Volume approximation of smooth convex bodies by three-polytopes of restricted number of edges

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Dedicated to Peter M. Gruber on his 65th birthday

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Abstract

For a given convex body K in $\mathbb{R}^3$ with C^2 boundary, let P_n^c be the circumscribed polytope of minimal volume with at most n edges, and let P_n^i be the inscribed polytope of maximal volume with at most n edges. Besides presenting an asymptotic formula for the volume difference as n tends to infinity in both cases, we prove that the typical faces of P_n^c and P_n^i are asymptotically regular triangles and squares, respectively, in a suitable sense.

1 Introduction

Let K be a convex body in $\mathbb{R}^d$ with C^2 boundary. For $x \in \partial K$, we write Q_x to denote the second fundamental form at x, and $\kappa(x)$ to denote the Gauß-Kronecker curvature $\det Q_x$ (see Section 2). We always integrate on ∂K with respect to the $(d-1)$-dimensional Hausdorff-measure. In addition for functions f and g of positive integers, we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

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For an integer $0 \leq k \leq d-1$, best approximation of K by polytopes of restricted number of k–faces has been intensively investigated in the last thirty years (see P.M. Gruber [15], [18], [19] and [20]). To indicate typical results, we write $\delta_S(K,M)$ to denote the symmetric difference metric of K and another convex body M in $\mathbb{R}^d$; that is, the volume of the symmetric difference $K\Delta M$ of K and M. Let P_n be a polytope minimizing $\delta_S(K,P_n)$ under the condition that the number of k–faces is at most n. If $k = 0, d-1$; namely, if the number of vertices or facets is restricted then the work of many people like L. Fejes Tóth, R. Vitale, R. Schneider, P.M. Gruber, S. Glasauer, M. Ludwig, K.J. Böröczky leads to the asymptotic formula

$$\delta_S(K,P_n) \sim c \cdot \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, dx \right)^{\frac{d+1}{d-1}} \cdot n^{\frac{-2}{d-1}} \tag{1}$$

as $n \to \infty$ where $c > 0$ depends only on k and d. It has been also shown that the projections of the vertices or facets, respectively, of P_n onto ∂K are uniformly distributed on ∂K with respect to the density function $\kappa(x)^{\frac{1}{d+1}}$. Moreover if $d = 3$ and $k = 0, 3$ then even the asymptotic shape of the typical faces of P_n is known. More precisely the typical faces are asymptotically regular hexagons or regular triangles in a suitable sense if the number of faces or vertices, respectively, is restricted. Finally we note that the analogues of all these results are known if P_n is assumed to be either inscribed or circumscribed.

However no asymptotic formula was known if $1 \leq k \leq d-2$. A partial result follows from combining the papers I. Bárány [2] and K.J. Böröczky [5]: If n is large then

$$c_1 \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, dx \right)^{\frac{d+1}{d-1}} n^{\frac{-2}{d-1}} < \delta_S(K,P_n) < c_2 \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, dx \right)^{\frac{d+1}{d-1}} n^{\frac{-2}{d-1}} \tag{2}$$

where $c_1, c_2 > 0$ depend only on k and d. We note that the corresponding asymptotic formulae are known in the case of random polytopes (see I. Bárány [2]). The goal of this paper is to extend the known results about best approximating polytopes to the case when $d = 3$ and the number of edges are restricted.

In our theorems, we will have the convex body K in $\mathbb{R}^3$ with C^2 boundary, and some sequence $\{L_n\}$ of polytopes such that the number $f(n)$ of faces of L_n tends to infinity with n. Given $k \geq 3$ and $\alpha > 0$, we will state at some point that the typical faces of L_n are asymptotically regular k–gons with respect to the density function $\kappa(x)^\alpha$. Let us introduce some notions that clarify the meaning of this statement.

For $\varepsilon > 0$ and convex compact sets F and Π, we say that F is ε–close to Π if there exist $x \in F$ and $y \in \Pi$ satisfying

$$(1+\varepsilon)^{-1} \cdot (\Pi - y) \subset F - x \subset (1+\varepsilon) \cdot (\Pi - y).$$

Let $u(x)$ denote the exterior unit normal to ∂K at $x \in \partial K$. A face F of L_n is called *proper* if there exits an $x_F \in \partial K$ such that $u(x_F)$ is an exterior normal also to F, and Q_{x_F} is positive definite (hence x_F is unique). For real $\alpha > 0$ and integer $m \geq 3$, we say that the typical faces of L_n are asymptotically regular m–gons with respect to the density function $\kappa(x)^\alpha$ if the following properties hold. There exists $\nu(n) > 0$ with $\lim_{n\to\infty} \nu(n) = 0$ such that for all but $\nu(n)$ percent of the faces F of L_n, F is a proper m–gon, and F is $\nu(n)$–close to some m–gon which is regular with respect to Q_{x_F} and is of area $\frac{\int_{\partial K} \kappa(x)^\alpha dx}{f(n)\cdot \kappa(x_F)^\alpha}$.

We will also discuss the distribution of the edges of L_n. Let $\mathcal{E}_n$ denote the family of edges of L_n, and let $\pi_{\partial K}$ denote the closest point map onto ∂K (see Section 2). We say that the projections of the edges of L_n are uniformly distributed on ∂K with respect to the density function $\kappa(x)^\alpha$ if for any Jordan measurable $Z \subset \partial K$, we have

$$\lim_{n\to\infty} \frac{\#\{e \in \mathcal{E}_n : \pi_{\partial K} e \subset Z\}}{n} = \lim_{n\to\infty} \frac{\#\{e \in \mathcal{E}_n : \pi_{\partial K} e \cap Z \neq \emptyset\}}{n} = \frac{\int_Z \kappa(x)^\alpha dx}{\int_{\partial K} \kappa(x)^\alpha dx}.$$

Here the cardinality of a finite set $\mathcal{S}$ is denoted by $\#\mathcal{S}$.

In Theorem 1.1, K is any convex body in $\mathbb{R}^3$ with C^2 boundary. We define P_n to be a best approximating polytope with at most n edges such that $\delta_S(K, P_n)$ is minimal. In addition let P_n^i be a polytope contained in K with at most n edges and of maximal volume, and let P_n^c be a polytope containing K with at most n edges and of minimal volume. The existence of these extremal polytopes follows from the Blaschke Selection Theorem and the continuity of the volume.

THEOREM 1.1 *The typical faces of P_n^i are asymptotically squares with respect to the density function $\kappa(x)^{1/4}$, and the typical faces of both P_n^c and P_n are asymptotically regular triangles with respect to the density function $\kappa(x)^{1/4}$.*

Remark: Theorem 1.1 in the case of inscribed polytopes provides the first example when not triangles or hexagons but quadrilaterals are the asymptotically optimal faces of the extremal polytopes in the case of volume approximation. We also have the asymptotic formulae

$$\delta_S(K, P_n^i) \sim \frac{1}{3}\left(\int_{\partial K} \kappa(x)^{1/4} dx\right)^2 \cdot \frac{1}{n}; \tag{3}$$

$$\delta_S(K,P_n) \sim \left(\frac{1}{4\sqrt{3}} - \frac{3}{16\pi}\right)\left(\int_{\partial K}\kappa(x)^{1/4}dx\right)^2 \cdot \frac{1}{n}; \tag{4}$$

$$\delta_S(K,P_n^c) \sim \frac{1}{4\sqrt{3}}\left(\int_{\partial K}\kappa(x)^{1/4}dx\right)^2 \cdot \frac{1}{n}. \tag{5}$$

In addition each extremal polytope in Theorem 1.1 has $n+o(n)$ edges, P_n^i has $\frac{n}{2}+o(n)$ faces, and both P_n^c and P_n have $\frac{2n}{3}+o(n)$ faces. Moreover the projections of the edges of either extremal polytope in Theorem 1.1 are uniformly distributed on ∂K with respect to the density function $\kappa(x)^{1/4}$. These properties follow from combining the proof of Theorem 1.1 with well-known methods (see say K. Böröczky, Jr. [4]), and we do not provide the arguments.

Let us turn to approximation with respect to the L_1–metric. If C is a convex body in $\mathbb{R}^3$ then its support function h_C is

$$h_C(u) = \max_{x\in C}\langle x,u\rangle \text{ for } u \in \mathbb{R}^3,$$

and the mean width of C is $M(C) = \frac{1}{4\pi}\int_{S^2}\{h_C(u)+h_C(-u)\}du = \frac{1}{2\pi}\int_{S^2}h_C(u)du$. Now the L_1–distance of the convex bodies C and M in $\mathbb{R}^3$ is defined by

$$\delta_1(C,M) = \int_{S^2}|h_C(u)-h_M(u)|\,du.$$

In particular if $M \subset C$ then $\delta_1(C,M)$ is proportional to the difference of the mean widths of C and M. The papers S. Glasauer, P.M. Gruber [11], M. Ludwig [22] and K. Böröczky, Jr. [4] provided asymptotic formulae for polytopal approximation of smooth convex bodies with respect to the L_1 metric if the number of vertices or facets are restricted.

In Theorem 1.2, K is any convex body in $\mathbb{R}^3$ with C^2 boundary. We define W_n to be a best approximating polytope with at most n edges such that $\delta_1(K,W_n)$ is minimal. In addition let W_n^i be a polytope contained in K with at most n edges and of maximal mean width, and let W_n^c be the polytope containing K with at most n edges and of minimal mean width.

THEOREM 1.2 *The typical faces of W_n^c are asymptotically squares with respect to the density function $\kappa(x)^{3/4}$, and the typical faces of both W_n^i and W_n are asymptotically regular hexagons with respect to the density function $\kappa(x)^{3/4}$.*

Remark: We also have the asymptotic formulae

$$\delta_1(K,W_n^c) \sim \frac{1}{3}\left(\int_{\partial K}\kappa(x)^{3/4}dx\right)^2 \cdot \frac{1}{n}; \tag{6}$$

$$\delta_1(K,W_n) \sim \left(\frac{1}{4\sqrt{3}}-\frac{3}{16\pi}\right)\left(\int_{\partial K}\kappa(x)^{3/4}dx\right)^2\cdot\frac{1}{n}; \tag{7}$$

$$\delta_1(K,W_n^i) \sim \frac{1}{4\sqrt{3}}\left(\int_{\partial K}\kappa(x)^{3/4}dx\right)^2\cdot\frac{1}{n}. \tag{8}$$

We note that each extremal polytope in Theorem 1.2 has $n+o(n)$ edges, W_n^c has $\frac{n}{2}+o(n)$ faces, and both W_n^i and W_n have $\frac{n}{3}+o(n)$ faces. In addition the projections of the edges of W_n^c, W_n^i and W_n are uniformly distributed on ∂K with respect to the density function $\kappa(x)^{3/4}$. These properties follow from combining the proof of Theorem 1.1 with well-known methods (see say K. Böröczky, Jr. [4]), and we do not provide the arguments.

