0, 2^{-22})$. *Using the notation as above, let* $\alpha_i' = \alpha_i$ *if* $\alpha_i \le 0$, *and let* $\alpha_i \le \alpha_i' \le (1+\varepsilon)\alpha_i$ *if* $\alpha_i > 0$, $i = 1,\ldots,k$. *In addition we assume that*

$$(1+\varepsilon)^{-1} Q_x \le q_y \le (1+\varepsilon)\, Q_x \ \ \textit{for any } y \in \widetilde{C},$$

moreover if $y \in C$ *and* $g_i(y-a_i) \le \alpha_i$ *for* $i \le k'$ *then* $y \in \mathrm{relint}\, C$. *Then*

$$V(\Omega) \ge (1 - 2^{21}\varepsilon) \cdot \sum_{i=1}^{k'} \int_{\Pi_i} |\tfrac{1}{2} Q_x(y-a_i) - \alpha_i'|\, dy.$$

Proof: We may assume that $Q_x(z) = 2\langle z,z\rangle = 2z^2$. It follows by the Taylor formula (6) that for any $y \in C$ and $i = 1,\ldots,k$, we have

$$(1+\varepsilon)^{-1}(y-a_i)^2 \le g_i(y-a_i) \le (1+\varepsilon)(y-a_i)^2.$$

For $i = 1,\ldots,k'$, if $\alpha_i \le 0$ then we define $D_i = \emptyset$, and if $\alpha_i > 0$ then we define $r_i = \sqrt{\alpha_i}$ and

$$D_i = \left\{ y \in \mathbb{R}^2 : g_i(y-a_i) \le \tfrac{\alpha_i}{2} \right\}.$$

The conditions in the Lemma yield that if $\alpha_i > 0$ then

$$a_i + \tfrac{r_i}{2} B^2 \subset D_i \subset \mathrm{relint}\, C. \tag{12}$$

In addition let

$$C^* = \cup_{i=1}^{k'} (\Pi_i \cup D_i),$$

and for any $i = 1,\ldots,k'$, let

$$\Omega_i = \{ y \in C^* : \forall j = 1,\ldots,k',\ g_i(y-a_i) - \alpha_i \le g_j(y-a_j) - \alpha_j \},$$

hence $\Pi_i \subset \Omega_i$ according to (7). The core of the proof of Lemma 2.1 is to prove the estimates

$$\sum_{i=1,\dots,k'} \int_{\Omega_i} g_i(y-a_i)\,dy \leq 2^{20} \sum_{i=1}^{k'} \int_{\Omega_i} |g_i(y-a_i)-\alpha_i|\,dy; \tag{13}$$

$$\sum_{\substack{1\leq i\leq k' \\ \alpha_i>0}} \int_{\Omega_i} \alpha_i\,dy \leq 2^{20} \sum_{i=1}^{k'} \int_{\Omega_i} |g_i(y-a_i)-\alpha_i|\,dy. \tag{14}$$

If $\alpha_i \leq 0$ for all $i = 1,\dots,k'$ then (13) and (14) readily follow, therefore we assume that $\alpha_1 \geq \dots \geq \alpha_{k'}$, and $\alpha_m > 0$ for some $1 \leq m \leq k'$, moreover $\alpha_i \leq 0$ if $i > m$. For $i \leq m$, we define $D'_i = a_i + 2r_i B^2$ and $\widetilde{D}_i = a_i + 8r_i B^2$.

Next let $l_1 = 1$, and we define $1 = l_1 < \dots < l_{m'} \leq m$. If l_j is known and all D'_i, $i \leq m$, intersect at least one of $D'_{l_1},\dots,D'_{l_j}$ then let $j = m'$. Otherwise let l_{j+1} be the smallest index such that $D'_{l_{j+1}}$ does not intersect $D'_{l_1},\dots,D'_{l_j}$. It follows that

$$\bigcup_{i=1}^{m} D'_i \subset \bigcup_{j=1}^{m'} \widetilde{D}_{l_j}. \tag{15}$$

If $i = 1,\dots,m$ and $y \in \Omega_i \backslash D'_i$ then $\alpha_i \leq \frac{1}{2} g_i(y-a_i)$, hence

$$\begin{aligned} \alpha_i &\leq g_i(y-a_i) - \alpha_i; \\ g_i(y-a_i) &\leq 2 \cdot [g_i(y-a_i) - \alpha_i]. \end{aligned} \tag{16}$$

However if $y \in \Omega_i \cap D'_i$ then let j be the smallest index such that $y \in \widetilde{D}_{l_j}$, hence $l_j \leq i$. We deduce

$$\alpha_i \leq \alpha_{l_j}, \tag{17}$$

which fact combining with $g_i(y-a_i) - \alpha_i \leq g_{l_j}(y-a_{l_j}) - \alpha_{l_j}$ leads to

$$g_i(y-a_i) \leq g_i(y-a_i) - \alpha_i + \alpha_{l_j} \leq g_{l_j}(y-a_{l_j}) \leq 2 \cdot (y-a_{l_j})^2.$$

It follows by using (12) and (15) that

$$\begin{aligned} \sum_{i=1}^{m} \int_{\Omega_i \cap D'_i} g_i(y-a_i)\,dy &\leq \sum_{j=1}^{m'} \int_{\widetilde{D}_{l_j}} 2 \cdot (y-a_{l_j})^2\,dy \\ &\leq 2^{17} \sum_{j=1}^{m'} \int_{D_{l_j}} (y-a_{l_j})^2\,dy \end{aligned}$$

$$\begin{aligned} &\leq 2^{18}\sum_{j=1}^{m'}\int_{D_{l_j}} g_{l_j}(y-a_{l_j})\,dy \\ &\leq 2^{18}\sum_{j=1}^{m'}\int_{D_{l_j}} |g_{l_j}(y-a_{l_j})-\alpha_{l_j}|\,dy. \end{aligned}$$

In addition if $y \in D_{l_j} \cap \Omega_t$ for some $j = 1,\dots,m'$ and $t = 1,\dots,k'$ then we have $g_t(y-a_t)-\alpha_t \leq g_{l_j}(y-a_{l_j})-\alpha_{l_j} < 0$, hence

$$\sum_{i=1}^{m}\int_{\Omega_i\cap D_i'} g_i(y-a_i)\,dy \leq 2^{18}\sum_{t=1}^{k'}\int_{\Omega_t} |g_t(y-a_t)-\alpha_t|\,dy.$$

Therefore we deduce by (16) that

$$\sum_{i=1}^{m}\int_{\Omega_i} g_i(y-a_i)\,dy \leq 2^{19}\sum_{t=1}^{k'}\int_{\Omega_t} |g_t(y-a_t)-\alpha_t|\,dy,$$

which in turn yields (13). Turning to (14), we use the notation as above. It follows by (17) that

$$\sum_{i=1}^{m}\int_{\Omega_i\cap D_i'} \alpha_i\,dy \leq \sum_{j=1}^{m'}\int_{\widetilde{D}_{l_j}} \alpha_{l_j}\,dy \leq \sum_{j=1}^{m'}\int_{\widetilde{D}_{l_j}} 2\cdot(y-a_{l_j})^2\,dy,$$

hence the rest of the argument for (14) is similar to the proof (13).

Next we claim that if $y \in \Omega_i \backslash \Pi_i$ for $i = 1,\dots,k'$ and $y \in \Pi_j$ for $j = 1,\dots,k$ then

$$|g_i(y-a_i)-\alpha_i| \leq |g_j(y-a_j)-\alpha_j|. \tag{18}$$

To prove (18), we observe that $\alpha_i > 0$ and $y \in D_i$, hence $g_j(y-a_j)-\alpha_j \leq g_i(y-a_i)-\alpha_i < 0$. In turn we conclude (18).

Finally we define $\alpha_i^* = \max\{\alpha_i, 0\}$, we deduce by (13), (14) and (18) that

$$\begin{aligned} \sum_{i=1}^{k'}\int_{\Omega_i} |(y-a_i)^2-\alpha_i'|\,dy &\leq \sum_{i=1}^{k'}\int_{\Omega_i} |g_i(y-a_i)-\alpha_i|\,dy + \\ &\quad \varepsilon\cdot\sum_{i=1}^{k'}\int_{\Omega_i}\{g_i(y-a_i)+\alpha_i^*\}\,dy \\ &\leq (1+2^{21}\varepsilon)\cdot\sum_{i=1}^{k'}\int_{\Omega_i} |g_i(y-a_i)-\alpha_i|\,dy \\ &\leq (1+2^{21}\varepsilon)\cdot\sum_{i=1}^{k}\int_{\Pi_i} |g_i(y-a_i)-\alpha_i|\,dy. \end{aligned}$$

In turn we conclude Lemma 2.1 by $\Pi_i \subset \Omega_i$ for $i = 1, \dots, k'$. $\square$

3 Some extremal properties of regular polygons

The discussion in Section 2 shows that the symmetric difference metric can be estimated from below by sums of integrals of the form $\int_\Pi |q(y) - \alpha|\, dy$ where Π is a k–gon, $\alpha \in \mathbb{R}$ and q is a positive definite quadratic form. It has been known that given k, q and $|\Pi|$, if the integral above is minimal then Π is regular with respect to q. In this section we prove stability versions of this property if $k \le 6$.

First we present some auxiliary statements that will be useful in the proofs of Lemmae 3.3, 3.8 and 3.13, moreover later in the proofs of Theorems 1.1 and 1.2. We will need that certain type of functions are concave or monotonic:

PROPOSITION 3.1 *Let* $f(t) = \tan t + \frac{\omega}{\tan t}$ *for given* $\omega \in [\frac{1}{3}, 3]$.

(i) $f(t)^{-1}$ *is concave on* $(0, \frac{\pi}{2})$, *and* $(f(t)^{-1})'' < -0.03$ *if* $t \in (\frac{\pi}{7}, \frac{5\pi}{12})$;

(ii) $t \cdot f(t)$ *is increasing on* $(0, \frac{\pi}{2})$, *and* $(t \cdot f(t))' > 0.07$ *if* $t \in (\frac{\pi}{7}, \frac{\pi}{2})$.

Proof: If $t \in (0, \frac{\pi}{2})$ then

$$(f(t)^{-1})'' = -(\tan^2 t \cdot (3\omega - 1) + 3\omega - \omega^2) \cdot \frac{2(\tan t)(1 + \tan^2 t)}{(\omega + \tan^2 t)^3} < 0,$$

hence $f(t)^{-1}$ is concave. In addition the function $\tan^2 t \cdot (3\omega - 1) + 3\omega - \omega^2$ is concave in ω for fixed t, thus it attains its minimum at $\omega = \frac{1}{3}$ or at $\omega = 3$. Therefore if $t \in (\frac{\pi}{7}, \frac{5\pi}{12})$ then

$$(f(t)^{-1})'' \le -\frac{\min\{8 \tan^2 \frac{\pi}{7}, \frac{8}{9}\} \cdot 2(\tan^2 \frac{\pi}{7})(1 + \tan^2 \frac{\pi}{7})}{(3 + \tan^2 \frac{5\pi}{12})} < -0.03.$$

Turning to (ii), let $t \in (0, \frac{\pi}{2})$. It follows by $\tan t > t + \frac{1}{3} t^3$ that

$$\begin{aligned} (t \cdot f(t))' &= t + \tan t + t \cdot \tan^2 t + \omega \left(\frac{1}{\tan t} - \frac{t}{\tan^2 t} - t \right) \\ &\ge t + \tan t + t \cdot \tan^2 t + \omega \left(\frac{t^3}{3 \tan^2 t} - t \right). \end{aligned}$$

Thus $\omega = 3$ can be assumed, hence $x + \frac{1}{x^2} - 2 > 1 - x$ for $x > 1$ yields

$$(t \cdot f(t))' \geq t \cdot \left(\frac{\tan t}{t} + \frac{t^2}{\tan^2 t} - 2 \right) + t \cdot \tan^2 t \geq t - \tan t + t \cdot \tan^2 t.$$

Since $t - \tan t + t \cdot \tan^2 t$ is a strictly increasing function of $t \in [0, \frac{\pi}{2})$, we have $(t \cdot f(t))' > 0$ for $t \in (0, \frac{\pi}{2})$, and even $(t \cdot f(t))' > 0.07$ for $t \in (\frac{\pi}{7}, \frac{\pi}{2})$. $\square$

If f is a C^2 function on (a,b) and $t, t_0 \in (a,b)$ then the Taylor formula says that

$$f(t) = f(t_0) + f'(t_0) \cdot (t - t_0) + \tfrac{1}{2} f''(t_0 + s(t - t_0)) \cdot (t - t_0)^2 \tag{19}$$

where $s \in (0,1)$. The Taylor formula yields simple stability properties of the quadratic function and concave functions. We state these properties in the form how we intend to use them. First if $\frac{t_1 + \ldots + t_n}{n} = t_0$ and the number of t_i with $|t_i - t_0| \geq \varepsilon$ is m for $\varepsilon > 0$ then

$$\frac{t_1^2 + \ldots + t_n^2}{n} \geq t_0^2 + \tfrac{m}{n} \cdot \varepsilon^2. \tag{20}$$

Secondly we have the following property of concave functions:

PROPOSITION 3.2 *Let $\omega > 0$, and let f be a concave function on $[a,b]$ satisfying $f''(t) \leq -\omega$ for all $t \in [a,b]$ with $|t - t_0| < \varepsilon_0$ for $t_0 \in (a,b)$ and $\varepsilon_0 > 0$. If $t_0 = \frac{t_1 + \ldots + t_n}{n}$ for $t_1, \ldots, t_n \in [a,b]$, and the number of t_i with $|t_i - t_0| \geq \varepsilon$ is m for $\varepsilon \in (0, \varepsilon_0)$ then*

$$\frac{f(t_1) + \ldots + f(t_n)}{n} \leq f(t_0) - \tfrac{\omega}{2} \cdot \tfrac{m}{n} \cdot \varepsilon^2.$$

We will also use the following consequence of Cauchy–Schwartz inequality: If $\gamma_i, A_i > 0$ for $i = 1, \ldots, m$ then

$$\sum_{i=1}^{m} \gamma_i A_i^2 \geq \left(\sum_{i=1}^{m} \frac{1}{\gamma_i} \right)^{-1} \left(\sum_{i=1}^{m} A_i \right)^2. \tag{21}$$

Finally we introduce a notation that will be used thorough Section 3. For $t \in (0, \frac{\pi}{2})$, let $R(t)$ be the triangle with a right angle such that o is a vertex, the angle at o is t, and the longest side is of length one.

3.1 Properties related to circumscribed polytopes

Most of the results of this section are hidden in P.M. Gruber [12] or in G. Fejes Tóth [4]. Still we provide proofs because the statements are not stated exactly as we need. For $t \in (0, \frac{\pi}{2})$, we define

$$\gamma(t) = \frac{\int_{R(t)} x^2\, dx}{|R(t)|^2} = \frac{1}{\tan t} + \frac{\tan t}{3}. \tag{22}$$

In particular

$$\frac{\gamma(\frac{\pi}{6})}{12} = \frac{5}{18\sqrt{3}}. \tag{23}$$

We note that (24) in Lemma 3.3 is due to L. Fejes Tóth (see say [5]).

LEMMA 3.3 *If q is a positive definite quadratic form on* $\mathbb{R}^2$, $\alpha \leq 0$ *and* Π *is a polygon of at most k sides then*

$$\int_\Pi \{q(x) - \alpha\}\, dx \geq \frac{\gamma(\frac{\pi}{k})}{2k} \cdot |\Pi|^2 \sqrt{\det q}. \tag{24}$$

If $k \leq 6$ *and* $\int_\Pi \{q(x) - \alpha\}\, dx \leq \frac{(1+\varepsilon)\gamma(\frac{\pi}{k})}{2k} |\Pi|^2 \sqrt{\det q}$ *for* $\varepsilon \in (0, \varepsilon_0)$ *then* Π *is a k–gon, and there exists some k–gon* Π_0 *that is regular with respect to q, has o as its centroid, and satisfies*

$$(1 + \vartheta\sqrt{\varepsilon})^{-1}\Pi_0 \subset \Pi \subset (1 + \vartheta\sqrt{\varepsilon})\Pi_0$$

where ε_0 *and* ϑ *are positive absolute constants.*

To prove Lemma 3.3, we need four simple auxiliary statements. The first two; namely, Propositions 3.4 and 3.5 are consequences of Proposition 3.1.

PROPOSITION 3.4 $t\gamma(t)$ *is increasing on* $(0, \frac{\pi}{2})$, *and* $(t\gamma(t))' > 0.07$ *for* $t \in (\frac{\pi}{7}, \frac{\pi}{2})$.

PROPOSITION 3.5 $\gamma(t)^{-1}$ *is concave on* $(0, \frac{\pi}{2})$, *and* $(\gamma(t)^{-1})'' < -0.03$ *for* $t \in (\frac{\pi}{7}, \frac{5\pi}{12})$.