Most of the paper is dedicated to the proof of Theorem 1.1. The fundamental facts, notions and notation that are used thorough the paper are presented in Section 2. The proof of Theorem 1.1 needs "momentum lemma type" estimates for planar polygons, which are subject of Section 3. The argument to prove Theorem 1.1 presented only for P_n^i in detail, and the necessary changes in the argument for P_n^c and P_n are outlined in Section 6. In Section 4 we construct inscribed polytopes with at most n edges that are asymptically best approximating. First we approximate small "round" parts of ∂K by pieces of paraboloids whose size tends to zero with n. For each paraboloidal piece, we transfer a suitable planar parallelogram tiling to a polytopal surface inscribed into the paraboloid using power diagrams. Finally the proof of Theorem 1.1 is completed in Section 5 by applying the estimates of Section 3 to the faces of P_n^i. Theorem 1.2 follows from combining the proof of Theorem 1.1 with an ingenious method based on duality developed by R. Schneider and the paper S. Glasauer and P.M. Gruber [11]. We sketch the argument in Section 7.

2 Some basic notions, notations and facts

Concerning notions for convex bodies and polytopes in this paper, consult the beautiful monographs R. Schneider [25] and P.M. Gruber [20]. In this paper we concentrate on $\mathbb{R}^3$. However we present some of the notions in $\mathbb{R}^d$, $d\geq 4$, because Section 1 quotes some results in higher dimensions.

For a compact convex set C in $\mathbb{R}^d$, we write $\mathrm{aff}\,C$ to denote its affine hull, ∂C to denote its relative boundary in $\mathrm{aff}\,C$, and $\mathrm{relint}\,C$ to denote its relative interior in $\mathrm{aff}\,C$. In addition C is called a convex body if $\mathrm{int}\,C\neq\emptyset$, and a convex disc if $\mathrm{aff}\,C$ is of dimension two. Let o be the origin in $\mathbb{R}^d$, and B^d be the unit ball in $\mathbb{R}^d$ centred at o. Volume in $\mathbb{R}^d$ is denoted by $V(\cdot)$, and the two-dimensional Hausdorff–measure is denoted by $|\cdot|$. For any objects $X_1,\dots,X_k$ in $\mathbb{R}^d$, we write $[X_1,\dots,X_k]$ to denote the convex hull of their union.

If A is any affine plane in $\mathbb{R}^d$ then the orthogonal projection into A is denoted by $p_A(\cdot)$.

Let K be a convex body in $\mathbb{R}^d$ with C^2 boundary, which we fix for the rest of the section. For $x \in \partial K$, we write T_x to denote the tangent hyperplane at x, and as before, write $u(x)$ unit exterior normal at x. For $C = p_{T_x}K$, let f be the convex C^2 function on relint$C$ whose graph is contained in ∂K; namely, $y - f(y)u(x) \in \partial K$ for $y \in \mathrm{relint}C$. For $y \in \mathrm{relint}C$, we write l_y to denote the linear form representing the derivative of f at y, and q_y to denote the quadratic form representing the second derivative of f at y on $T_x - x$. According to the Taylor formula, if $a, y \in \mathrm{relint}C$ then there exists $\xi(a,y) \in [a,y]$ satisfying

$$f(y) = f(a) + l_a(y)(y-a) + \tfrac{1}{2}\, q_{\xi(a,y)}(y-a). \tag{9}$$

Naturally $q_{\xi(a,y)}(y-a)$ is continuous as a function of y. Next we consider the function $\psi(y) = f(a) + l_a(y)(y-a) + \alpha$, $\alpha \in \mathbb{R}$, of $y \in T_x$. In particular the graph of ψ is parallel to the tangent plane to ∂K at $a - f(a)u(x)$. For a Jordan measurable $\Pi \subset \mathrm{relint}C$, writing

$$\begin{aligned} \Psi &= \{(1-t)f(y) + t\psi(y) : y \in \Pi \text{ and } t \in [0,1]\}, \quad \text{we have} \\ V(\Psi) &= \int_{\Pi} \left|\tfrac{1}{2}\, q_{\xi(a,y)}(y-a) - \alpha\right| dy. \end{aligned} \tag{10}$$

The second fundamental form Q_x at the $x \in \partial K$ is defined by $Q_x = q_x$. It is a positive semi-definite quadratic form on $T_x - x$. Its $d-1$ eigenvalues are the principal curvatures at x, and its determinant (the product of the principal curvatures) is the Gauß-Kronecker curvature at x. Readily $\kappa(x) \geq 0$ for $x \in \partial K$.

Let $d = 3$ for the rest of the section. We write cl$Y$ to denote the closure of any $Y \subset \mathbb{R}^3$, and consider ∂K with the subspace topology as a subset of $\mathbb{R}^3$. We say that $Y \subset \partial K$ is Jordan measurable if the relative boundary ∂Y of Y on ∂K is of two-dimensional Hausdorff–measure zero.

Since ∂K is C^2, there exists $\eta > 0$ such that balls of radius 2η roll from inside along ∂K (see also K. Leichtweiß [21]). More precisely, for any $x \in \partial K$, the 3-ball of radius 2η and of centre $x - 2\eta\, u(x)$ is contained in K. We write $K_{-\eta}$ to denote the family of points z such that $z + \eta B^d \subset K$. The existence of the rolling ball yields that if $x \in \partial K$ and $y_1, y_2 \in T_x$ then

$$Q_x(y_1 - y_2) \leq \eta^{-1}\|y_1 - y_2\|^2; \tag{11}$$

$$\kappa(x) \leq \eta^{-2}. \tag{12}$$

In addition if $\|x-z\| \le \eta$ for $x,z \in \partial K$ then

$$\langle u(x), u(z)\rangle \ge 1 - \tfrac{1}{\eta^2}\,\|x-y\|^2. \tag{13}$$

If $w \in \mathbb{R}^3 \backslash K_{-\eta}$ then there exists a unique closest point $\pi_{\partial K}(w)$ of ∂K to w. In particular $\pi_{\partial K}(w) = x$ if and only if $x \in \partial K$ and $w = x + su(x)$ for $s > -\eta$. The closest point map satisfies

$$\|\pi_{\partial K}(w_1) - \pi_{\partial K}(w_2)\| \le 2\|w_1 - w_2\| \;\text{ for } w_1, w_2 \in \mathbb{R}^3 \backslash K_{-\eta}. \tag{14}$$

We say that $Z \subset \mathbb{R}^3$ is a polytopal surface, if it is the union of certain faces of some polytope $P \subset \mathbb{R}^3$. The faces and vertices of Z are the faces and vertices, respectively, of P that lie in Z. These notions do not depend on the actual choice of P. It follows from (14) that if $Z \subset \mathbb{R}^3 \backslash K_{-\eta}$ is a polytopal surface then $\pi_{\partial K}(Z)$ is Jordan measurable.

Next let $Y \subset \partial K$ be closed such that $\kappa(x) > 0$ for $x \in Y$. Then there exist $\rho, R > 0$ depending on Y and K with the following properties. If $x \in Y$ and $y_1, y_2 \in T_x$ then

$$\begin{aligned} \partial K \cap (x + \rho B^3) &\subset x - Ru(x) + RB^3; & (15)\\ Q_x(y_1 - y_2) &\ge R^{-1}\|y_1 - y_2\|^2. & (16) \end{aligned}$$

We frequently measure the distance of K from another convex body M by the Hausdorff distance $\delta_H(K,M)$. For $x \in \mathbb{R}^3$ and compact $Z \subset \mathbb{R}^3$, we write $d(x,Z)$ to denote the distance of x from Z. Therefore

$$\delta_H(K,M) = \max\{\max_{x\in K} d(x,M), \max_{y \in M} d(y,K)\}.$$

In the final part of the section, we present the rough estimate (17), which will be useful to estimate the volume between a polytopal surface and ∂K. Let $Y \subset \partial K$ be Jordan measurable with $|Y| > 0$. If K contains a ball of radius t then

$$\begin{aligned} V(\{x + su(x) : x \in Y \text{ and } s \in [0,t]\}) &\le |Y|\cdot t + 2\pi M(K)\cdot t^2 + \tfrac{4\pi}{3}\cdot t^3;\\ V(\{x - su(x) : x \in Y \text{ and } s \in [0,t]\}) &\le |Y|\cdot t. \end{aligned}$$

In particular there exists $\eta_* \in (0,\eta)$ depending only on K with the following property. If $t \in (0,\eta_*)$ then

$$V(\{x + su(x) : x \in Y \text{ and } s \in [-t,t]\}) \le 3\cdot|Y|\cdot t. \tag{17}$$

3 Planar formulae and estimates

The main goal of the section is to prove Lemmae 3.2, 3.3 and 3.4 that resemble the classical Momentum Lemma of L. Fejes Tóth [10]. But before this we prove Proposition 3.1, which helps to bound a convex polygon in terms of a positive definite quadratic form.

PROPOSITION 3.1 *Let q be a positive definit quadratic form on $\mathbb{R}^2$, let Π be a polygon in $\mathbb{R}^2$, and let $\theta > 0$. If there exist $\beta \in \mathbb{R}$, a linear form l on $\mathbb{R}^2$ and a C^2 function f on Π with q_x being the quadratic from representing the second derivative of f at $x \in \Pi$ such that*

$$\tfrac{1}{2} q \le q_x \le 2q \ \textit{ and } \ |f(x)+l(x)+\beta| \le \theta \ \textit{ for } x \in \Pi,$$

then $q(x-a) \le 128\theta$ for $a, x \in \Pi$.