PROPOSITION 3.6 *If T is a triangle that has an angle t at the vertex o for* $t \in (0, \pi/2)$, *and T has an obtuse angle then*

$$\int_T x^2\, dx \geq \gamma(t) \cdot |T|^2.$$

Proof: We may assume that $|T| = |R(t)|$, and T is positioned in a way such that T and $R(t)$ share their angle t at o, and their longest sides are collinear. Since in this case all points of $R(t)\backslash T$ are closer to o than any point of $T\backslash R(t)$, we conclude Proposition 3.6. $\square$

PROPOSITION 3.7 *If* Π *is a convex disc with* $o \notin \mathrm{relint}\,\Pi$*, and* $k \geq 3$ *then*

$$\int_\Pi x^2\,dx \geq 1.1 \cdot \frac{\gamma^c(\frac{\pi}{k})}{2k} \cdot |\Pi|^2.$$

Proof: Since there exists a half plane containing Π such that o lies on the boundary of the half plane, we may assume that Π is a semi circular disc centred at o. In this case direct calculations and Proposition 3.4 yield

$$\int_\Pi x^2\,dx \geq 1.1 \cdot \frac{\gamma^c(\frac{\pi}{3})}{6} \cdot |\Pi|^2 \geq 1.1 \cdot \frac{\gamma^c(\frac{\pi}{k})}{2k} \cdot |\Pi|^2. \quad \square$$

Proof of Lemma 3.3: We may assume that $q(z) = z^2$. Let Π be a polygon with at most k sides. We may assume $o \in \mathrm{relint}\,\Pi$ according to Proposition 3.7. We dissect Π into triangles. We consider all non-degenerate triangles of the from $[o,v,w]$ where v is the closest point of some side e of Π to o, and w is an endpoint of e. We write $R_1,\dots,R_l$ to denote these triangles, hence $R_1,\dots,R_l$ tile Π. It follows that the angle s_i of R_i at o is acute, and R_i has an angle which is at least $\frac{\pi}{2}$, $i = 1,\dots,l$. Naturally $l \leq 2k$, and in addition $l \geq 5$ because all s_i are acute. We deduce

$$\int_\Pi x^2 \geq \sum_{i=1}^{l} \gamma^c(s_i)|R_i|^2 \geq \left(\sum_{i=1}^{l} \frac{1}{\gamma^c(s_i)}\right)^{-1} \left(\sum_{i=1}^{l} |R_i|\right)^2 \geq \frac{\gamma^c(\frac{2\pi}{l})}{l} \cdot |\Pi|^2 \geq \frac{\gamma^c(\frac{\pi}{k})}{2k} \cdot |\Pi|^2$$

by Propositions 3.4, 3.5 and 3.6, moreover by the Cauchy–Schwartz inequality (21). Therefore let $k \leq 6$, and let $\int_\Pi x^2\,dx \leq \frac{(1+\varepsilon)\gamma^c(\frac{\pi}{k})}{2k} \cdot |\Pi|^2$. It follows by Proposition 3.4 that if ε_0 is small enough then $l = 2k$. In particular each R_i has a right angle at a vertex that is not the vertex of Π. Combining Propositions 3.2 and 3.5 yields that $|s_i - \frac{\pi}{k}| \leq \tau\sqrt{\varepsilon}$ for $i = 1,\dots,2k$ where $\tau > 0$ is an absolute constant. In turn we conclude Lemma 3.3. $\square$

3.2 Properties related to inscribed polytopes

For $t \in (0, \frac{\pi}{2})$, we define

$$\gamma^i(t) = \frac{\int_{R(t)} \{1 - x^2\}\, dx}{|R(t)|^2} = \frac{1}{\tan t} + \frac{5\tan t}{3}. \tag{25}$$

In particular

$$\frac{\gamma^i(\frac{\pi}{3})}{6} = \frac{1}{\sqrt{3}}. \tag{26}$$

We note that a restricted version of (27) in Lemma 3.8 is due to P.M. Gruber [8].

LEMMA 3.8 *If q is a positive definite quadratic form on $\mathbb{R}^2$, $\alpha > 0$ and Π is a triangle such that $q(x) \le \alpha$ for $x \in \Pi$ then*

$$\int_\Pi \{\alpha - q(x)\}\, dx \ge \frac{\gamma^i(\frac{\pi}{3})}{6} \cdot |\Pi|^2 \sqrt{\det q}. \tag{27}$$

If $\int_\Pi \{\alpha - q(x)\}\, dx \le \frac{(1+\varepsilon)\gamma^i(\frac{\pi}{3})}{6} |\Pi|^2 \sqrt{\det q}$ for $\varepsilon \in (0, \varepsilon_0)$ then there exists some triangle Π_0 that is regular with respect to q, has o as its centroid, and satisfies

$$(1 + \vartheta\sqrt{\varepsilon})^{-1} \Pi_0 \subset \Pi \subset (1 + \vartheta\sqrt{\varepsilon}) \cdot \Pi_0$$

where ε_0 and ϑ are positive absolute constants.

Let us prove the analogues of Propositions 3.4 to 3.7. Propositions 3.9 and 3.10 are consequences of Proposition 3.1.

PROPOSITION 3.9 *$t\gamma^i(t)$ is increasing on $(0, \frac{\pi}{2})$, and $(t\gamma^i(t))' > 0.07$ for $t \in (\frac{\pi}{7}, \frac{\pi}{2})$.*

PROPOSITION 3.10 *$\gamma^i(t)^{-1}$ is concave on $(0, \frac{\pi}{2})$, and $(\gamma^i(t)^{-1})" < -0.03$ for $t \in (\frac{\pi}{7}, \frac{5\pi}{12})$.*

The following statement is more general then the direct analogue of Proposition 3.6 because of applications in Proposition 3.12.

PROPOSITION 3.11 *Let $T \subset rB^2$ be a triangle, which has an angle t at the vertex o for $t \in (0, \pi/2)$, and has another angle that is at least $\frac{\pi}{2}$. If $\Pi \subset T$ is a convex disc then*

$$\int_\Pi \{r^2 - x^2\}\, dx \ge \gamma^i(t) |\Pi|^2.$$

Proof: We may assume that $r = 1$ and $T = R(t)$. Let $R(t) = [o,a,b]$ where $R(t)$ has a right angle at a, hence $\|b\| = 1$. We define $\mathcal{Q}$ to be the family of convex discs $Q \subset R(t)$ with $|Q| \geq |\Pi|$. There exists some $Q_0 \in \mathcal{Q}$ satisfying that $\frac{\int_{Q_0}\{1-x^2\}\,dx}{|Q_0|^2}$ is minimal, and Proposition 3.11 follows if

$$\frac{\int_{Q_0}\{1-x^2\}\,dx}{|Q_0|^2} \geq \frac{\int_{R(t)}\{1-x^2\}\,dx}{|R(t)|^2}. \tag{28}$$

We may assume that $Q_0 \neq R(t)$.

In the proof of (28), we will use that if $C_1, C_2 \in \mathcal{Q}$ with $|C_1| = |C_2|$ then

$$\int_{C_1}\{1-x^2\}\,dx \leq \int_{C_2}\{1-x^2\}\,dx \text{ if and only if } \int_{C_1} x^2\,dx \geq \int_{C_2} x^2\,dx. \tag{29}$$

Our main method for transforming elements of $\mathcal{Q}$ is the so-called Blaschke-Schüttelung (see T. Bonnesen and W. Fenchel [1]). Let us given a line l, a vector u not parallel to l, and a convex disc C that lies on one side of l. Then applying the Blaschke–Schüttelung parallel to u and with respect to l to C leads to some convex disc C' as follows. We translate any secant σ of C parallel to u into a segment σ', which intersects l in an endpoint, and lies on the same side of l where C lies. We define C' to be the union of all such σ'. Readily $|C'| = |C|$. In addition if

$$\max_{x\in\sigma'} \|x\| \geq \max_{x\in\sigma} \|x\| \tag{30}$$

holds for any secant σ of C then

$$\int_{C'} x^2\,dx \geq \int_C x^2\,dx, \tag{31}$$

with strict inequality if strict inequality holds in (30) for at least one secant σ.

After applying Blaschke–Schüttelung first parallel to a with respect to aff$\{a,b\}$, then parallel to $b-a$ with respect to aff$\{o,b\}$, we may assume the following by (31): There exist $\tilde{a} \in [a,b]$ and $\tilde{b} \in [o,b]$ such that $Q_0 \cap [a,b] = [\tilde{a},b]$ and $Q_0 \cap [o,b] = [\tilde{b},b]$, moreover the lines through $\tilde{a}$ and $\tilde{b}$ parallel to a and $b-a$, respectively, are supporting lines of Q_0.