Proof: It follows from the Taylor formula (9) that there exist a $\beta' \in \mathbb{R}$, a function g on Π and a vector $v \in \mathbb{R}^2$ such that

$$f(x)+l(x)+\beta = g(x)+\langle v, x-a\rangle + \beta'$$

where $\frac{1}{4}q(x-a) \le g(x) \le q(x-a)$ for $x \in \Pi$. Given $x \in \Pi$, let us consider the function

$$h(t) = f(a+t(x-a))+l(a+t(x-a))+\beta = g(a+t(x-a))+t\langle v, x-a\rangle + \beta'$$

for $t \in [0,1]$. Since $|h(0)| \le \theta$, we have $|\beta'| \le \theta$. In addition

$$9\theta \ge |h(1)-8h(\tfrac{1}{8})| = \left|g(x)-8g(a+\tfrac{1}{8}(x-a))-7\beta'\right|$$

yields $g(x)-8g(a+\frac{1}{8}(x-a)) \le 16\theta$. Since $g(x)-8g(a+\frac{1}{8}(x-a)) \ge \frac{1}{8}q(x-a)$, we conclude Proposition 3.1. □

The rest of the section with the Momentum Lemma type statements is based on K.J. Böröczky, P. Tick and G. Wintsche [7]. Let us recall some basic facts and notation. A simple stability property of the quadratic function is the following: If $\frac{t_1+\ldots+t_n}{n} = t_0$ and the number of t_i with $|t_i - t_0| \ge \varepsilon$ is m for $\varepsilon > 0$ then

$$\frac{t_1^2+\ldots+t_n^2}{n} \ge t_0^2 + \tfrac{m}{n}\cdot\varepsilon^2. \qquad (18)$$

We will also use the following consequence of Cauchy-Schwartz inequality: If $\gamma_i, A_i > 0$ for $i = 1, \dots, m$ then

$$\sum_{i=1}^{m} \gamma_i A_i^2 \geq \left(\sum_{i=1}^{m} \frac{1}{\gamma_i} \right)^{-1} \left(\sum_{i=1}^{m} A_i \right)^2. \tag{19}$$

For $t \in (0, \frac{\pi}{2})$, let $R(t)$ be the triangle with a right angle such that o is a vertex, the angle at o is t, and the longest side is of length one.

3.1 Estimates related to circumscribed and general polytopes

For $t \in (0, \frac{\pi}{2})$, we define

$$\gamma^c(t) = \frac{\int_{R(t)} x^2 \, dx}{|R(t)|^2} = \frac{1}{\tan t} + \frac{\tan t}{3}. \tag{20}$$

This function satisfies $\gamma^c(\frac{\pi}{k}) > \gamma^c(\frac{\pi}{3}) = \frac{2}{\sqrt{3}}$ for $k \geq 4$. In particular if Π is a triangle regular with respect to the positive definite quadratic form q and has o as its centroid then

$$\int_{\Pi} q(x) \, dx = \frac{1}{\sqrt{3}} \cdot \frac{|\Pi|^2}{3} \sqrt{q}. \tag{21}$$

Therefore Lemma 3.3 in K.J. Böröczky, P. Tick and G. Wintsche [7] yields

LEMMA 3.2 *If q is a positive definite quadratic form on $\mathbb{R}^2$, $\alpha \leq 0$ and Π is a k–gon then*

$$\int_{\Pi} \{q(x) - \alpha\} \, dx \geq \frac{1}{\sqrt{3}} \cdot \frac{|\Pi|^2}{k} \sqrt{\det q}.$$

If $\int_{\Pi} \{q(x) - \alpha\} \, dx \leq \frac{1+\varepsilon}{\sqrt{3}} \cdot \frac{|\Pi|^2}{k} \sqrt{\det q}$ for $\varepsilon \in (0, \varepsilon_0)$ then we have $\alpha \geq -\varepsilon |\Pi| \sqrt{\det q}$, Π is a triangle, and there exists some triangle Π_0 that is regular with respect to q, has o as its centroid, and satisfies

$$(1 + \vartheta\sqrt{\varepsilon})^{-1} \Pi_0 \subset \Pi \subset (1 + \vartheta\sqrt{\varepsilon}) \Pi_0$$

where ε_0 and ϑ are positive absolute constants.

For $t \in (0, \frac{\pi}{2})$, we define

$$\gamma(t) = \frac{\min_{\alpha \in \mathbb{R}} \int_{R(t)} |x^2 - \alpha| \, dx}{|R(t)|^2}.$$

If $t \in (0, \frac{\pi}{3}]$ then

$$\gamma(t) = \frac{1}{\tan t} + \frac{\tan t}{3} - \frac{1}{2t}. \tag{22}$$

This function satisfies $\gamma(\frac{\pi}{k}) > \gamma(\frac{\pi}{3}) = \frac{2}{\sqrt{3}} - \frac{3}{2\pi}$ for $k \geq 4$. In particular if Π is a triangle regular with respect to the positive definite quadratic form q and has o as its centroid, and $\alpha > 0$ satisfies $|\{x \in \Pi : q(x) \leq \alpha\}| = \frac{1}{2}|\Pi|$ then

$$\int_{\Pi} |q(x) - \alpha|\, dx = \left(\frac{1}{\sqrt{3}} - \frac{3}{4\pi}\right) \frac{|\Pi|^2}{3} \sqrt{q}. \tag{23}$$

Therefore Lemma 3.13 in K.J. Böröczky, P. Tick and G. Wintsche [7] yields

LEMMA 3.3 *There exist absolute constants $\varepsilon_0, \vartheta > 0$ with the following properties: If q is a positive definite quadratic form on $\mathbb{R}^2$, $\alpha \in \mathbb{R}$ and Π is a k-gon then*

$$\int_{\Pi} |q(x) - \alpha|\, dx \geq \left(\frac{1}{\sqrt{3}} - \frac{3}{4\pi}\right) \frac{|\Pi|^2}{k} \sqrt{\det q}.$$

If $\int_{\Pi} |q(x) - \alpha|\, dx \leq (1+\varepsilon)(\frac{1}{\sqrt{3}} - \frac{3}{4\pi})\frac{|\Pi|^2}{k}\sqrt{\det q}$ for $\varepsilon \in (0, \varepsilon_0)$ then Π is a triangle, and there exists some triangle Π_0 that is regular with respect to q, has o as its centroid, and satisfies

$$(1+\vartheta\sqrt{\varepsilon})^{-1}\Pi_0 \subset \Pi \subset (1+\vartheta\sqrt{\varepsilon}) \cdot \Pi_0;$$
$$\tfrac{1-\vartheta\sqrt{\varepsilon}}{2}|\Pi| < |\{x \in \Pi : q(x) \leq \alpha\}| < \tfrac{1+\vartheta\sqrt{\varepsilon}}{2}|\Pi|.$$

3.2 Estimates related to inscribed polytopes

For $t \in (0, \frac{\pi}{2})$, we define

$$\gamma^i(t) = \frac{\int_{R(t)} \{1 - x^2\}\, dx}{|R(t)|^2} = \frac{1}{\tan t} + \frac{5 \tan t}{3}.$$

The analogue of Lemmae 3.2 and 3.3 can be proved for γ^i as well, but in this case the optimal regular polygon is a pentagon. The reason is that among integers $k \geq 3$, the minimum of $\gamma^i(\frac{\pi}{k})$ is attained for $k = 5$. Since regular pentagons do not tile the plane, we need another approach. We will eventually show that typical faces of P_n^i are suitable parallelograms by considering stars of vertices around P_n^i. For $t \in (0, \frac{\pi}{2})$, let $\widetilde{R}(t)$ be a triangle with right angle such that o is a vertex, the longest side is of length 1, and the angle at o is $\frac{\pi}{2}-t$. In particular the other acute angle of $\widetilde{R}(t)$ is t. We define

$$\tilde{\gamma}^i(t)=\gamma^i(\tfrac{\pi}{2}-t)=\frac{\int_{\widetilde{R}(t)}\{1-x^2\}\,dx}{|\widetilde{R}(t)|^2}=\tan t+\frac{5}{3\tan t}. \tag{24}$$

We observe that $\tilde{\gamma}^i(t)$ is strictly convex on $t\in(0,\frac{\pi}{2})$, and attains its minimum at $t=\arctan\frac{\sqrt{5}}{\sqrt{3}}$. It follows that if $t\leq\frac{2\pi}{9}$ then

$$\tilde{\gamma}^i(t)\geq\tilde{\gamma}^i(\tfrac{2\pi}{9})>\tilde{\gamma}^i(\tfrac{\pi}{4}). \tag{25}$$

In addition we have

$$\tilde{\gamma}^i(\tfrac{\pi}{3}) \;>\; \tilde{\gamma}^i(\tfrac{\pi}{4})=\tfrac{8}{3}; \tag{26}$$

$$\tilde{\gamma}^i(t) \;\geq\; \tfrac{2\sqrt{5}}{\sqrt{3}} \text{ for } t\in(0,\tfrac{\pi}{2}). \tag{27}$$

Let Π be a parallelogram centred at o that is a square with respect to a positive definite quadratic form q. Since $\gamma^i(\frac{\pi}{4})=\tilde{\gamma}^i(\frac{\pi}{4})=\frac{8}{3}$, if $\max_{x\in\Pi}q(x)=\alpha$ then

$$\int_{\Pi}\{\alpha-q(x)\}dx=\frac{1}{3}\,|\Pi|^2\sqrt{\det q}. \tag{28}$$

Let q be a positive definite quadratic form in two variables. We say that Σ is a *decorated cell complex* with respect to q if Σ is the set of finitely many planar polygons (cells) with the following properties. The intersection of any two elements of Σ is either a common side, a common vertex, or the empty set. In addition an $a(\Pi)\in\mathbb{R}^2$ and an $\alpha(\Pi)>0$ are assigned to any $\Pi\in\Sigma$ such that

$$q(x-a(\Pi))\leq\alpha(\Pi) \text{ for } x\in\Pi.$$

Next we define the schemes associated to the decorated cell complex Σ. We assign a scheme R to any triple (Π,e,v) where $\Pi\in\Sigma$, e is a side of Π, and v is an endpoint of e. Let w be the point of e minimizing $q(w-a(\Pi))$. If $[a(\Pi),w,v]$ is a (non-degenerate) triangle that intersects $\operatorname{relint}\Pi$ then the associated scheme is $R=[a(\Pi),w,v]$. Otherwise the associated scheme is the point $R=\{v\}$. We call R a non-degenerate associated scheme if it is a triangle. In this case we define $a(R)=a(\Pi)$, $\alpha(R)=\alpha(\Pi)$, $\Pi(R)=\Pi$ and $v(R)=v$. There exists some linear transformation A such that $q(x)=\|Ax\|^2$ for $x\in\mathbb{R}^2$. Then the angle of AR at Aw

is at least $\frac{\pi}{2}$, and we write $t(R)$ to denote the (acute) angle of AR at Av. We note that $t(R)$ does not depend on the choice of A.