We suppose that $\tilde{a} \neq a$, and seek a contradiction. Let $c \in [o,b]$ satisfy that $\tilde{a} - c$ is parallel to a, hence $c \neq o$. Since $\langle x-b, c\rangle < 0$ for $x \in Q_0 \backslash \{b\}$, we have $(b-c)^2 - (x-c)^2 < b^2 - x^2 = 1 - x^2$, thus

$$\int_{Q_0}\{(a-c)^2 - (x-c)^2\}\,dx < \int_{Q_0}\{1-x^2\}\,dx.$$

Now $\widetilde{Q} \in \mathcal{Q}$ for $\widetilde{Q} = b + \frac{1}{\|b-c\|}(Q_0 - b)$, and

$$\frac{\int_{\widetilde{Q}} \{1-x^2\}\,dx}{|\widetilde{Q}|^2} = \frac{\int_{Q_0} \{(a-c)^2-(x-c)^2\}\,dx}{|Q_0|^2} < \frac{\int_{Q_0} \{1-x^2\}\,dx}{|Q_0|^2}.$$

It is absurd, therefore $\tilde{a} = a$.

Next we define $a_0 \in [o,a]$ and $b_0 \in [o,a]$ by the properties that $\|a_0\| = \|b_0\|$ and the segment $[a_0, b_0]$ touches Q_0. After applying Blaschke–Schüttelung parallel to $a_0 - b_0$ with respect to $\mathrm{aff}\{o,b\}$, we may assume $b_0 = \tilde{b} \in Q_0$. We suppose that $a_0 \notin Q_0$, and seek a contradiction. We define $a' \in [o,a]$ by $Q_0 \cap [o,a] = [a',a]$, and $b' \in [o,b]$ by $\|b'\| = \|a'\|$. In addition we choose $c' \in [b_0, b']$ with $c' \neq b_0, b'$. The line $\mathrm{aff}\{a',c'\}$ dissects Q_0 into two convex discs, the polygon M containing b, and the convex disc N containing b_0. Let N' be the image of N by the Blaschke–Schüttelung parallel to $a'-c'$ with respect to $\mathrm{aff}\{o,a\}$. Then $Q' = M \cup N' \in \mathcal{Q}$, $|Q'| = |Q_0|$ and

$$\int_{Q'} x^2\,dx = \int_M x^2\,dx + \int_{N'} x^2\,dx > \int_M x^2\,dx + \int_N x^2\,dx = \int_{Q_0} x^2\,dx,$$

that is absurd. Therefore $a_0 \in Q_0$, which in turn yields $Q_0 = [a, a_0, b_0, b]$.

Let s be the area of the isosceles triangle $[o, a_0, b_0]$, hence

$$\frac{\int_{Q_0} \{1-x^2\}\,dx}{|Q_0|^2} = \frac{|R(t)| - \gamma^c(t)|R(t)|^2 - s + \frac{\gamma^c(t/2)}{2}\cdot s^2}{(|R(t)|-s)^2}.$$

As t is fixed, we write $f(s)$ to denote the right hand side above as a function of s, which function satisfies

$$f'(s) = \frac{\{1-|R(t)|\cdot\gamma^c(t/2)\}(|R(t)|-s) - \{2\gamma^c(t)-\gamma^c(t/2)\}\cdot|R(t)|^2}{(|R(t)|-s)^3}.$$

Now $s \leq \frac{\|a\|^2 \sin t}{2} = |R(t)|\cos t$ yields $|R(t)| - s \geq (1-\cos t)|R(t)|$, moreover elementary calculations and using the formula (22) for γ^c lead to

$$\left\{1-|R(t)|\gamma^c(\tfrac{t}{2})\right\}(1-\cos t) - \left\{2\gamma^c(t)-\gamma^c(\tfrac{t}{2})\right\}\cdot|R(t)| = (1-\tfrac{2}{3}\sin^2\tfrac{t}{2})(1-\cos t)^2.$$

Since $2\gamma^c(t) - \gamma^c(t/2) \geq 0$ according to Proposition 3.4, it follows that $f'(s) > 0$ for all $s \leq |R(t)|\cos t$. We conclude (28), and in turn Proposition 3.11. $\square$

Finally we present the analogue of Proposition 3.7. Unfortunately this fact is not as trivial as Proposition 3.7 because Proposition 3.12 does not hold for any convex disc as Π; for example, if Π is a semi circular disc with centre o and radius one then $\int_\Pi \{1-x^2\}\,dx < \frac{\gamma^i(\frac{\pi}{3})}{6}\cdot|\Pi|^2$.

PROPOSITION 3.12 *If* $\Pi\subset rB^2$ *is a triangle with* $o\notin \mathrm{relint}\,\Pi$ *then*

$$\int_\Pi \{r^2-x^2\}\,dx \geq 1.1\cdot\frac{\gamma^i(\frac{\pi}{3})}{6}\cdot|\Pi|^2.$$

Proof: We say that a side e of Π is a dark side if $o\notin e$ and e is a common side of Π and $[o,\Pi]$. We consider all non-degenerate triangles of the from $[o,v,w]$ where v is the closest point of some dark side e of Π to o, and w is an endpoint of e. Let $R_1,\ldots,R_l$ be the resulting triangles, hence $\Pi\cap R_j$, $j=1,\ldots,l$, form a tiling of Π. We observe that $l\leq 4$, moreover if $j=1,\ldots,l$ then the angle s_j of R_j at o is acute, and R_j has an angle that is at least $\frac{\pi}{2}$. Writing $s^*=\frac{s_1+\ldots+s_l}{l}$, it follows by (25), (26), Proposition 3.11 and by the Cauchy–Schwartz inequality (21) that

$$\begin{aligned}\int_\Pi \{r^2-x^2\}\,dx &\geq \sum_{j=1}^{l}\gamma^i(s_j)\cdot|R_j\cap\Pi|^2 \geq \left(\sum_{j=1}^{l}\gamma^i(s_j)^{-1}\right)^{-1}\left(\sum_{j=1}^{l}|R_j\cap\Pi|\right)^2\\ &\geq \frac{\gamma^i(s^*)}{l}\cdot|\Pi|^2 \geq \frac{2\sqrt{\frac{5}{3}}}{4}\cdot|\Pi|^2 > 1.1\cdot\frac{\gamma^i(\frac{\pi}{3})}{6}\cdot|\Pi|^2. \quad \square\end{aligned}$$

Based on Propositions 3.9 to 3.12, Lemma 3.8 can be proved analogously to Lemma 3.3. $\square$

3.3 Properties related to general polytopes

This section builds on K. Böröczky, Jr. and M. Ludwig [3]. For $t\in(0,\frac{\pi}{2})$, we define

$$\gamma(t)=\frac{\min_{\alpha\in\mathbb{R}}\int_{R(t)}|x^2-\alpha|\,dx}{|R(t)|^2}.$$

According to K. Böröczky, Jr. and M. Ludwig [3],

$$\gamma(t)=\frac{1}{\tan t}+\frac{\tan t}{3}-\frac{1}{2t}\ \text{ if } t\in(0,1.05], \tag{32}$$

where $\frac{\pi}{3} < 1.05 < \frac{\pi}{2}$ and $\tan 1.05 < 2 \cdot 1.05$. Therefore

$$\frac{\gamma(\frac{\pi}{3})}{6} = \frac{1}{3\sqrt{3}} - \frac{1}{4\pi}; \tag{33}$$

$$\frac{\gamma(\frac{\pi}{6})}{12} = \frac{5}{18\sqrt{3}} - \frac{1}{4\pi}. \tag{34}$$

The estimate (35) in Lemma 3.13 is a restatement of Theorem 3 in K. Böröczky, Jr. and M. Ludwig [3]. We note that the proof Lemma 3.13 is more complicated than the proof of Lemma 3.3 because instead of Proposition 3.6, we have Proposition 3.16.

LEMMA 3.13 *There exist absolute constants $\varepsilon_0, \vartheta > 0$ with the following properties: If q is a positive definite quadratic form on $\mathbb{R}^2$, $\alpha \in \mathbb{R}$ and Π is a polygon of at most k sides then*

$$\int_\Pi |q(x) - \alpha|\, dx \geq \frac{\gamma(\frac{\pi}{k})}{2k} \cdot |\Pi|^2 \sqrt{\det q}. \tag{35}$$

In addition if $k \leq 6$ and $\int_\Pi |q(x) - \alpha|\, dx \leq \frac{(1+\varepsilon)\gamma(\frac{\pi}{k})}{2k} |\Pi|^2 \sqrt{\det q}$ for $\varepsilon \in (0, \varepsilon_0)$ then Π is a k–gon, and there exists some k–gon Π_0 that is regular with respect to q, has o as its centroid, and satisfies

$$(1 + \vartheta\sqrt{\varepsilon})^{-1}\Pi_0 \subset \Pi \subset (1 + \vartheta\sqrt{\varepsilon}) \cdot \Pi_0;$$

$$\tfrac{1-\vartheta\sqrt{\varepsilon}}{2} |\Pi| < |\{x \in \Pi : q(x) \leq \alpha\}| < \tfrac{1+\vartheta\sqrt{\varepsilon}}{2} |\Pi|.$$

To prove Lemma 3.13, we need several auxiliary statements. Proposition 3.1 yields directly Proposition 3.14.

PROPOSITION 3.14 *$t\gamma(t)$ is increasing on $(0, 1.05)$, and if $t \in (\frac{\pi}{7}, 1.05)$ then $(t\gamma(t))' > 0.07$.*

Let us recall some results of [3]. We note that there exists a unique $t^* \in (1.05, \frac{\pi}{2})$ such that $\tan t^* = 2t^*$. Lemma 4 of [3] states that $\gamma(t)^{-1}$ is concave on $(0, t^*)$. Its proof actually verifies that $(\gamma(t)^{-1})$" is continuous and negative on $(0, t^*)$. Next let $l(t)$ be the linear function whose graph is tangent to the graph of $\gamma(t)^{-1}$ at $\frac{\pi}{3}$. Lemma 5 of [3] states that $\gamma(t)^{-1} < l(t)$ for $t \in (\frac{\pi}{3}, \frac{\pi}{2})$. We deduce

PROPOSITION 3.15 *There exists a concave function $\theta(t) \geq \gamma(t)^{-1}$ on $(0,\pi/2)$ such that $\theta(t) = \gamma(t)^{-1}$ for $t \in (0, \frac{\pi}{3}]$. In addition $(\gamma(t)^{-1})'' < -\xi$ for $t \in (\frac{\pi}{7}, 1.05)$ where $\xi > 0$ is an absolute constant.*

Remark: Since the resulting $\theta(t)$ is linear if $t \geq \frac{\pi}{3}$, we cannot apply Proposition 3.2 if $\sum_{i=1}^{6} t_i = 2\pi$ for acute $t_1, \ldots, t_6$. In this case the Taylor formula (19) yields

$$\sum_{i=1}^{6} \theta(t_i) \leq \left(\sum_{i=1}^{6} l(t_i) \right) - \frac{\xi}{2} \left(\frac{\pi}{3} - \min_{i=1,\ldots,6} t_i \right)^2 \leq 6\,\theta(\tfrac{\pi}{6}) - \frac{\xi}{50} \max_{i=1,\ldots,6} (\tfrac{\pi}{3} - t_i)^2. \quad (36)$$

Next we restate Lemma 3 of [3].

PROPOSITION 3.16 *If $\alpha \in \mathbb{R}$ and T is a triangle that has an angle $2t$ at the vertex o for $t \in (0,\pi/2)$ then*

$$\int_T |x^2 - \alpha|\, dx \geq \tfrac{\gamma(t)}{2} \cdot |T|^2.$$

Finally combining Lemma 2 in [3] and Proposition 3.14 leads to

PROPOSITION 3.17 *If $\alpha \in \mathbb{R}$, Π is a polygon with at most k sides, and $o \notin$ $\mathrm{relint}\,\Pi$ then*

$$\int_\Pi |x^2 - \alpha|\, dx \geq 1.1 \cdot \frac{\gamma(\frac{\pi}{3})}{6} \cdot |\Pi|^2 \geq 1.1 \cdot \frac{\gamma(\frac{\pi}{k})}{2k} \cdot |\Pi|^2.$$

Proof of Lemma 3.13: We may assume that $q(z) = z^2$. Since (35) coincides with Theorem 3 in [3], we assume that the m–gon Π for $m \leq k \leq 6$ and $\alpha \in \mathbb{R}$ satisfy $\int_\Pi |x^2 - \alpha|\, dx \leq \frac{(1+\varepsilon)\gamma(\frac{\pi}{k})}{2k} \cdot |\Pi|^2$. If ε_0 is small enough then $o \in \mathrm{relint}\,\Pi$ according to Proposition 3.17.

We dissect Π into the triangles $T_1, \ldots, T_m$ by connecting o to the vertices of Π, and write e_i to denote the side of T_i opposite to o. Next we assign two triangles R_{i1} and R_{i2} to each T_i. If both angles of T_i at the endpoints of e_i are acute then let w_i be the closest point of e_i to o, and let R_{i1} and R_{i2} be the two triangles, which tile T_i and intersect in the common side $[o, w_i]$. In this case both R_{i1} and R_{i2} have a right angle at w_i, and we write t_{ij} to denote the angle of R_{ij} at o, $j = 1, 2$. Otherwise we call T_i skew, and let $t_{i1} = t_{i2}$ be half of the angle of T_i at o, moreover let R_{ij} be a rescaled copy of $R(t_{ij})$ with $|R_{ij}| = \frac{1}{2}|T_i|$ for $j = 1, 2$. In both cases $t_{i1} + t_{i2}$ is the angle of T_i at o, $i=1,\dots,m$. We apply Proposition 3.16 to all skew T_i, and deduce by Proposition 3.15 and the Cauchy–Schwartz inequality (21) that

$$\frac{(1+\varepsilon)\gamma(\frac{\pi}{k})}{2k}\cdot|\Pi|^2 \geq \sum_{\substack{i=1,\dots,m\\ j=1,2}}\gamma(t_{ij})|R_{ij}|^2 \tag{37}$$

$$\geq \left(\sum_{\substack{i=1,\dots,m\\ j=1,2}}\frac{1}{\gamma(t_{ij})}\right)^{-1}\left(\sum_{\substack{i=1,\dots,m\\ j=1,2}}|R_{ij}|\right)^2$$

$$\geq \left(\sum_{\substack{i=1,\dots,m\\ j=1,2}}\theta(t_{ij})\right)^{-1}|\Pi|^2 \geq (2m\cdot\theta(\tfrac{\pi}{m}))^{-1}|\Pi|^2. \tag{38}$$

It follows by Proposition 3.14 that $m=k$ if ε_0 is small enough.

During the rest of the argument, we write $\vartheta_1,\vartheta_2,\dots$ to denote positive absolute constants. We apply Propositions 3.2 and 3.15 if $k\geq 4$, and (36) if $k=3$ to (38), and obtain

$$|t_{ij}-\tfrac{\pi}{k}|\leq\vartheta_1\sqrt{\varepsilon}\ \text{ for } i=1,\dots,k \text{ and } j=1,2. \tag{39}$$

If no T_i is skew then (39) readily yields the existence of Π_0 in Lemma 3.13.

Therefore we suppose that there is a skew T_l for suitably small ε_0, and seek a contradiction. We deduce by (39) that $\gamma(t_{ij})\geq(1-\vartheta_2\sqrt{\varepsilon})\gamma(\frac{\pi}{k})$, thus (37) and $2(|R_{i1}|^2+|R_{i2}|^2)\geq|T_i|^2$ yield that $k\sum_{i=1}^k|T_i|^2\leq(1+\vartheta_3\sqrt{\varepsilon})\left(\sum_{i=1}^k|T_i|\right)^2$. Using the convexity of t^2 (compare (20)), we obtain

$$1-\vartheta_4\sqrt[4]{\varepsilon}\leq\frac{|T_i|}{|\Pi|/k}\leq 1+\vartheta_4\sqrt[4]{\varepsilon}\ \text{ for } i=1,\dots,k. \tag{40}$$

Let $v\neq o$ be the vertex of T_l where the angle α_l of T_l is at least $\frac{\pi}{2}$, and let T_p the other triangle that has v as a vertex. If α_p is the angle of T_p at v then combining (39) and (40) yields that $|\alpha_l-\alpha_p|\leq\vartheta_5\sqrt[4]{\varepsilon}$. It follows by $\alpha_l\geq\frac{\pi}{2}$ that $\alpha_p\geq\frac{5\pi}{12}$ if ε_0 is small enough, moreover $\alpha_p<\frac{\pi}{2}$ by the convexity of Π. In addition the angle of T_p at o is at least $\frac{\pi}{4}$ by $k\leq 6$, hence the third angle of T_p is at most $\frac{\pi}{3}$. Therefore T_p is not skew, and $|t_{p1}-t_{p2}|\geq\alpha_p-\frac{\pi}{3}\geq\frac{\pi}{12}$. It contradicts (39) for suitably small ε_0, thus no $T_1,\dots,T_k$ is skew. In turn we conclude the existence of suitable Π_0.