Two simple properties of a decorated cell complex Σ will be of very good use. Let $\Pi \in \Sigma$. If $R_1,\ldots,R_m$ are the non-degenerate associated schemes with $\Pi(R_i) = \Pi$ then

$$R_1 \cap \Pi, \ldots, R_m \cap \Pi \ \text{ tile } \Pi. \tag{29}$$

In addition if v is a vertex of Π, and both associated schemes R with $v = v(R)$ and $\Pi = \Pi(R)$ are non-degenerate then

$$a(\Pi) - v \in \operatorname{relint}\operatorname{pos}(\Pi - v). \tag{30}$$

LEMMA 3.4 *There exist absolute constants $\varepsilon_0, \vartheta > 0$ with the following properties. Let Σ be a decorated cell complex in $\mathbb{R}^2$ with respect to a positive definite quadratic form q. In addition let v be the vertex of exactly k cells of Σ such that these k cells cover a neighbourhood of v, and there exist all together $m \geq 1$ non-degenerate associated schemes $R_1,\ldots,R_m$ with $v(R_i) = v$. Then*

$$\sum_{i=1}^{m} \int_{R_i \cap \Pi(R_i)} \{\alpha(R_i) - q(x - a(R_i))\}dx \geq \frac{4}{3} \cdot \frac{(\sum_{i=1}^{m} |R_i \cap \Pi(R_i)|)^2}{k} \sqrt{\det q}. \tag{31}$$

In addition if $\varepsilon \in (0, \varepsilon_0)$ and

$$\sum_{i=1}^{m} \int_{R_i} \{\alpha(R_i) - q(x - a(R_i))\}dx \leq \frac{(1+\varepsilon)4}{3} \cdot \frac{(\sum_{i=1}^{m} |R_i \cap \Pi(R_i)|)^2}{k} \sqrt{\det q} \tag{32}$$

then $k = 4$ and $m = 2k = 8$, moreover $\alpha(R_i) \leq (1 + \vartheta\varepsilon) \cdot q(v - a(R_i))$ and

$$|t(R_i) - \tfrac{\pi}{4}| < \vartheta\sqrt{\varepsilon} \ \textit{ hold for } i = 1,\ldots,m. \tag{33}$$

To prove Lemma 3.4, we need auxiliary some statements based on some results in K.J. Böröczky, P. Tick, G. Wintsche [7]. Proposition 3.1 in [7] yields

PROPOSITION 3.5 *$\tilde{\gamma}^i(t)^{-1}$ is concave on $(0, \frac{\pi}{2})$, and $(\tilde{\gamma}^i(t)^{-1})$" < -0.03 if $t \in (\frac{\pi}{7}, \frac{5\pi}{12})$.*

In turn we have (compare Proposition 3.2 in [7]):

PROPOSITION 3.6 *If $\frac{t_1+\ldots+t_8}{8} = \frac{\pi}{4}$ for $t_1,\ldots,t_8 \in [0, \frac{\pi}{2}]$, and $\max_{i=1}^{8} |t_i - \frac{\pi}{4}| \geq \varepsilon$ for $\varepsilon \in (0, \frac{\pi}{16})$ then*

$$\frac{\tilde{\gamma}^i(t_1)^{-1} + \ldots + \tilde{\gamma}^i(t_8)^{-1}}{8} \leq \tilde{\gamma}^i(\tfrac{\pi}{4})^{-1} - 0.001 \cdot \varepsilon^2.$$

Next we deduce the following by Proposition 3.11 in [7].

PROPOSITION 3.7 *If $t \in (0, \pi/2)$ and $\Pi \subset \widetilde{R}(t)$ is a convex disc then*

$$\int_{\Pi} \{1 - x^2\}\, dx \geq \tilde{\gamma}^i(t) |\Pi|^2.$$

In turn we prove

PROPOSITION 3.8 *Let $R = [a, w, v]$ a triangle that has an angle at least $\frac{\pi}{2}$ at w, and let $C \subset R$ be a polygon such that $[w, v]$ is a side of C. If the angle of C at v is s then*

$$\int_{C} \{(v - a)^2 - (x - a)^2\}\, dx \geq \tilde{\gamma}^i(s) |C|^2.$$

Proof: Let $a' \in [a, w]$ be the point such that $[a', v]$ contains the side of C at v different from $[w, v]$, and let $R' = [a', w, v]$. We deduce by Proposition 3.7 that

$$\int_{C} \{(v - a')^2 - (x - a')^2\}\, dx \geq \tilde{\gamma}^i(s) |C|^2.$$

Since $[(v - a)^2 - (x - a)^2] - [(v - a')^2 - (x - a')^2] = 2\langle a' - a, v - x\rangle \geq 0$ for $x \in C$, we conclude Proposition 3.8. $\square$

Proof of Lemma 3.4: We may assume that $q(x) = x^2$. For $i = 1, \ldots, m$, we define s_i to be the angle of $\Pi(R_i)$ at v if it is less than $t(R_i)$, and $s_i = t(R_i)$ otherwise. In particular each s_i is acute, and $\sum_{i=1}^m s_i \leq 2\pi$. In addition we define $C_i = R_i \cap \Pi(R_i)$, $i = 1, \ldots, m$. We note that $m \leq 2k$, and if $m = 2k$ then $s_i = t(R_i)$ for $i = 1, \ldots, 2k$ according to (30), hence

$$s_1 + \ldots + s_{2k} = 2\pi. \tag{34}$$

Even if $m < 2k$, it follows from Proposition 3.8, the Cauchy-Schwartz inequality and concavity of $\tilde{\gamma}^i(t)^{-1}$ (see Proposition 3.5) that

$$\begin{aligned}
\sum_{i=1}^{m} \int_{C_i} \{\alpha(R_i) - (x - a(R_i))^2\} dx \quad &\geq \quad \sum_{i=1}^{m} \tilde{\gamma}^i(s_i) |C_i|^2 &(35)\\
&\geq \quad \left(\sum_{i=1}^{m} \tilde{\gamma}^i(s_i)^{-1}\right)^{-1} \left(\sum_{i=1}^{m} |C_i|\right)^2 &(36)\\
&\geq \quad \frac{\tilde{\gamma}^i\left(\frac{\sum_{i=1}^m s_i}{m}\right)}{m} \cdot \left(\sum_{i=1}^{m} |C_i|\right)^2. &(37)
\end{aligned}$$

First we verify (31). If $m \geq 9$ then $\frac{\sum_{i=1}^m s_i}{m} \leq \frac{2\pi}{9}$, hence $m \leq 2k$ and (25) yield

$$\sum_{i=1}^m \int_{C_i} \{\alpha(R_i) - q(x - a(R_i))\} dx \geq \frac{\tilde{\gamma}^i\left(\frac{2\pi}{9}\right)}{2k} \left(\sum_{i=1}^m |C_i|\right)^2 > \frac{4}{3k} \left(\sum_{i=1}^m |C_i|\right)^2.$$

Therefore we assume $m \leq 8$. If $m < 2k$ then $\frac{k}{m} \geq \frac{4}{7}$, thus it follows from (27), (37) and $\frac{2\sqrt{5}}{\sqrt{3}} \cdot \frac{4}{7} > \frac{4}{3}$ that

$$\sum_{i=1}^m \int_{C_i} \{\alpha(R_i) - q(x - a(R_i))\} dx \geq \frac{2\sqrt{5}}{\sqrt{3}} \cdot \frac{4}{7} \cdot \frac{(\sum_{i=1}^m |C_i|)^2}{k} > \frac{4}{3} \cdot \frac{(\sum_{i=1}^m |C_i|)^2}{k}.$$

In particular the last remaining cases are when $k = 3, 4$ and $m = 2k$, hence (34) yields $\sum_{i=1}^m s_i = 2\pi$. In these cases (31) is a consequence of (26) and (37).

Next we assume (32). If follows from the proof of (31) that if ε_0 is small enough then $m = 2k$ and $k = 4$. In particular $s_i = t(R_i)$, $i = 1, \ldots, m$, and $\sum_{i=1}^m s_i = 2\pi$. Since $\tilde{\gamma}^i\left(\frac{\sum_{i=1}^8 s_i}{8}\right)^{-1} \leq (1 + \varepsilon) \cdot \frac{\sum_{i=1}^8 \tilde{\gamma}^i(s_i)^{-1}}{8}$, we conclude (33) by Propositions 3.6 and 3.5.

Finally, to prove the upper bound on $\alpha(R_i)$, we write $\vartheta_1, \vartheta_2 \ldots$ to denote positive absolute constants. We deduce $\tilde{\gamma}^i(s_i) \geq \frac{(1-\vartheta_1\sqrt{\varepsilon})8}{3}$ in (35) for $i = 1, \ldots, 8$ by (33). Thus (32) yields $\frac{\sum_{i=1}^8 |C_i|^2}{8} \leq (1 + \vartheta_2\sqrt{\varepsilon}) \left(\frac{\sum_{i=1}^8 |C_i|}{8}\right)^2$, and hence if ϑ_0 is small enough then $\frac{|C_i|}{|C_j|} \leq 2$ for $i, j = 1, \ldots, 8$ according to (18). Now we observe that (35) holds even if we replace $\alpha(R_i)$ by $(v - a(R_i))^2$, $i = 1, \ldots, 8$. Therefore (32) yields

$$\alpha(R_i) - (v - a(R_i))^2 \leq \vartheta_3 \varepsilon |C_i|, \; i = 1, \ldots, 8.$$

Since $(v - a(R_i))^2 \geq |C_i|$, we conclude $\alpha(R_i) \leq (1 + \vartheta_3\varepsilon) \cdot (v - a(R_i))^2$. $\square$

4 Constructing a well approximating inscribed polytope for volume approximation

Let K be the convex body in $\mathbb{R}^3$ with C^2 boundary of Theorem 1.1. The whole section is dedicated to prove the following statement:

LEMMA 4.1 *Given $\varepsilon \in (0, \varepsilon_0)$, for any $n > n_0$ there exists a polytope $\widetilde{P}_n^i \subset K$ that has at most $(1+\vartheta\varepsilon)n$ edges, and*

$$\delta_S(K, \widetilde{P}_n^i) \leq (1+\vartheta\varepsilon) \cdot \frac{1}{3} \left(\int_{\partial K} \kappa(x)^{1/4} dx \right)^2 \cdot \frac{1}{n}$$

where $\vartheta, \varepsilon_0 > 0$ depend on K, and n_0 depends on K and ε.

The asymptotic shape of the typical faces of $\widetilde{P}_n^i$ should be parallelograms, hence we will use power diagrams (see say F. Aurenhammer [1]). To indicate the main idea, let us consider a parallelogram Π that is regular with respect to a positive definite quadratic form q. The side to side tiling of the plane by translates of Π can be lifted to an unbounded polytopal surface inscribed into the graph of q (that is a paraboloid) in a way such that the faces of the unbounded polytopal surface project to the tiles. Therefore we replace the "non-flat" part of ∂K with pieces of paraboloids, and after then we build the boundary of the well-approximating polytopes using these pieces.

To prove Lemma 4.1, first we separate the "flat part" of ∂K. For $\mu > 0$, we write $X(\mu)$ to denote the set of $x \in \partial K$ with $\kappa(x) > \mu$. If $X(\mu)$ is not Jordan measurable then there exists a rational number r_μ with $|X(\mu)| < r_\mu < |\mathrm{cl} X(\mu)|$, therefore $X(\mu)$ is Jordan measurable for all but countably many μ. We recall Lemma 1 in K.J. Böröczky [4] in the following form where ε and K come from Lemma 4.1:

PROPOSITION 4.2 *There exist $\mu_0, m_0 > 0$ depending on K and ε such that for $m > m_0$, one finds a polytopal surface $Z \subset K$ with at most m vertices satisfying*

$$\begin{aligned} \partial K \backslash X(\mu_0) &\subset \pi_{\partial K}(Z) \\ \max_{x \in Z} d(x, \partial K) &\leq \frac{\varepsilon^2}{m}. \end{aligned}$$

We choose small $\mu > 0$ such that $\mu < \mu_0$ for the μ_0 of Proposition 4.2, moreover $X(\mu)$ is Jordan measurable, and satisfies

$$\mu < \varepsilon^4; \tag{38}$$

$$\int_{X(\mu)} \kappa(x)^{1/4} dx \geq (1-\varepsilon) \int_{\partial K} \kappa(x)^{1/4} dx. \tag{39}$$

Let X and X' be open Jordan measurable subsets of ∂K such that $\mathrm{cl} X(\mu) \subset X$ and $\mathrm{cl} X \subset X'$, moreover $\kappa(x) > 0$ holds for $x \in \mathrm{cl} X'$.

During the argument, the implied constant in $O(\cdot)$ depends only on K. Let us recall from Section 2 that T_x is the tangent plane at $x \in \partial K$. We may choose $\delta > 0$ with the following properties: The minimal distance between any two of the sets $\partial X(\mu)$, ∂X and $\partial X'$ is larger than 6δ. Moreover given $x \in X'$,

(i) if $w \in \partial K$ near x with $\|p_{T_x} w - x\| \leq \delta$ then

$$\langle u(w), u(x) \rangle \geq (1+\varepsilon^3)^{-1}; \tag{40}$$

(ii) if f is the convex function on $T_x \cap (x + \delta B^3)$ whose graph is part of ∂K, and q_y is the quadratic form representing the second derivative of f at $y \in T_x \cap (x + \delta B^3)$ (hence $Q_x = q_x$) then

$$(1+\varepsilon^3)^{-1} Q_x \leq q_y \leq (1+\varepsilon^3) Q_x.$$

Let $x \in X'$. We now discuss the consequences (41), (42) and (43) of (i) and (ii). If $w \in \partial K$ near x with $\|p_{T_x} w - x\| \leq \frac{\delta}{2}$ then (i) yields

$$(1+\varepsilon^3)^{-1} |t| \ \leq \ d(w + t\,u(x), \partial K) \leq |t| \quad \text{for } t \in (-\tfrac{\delta}{2}, \tfrac{\delta}{2}). \tag{41}$$

Next let C be a convex disc such that $C \cup (K \cap \text{aff} C) \subset x + \delta B^3$. First (i) and (ii) yield

$$\int_{\pi_{\partial K} C} \kappa(z)^{\frac{1}{4}} dz = (1 + O(\varepsilon)) \kappa(x)^{\frac{1}{4}} |C|. \tag{42}$$

Secondly for $\Pi = p_{T_x} C$ and $y_1, y_2 \in \Pi$, combining Proposition 3.1, (ii) and (41) leads to

$$Q_x(y_1 - y_2) \leq 256 \cdot \max_{z \in C \cup (K \cap \text{aff} C)} d(z, \partial K). \tag{43}$$

We note that the typical faces of the well-approximating polytopes near an $x \in X$ are of area of order $\frac{\int_{\partial K} \kappa(w)^{1/4} dw}{n\kappa(x)^{1/4}}$. Therefore we will use pieces of paraboloids near x whose area is of order $\frac{\int_{\partial K} \kappa(w)^{1/4} dw}{\varepsilon^2 n\kappa(x)^{1/4}}$.

We always assume that $\varepsilon_0 > 0$ depending on K is small enough, and n_0 depending on K and $\varepsilon \in (0, \varepsilon_0)$ is large enough to satisfy the estimates below. If $Z \subset \mathbb{R}^3 \backslash K_{-\eta}$ (see Section 2 for the definition of η) is a polytopal surface then let

$$\Delta_K(Z) = \{(1-t)z + t\pi_{\partial K} z : z \in Z \text{ and } t \in [0,1]\}.$$

To simplify formulae, we define

$$I=\int_{\partial K}\kappa(w)^{1/4}dw.$$

For $x\in X'$ and $\lambda>0$, we define the ellipse $\mathcal{E}(x,\lambda)=\{y\in T_x: Q_x(y-x)\leq\lambda^2\}$. In particular $|\mathcal{E}(x,\lambda)|=\lambda^2\kappa(x)^{-\frac{1}{2}}\pi$. We observe that if n_0 is large then all ellipses $\mathcal{E}(x,\frac{4\kappa(x)^{\frac{1}{8}}I^{\frac{1}{2}}}{\varepsilon\sqrt{n}})$, $x\in X'$, are of the diameter at most δ.

For large n, let Ω be a family of points in X such that the sets $\pi_{\partial K}\mathcal{E}(x,\frac{\kappa(x)^{\frac{1}{8}}I^{\frac{1}{2}}}{\varepsilon\sqrt{n}})$ are pairwise disjoint for $x\in\Omega$, but for any $y\in X$ there exists an $x\in\Omega$ such that $y\in\pi_{\partial K}\mathcal{E}(x,\frac{3\kappa(x)^{\frac{1}{8}}I^{\frac{1}{2}}}{\varepsilon\sqrt{n}})$.

Let $x\in\Omega$. Our first goal is to construct a paraboloidal piece very closely approximating ∂K near x. For the function $\psi(y)=\frac{1+\varepsilon^3}{2}Q_x(y-x)$ of $y\in T_x$, we define

$$\Xi_x=\left\{y-\psi(y)u(x): y\in\mathcal{E}\left(x,\frac{4\kappa(x)^{\frac{1}{8}}I^{\frac{1}{2}}}{\varepsilon\sqrt{n}}\right)\right\},$$

and let Ξ'_x be the paraloid containing Ξ_x (the graph of ψ above T_x). It follows from the definition of the function ψ, (ii) and the Taylor formula (9) that

$$\Xi\ \subset\ K \tag{44}$$

$$d(w,\partial K)\ \leq\ \frac{16I\kappa(x)^{\frac{1}{4}}\varepsilon}{n}\quad\text{for } w\in\Xi_x. \tag{45}$$

Next we define M to be the (possibly undounded) polyhedral set determined by the tangent planes to K at all $x\in\Omega$. For the face C'_x of M touching at $x\in\Omega$, we set $C_x=C'_x\cap\mathcal{E}(x,\frac{3\kappa(x)^{\frac{1}{8}}I^{\frac{1}{2}}}{\varepsilon\sqrt{n}})$. If n is large enough then the sets $\Pi_{\partial K}(C_x)$, $x\in\Omega$, do not overlap and cover X. In addition if $x\in\Omega$ then

$$\mathcal{E}\left(x,\frac{\kappa(x)^{\frac{1}{8}}I^{\frac{1}{2}}}{2\varepsilon\sqrt{n}}\right)\subset C_x. \tag{46}$$

After these preparations, we start to construct the vertices of $\widetilde{P}_n^i$. The idea is to construct independently a polytopal surface incribed into Ξ_x for each $x\in\Omega$. Along the way, we also establish some key properties of these polytopal surfaces.

Let $x\in\Omega$. We define $\mathcal{T}_x$ to be a side to side tiling of T_x by parallelograms that are regular with respect to Q_x, and their common area is $\frac{2\int_{\partial K}\kappa(w)^{1/4}dw}{n\kappa(x)^{1/4}}$. For any tile Π of $\mathcal{T}_x$, let a_Π be the centre of Π, and let

$$\alpha_x = \max_{y\in\Pi} \tfrac{1+\varepsilon^3}{2}\, Q_x(y-a_\Pi) = \frac{1+\varepsilon^3}{4}\cdot\frac{2I}{n\kappa(x)^{1/4}}\sqrt{\det Q_x} = \frac{(1+\varepsilon^3)I\kappa(x)^{1/4}}{2n}. \quad (47)$$

We define the convex unbounded polyhedron Λ'_x in a way such that for any tile Π of $\mathcal{T}_x$, Λ'_x has a face F with $p_{T_x}F = \Pi$, aff$F$ is parallel to the tangent plane to Ξ'_x at $a_\Pi - \psi(a_\Pi)u(x)$, and passes through $a_\Pi - (\psi(a_\Pi)+\alpha_x)u(x)$. In particular writing $\mathcal{V}'_x$ to denote the family of $w\in\Xi'_x$ such that $p_{T_x}w$ is a vertex of $\mathcal{T}_x$, we have $\Lambda'_x = [\mathcal{V}'_x]$. We define Λ_x to be the union of the faces F of Λ'_x such that $\pi_{\partial K}F\cap\pi_{\partial K}C_x\neq\emptyset$.

To approximate the flat part, we apply Proposition 4.2 with $m=\varepsilon n$, and obtain a polytopal surface $Y'\subset K$ with at most εn vertices such that

$$\partial K\backslash X(\mu) \subset \pi_{\partial K}(Y'); \quad (48)$$

$$\max_{x\in Y'} d(x,\partial K) \leq \frac{\varepsilon}{n}. \quad (49)$$

Let Y be the union of the faces F of Y' with $\pi_{\partial K}F\cap(\partial K\backslash X)\neq\emptyset$. We define

$$\widetilde{P}_n^i = [Y, \cup_{x\in\Omega}\Lambda_x], \quad (50)$$

which readily satifies $\widetilde{P}_n^i\subset K$.