Finally we define $\Pi_\alpha^+=\{x\in\Pi: x^2\geq\alpha\}$ and $\Pi_\alpha^-=\{x\in\Pi: x^2\leq\alpha\}$, hence the formula

$$\frac{\partial}{\partial\alpha}\int_\Pi|x^2-\alpha|\,dx=|\Pi_\alpha^-|-|\Pi_\alpha^+|$$

completes the proof of Lemma 3.13. □

4 The proof of Theorem 1.1

We only prove Theorem 1.1 for $P_{(n)}$ in detail, and sketch the necessary changes for the case of $P^c_{(n)}$ at the end of the proof. For $P_{(n)}$, it is sufficient to prove the following statement.

For a given convex body K in $\mathbb{R}^3$ with C^2 boundary, let $P_{(n)}$ be a polytope with at most n faces such that $\delta_S(K,P_{(n)})$ is minimal. For $\nu \in (0,\nu_0)$, if $g(n)$ is number of faces F of $P_{(n)}$ such that F is a proper hexagon, and F is $\vartheta\nu$–close to some hexagon that is regular with respect to Q_{x_F} and is of area $\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{n\cdot\kappa(x_F)^{1/4}}$ then

$$g(n) > (1-\tilde{\vartheta}\nu)n \quad \text{for } n > n_0 \tag{41}$$

where ϑ and $\tilde{\vartheta}$ are positive absolute constants, and $\nu_0 > 0$ depends on K, moreover n_0 depends on ν and K.

We recall that for any $x \in \partial K$, $u(x)$ is the exterior unit normal to ∂K at x. It is well–known (see say K. Leichtweiß [16]) that there exists $\eta > 0$ such that balls of radius η roll from inside on ∂K. In other words for any $x \in \partial K$, the three–ball of radius η and of centre $x - \eta\, u(x)$ is contained in K. Let $K_{-\eta}$ be the family of points z such that $z + \eta B^3 \subset K$. Now if $y \in \mathbb{R}^3 \backslash K_{-\eta}$ then there exists a unique closest point of ∂K to y, and we write $\pi(y)$ to denote this point.

We write $\mathrm{cl}\, Y$ to denote the closure of any $Y \subset \mathbb{R}^3$, and consider ∂K with the subspace topology as a subset of $\mathbb{R}^3$. We say that $Y \subset \partial K$ is Jordan measurable if the relative boundary of Y on ∂K is of two–dimensional Hausdorff–measure zero. Let X_0, X' and X be relatively open Jordan measurable subsets of ∂K such that $\mathrm{cl}\, X_0 \subset X$, $\mathrm{cl}\, X \subset X'$, $\kappa(x) > 0$ for $x \in \mathrm{cl}\, X'$, and

$$\int_{X_0} \kappa(x)^{1/4} dx \geq (1-\mu\nu^2) \int_{\partial K} \kappa(x)^{1/4} dx.$$

It is practical to define

$$\mu = \nu^6.$$

We have $\delta > 0$ with the following properties: $(X_0 + 2\delta B^3) \cap \partial K \subset X$ and $(X + 2\delta B^3) \cap \partial K \subset X'$. Moreover if C is a convex disc that touches K in $x \in X$ and C is of diameter at most δ then

(i) writing C' to denote the orthogonal projection of $\pi(C)$ into aff$C$, we have

$$x+(1-\mu\nu^2)(C-x)\subset C'\subset x+(1-\mu\nu^2)^{-1}(C-x);$$

(ii) if $w\in\pi(C)$ then $\langle u(w),u(x)\rangle\geq 1-\mu\nu^2$;

(iii) if f is the convex function on C such that its graph is the part of ∂K, and q_y is the quadratic form representing the second derivative of f at $y\in C$ (hence $Q_x=q_x$) then

$$(1+\mu\nu^2)^{-1}Q_x\leq q_y\leq(1+\mu\nu^2)Q_x.$$

During the proof of (41), $\vartheta_1,\vartheta_2,\ldots$ denote positive absolute constants, moreover $\omega_1,\omega_2,\ldots$ denote positive constants that depend on K, ν and μ. Now there exists a convex polytope M circumscribed around K such that diam$G<\delta$ holds for each face G of M with $\pi(G)\cap X\neq\emptyset$. We write $\mathcal{M}$ to denote the family of faces of M that touch K in a point of X, and let $G\in\mathcal{M}$ touch K in x_G. Therefore

$$\sum_{G\in\mathcal{M}}\kappa(x_G)^{1/4}|G|\geq(1-\vartheta_1\mu\nu^2)\int_{\partial K}\kappa(x)^{1/4}dx. \tag{42}$$

We start to investigate $P_{(n)}$. We define

$$\tilde{\gamma}=\frac{5}{36\sqrt{3}}-\frac{1}{8\pi}=\frac{1}{2}\cdot\frac{\gamma(\frac{\pi}{6})}{12}.$$

According to (4), if n is large then

$$\delta_S(K,P_{(n)})<(1+\mu\nu^2)\cdot\tilde{\gamma}\cdot\left(\int_{\partial K}\kappa(x)^{1/4}dx\right)^2\cdot\frac{1}{n}. \tag{43}$$

It follows by (43) and the existence of the rolling ball of radius η that

$$\delta_H(K,P_{(n)})\leq\omega_1 n^{-1/2}. \tag{44}$$

Therefore if n_0 is large enough then $K_{-\eta}\subset\mathrm{int}P_{(n)}$. Since the infimum of the principal curvatures at the points of X' is positive, we deduce that if F is a face of $P_{(n)}$ such that $\pi(F)\subset X'$ then

$$\mathrm{diam}F\leq\omega_2 n^{-1/4}. \tag{45}$$

Recalling that $G \in \mathcal{M}$ touches K in x_G, we write $\widetilde{\mathcal{M}}$ to denote the family of convex discs of the form

$$(1-2\mu\nu^2)(G-x_G)+x_G$$

as G runs through the elements of $\mathcal{M}$. In turn for $C \in \widetilde{\mathcal{M}}$, we write x_C to denote the point where C touches K, and define

$$C' = (1-\mu\nu^2)(C-x_C)+x_C.$$

In addition let $\mathcal{F}_C$ denote the family of faces of $P_{(n)}$ near C whose orthogonal projection to affC intersects relint$C$. We deduce by (i) and (45) that if n_0 is large enough then the families $\mathcal{F}_C$ for $C \in \widetilde{\mathcal{M}}$ are pairwise disjoint, and by (42) that

$$\sum_{C\in\widetilde{\mathcal{M}}} \kappa(x_C)^{1/4}|C'| \geq (1-\vartheta_2\mu\nu^2)\int_{\partial K}\kappa(x)^{1/4}dx. \tag{46}$$

For any plane L in $\mathbb{R}^3$, we write p_L to denote the orthogonal projection into L. Let $C \in \widetilde{\mathcal{M}}$. We write $\mathcal{F}'_C$ to denote the family of all $F \in \mathcal{F}_C$ such that $p_{\mathrm{aff}\,C}(F)$ intersects relintC'. Again if n_0 is large enough then (44) yields for any $F \in \mathcal{F}'_C$ that

$$p_{\mathrm{aff}\,C}(K\cap \mathrm{aff}\,F) \subset \mathrm{relint}\,C. \tag{47}$$

We recall that for any $F \in \mathcal{F}_C$, x_F denotes the point of ∂K such that $u(x_F)$ is an exterior unit normal to F, and write $a_F = p_{\mathrm{aff}\,C}(x_F)$. In addition let $z_F \in \mathrm{aff}F$ satisfy $p_{\mathrm{aff}\,C}(z_F) = a_F$, and let $\alpha_F = \langle u(x_C), x_F - z_F\rangle$. For any $F \in \mathcal{F}'_C$, we define

$$\Pi_F = C' \cap p_{\mathrm{aff}\,C}(F). \tag{48}$$

It follows by (iii) and (47) that we may apply Lemma 2.1 to each $C \in \widetilde{\mathcal{M}}$ with $\varepsilon = \mu\nu^2$, and we obtain (see also (10))

$$\delta_S(K,P_{(n)}) \geq (1-\vartheta_3\mu\nu^2)\sum_{C\in\widetilde{\mathcal{M}}}\sum_{F\in\mathcal{F}'_C}\int_{\Pi_F}|\tfrac{1}{2}Q_{x_C}(y-a_F)-\alpha_F|\,dy. \tag{49}$$

For any $F \in \mathcal{F}'_C$, we define $k(F)$ to be the number of sides of Π_F, and

$$I(F) = \kappa(x_C)^{1/4}|\Pi_F|. \tag{50}$$

Next we decompose $\cup_{C\in\widetilde{\mathcal{M}}}\mathcal{F}'_C$ into the families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$. Let $F\in\mathcal{F}'_C$ for $C\in\widetilde{\mathcal{M}}$. We put F into $\mathcal{F}_4$ if $k(F)\neq 6$, and into $\mathcal{F}_3$ if Π_F is a hexagon that is not ν–close to any hexagon that is regular with respect Q_{x_C}. Therefore $F\in\mathcal{F}_1\cup\mathcal{F}_2$ if Π_F is a hexagon, which is ν–close to some hexagon that is regular with respect Q_{x_C}. Assuming this, we have

$$F\in\mathcal{F}_1 \quad \text{if} \quad \left|\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{n\cdot I(F)}-1\right|\leq\nu;$$
$$F\in\mathcal{F}_2 \quad \text{if} \quad \left|\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{n\cdot I(F)}-1\right|>\nu.$$

We write n_j to denote the cardinality of $\mathcal{F}_j$. Using (43) and (46) to get an upper bound on $\delta_S(K,P_{(n)})$, and (49) and Lemma 3.13 to get a lower bound on $\delta_S(K,P_{(n)})$, we obtain

$$\begin{aligned}(1+\vartheta_4\mu\nu^2)\cdot\tilde{\gamma}\cdot\left(\sum_{F\in\cup_{j=1}^4\mathcal{F}_j}I(F)\right)^2\frac{1}{n} \;\geq\; & \tilde{\gamma}\cdot\left(\sum_{F\in\cup_{j=1}^3\mathcal{F}_j}I(F)^2\right)\\ & +\vartheta_5\nu^2\left(\sum_{F\in\mathcal{F}_3}I(F)^2\right)\\ & +\frac{1}{2}\sum_{F\in\mathcal{F}_4}\frac{\gamma(\frac{\pi}{k(F)})}{2k(F)}\cdot I(F)^2.\end{aligned} \tag{51}$$

We claim that last term above satisfies

$$\frac{1}{2}\sum_{F\in\mathcal{F}_4}\frac{\gamma(\frac{\pi}{k(F)})}{2k(F)}\cdot I(F)^2\geq(1+\vartheta_6)\cdot\tilde{\gamma}\cdot\left(\sum_{F\in\mathcal{F}_4}I(F)\right)^2\cdot\frac{1}{n_4}. \tag{52}$$

It follows by the Cauchy–Schwartz inequality (21) that

$$\begin{aligned}\sum_{F\in\mathcal{F}_4}\frac{\gamma(\frac{\pi}{k(F)})}{2k(F)}\cdot I(F)^2 \;=\;& \sum_{F\in\mathcal{F}_4}2k(F)\cdot\gamma(\tfrac{\pi}{k(F)})\cdot\left(\frac{I(F)}{2k(F)}\right)^2\\ \geq\;& \left(\sum_{F\in\mathcal{F}_4}2k(F)\cdot\gamma(\tfrac{\pi}{k(F)})^{-1}\right)^{-1}\cdot\left(\sum_{F\in\mathcal{F}_4}I(F)\right)^2.\end{aligned}$$

Since C' is tiled by Π_F as F runs through $\mathcal{F}'_C$, and all tiles have small diameter for large n according to (45), the average number of sides of all Π_F, $F \in \mathcal{F}'_C$, is at most six (see Lemma 4.1 below). In particular the average of all $k(F)$, $F \in \mathcal{F}_4$, is at most six. If the average is at least 5.5 then we use Proposition 3.2 to the concave $\gamma(t)^{-1}$ (compare Proposition 3.15), and after that use the monotonicity of $t\gamma(t)$ (compare Proposition 3.14) to obtain

$$\begin{aligned}\left(\sum_{F\in\mathcal{F}_4} 2k(F)\cdot\gamma(\tfrac{\pi}{k(F)})^{-1}\right)^{-1} &\geq \frac{1+\vartheta_7}{n_4}\cdot\frac{n_4}{\sum_{F\in\mathcal{F}_4}2k(F)}\cdot\gamma\left(\tfrac{n_4 2\pi}{2\sum_{F\in\mathcal{F}_4}k(F)}\right)\\ &\geq \frac{1+\vartheta_7}{n_4}\cdot\frac{\gamma(\frac{\pi}{6})}{12}=\frac{1+\vartheta_7}{n_4}\cdot 2\tilde{\gamma}.\end{aligned}$$

If the average of all $k(F)$, $F \in \mathcal{F}_4$, is less than 5.5 then first we simply use the concavity of $\gamma(t)^{-1}$, and after that we obtain an error term from the monotonicity of $t\gamma(t)$ (compare Proposition 3.14, and observe that the average of all $k(F)$ is at least three):

$$\begin{aligned}\left(\sum_{F\in\mathcal{F}_4} 2k(F)\cdot\gamma(\tfrac{\pi}{k(F)})^{-1}\right)^{-1} &\geq \frac{1}{n_4}\cdot\frac{n_4}{\sum_{F\in\mathcal{F}_4}2k(F)}\cdot\gamma\left(\tfrac{n_4\pi}{\sum_{F\in\mathcal{F}_4}k(F)}\right)\\ &\geq \frac{1+\vartheta_8}{n_4}\cdot\frac{\gamma(\frac{\pi}{6})}{12}=\frac{1+\vartheta_8}{n_4}\cdot 2\tilde{\gamma}.\end{aligned}$$

In turn we deduce the claim (52).

Now by applying the inequality for quadratic mean to $\sum_{F\in\mathcal{F}_3} I(F)^2$, we deduce

$$\begin{aligned}(1+\vartheta_4\mu\nu^2)\cdot\tilde{\gamma}\left(\sum_{F\in\cup_{j=1}^4\mathcal{F}_j} I(F)\right)^2\cdot\frac{1}{n} \geq\ & \tilde{\gamma}\cdot\left(\sum_{F\in\cup_{j=1}^3\mathcal{F}_j} I(F)^2\right)\\ &+\vartheta_5\nu^2\left(\sum_{F\in\mathcal{F}_3} I(F)\right)^2\cdot\frac{1}{n_3}\\ &+(1+\vartheta_6)\cdot\tilde{\gamma}\cdot\left(\sum_{F\in\mathcal{F}_4} I(F)\right)^2\cdot\frac{1}{n_4}.\end{aligned} \tag{53}$$

First we show that the contribution coming from faces in $\mathcal{F}_3$ and $\mathcal{F}_4$ is negligible. Applying the inequality for quadratic mean and the Cauchy–Schwartz inequality

(21) in (53) leads to

$$(1+\vartheta_4\mu\nu^2)\cdot\tilde{\gamma}\left(\sum_{F\in\cup_{j=1}^4\mathcal{F}_j} I(F)\right)^2\cdot\frac{1}{n} \;\;\geq\;\; \tilde{\gamma}\cdot\left(\sum_{F\in\cup_{j=1}^4\mathcal{F}_j} I(F)\right)^2\cdot\frac{1}{\sum_{j=1}^4 n_j}$$
$$+\vartheta_9\nu^2\left(\sum_{F\in\mathcal{F}_3\cup\mathcal{F}_4} I(F)\right)^2\frac{1}{n_3+n_4}.$$

Since $\sum_{j=1}^4 n_j \leq n$, it follows that

$$\sum_{F\in\mathcal{F}_3\cup\mathcal{F}_4} I(F) \leq \vartheta_{10}\sqrt{\mu}\cdot\sum_{F\in\cup_{j=1}^4\mathcal{F}_j} I(F). \tag{54}$$

Thus (46) and (53) yield

$$\left(\sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} I(F)\right)^2\cdot\frac{1}{n} \;\;\geq\;\; (1-O(\sqrt{\mu}))\left(\sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} I(F)^2\right); \tag{55}$$

$$\sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} I(F) \;\;=\;\; (1+O(\sqrt{\mu}))\cdot\int_{\partial K}\kappa(x)^{1/4}dx. \tag{56}$$

Applying the inequality for quadratic mean in (55) leads to $n_1+n_2=(1+O(\sqrt{\mu}))n$, hence (56) shows that

$$I_0=\frac{\sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} I(F)}{n_1+n_2}=(1+O(\sqrt{\mu}))\cdot\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{n}.$$

Therefore if ν_0 after (41) is small enough then we apply (20) with $t_0=I_0$ and $\varepsilon=\frac{\nu\cdot\int_{\partial K}\kappa(x)^{1/4}dx}{2n}$ to obtain

$$\sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} I(F)^2 \geq \left(1+\frac{\vartheta_{11}\nu^2 n_2}{n_1+n_2}\right)\left(\sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} I(F)\right)^2\cdot\frac{1}{n_1+n_2}.$$

Comparing to (55) leads to $\frac{n_2}{n_1+n_2}=O(\frac{\sqrt{\mu}}{\nu^2})=O(\nu)$, hence

$$n_1 \geq (1-\vartheta_{12}\nu)\cdot n. \tag{57}$$

We are not ready because some Π_F is not the projection of F. We call $F\in\mathcal{F}_1$ a border face if assuming $F\in\mathcal{F}_C$, $C\in\widetilde{\mathcal{M}}$, Π_F meets the relative boundary of C'.