To prove Lemma 4.1 for $\widetilde{P}_n^i$, first we establish some simple properties of $\widetilde{P}_n^i$ and Λ_x for $x\in\Omega$. In particular (47) and (45) together with (i) and (ii) yield that $\widetilde{P}_n^i$ is close K in the following sense. For $x\in\Omega$, if $w\in\partial\widetilde{P}_n^i$ such that $\langle u(x),u(w)\rangle>0$ and $p_{T_x}w\in\mathcal{E}\left(x,\frac{4\kappa(x)^{\frac{1}{8}}I^{\frac{1}{2}}}{\varepsilon\sqrt{n}}\right)$ then

$$d(w,\partial K)\leq\frac{I\kappa(x)^{1/4}}{n}. \quad (51)$$

A consequence of (i) is that if $w\in C_i\cup\Lambda_i$ then

$$\sqrt{Q_x\left(p_{T_x}(w)-p_{T_x}(\pi_{\partial K}w)\right)}\leq\frac{\varepsilon\kappa(x)^{\frac{1}{8}}I^{\frac{1}{2}}}{\sqrt{n}}. \quad (52)$$

Next let $x\in\Omega$, which we keep fixed until (56) . It follows from (46) and (47) that for any tile Π in $\mathcal{T}_x$, we have

$$\Pi - a_\Pi \subset x + 4\varepsilon(C_x - x). \quad (53)$$

In turn we deduce

$$x+(1-8\varepsilon)(C_x-x)\subset p_{T_x}\Lambda_x\subset x+(1+8\varepsilon)(C_x-x). \tag{54}$$

To prove (55) and (56), we use that if Π is a tile of $\mathcal{T}_x$ then $|\Pi|=\frac{2\int_{\partial K}\kappa(w)^{1/4}dw}{n\kappa(x)^{1/4}}$. Writing m_x to denote the number of faces of Λ_x, we deduce from (42) and (54) that

$$m_x\leq(1+O(\varepsilon))\,\frac{n\kappa(x)^{1/4}|C_x|}{2\int_{\partial K}\kappa(w)^{1/4}dw}\leq\frac{(1+O(\varepsilon))n}{2}\cdot\frac{\int_{\pi_{\partial K}C_x}\kappa(w)^{\frac{1}{4}}dw}{\int_{\partial K}\kappa(w)^{1/4}dw}. \tag{55}$$

Finally we estimate the volume difference $V(\Delta_K(F))$ corresponding to a face F of Λ_i. For $\Pi=p_{T_x}F$, the first inequality in (56) follows from (10), (45) and (52), and the second from (28) and (47). In particular we have

$$\begin{aligned} V(\Delta_K(F)) &\leq (1+O(\varepsilon))\int_{\Pi}\{\alpha_x-\tfrac{1+\varepsilon^3}{2}Q_x(x)\}dx\leq\frac{1+O(\varepsilon)}{6}\cdot|\Pi|^2\kappa(x)^{\frac{1}{2}}\\ &= \frac{2(1+O(\varepsilon))}{3n^2}\left(\int_{\partial K}\kappa(w)^{1/4}dw\right)^2. \end{aligned} \tag{56}$$

We are ready to verify the estimates of Lemma 4.1 for $\widetilde{P}_n^i$. We divide the faces of $\widetilde{P}_n^i$ into two groups. Let Ω^* be the family of $x\in\Omega$ with $\pi_{\partial K}\Lambda_x\cap X(\mu)\neq\emptyset$, let Z be the union of the faces of $\widetilde{P}_n^i$ that are also faces of some Λ_x for $x\in\Omega^*$, and let Z' be the union of the rest of the faces.

First we consider Z. It follows from (55) and (56) that

$$\begin{aligned} V(\Delta_K(Z)) &\leq \sum_{x\in\Omega^*}V(\Delta_K(\Lambda_x))\\ &\leq (1+O(\varepsilon))\sum_{x\in\Omega^*}\frac{n\cdot\int_{\pi_{\partial K}C_x}\kappa(w)^{\frac{1}{4}}dw}{2\cdot\int_{\partial K}\kappa(w)^{1/4}dw}\cdot\frac{2}{3n^2}\left(\int_{\partial K}\kappa(w)^{1/4}dw\right)^2\\ &\leq \frac{1+O(\varepsilon)}{3n}\left(\int_{\partial K}\kappa(w)^{1/4}dw\right)^2. \end{aligned} \tag{57}$$

To estimate the number $e(Z)$ of edges of Z, we simply count the number of edges Λ_x for every $x\in\Omega^*$. Any face of Λ_x is a quadrilateral, and if $p_{T_x}s\subset\mathrm{relint}(x+(1-8\varepsilon)(C_x-x))$ for an edge s of Λ_x then it is contained in two faces of Λ_x according to (54). Therefore combining (53) and (54) with the estimate (55) on m_x leads to

$$e(Z)\leq\sum_{x\in\Omega^*}(1+O(\varepsilon))2m_x\leq(1+O(\varepsilon))\sum_{x\in\Omega^*}\frac{n\cdot\int_{\pi_{\partial K}C_x}\kappa(w)^{\frac{1}{4}}dw}{\int_{\partial K}\kappa(w)^{1/4}dw}\leq(1+O(\varepsilon))n. \tag{58}$$

Let us turn to Z'. For $x \in \Omega$, the observation (59) helps to locate faces of $\widetilde{P}_n^i$ near Λ_x. If $x' \in \Omega \backslash x$ with $\langle u(x), u(x') \rangle > 0$ and F is a face of $\widetilde{P}_n^i$ intersecting $\Lambda_{x'}$ then combining (54) for x' with (43) and (51) leads to

$$x + (1 - \gamma\varepsilon)(C_x - x) \cap p_{T_x} F = \emptyset \tag{59}$$

for some absolute constant $\gamma > 0$. To estimate $V(\Delta_K(Z'))$, we define Z'_x to be the union of the faces of Z' that intersect Λ_x for $x \in \Omega^*$, and Z'_0 to be the union of the faces F of Z' that intersect either Y or some Λ_x with $x \in \Omega \backslash \Omega^*$. It follows from (59) and the definition of Y that if n is large then $Z'_x \cap Y = \emptyset$ for $x \in \Omega^*$. Therefore (54) and (59) yield that

$$p_{T_x} Z'_x \subset [x + (1 + \gamma_1 \varepsilon)(C_x - x)] \backslash [x + (1 - \gamma_2 \varepsilon)(C_x - x)] \tag{60}$$

for $x \in \Omega^*$ where $\gamma_1, \gamma_2 > 0$ are absolute constants. In particular $|p_{T_x} Z'_x| = O(\varepsilon)|C_x|$. Now we apply (17) using (51), and after that use (42) to obtain

$$V(\Delta_K(Z'_x)) \leq \frac{O(\varepsilon)\kappa(x)^{\frac{1}{4}}|C_x|}{n} \leq \frac{O(\varepsilon)\int_{\pi_{\partial K} C_x} \kappa(z)^{\frac{1}{4}} dz}{n}. \tag{61}$$

For Z'_0, we claim that if $w \in Z'_0$ then

$$d(w, \partial K) \leq \frac{O(\varepsilon)}{n}. \tag{62}$$

If $p_{T_x} w \in \mathcal{E}(x, \frac{4\kappa(x)^{\frac{1}{8}} I^{\frac{1}{2}}}{\varepsilon\sqrt{n}})$ for some $x \in \Omega \backslash \Omega^*$ then (62) follows from (38) and (51). Otherwise (43), (51) and (54) imply that $w \in Y$, hence (62) follows from (49). In turn combining (17) and (62) leads to $V(\Delta_K(Z'_0)) \leq \frac{O(\varepsilon)}{n}$. We conclude by (61) the estimate

$$\begin{aligned} V(\Delta_K(Z')) &\leq V(\Delta_K(Z'_0)) + \sum_{x \in \Omega^*} V(\Delta_K(Z'_x)) \\ &\leq \frac{O(\varepsilon)}{n} + \sum_{x \in \Omega^*} \frac{O(\varepsilon)\int_{\pi_{\partial K} C_x} \kappa(z)^{\frac{1}{4}} dz}{n} \leq \frac{O(\varepsilon)}{n}. \end{aligned} \tag{63}$$

Finally we show that Z' has only a few edges. Actually we start with estimating the number of vertices of Z'. For $x \in \Omega^*$, we write $\mathcal{V}'_x$ to denote the family of vertices of Z' that are vertices of Λ_x, as well. In addition we write $\mathcal{V}'_0$ to denote the family of vertices of Z' that are vertices either of some Λ_x with $x \in \Omega \backslash \Omega^*$ or of Y. For $x \in \Omega^*$, we have $\mathcal{V}_x \subset Z_x$, and if $\Pi = p_{T_x}F$ for a face F of Λ_x that has a vertex in $\mathcal{V}_x$ then Π is a square with respect to Q_x with $|\Pi| = \frac{2\int_{\partial K}\kappa(w)^{1/4}dw}{n\kappa(x)^{1/4}}$. We deduce first by (60), and secondly by (42) that

$$\#\mathcal{V}_x \leq O(\varepsilon)\kappa(x)^{\frac{1}{4}}|C_x| \cdot n \leq O(\varepsilon)\int_{\pi_{\partial K}C_x}\kappa(z)^{\frac{1}{4}}dz \cdot n. \tag{64}$$

For $\mathcal{V}_0$, we simply use the estimate (55) and that Y has at most εn vertices. According to (39), we have

$$\begin{aligned}\#\mathcal{V}_0 &\leq \varepsilon n + \sum_{x\in\Omega\setminus\Omega^*} m_x \leq \varepsilon n + \sum_{x\in\Omega\setminus\Omega^*} \frac{\int_{\pi_{\partial K}C_x}\kappa(w)^{\frac{1}{4}}dw}{\int_{\partial K}\kappa(w)^{1/4}dw}\cdot n \\ &\leq \varepsilon n + \frac{\int_{\partial K\setminus X(\mu)}\kappa(w)^{\frac{1}{4}}dw}{\int_{\partial K}\kappa(w)^{1/4}dw} \leq 2\varepsilon n.\end{aligned}$$

This estimate together with (64) for all $x \in \Omega^*$ implies that the number $v(Z')$ of vertices of Z' satisfies $v(Z') = O(\varepsilon)n$. Since the edge graph of Z' is planar, $e(Z') < 3v(Z')$ holds for the number $e(Z')$ of edges of Z' (see say P. Brass, W. Moser, J. Pach [8]). In particular

$$e(Z') \leq O(\varepsilon)n. \tag{65}$$

To conclude Lemma 4.1 for $\widetilde{P}_n^i$, we observe that (58) and (65) yield that the number of edges of $\widetilde{P}_n^i$ is at most $(1+O(\varepsilon))n$. Therefore combining (57) and (63) completes the argument. □

5 The proof of Theorem 1.1 for P_n^i

Let K be a convex body in $\mathbb{R}^3$ with C^2 boundary. We use the notation as it was set up before Theorem 1.1. We write $f(n)$ to denote the number of faces of P_n^i. Having Lemma 4.1, Theorem 1.1 and the asymptotic formula after it are consequences of (66) and (67) below.