Otherwise we call $F \in \mathcal{F}_1$ an inner face; namely, if $\Pi_F \subset \mathrm{relint} C'$. We observe that if F is an inner face then Π_F is the projection of F, hence F is $\vartheta_{13}\nu$–close to some hexagon that is regular with respect to the positive definite Q_F, and is of area $\frac{\int_{\partial K} \kappa(x)^{1/4} dx}{n \cdot \kappa(x_F)^{1/4}}$. However if F is a border face and $F \in \mathcal{F}_C$ then Π_F lies in a $\omega_3 n^{-1/2}$ neighbourhood of the relative boundary of C' in aff$C$. Since any border face F is in $\mathcal{F}_1$, we have $|\Pi_F| > \frac{\omega_4}{n}$, therefore the number of border faces is at most $\omega_5\sqrt{n}$. After choosing n_0 large enough, the number of border faces is less than $\nu \cdot n$, hence $g(n) \geq (1 - \vartheta_{14}\nu) \cdot n$. Therefore we conclude (41), and in turn Theorem 1.1 in the case of $P_{(n)}$. □

To prove Theorem 1.1 in the case of $P^c_{(n)}$, only two changes are needed in the argument. First all α_F in (49) satisfy $\alpha_F \leq 0$ (compare (8)). Secondly we use Lemma 3.3 instead of Lemma 3.13. □

No face can be added to $P_{(n)}$ or $P^c_{(n)}$, and no face of $P_{(n)}$ or $P^c_{(n)}$ can be varied in a way such that $\delta_S(P_{(n)}, K)$ or $\delta_S(P^c_{(n)}, K)$, respectively, decreases, hence we deduce the Remark after Theorem 1.1. □

In the proof of Theorem 1.1, we used the fact that the average number of sides of the tiles of a suitable tiling is at most six.

LEMMA 4.1 *For any convex polygon Π there exists $\delta > 0$ with the following property: If the convex polygons $\Pi_1, \ldots, \Pi_n$ form a side to side tiling of Π, and each Π_i is of diameter at most δ then writing k_i to denote the number of sides of Π_i, we have $k_1 + \ldots + k_n < 6n$.*

Proof: We write m to denote the number of sides of Π, and p to denote the perimeter of Π. Let $\delta = \frac{p}{2m}$. If e is the number of edges, and v is the number of vertices in the tiling $\Pi_1, \ldots, \Pi_n$ of Π as above then the Euler formula says

$$v - e + n = 1 > 0. \tag{58}$$

Since at least two edges of the tiling meet at any vertex of Π, and at least three edges of the tiling meet at any other vertices of the tiling, summing up the degrees of the vertices of the tiling leads to $3v \leq 2e + m$. It follows by (58) that $e < 3n + m$. In addition let b be the number of segments that are sides of some Π_i and are contained in $\partial\Pi$, hence $b \geq 2m$ by the choice of δ. Therefore

$$k_1 + \ldots + k_n = 2e - b < 6n + 2m - b \leq 6n. \quad \square$$

5 The proof of Theorem 1.2

Since the proof of Theorem 1.2 is very similar to the proof of Theorem 1.1, we only provide a sketch about the necessary changes.

We start with the case of P_n, which has at most $2n$ faces according to Euler the formula. The main changes compared to (41) are that now $g(n)$ counts the number of triangular faces, which are close to regular in the suitable sense, and we prove

$$g(n) > (1-\tilde{\vartheta}\nu)2n. \tag{59}$$

We define X, M and $\tilde{\mathcal{M}}$ as in Section 4. Instead of $\tilde{\gamma}$, we use

$$\gamma^* = \frac{1}{12\sqrt{3}} - \frac{1}{16\pi} = \frac{1}{4} \cdot \frac{\gamma(\frac{\pi}{3})}{6}.$$

Here we have the factor $\frac{1}{4}$ unlike the factor $\frac{1}{2}$ in the definition of $\tilde{\gamma}$ because P_n has asymptotically twice as many faces as $P_{(n)}$.

An essential change in the argument that first we triangulate ∂P_n by triangulating any non–triangular face by diagonals from a fixed vertex of the face. We write Σ to denote the resulting triangular complex, which has the same family of vertices as P_n. For $C \in \tilde{\mathcal{M}}$, we write $\mathcal{F}_C$ to denote the family of all faces F of Σ that lies near C and $p_{\mathrm{aff}C}(F)$ intersects relint$C$, moreover $\mathcal{F}_C'$ to denote the family of all $F \in \mathcal{F}_C$ such that $p_{\mathrm{aff}C}(F)$ intersects relintC'. For any $F \in \mathcal{F}_C$, we define $\Pi_F = p_{\mathrm{aff}C}(F)$ (hence we do not intersect with C' as in (48)). In addition, we define a_F, α_F and $I(F)$ analogously as in Section 4.

Other changes compared to the argument in Section 4 are concerned with the definitions of $\mathcal{F}_j$ after (50). We decompose $\cup_{C\in\widetilde{\mathcal{M}}}\mathcal{F}_C'$ into only three families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$. Let $F \in \mathcal{F}_C'$ for $C \in \widetilde{\mathcal{M}}$. We put F into $\mathcal{F}_1$ if $\left|\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{2n\cdot I(F)} - 1\right| \leq \nu$, and there exists a triangle T whose centroid is a_F, which is regular with respect Q_{x_C}, and

$$(1+\nu)^{-1}(T-a_F) \subset \Pi_F - a_F \subset (1+\nu)(T-a_F).$$

We put F into $\mathcal{F}_2$ if such a T exists but $\left|\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{2n\cdot I(F)} - 1\right| > \nu$. Finally $F \in \mathcal{F}_3$ if no such T exists. As in Section 4, let n_i denote the cardinality of $\mathcal{F}_i$.

We deduce the analogue of (51) without the last term concerning $\mathcal{F}_4$, which yields right away the analogue (53). Continuing with essentially the same argument as in Section 4 (keeping only $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$) proves the analogue of (57); namely, $n_1 \geq (1-\vartheta^*\nu)\cdot 2n$ where ϑ^* is a positive absolute constant. We are not ready because P_n may have some faces that not triangles. If G is a face of P_n such that $F \subset G$ for $F \in \mathcal{F}_1$ then G is near some $C \in \widetilde{\mathcal{M}}$, and $p_{\mathrm{aff}\,C}(x(G)) = a_F \in \mathrm{relint}\Pi_F$. In particular there is no other $F' \in \mathcal{F}_1$ with $F' \subset G$. Since Σ has at most $2n$ elements, the number of $F \in \mathcal{F}_1$ that are not faces of P_n is at most $\vartheta^*\nu \cdot 2n$, therefore $g(n) \geq (1-2\vartheta^*\nu)\cdot 2n$. $\square$

The proof in the case of P_n^i runs closely as for P_n, the main difference is that one uses Lemma 3.8 instead of Lemma 3.13. There is one additional change in the argument. For each $F \in \mathcal{F}'_C$, we define

$$\alpha'_F = (1+\mu\nu^2)\cdot\alpha_F.$$

Therefore $\frac{1}{2}\,Q_{x_C}(y-a_F) \leq \alpha'_F$ for any $y \in \Pi_F$ (see (9)), and (49) is replaced by

$$\delta_S(K,P_n^i) \geq (1-\vartheta_3\mu\nu^2)\sum_{C\in\widetilde{\mathcal{M}}}\sum_{F\in\mathcal{F}'_C}\int_{\Pi_F}\left\{\alpha'_F - \tfrac{1}{2}\,Q_{x_C}(y-a_F)\right\}dy.$$

The arguments just sketched complete the proof of Theorem 1.2. $\square$

Concerning the Remark after Theorem 1.2, both P_n and P_n^i have at most $2n-4$ faces according to the Euler formula, hence the numbers of faces of both P_n and P_n^i are $2n-o(n)$ by (59). Readily all vertices of P_n^i lie on ∂K. To prove the property of the typical faces of P_n, we force the following extra condition on any element $F \in \mathcal{F}'_C$ of $\mathcal{F}_1$ or $\mathcal{F}_2$. Any such F should satisfy

$$\tfrac{1-\nu}{2}\,|\Pi_F| < \left|\{y \in \Pi_F : \tfrac{1}{2}\,Q_{x_C}(y-a_F) \leq \alpha_F\}\right| < \tfrac{1+\nu}{2}\,|\Pi_F|. \quad \square$$

Acknowledgement: We are grateful to Ákos G. Horváth for helpful observations, to Endre Makai for pointing out various errors in an earlier version, and to an unknown referee whose remarks considerably improved the paper.

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