There exist positive ν_0, ϑ depending on K with the following properties. Let $\nu \in (0, \nu_0)$. We write $g(n)$ to denote the number of faces F of P_n^i such that F is a proper quadrilateral, and F is $\vartheta\nu$–close to some quadrilateral that is regular with respect to Q_{x_F} and is of area $\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{f(n)\cdot\kappa(x_F)^{1/4}}$. Then for any $n > n_0$ where n_0 depends on ν and K, we have the estimates

$$g(n) \;>\; (1-\vartheta\nu)f(n); \tag{66}$$

$$\delta_S(K,P_n^i) \;\geq\; (1-\vartheta\nu)\cdot\frac{1}{3}\left(\int_{\partial K}\kappa(x)^{1/4}dx\right)^2\cdot\frac{1}{n}. \tag{67}$$

In addition, P_n^i has $n+o(n)$ edges, and $f(n)=\frac{n}{2}+o(n)$.

It is practical to define

$$\mu = \nu^3.$$

Let X_0, X' and X be relatively open Jordan measurable subsets of ∂K such that $\mathrm{cl}X_0 \subset X$, $\mathrm{cl}X \subset X'$, $\kappa(x) > 0$ for $x \in \mathrm{cl}X'$, and

$$\int_{X_0}\kappa(x)^{1/4}dx \geq (1-\mu^2\nu^2)\int_{\partial K}\kappa(x)^{1/4}dx.$$

We have $\delta > 0$ with the following properties: $(X_0+2\delta B^3)\cap\partial K \subset X$ and $(X+2\delta B^3)\cap\partial K \subset X'$. Moreover if C is a convex disc that touches K in $x\in X$ and C is of diameter at most δ then

(i) writing C' to denote the orthogonal projection of $\pi_{\partial K}(C)$ into aff$C$, we have

$$x+(1-\mu^2\nu^2)(C-x) \subset C' \subset x+(1-\mu^2\nu^2)^{-1}(C-x);$$

(ii) if $w\in\pi_{\partial K}(C)$ then $\langle u(w),u(x)\rangle \geq 1-\mu^2\nu^2$;

(iii) if φ is the convex function on C such that its graph is part of ∂K, and q_y is the quadratic form representing the second derivative of φ at $y\in C$ (hence $Q_x = q_x$) then

$$(1+\mu^2\nu^2)^{-1}Q_x \leq q_y \leq (1+\mu^2\nu^2)Q_x.$$

During the proof of (66), $\vartheta_1,\vartheta_2,\ldots$ denote positive constants depending only on K, and the implied constant in $O(\cdot)$ depends only on K. There exists a convex polytope M circumscribed around K such that diam$G < \delta$ holds for each face G of M with $\pi_{\partial K}(G)\cap X \neq \emptyset$. We write $\mathcal{M}$ to denote the family of faces of M that touch K in a point of X, and let $G\in\mathcal{M}$ touch K in x_G. Therefore

$$\sum_{G\in\mathcal{M}}\kappa(x_G)^{1/4}|G| \geq (1-\vartheta_1\mu^2\nu^2)\int_{\partial K}\kappa(x)^{1/4}dx. \tag{68}$$

According to Lemma 4.1, if n is large then

$$\delta_S(K,P_n^i) < (1+\mu^2\nu^2)\cdot\frac{1}{3}\cdot\left(\int_{\partial K}\kappa(x)^{1/4}dx\right)^2\cdot\frac{1}{n}. \tag{69}$$

It follows from (69) and the existence of the rolling ball of radius η (see Section 2) that

$$\delta_H(K,P_n^i) \leq \vartheta_2 n^{-1/2}. \tag{70}$$

Therefore if n_0 is large enough then $K_{-\eta}\subset \mathrm{int}P_n^i$. We deduce by (15) and Proposition 3.1 that if F is a face of P_n^i with $\pi_{\partial K}(F)\subset X'$ then

$$\mathrm{diam}\,F \leq \omega n^{-1/4} \tag{71}$$

where $\omega>0$ depends on ν and K.

Recalling that $G\in\mathcal{M}$ touches K in x_G, we write $\widetilde{\mathcal{M}}$ to denote the family of convex discs of the form

$$(1-2\mu^2\nu^2)(G-x_G)+x_G$$

as G runs through the elements of $\mathcal{M}$. In turn for $C\in\widetilde{\mathcal{M}}$, we write x_C to denote the point where C touches K, and define

$$C' = (1-\mu^2\nu^2)(C-x_C)+x_C.$$

In addition let $\mathcal{F}_C$ denote the family of faces of P_n^i near C whose orthogonal projection to affC intersects relint$C$. We deduce by (i) and (71) that if n_0 is large enough then the families $\mathcal{F}_C$ for $C\in\widetilde{\mathcal{M}}$ are pairwise disjoint, and by (68) that

$$\sum_{C\in\widetilde{\mathcal{M}}}\kappa(x_C)^{1/4}|C'| = (1+O(\mu^2\nu^2))\int_{\partial K}\kappa(x)^{1/4}dx. \tag{72}$$

Let $C\in\widetilde{\mathcal{M}}$. We write $\mathcal{F}_C'$ to denote the family of all $F\in\mathcal{F}_C$ such that $p_{\mathrm{aff}C}(F)$ intersects relintC'. Again if n_0 is large enough then (70) yields for any $F\in\mathcal{F}_C'$ that

$$p_{\mathrm{aff}C}(K\cap\mathrm{aff}F)\subset\mathrm{relint}\,C. \tag{73}$$

We recall that for any $F\in\mathcal{F}_C$, x_F denotes the point of ∂K such that $u(x_F)$ is an exterior unit normal to F, and write $a_F = p_{\mathrm{aff}C}(x_F)$. In addition let $z_F\in\mathrm{aff}F$

satisfy $p_{\mathrm{aff}\,C}(z_F) = a_F$, let $\alpha_F = \langle u(x_C), x_F - z_F\rangle$, and let $\Pi_F = p_{\mathrm{aff}\,C}(F)$. We observe that

$$\tfrac{1}{2}\,Q_{x_C}(y - a_F) \le (1+\varepsilon)\alpha_F \ \text{ if } y \in \Pi_F.$$

We observe that $\alpha_F > 0$ and define $\alpha'_F = (1+\varepsilon)\alpha_F$. It follows from (iii) and (73) that we may apply Lemma 2.1 in K.J. Böröczky, P. Tick, G. Wintsche [7]) to each $C \in \widetilde{\mathcal{M}}$ with $\varepsilon = \mu^2\nu^2$, and we obtain

$$\delta_S(K, P_n^i) \ge (1 - \vartheta_3\mu^2\nu^2) \sum_{C\in\widetilde{\mathcal{M}}} \sum_{F\in\mathcal{F}'_C} \int_{\Pi_F} |\tfrac{1}{2}\,Q_{x_C}(y - a_F) - \alpha'_F|\, dy. \tag{74}$$

For any $F \in \mathcal{F}_C$, $C \in \widetilde{\mathcal{M}}$, we define $k(F)$ to be the number of sides of Π_F, and

$$I(F) = \kappa(x_C)^{1/4}|\Pi_F|. \tag{75}$$

We write $\mathcal{F} = \cup_{C\in\widetilde{\mathcal{M}}}\mathcal{F}'_C$. In addition we write $e(n)$ to denote the number of edges of P_n^i, hence $e(n) \le n$.

Let $C \in \widetilde{\mathcal{M}}$. For any $F \in \mathcal{F}_C$, by assigning a_F and α'_F to Π_F, we obtain a decorated cell complex Σ_C with respect to $\frac{1}{2}\,Q_{x_C}$. We write $\mathcal{V}_C$ to denote the family of vertices v of Σ_C such that the cells of Σ_C cover a neighbourhood of v. For any $v \in \mathcal{V}_C$, let $\mathcal{R}_v$ be the family of all associated schemes R with $v(R) = v$, and let $S_v = \cup\mathcal{R}_v$. We define $d(v)$ to be the degree of v, and

$$I(v) = \kappa(x_C)^{1/4}|S_v|.$$

Finally we write $\mathcal{V} = \cup_{C\in\widetilde{\mathcal{M}}}\mathcal{V}_C$. If follows from (72) that

$$\sum_{v\in\mathcal{V}} I(v) = (1 + O(\mu^2\nu^2)) \int_{\partial K} \kappa(x)^{1/4} dx. \tag{76}$$

Since the sum of the degrees of the vertices of P_n^i is $2e(n)$, we have $\sum_{v\in\mathcal{V}} d(v) \le 2n$. Applying Lemma 3.4 to (74), and then the Cauchy-Schwartz inequality (19) and (76), we deduce (67) by

$$\begin{aligned}
\delta_S(K, P_n^i) &\ge (1 - \vartheta_{13}\mu^2\nu^2)\frac{2}{3} \cdot \sum_{v\in\mathcal{V}} \frac{I(v)^2}{d(v)} \\
&\ge (1 - \vartheta_{13}\mu^2\nu^2)\frac{2}{3} \cdot \frac{\left(\sum_{v\in\mathcal{V}} I(v)\right)^2}{\sum_{v\in\mathcal{V}} d(v)} \\
&\ge (1 - \vartheta_{14}\mu^2\nu^2)\frac{1}{3} \cdot \frac{\int_{\partial K}\kappa(x)^{1/4}dx}{n}.
\end{aligned}$$

To prove (66), we divide $\mathcal{V}$ into $\mathcal{V}_1$ and $\mathcal{V}_2$. For $v \in \mathcal{V}_C$, $C \in \widetilde{\mathcal{M}}$, we put v into $\mathcal{V}_1$ if $d(v) = 4$, $\mathcal{R}_v$ has eight non-degenerate elements, and $|t(R) - \frac{\pi}{4}| \le \nu$ for $R \in \mathcal{R}_v$. Otherwise we put v into $\mathcal{V}_2$. Applying (76) to the upper bound of (69) for $\delta_S(K, P_n^c)$, and Lemma 3.4 and the Cauchy-Schwartz inequality (19) to the lower bound (74) for $\delta_S(K, P_n^c)$, we have

$$\frac{(\sum_{v\in\mathcal{V}} I(v))^2}{3n} \ge \frac{(1-\vartheta_{15}\mu^2\nu^2)2}{3} \cdot \left(\sum_{v\in\mathcal{V}} \frac{I(v)^2}{d(v)}\right) + \vartheta_{16}\nu^2 \sum_{v\in\mathcal{V}_2} \frac{I(v)^2}{d(v)} \tag{77}$$

$$\ge \frac{(1-\vartheta_{15}\mu^2\nu^2)2}{3} \cdot \frac{(\sum_{v\in\mathcal{V}} I(v))^2}{\sum_{v\in\mathcal{V}} d(v)} + \vartheta_{16}\nu^2 \frac{\left(\sum_{v\in\mathcal{V}_2} I(v)\right)^2}{\sum_{v\in\mathcal{V}_2} d(v)}. \tag{78}$$

Since $\sum_{v\in\mathcal{V}} d(v) \le 2n$, (78) yields

$$\sum_{v\in\mathcal{V}_2} I(v) \le \vartheta_{17}\mu \sum_{v\in\mathcal{V}} I(v), \text{ and in turn} \tag{79}$$

$$\sum_{v\in\mathcal{V}_1} I(v) \ge (1-\vartheta_{17}\mu) \sum_{v\in\mathcal{V}} I(v) \ge (1-\vartheta_{18}\mu) \int_{\partial K} \kappa(x)^{1/4} dx. \tag{80}$$

In addition (76) and (77) imply

$$(1-\vartheta_{19}\mu)\tfrac{n}{2} \le \#\mathcal{V}_1 \le \tfrac{e(n)}{2} \le \tfrac{n}{2}. \tag{81}$$

We write $\mathcal{F}_1$ to denote the family of all $F \in \mathcal{F}$ such that all vertices of Π_F come from $\mathcal{V}_1$. It follows that for any $F \in \mathcal{F}_1 \cap \mathcal{F}'_C$, $C \in \widetilde{\mathcal{M}}$, there exists quadrilateral D that is regular with respect to Q_{x_C}, whose centroid is a_F, and

$$(1+\vartheta_{20}\nu)^{-1}(D - a_F) \subset \Pi_F - a_F \subset (1+\vartheta_{20}\nu)(D - a_F).$$

We deduce by (79), (80) and (81) that

$$(1-\vartheta_{21}\mu)\tfrac{n}{2} \le \#\mathcal{F}_1 \le f(n) \le (1+\vartheta_{21}\mu)\tfrac{n}{2}; \tag{82}$$

$$\sum_{F\in\mathcal{F}_1} I(F) = (1+O(\mu)) \int_{\partial K} \kappa(x)^{1/4} dx. \tag{83}$$

Therefore $\frac{\sum_{F\in\mathcal{F}_1} I(F)}{\#\mathcal{F}_1} = (1+O(\mu))\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{n/2}$, and (77) yields

$$\frac{(\sum_{F\in\mathcal{F}_1} I(F))^2}{\#\mathcal{F}_1} \ge (1-\vartheta_{22}\mu) \sum_{F\in\mathcal{F}_1} I(F)^2. \tag{84}$$

Let $g_1(n)$ be the number of $F \in \mathcal{F}_1$ such that $|\frac{\int_{\partial K}\kappa(x)^{1/4}dx}{(n/2)\cdot I(F)} - 1| \le \nu$. It follows from applying (18) to (84) that $g_1(n) \ge (1-\vartheta_{23}\frac{\mu}{\nu^2})\#\mathcal{F}_1 \ge (1-\vartheta_{24}\nu)f(n)$, concluding the proof of (66).

6 Changes in the proof of Theorem 1.1 for P_n and P_n^c

For the construction (analogue of Lemma 4.1) in the cases of P_n and P_n^c, let us sketch the essential changes in the argument.

To approximate the flat part part, we may assume that $rB^3 \subset K \subset RB^3$ for some $R, r > 0$. Given $\varepsilon > 0$, we use the μ_0 of Proposition 4.2, and choose positive $\mu < \min\{\mu_0, \varepsilon^4\}$. We construct the polytopal surface $Z \subset K$ with at most εn vertices such that $\partial K \backslash X(\mu_0) \subset \pi_{\partial K}(Z)$ and $d(x, \partial K) \leq \frac{r\varepsilon}{2Rn}$ for $x \in Z$. We now use $Y' = (1 + \frac{\varepsilon}{Rn})Z$ both for P_n and P_n^c, which for large n avoids int$K$, and satisfies $\partial K \backslash X(\mu) \subset \pi_{\partial K}(Y')$ and $d(x, \partial K) \leq \frac{\varepsilon}{n}$ for $x \in Y'$.

Let us turn to the "round" part. For $x \in \Omega$, now $\mathcal{T}_x$ is a side to side tiling of T_x by triangles that are regular with respect to Q_x, and their common area is $\frac{3\int_{\partial K} \kappa(w)^{1/4} dw}{2n\kappa(x)^{1/4}}$. We define $\psi(y) = \frac{1-\varepsilon^3}{2} Q_x(y-x)$, hence Ξ_x avoids int$K$. For P_n^c, we have $\alpha_x = 0$, thus the faces of Λ_x' touch the paraboloid Ξ_x'. For P_n, we choose α_x in a way such that the area of the ellipse $\{y \in T_x : \frac{1-\varepsilon^3}{2} Q_x(y-x) \leq \alpha_x\}$ is $\frac{1}{2}|\Pi|$ for any tile Π of $\mathcal{T}_x$. The rest of construction goes through with the obvious changes.

The other half of the proof of Theorem 1.1 for P_n^c and P_n is quite different from the argument in Section 5 for P_n^i. Instead now one uses an argument based on Lemmae 3.2 and 3.3 that is very similar to the corresponding argument in Böröczky, P. Tick and G. Wintsche [7].

7 Sketch of the proof of Theorem 1.2

Let K be a convex body with C^2 boundary in $\mathbb{R}^d$. We use the notation set up in Sections 1 and 2. First we review some additional properties of the extremal polytopes with respect to the volume which properties follow from the proof of Theorem 1.1 (compare Lemmae 3.2, 3.3 and 3.4). We write L_n to denote either of the extremal polytopes in Theorem 1.1. To describe the position of the typical face of L_n, let c_F denote the centroid of a convex disc F. There exists positive $\mu(n)$ satisfying $\lim_{n\to\infty} \mu(n) = 0$ such that for all but $\mu(n)$ percent of the vertices v of L_n, if F is a face of L_n then F is proper and satisfies the properties in Theorem 1.1. In addition we have the following properties.

(i) If $L_n = P_n^c$ then v is of degree six, and for any face F of P_n^c containing v, the distance of c_F from K is at most $\mu(n)|F|\kappa(x_F)^{\frac{1}{2}}$, and the distance of the vertices of F from K is at least $\frac{1}{4}|F|\kappa(x_F)^{\frac{1}{2}}$.

(ii) If $L_n = P_n$ then v is of degree six, and for any face F of P_n containing v, we have $\frac{1-\mu(n)}{2}|F| < |F \cap K| < \frac{1+\mu(n)}{2}|F|$.

(iii) If $L_n = P_n^i$ then v is of degree four, and for any face F of P_n^i containing v, the distance of c_F from ∂K is at least $\frac{1}{4}|F|\kappa(x_F)^{\frac{1}{2}}$, and the distance of the vertices of F from ∂K is at most $\mu(n)|F|\kappa(x_F)^{\frac{1}{2}}$.

Following a suggestion of R. Schneider, the paper S. Glasauer and P.M. Gruber [11] introduced an ingenious method to translate a result about polytopal approximation with respect to δ_S into a "dual" result with respect to δ_1, which we now sketch. The paper [11] discussed only the case when $\kappa(x) > 0$ for all $x \in \partial K$ (see also M. Ludwig [22]), but this restriction is not necessary (see K. Böröczky, Jr. [4]). We recall that if M is a convex body with $o \in \mathrm{int}M$ then its dual (or polar) is

$$M^* = \{x \in \mathbb{R}^3 : \langle x, y \rangle \leq 1 \text{ for all } y \in M\}.$$

If N is a convex body with $M \subset N$ then $N^* \subset M^*$. In addition if M is a polytope then faces of M correspond to vertices of M^*. More precisely if v is a vertex M^*, and the affine hulls of the k faces of M^* containing v are of the form $\{x \in \mathbb{R}^3 : \langle x, u_i \rangle = 1\}$ for $i = 1, \ldots, k$ then the vertices of the face of M corresponding to v are $u_1, \ldots, u_k$.

To describe the main idea in S. Glasauer and P.M. Gruber [11], we assume that $o \in \mathrm{int}K$. Let $X \subset \partial K$ be Jordan measurable such that $\kappa(x) > 0$ for $x \in \mathrm{clX}$, and let Q be a polytope with $o \in \mathrm{int}Q$. We define $\Sigma = \{u(x) : x \in X\} \subset S^2$, and

$$Z = \{z \in \partial Q : \exists x \in X \text{ such that } u(x) \text{ is an exterior normal to } \partial Q \text{ at } z\}.$$

Now the parts of ∂K^* and ∂Q^* corresponding to X and Z are $X^* = \{\frac{1}{h_K(u)} u : u \in \Sigma\}$ and $Z^* = \{\frac{1}{h_Q(u)} u : u \in \Sigma\}$, respectively. We define the part of $\mathbb{R}^3$ "between X^* and Z^*" to be

$$\Delta = \left\{ \left(\frac{\lambda}{h_Q(u)} + \frac{1-\lambda}{h_K(u)} \right) u : u \in \Sigma \text{ and } \lambda \in [0,1] \right\}.$$

The main observation in S. Glasauer and P.M. Gruber [11] is that

$$\int_\Sigma |h_Q(u) - h_K(u)|\, du = \int_\Delta \|y\|^{-4} dy. \tag{85}$$

In particular approximation of K with respect to δ_1 translates into "weighted volume approximation" of K^*. We write $\kappa^*(y)$ to denote the Gauß-Kronecker curvature at $y \in \partial K^*$. On a small neighbourhood of $y \in X^*$, the function $\|y\|^{-4}$ is essencially constant, therefore locally we can use the argument developed for Theorem 1.1. In particular say the argument in Section 5 shows that when estimating the "weighted volume" difference, the integrand on X^* contains a factor that is the square root of the weight function $\|y\|^{-4}$. Calculating the Jacobian of the map $\varphi : X \to X^*$ defined by $\varphi(x) = \frac{1}{h_K(u(x))} u(x)$, we deduce

$$\int_{X^*} \kappa^*(y)^{\frac{1}{4}} \|y\|^{-2} dy = \int_X \kappa(x)^{\frac{3}{4}} dx.$$

With the help of these observations, one obtains the asymptotic formulae after Theorem 1.2. To determine the asymptotic shape of the typical faces of W_n^i, W_n and W_n^c, we use the method developed for Theorem 1.1 to describe the typical faces of P_n^c, P_n and P_n^i, respectively, together with the properties (i), (ii) and (ii) above.

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